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CHERN CURVATURE AND Q-CONVEXITY PROPERTIES OF HERMITEAN HOLOMORPHIC VECTOR BUNDLES

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LECTURE NOTES

TABLE OF CONTENTS

I. LINEAR CONNECTIONS ON VECTOR BUNDLES

LINEAR CONNECTIONS ON VECTOR BUNDLES

§1. Let M be a smooth manifold of dimension m and let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be the scalar field.

VECTOR BUNDLE

Def. A (real, complex) *vector bundle* of rank r over M is a $\mathcal{C}^\infty$ manifold E, together with

(i) a $\mathcal{C}^\infty$ projection map $\pi : E \to M$,

(ii) a $\mathbb{K}$-vector space structure of dimension r on each fiber $E_x = \pi^{-1}(x)$; $x \in M$,

(Thus, $E_x \approx \mathbb{K}^r$).

such that the vector space structure is *locally trivial*:

LOCAL TRIV.

$\exists$ $\mathcal{V} = \{V_\alpha\}_{\alpha \in I}$ = covering of M and

$\exists$ $\mathcal{C}^\infty$-diffeomorphisms

$$\boxed{\varphi_\alpha : E|_{V_\alpha} \to V_\alpha \times \mathbb{K}^r} \quad (E|_{V_\alpha} = \pi^{-1}(V_\alpha))$$

such that for every $x \in V_\alpha$,

$$\varphi_{\alpha,x} : E_x \longrightarrow \{x\} \times \mathbb{K}^r = \mathbb{K}^r$$

is a $\mathbb{K}$-linear isomorphism.

Transition maps: $\alpha, \beta \in I$;

$$\boxed{\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : V_{\alpha\beta} \times \mathbb{K}^r \to V_{\alpha\beta} \times \mathbb{K}^r} \quad \boxed{V_{\alpha\beta} = V_\alpha \cap V_\beta}$$

is a linear automorphism on each fiber.

Thus: $\varphi_{\alpha\beta}(x,\xi) = (x, g_{\alpha\beta}(x)\cdot\xi)$

$g_{\alpha\beta} : V_{\alpha\beta} \to GL_n(\mathbb{K})$

$$\begin{tikzcd} & E|V_{\alpha\beta} \arrow[dl, "\varphi_\alpha"'] \arrow[dr, "\varphi_\beta"] & \\ V_{\alpha\beta}\times\mathbb{K}^n \arrow[rr, leftrightarrow] & \circlearrowright & V_{\alpha\beta}\times\mathbb{K}^n \end{tikzcd}$$

$\varphi_{\alpha\beta} = \varphi_\alpha\circ\varphi_\beta^{-1}$.

Cocycle condition:

$$\boxed{g_{\alpha\beta}\, g_{\beta\gamma} = g_{\alpha\gamma} \text{ on } V_{\alpha\beta\gamma}}$$

Conversely: Every 1-cocycle on $\mathcal{V}$ with values in the sheaf of maps $M \to GL_n(\mathbb{K})$ determines a vector bundle on M with these transition maps.

Examples: 1° $E = M\times\mathbb{K}^n$ = the trivial bundle.

2° TM = the tangent bundle of M.

$\tau_\alpha : V_\alpha \to U_\alpha \subset \mathbb{R}^m$ local coordinates and

$\tau_{\alpha\beta} = \tau_\alpha\circ\tau_\beta^{-1}$

$g_{\alpha\beta}(x) = d\tau_{\alpha\beta}(\tau_\beta(x))$

= the differential of the transition map $\tau_{\alpha\beta}$

= the transition maps of TM.

3° T^*M = the cotangent bundle of M

= the dual bundle of TM.

4° operations: $\otimes$, Λ^k, Hom, ...

Sections, frames:

A *section* $s : M \to E$ is a map satisfying $\pi \circ s = \mathrm{id}_M$.

$\mathcal{C}^k(\Omega, E)$ = the space of all $\mathcal{C}^k$-sections of $E|_\Omega$, $\Omega \subset M$.

$\varphi : E|_V \to V \times \mathbb{K}^r$ local trivialization

The corresponding *local frame*:

$$\boxed{e_j(x) = \varphi^{-1}(x; (0, \ldots, \underset{\substack{\uparrow \\ j\text{th spot}}}{1}, \ldots 0)) \; ; \quad 1 \le j \le r.}$$

Then for every section $s : V \to E|_V$:

$$\boxed{s(x) = \sum_{\lambda=1}^{r} \sigma_\lambda(x) \cdot e_\lambda(x) \; ; \quad \sigma_\lambda \in \mathcal{C}^k(V, \mathbb{K}).}$$

Let $\mathcal{V} = \{V_\alpha\}$ covering of M,

$\{\varphi_\alpha\}$ corresponding trivializations of E,

$s : M \to E$ ... a section

$\varphi_\alpha \circ s = \sigma^\alpha = (\sigma_1^\alpha, \ldots, \sigma_r^\alpha)$ local components of s in the φ_α-trivialization

Then:
$$\boxed{\sigma^\alpha = g_{\alpha\beta} \cdot \sigma^\beta \text{ on } V_{\alpha\beta}} \; -$$

This is the change of frame formula for the component vector.

For frames:
$$\boxed{f^\alpha \cdot g_{\alpha\beta} = f^\beta}$$
(reversed order!)

§2. What is a connection on a vector bundle?

A rule on how to differentiate sections $s : M \to E$ along vector fields $\xi \in \mathfrak{X}(M) = \Gamma(M, TM)$ such that the result $D_\xi \cdot s$ is again a section of E.

main example: $E = M \times \mathbb{R}^n$ (or $M \times \mathbb{C}^n$)

$$s : M \to E$$

$$s(x) = (x, \sigma(x)); \quad \sigma : M \to \mathbb{R}^n$$

vector valued function

If ξ is a vector field on M, can define

$$(D_\xi \cdot s)(x) = d\sigma(x) \cdot \xi \quad \text{(the differential of } \sigma \text{ applied to } \xi\text{)}$$

$$= \nabla_{\xi(x)} \sigma \quad \text{(= the directional derivative of } \sigma \text{ in the direction } \xi(x)\text{)}.$$

$$= \sum \xi_j \frac{\partial \sigma}{\partial x_j}$$

The main problem: if the bundle E is nontrivial, the above procedure is not globally well defined since we must also differentiate the transition functions:

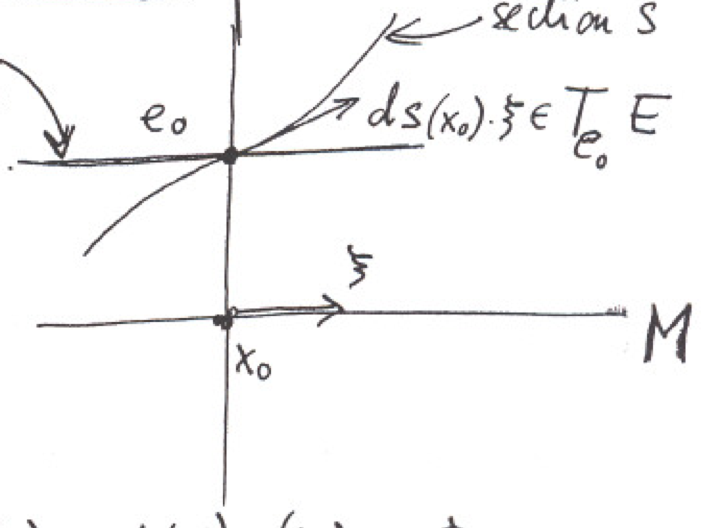
$$\sigma^\alpha = g_{\alpha\beta} \cdot \sigma^\beta$$

$$d\sigma^\alpha = \underset{\substack{\uparrow \\ \text{extra term!}}}{dg_{\alpha\beta}} \cdot \sigma^\beta + g_{\alpha\beta} \cdot d\sigma^\beta$$

OK if $g_{\alpha\beta}$ is (locally) constant; such E is said to be a flat vector bundle.

<u>A solution</u>: (<u>Ehresmann connection</u>) $E_{x_0} = \pi^{-1}(x_0)$

horizontal $\to H_{e_0}$



$s: M \to E$ section

$x_0 \in M$, $e_0 = s(x_0) \in E_{x_0}$.

$\xi \in T_{x_0}M$.

$v = ds_{x_0}(\xi) \in T_{e_0}E$

$$d\pi \circ ds_{x_0}(\xi) = d(\pi \circ s)_{x_0}(\xi) = d(id)_{x_0}(\xi) = \xi.$$

<u>Therefore</u>, the vector v projects under π back to ξ: $d\pi \circ ds = id$.

<u>Suppose</u> that $H_{e_0} \subset T_{e_0}E$ is a linear subspace such that

$$d\pi_{e_0} : H_{e_0} \xrightarrow{\approx} T_{x_0}M$$

is an isomorphism. Call H_{e_0} a <u>horizontal space</u>.

We also have a well-defined <u>vertical tangent space</u>:

$$VT_{e_0}E = T_{e_0}(E_{x_0}) = \ker d\pi_{e_0} \approx E_{x_0}.$$

<u>Then</u>: $\boxed{T_{e_0}E = H_{e_0} \oplus VT_{e_0}E}$ = horizontal $\oplus$ vertical

Let

$$\underset{H_{e_0}}{\underset{\cap}{v}} = v^h \oplus \underset{VT_{e_0}E \approx E_{x_0}}{\underset{\cap}{w}}$$

w = the vertical component of v.

Assume that we have a "horizontal subbundle"

$H \subset TE$ such that

$$\begin{cases} TE = VT(E) \oplus H; \\ H|_M = TM \qquad (M = \text{the zero section of } E) \end{cases}$$

Let $\tau : TE \to VT(E)$ denote the projection with $\ker\tau = H$.

Define: $D_\xi s = \tau \circ ds(\xi)$

= the vertical component of $ds(\xi)$.

We can identify $D_\xi s$ with a section of E by taking the pull-back to M by s:

$$(D_\xi s)(x) = \tau_{s(x)}\left(ds_x(\xi)\right) \in VT_{s(x)} E_x \approx E_x .$$

We call D covariant derivative associated to the horizontal bundle $H \subset TE$.

$$\boxed{D : \mathcal{C}^\infty(M,E) \to \mathcal{C}_1^\infty(M,E) = \mathcal{C}^\infty(M, T^*M \otimes E)}$$

1-forms with values in E

$\forall \xi \in \mathfrak{X}(M)$: $D_\xi : \mathcal{C}^\infty(M,E) \to \mathcal{C}^\infty(M,E)$.

<u>Definition</u>: A <u>linear connection</u>, D, on a vector bundle $E \xrightarrow{\pi} M$ is a linear differential operator

$$D: \mathcal{C}^\infty(M,E) \longrightarrow \mathcal{C}^\infty_1(M,E) = \mathcal{C}^\infty(M, \underbrace{T^*M \otimes E)}_{\substack{\text{differential 1-forms} \\ \text{with values in } E}}$$

satisfying the Leibnitz rule:

$$\boxed{D(f \cdot s) = df \cdot s + f \cdot Ds}, \quad \left\{ \begin{array}{l} f \in \mathcal{C}^\infty(M) \\ s \in \mathcal{C}^\infty(M,E) \end{array} \right\}$$

Generalization (extension) of D to E-valued forms:

$$\left\{ \begin{array}{l} D: \mathcal{C}^\infty_q(M,E) \longrightarrow \mathcal{C}^\infty_{q+1}(M,E) \\ \\ D(f \wedge s) = df \wedge s + (-1)^{\deg f} f \wedge Ds \end{array} \right.$$

$\forall f \in \mathcal{C}^\infty_p(M, \mathbb{K})$ = diff. p-forms with scalar coeff.
$\forall s \in \mathcal{C}^\infty_q(M,E)$ = q-forms with values in E.

<u>In</u> <u>local frame</u> $f = (e_1, \dots, e_r)$

$$\boxed{De_\mu = \sum_{\lambda=1}^{r} \theta_{\lambda\mu} \otimes e_\lambda \quad ; \quad 1 \le \mu \le r}$$

where $\theta_{\lambda\mu}$ are 1-forms with scaler coefficients.

Any section s is written locally in this frame as

$$\boxed{s = \sum_{\lambda=1}^{r} \sigma_\lambda \cdot e_\lambda}$$

The Leibnitz rule gives:

$$
\begin{aligned}
Ds &= D\Big(\sum_{\lambda=1}^{n} \sigma_\lambda e_\lambda\Big) \\
&= \sum_{\lambda=1}^{n} d\sigma_\lambda \otimes e_\lambda + \sum_{\mu=1}^{n} \sigma_\mu De_\mu \\
&= \sum_{\lambda} d\sigma_\lambda \otimes e_\lambda + \sum_{\lambda,\mu} \sigma_\mu \cdot \theta_{\lambda\mu} \otimes e_\lambda \\
&= \sum_{\lambda=1}^{n} \Big(d\sigma_\lambda + \sum_\mu \theta_{\lambda\mu}\sigma_\mu \Big) \otimes e_\lambda
\end{aligned}
$$

Write $\theta = (\theta_{\lambda\mu})_{\lambda,\mu=1}^{n}$ = matrix-valued 1-form
= <u>the</u> <u>connection</u> <u>form</u>

Thus in the given frame we have

$$\boxed{Ds \simeq d\sigma + \theta\sigma \quad ; \quad s = f\cdot\sigma = (e_1, \dots, e_r)\begin{pmatrix}\sigma_1 \\ \vdots \\ \sigma_n\end{pmatrix}}$$

In general, when σ is a vector-valued form, we have

$$\boxed{Ds \simeq d\sigma + \theta\wedge\sigma} \quad \boxed{\theta = \text{connection form}}$$

<u>Conversely</u>: every operator D, given locally in this form, is a covariant derivative (i.e., a connection).

Change of gauge formula:

Suppose we have two local frames,

$$f = (e_1, \dots, e_n) \quad , \quad \tilde{f} = (\tilde{e}_1, \dots, \tilde{e}_n).$$

Then $s = \sum \sigma_\lambda e_\lambda = \sum \tilde{\sigma}_\lambda \cdot \tilde{e}_\lambda$ $(s = f \cdot \sigma = \tilde{f} \cdot \tilde{\sigma})$.

Let $g = (g_{\lambda\mu})$ be the transition matrix:

$$\tilde{\sigma} = g \cdot \sigma \quad (\Leftrightarrow f = \tilde{f} \cdot g).$$

Let $\tilde{\theta}$ denote the connection matrix in frame $\tilde{f}$:

$$Ds \underset{\tilde{f}}{\simeq} d\tilde{\sigma} + \tilde{\theta} \cdot \tilde{\sigma}$$

$$Ds \underset{f}{\simeq} g^{-1}(d\tilde{\sigma} + \tilde{\theta} \cdot \tilde{\sigma}) \quad \text{(using change of frame formula)}$$

$$= g^{-1}(d(g\sigma) + \tilde{\theta} \cdot g\sigma)$$

$$= g^{-1}(dg \cdot \sigma + g \cdot d\sigma + \tilde{\theta} g \sigma)$$

$$= d\sigma + (g^{-1}dg + g^{-1}\tilde{\theta} g) \cdot \sigma$$

Comparing with

$$Ds \underset{f}{\simeq} d\sigma + \theta \cdot \sigma$$

we conclude

$$\boxed{\theta = g^{-1}\tilde{\theta} g + g^{-1} dg} \quad \text{gauge transformation law}$$

$\underline{n=1}$:

$$\boxed{\theta = \tilde{\theta} + g^{-1}dg = \tilde{\theta} + d(\log g)}$$

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Change of gauge for the connection

(1°) $\tilde{f} = f \cdot g^{-1}$; $\tilde{\sigma} = g\sigma$ $(\tilde{f}\tilde{\sigma} = fg)$

$$Ds \underset{\text{def}}{=} d\sigma + \theta \cdot \sigma$$

$$\underset{\text{gauge}}{=} d\tilde{\sigma} + \tilde{\theta}\tilde{\sigma}$$

$$d\tilde{\sigma} + \tilde{\theta}\tilde{\sigma} = g \cdot (d\sigma + \theta\sigma)$$

$$d(g\sigma) + \tilde{\theta} \cdot g\sigma = g \cdot d\sigma + g \cdot \theta\sigma$$

$$dg \cdot \sigma + g d\sigma + \tilde{\theta} \cdot g\sigma = g d\sigma + g\theta \cdot \sigma$$

$$dg + \tilde{\theta} \cdot g = g\theta$$

$$\boxed{\theta = g^{-1}\tilde{\theta} g + g^{-1} dg.}$$

(2°) If $\tilde{f} = f \cdot g$, $\tilde{\sigma} = g^{-1}\sigma$, get

$$\theta = g\tilde{\theta}g^{-1} + g \cdot dg^{-1}$$

$$= g\tilde{\theta}g^{-1} + g(-g^{-1}dg \cdot g^{-1}) = g\tilde{\theta}g^{-1} - dg \cdot g^{-1}$$

$$g^{-1}\theta g = \tilde{\theta} - g^{-1}dg$$

$$\boxed{\tilde{\theta} = g^{-1}\theta g + g^{-1}dg}$$

§3 The curvature tensor

$$\mathcal{C}^\infty(M,E) \xrightarrow{D} \mathcal{C}_1^\infty(M,E) \xrightarrow{D} \mathcal{C}_2^\infty(M,E) \xrightarrow{D} \cdots$$

$$D^2 = D\circ D$$

$$\begin{aligned} D^2 s &\underset{\varepsilon}{\simeq} d(d\sigma + \theta\sigma) + \theta\wedge(d\sigma + \theta\sigma) \\ &= (d^2\sigma + d\theta\cdot\sigma - \theta\wedge d\sigma) + (\theta\wedge d\sigma + \theta\wedge\theta\cdot\sigma) \\ &= (d\theta + \theta\wedge\theta)\cdot\sigma \end{aligned}$$

We use that $d^2\sigma = 0$ and that two terms cancel out.

Definition: $\Theta = \Theta(D) \in \mathcal{C}_2^\infty(M, \mathrm{Hom}(E,E))$

$$\Theta(D) \underset{\varepsilon}{\simeq} d\theta + \theta\wedge\theta \quad (\text{in frame } \varepsilon)$$

This 2-form with values in $\mathrm{Hom}(E,E) = E^*\otimes E$ is called the curvature form associated to D.

Special case: $r = 1$, line bundle:

Now $\theta\wedge\theta = 0$, so

LINE BUNDLE

$$\boxed{\Theta \simeq d\theta}$$

Since we have $\theta = \tilde{\theta} + d(\log g)$ under change of frame, it follows that

$$\boxed{\Theta = d\theta = d\tilde{\theta}}$$

is a well-defined scalar-valued 1-form on M.

Indeed, $\mathrm{Hom}(E,E) \approx M\times\mathbb{K}$ when $r=1$ (homotheties).

Change of gauge formula for curvature

Let (θ, Θ) and $(\tilde{\theta}, \tilde{\Theta})$ be the connection/curvature with respect to a pair of frames f, $\tilde{f}$.

Let g be the change of frame matrix:

$$\tilde{f} = f \cdot g \,; \quad \tilde{\sigma} = g \cdot \sigma$$

where $s = f \cdot \sigma = \tilde{f} \cdot \tilde{\sigma}$.

Then: $\begin{cases} D^2 s \underset{f}{\simeq} \Theta \sigma \\ D^2 s \underset{\tilde{f}}{\simeq} \tilde{\Theta} \cdot \tilde{\sigma} \end{cases}$ and

From $\quad \tilde{\Theta}\tilde{\sigma} = g \cdot \Theta \sigma \quad$ and $\quad \tilde{\sigma} = g\sigma$

we get:

$$\tilde{\Theta} \cdot g \sigma = g \, \Theta \, \sigma \,;$$

$$\boxed{\Theta = g^{-1} \cdot \tilde{\Theta} \cdot g}$$

Direct calculation: see the following page.

Remark: The above calculation shows that

$$\Theta = g^{-1} \tilde{\Theta} g$$

is the change-of-frame formula provided that Θ is indeed a 2-form with values in $\mathrm{Hom}(E, E)$. This is justified by the direct calculation on the following page.

If $n = 1$, multiplication commutes and we get $\Theta = \tilde{\Theta}$.

Change of gauge formula for curvature.

$$\theta = g^{-1}\tilde{\theta}g + g^{-1}dg$$

$$d\theta = -g^{-1}dg\cdot g^{-1}\wedge\tilde{\theta}g + g^{-1}d\tilde{\theta}\wedge g - g^{-1}\tilde{\theta}\wedge dg$$
$$- g^{-1}dg\cdot g^{-1}\wedge dg$$

$$\cancel{\Theta + \theta\wedge\theta = d\theta + (g^{-1}\tilde{\theta}g)\wedge(g^{-1}\tilde{\theta}g) +}$$

$$\theta\wedge\theta = (g^{-1}\tilde{\theta}g + g^{-1}dg)\wedge(g^{-1}\tilde{\theta}g + g^{-1}dg)$$
$$= g^{-1}\tilde{\theta}\wedge\tilde{\theta}g + g^{-1}dg\wedge g^{-1}dg$$
$$+ g^{-1}dg\, g^{-1}\wedge\tilde{\theta}g + g^{-1}\tilde{\theta}\wedge dg$$

$\left\{\begin{array}{l} (1^{st} \text{ term in } d\theta) + (3^{rd} \text{ term in } \theta\wedge\theta) \text{ cancel,} \\ (3^{rd} \text{ term in } d\theta) + (4^{th} \text{ term in } \theta\wedge\theta) \text{ cancel,} \\ (4^{th} \text{ term in } d\theta) + (2^{nd} \text{ term in } \theta\wedge\theta) \text{ cancel.} \end{array}\right\}$

$$\Theta = d\theta + \theta\wedge\theta = g^{-1}d\tilde{\theta}\wedge g + g^{-1}\tilde{\theta}\wedge\tilde{\theta}\cdot g$$

$$= g^{-1}\left(d\tilde{\theta} + \tilde{\theta}\wedge\tilde{\theta}\right)\cdot g$$

$$\boxed{\Theta = g^{-1}\cdot\tilde{\Theta}\cdot g}$$

Change of gauge for the curvature tensor

$\underline{\underline{r=1}}$: $\left\{\begin{array}{l} \theta = \tilde{\theta} + d(\log g) \\ \Theta = d\theta = d\tilde{\theta} = \tilde{\Theta} \end{array}\right.$ ... global scalar-valued $\underline{\text{closed}}$ 2-form on M.

§4. Parallel transport and horizontal vector fields

Recall: $f = (e_1, \dots, e_n)$ local frame

$$s = f \cdot \sigma = \sum_j e_j \sigma_j \quad \text{local section}$$

$$Ds \underset{f}{\simeq} d\sigma + \theta \cdot \sigma$$

$$(Ds)_j = d\sigma_j + \sum_k \theta_j^k \cdot \sigma_k$$

$\theta = (\theta_j^k)$ $\quad j = $ row, $k =$ column.

In local coordinates $x = (x^1, .., x^m)$:

$$\theta_j^k = \sum_i \Gamma_{ij}^k \, dx^i \; ; \quad \Gamma_{ij}^k = \text{Schwarz-Christoffel}$$

$$\boxed{\left(D_{\frac{\partial}{\partial x_i}} s\right)_j = \frac{\partial \sigma_j}{\partial x_i} + \sum_{k=1}^{n} \Gamma_{ij}^k \cdot \sigma_k} \quad \text{connection in components.}$$

Given $e^\circ \underset{f}{=} \sigma^\circ \in E_{x_0}$ $\quad (x_0 = \pi(e^\circ))$,
the Horizontal space H_{e° is the tangent space to any section with

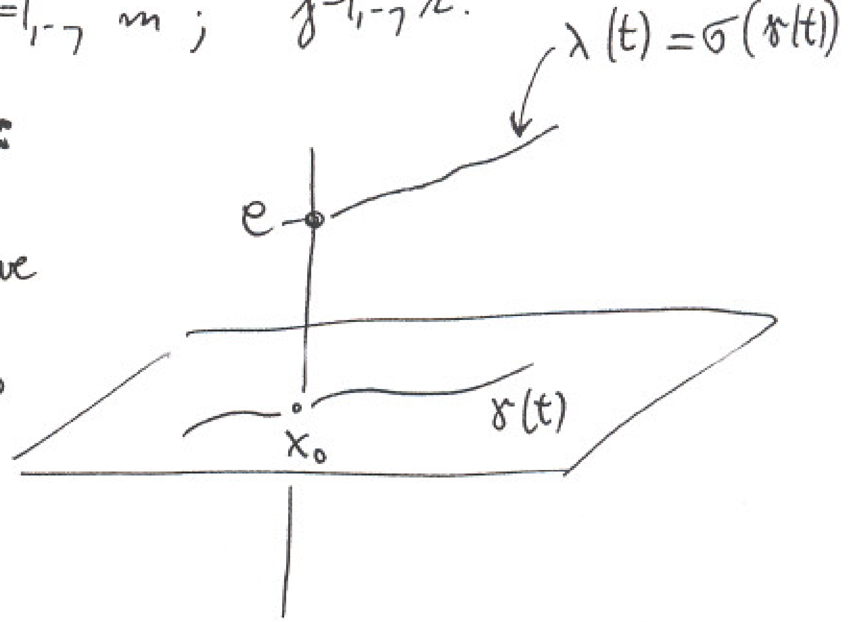
$$\begin{cases} \sigma(x_0) = \sigma^\circ \\ D\sigma\big|_{x_0} = d\sigma\big|_{x_0} + \theta(x_0) \cdot \sigma^\circ = 0. \end{cases}$$

<u>In components</u>:

$$\frac{\partial \sigma_j}{\partial x^i}(x_0) = -\sum_{k=1}^{n} \Gamma_{ij}^{k}(x_0)\cdot \sigma_k^0$$

$$i = 1, \ldots, m; \quad j = 1, \ldots, n.$$

<u>Parallel</u> <u>transport</u>:



Given a smooth curve $\gamma(t)$ in M, $\gamma(0) = x_0$

and a point
$e \in E_{x_0} = \pi^{-1}(x_0)$,

there exists a unique function $\lambda(t) \in \mathbb{R}^n$ such that the tangent vector

$$(\dot\gamma(t), \dot\lambda(t)) \in H_{(\gamma(t), \lambda(t))}$$

is <u>horizontal</u> for every t.

To get such λ, write it locally as $\lambda(t) = \sigma(\gamma(t))$; then the condition is

$$D_{\dot\gamma(t)}\sigma = d\sigma_{\gamma(t)}(\dot\gamma(t)) + \theta(\gamma(t), \dot\gamma(t))\cdot\sigma(\gamma(t)) = 0$$

or:
$$\begin{cases} \dot\lambda(t) + \theta(\gamma(t), \dot\gamma(t))\cdot\lambda(t) = 0. \\ \lambda(0) = \sigma^0 = \text{the component of } e \end{cases}$$

This system of linear ODE's has a unique solution.

Horizontal vector fields

For every vector field $\xi = \sum \xi_i \frac{\partial}{\partial x^i}$ on M there exists a unique horizontal lifting $\tilde{\xi}$, i.e., a vector field on E that is tangential to the horizontal distribution $H \subset TE$.

Let us calculate explicitly the horizontal liftings η_i of vector fields $\frac{\partial}{\partial x^i}$, $i = 1, \dots, m$.

Write $\theta = \sum_{i=1}^{m} \Gamma_i \, dx^i$; $\Gamma_i = \left(\Gamma_{ij}^{k}\right)_{j,k=1,\dots,n}$.

Then:
$$\boxed{D_{\frac{\partial}{\partial x^i}} \sigma = \frac{\partial \sigma}{\partial x^i} + \Gamma_i \cdot \sigma}$$

The section σ is tangent to η_i precisely when it is horizontal, which means that the above expression vanishes:

$$\boxed{\frac{\partial \sigma_j}{\partial x^i} + \sum_k \Gamma_{ij}^{k} \cdot \sigma_k = 0; \quad j = 1, \dots, n.}$$

Hence the j-th vertical component of η_i must equal
$-\sum_k \Gamma_{ij}^{k} \sigma_k$.

<u>Therefore</u>: the horizontal lifting of $\partial/\partial x^i$ equals

$$\boxed{\eta_i = \frac{\partial}{\partial x^i} - \sum_{j,k=1}^{n} \Gamma^k_{ij} \cdot \sigma_k \cdot \frac{\partial}{\partial \sigma_j}.}$$

(Non)integrability of the horizontal distribution $H \subset TE$ is measured by the commutators $[\eta_i, \eta_j]$; H is integrable iff $[\eta_i, \eta_j] = 0$ for all $i, j = 1, \ldots, m$.

Let us write in vector notation

$$\eta_i = \frac{\partial}{\partial x_i} - (\Gamma_i \cdot \sigma) \cdot \frac{\partial}{\partial \sigma}.$$

<u>Observe</u>: $$(d\pi)[\eta_i, \eta_j] = [\pi_* \eta_i, \pi_* \eta_j] = \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0.$$

Therefore, the commutators $[\eta_i, \eta_j]$ are <u>vertical fields</u> (tangential to the fibers of E).

We calculate them explicitly:

$$[\eta_\lambda, \eta_\mu] = \left[\frac{\partial}{\partial x^\lambda} - \sum_{j,k} \Gamma^k_{\lambda j}\sigma_k \cdot \frac{\partial}{\partial \sigma_j}, \frac{\partial}{\partial x^\mu} - \sum_{l,s} \Gamma^s_{\mu l}\sigma_s \cdot \frac{\partial}{\partial \sigma_l}\right]$$

$$= -\sum_{l,s} \frac{\partial \Gamma^s_{\mu l}}{\partial x^\lambda} \cdot \sigma_s \cdot \frac{\partial}{\partial \sigma_l} + \sum_{j,k} \frac{\partial \Gamma^k_{\lambda j}}{\partial x^\mu} \cdot \sigma_k \cdot \frac{\partial}{\partial \sigma_j}$$

$$+ \sum_{j,k,l} \Gamma^k_{\lambda j} \cdot \Gamma^j_{\mu l} \cdot \sigma_k \cdot \frac{\partial}{\partial \sigma_l}$$

$$- \sum_{j,k,s} \Gamma^s_{\mu l} \, \Gamma^l_{\lambda j} \cdot \sigma_s \cdot \frac{\partial}{\partial \sigma_j}$$

$$= \sum_{j,k}\left(\frac{\partial \Gamma^k_{\lambda j}}{\partial x^\mu} - \frac{\partial \Gamma^k_{\mu j}}{\partial x^\lambda} + \sum_l \left(\Gamma^k_{\lambda l}\,\Gamma^l_{\mu j} - \Gamma^k_{\mu l}\,\Gamma^l_{\lambda j}\right)\right)\sigma_k \cdot \frac{\partial}{\partial \sigma_j}$$

Replacements of indices made ~~in~~ the above calculations

$$\left\{\begin{array}{l} 1^{st} \text{ term}: \; l \to j, \; s \to k \\ 2^{nd} \text{ term}: \text{ none.} \\ 3^{rd} \text{ term}: \; l \to j, \; j \to l. \\ 4^{th} \text{ term}: \; s \to k \end{array}\right\}$$

We see that $[\eta_\lambda, \eta_\mu]$ is a ~~v~~ vertical vector field that is <u>linear</u> in $\sigma = (\sigma_k)$.

Introducing the matrix notation

$$\Gamma_\lambda = \left(\Gamma^k_{\lambda j} \right),$$

we have that the coefficients of $[\eta_\lambda, \eta_\mu]$ are given by

$$\underbrace{\left(\frac{\partial \Gamma_\lambda}{\partial x^\mu} - \frac{\partial \Gamma_\mu}{\partial x^\lambda} + \left(\Gamma_\mu \cdot \Gamma_\lambda - \Gamma_\lambda \Gamma_\mu \right) \right)}_{\text{the matrix of } [\eta_\lambda, \eta_\mu].} \cdot \sigma$$

$$= \left(\frac{\partial \Gamma_\lambda}{\partial x^\mu} - \frac{\partial \Gamma_\mu}{\partial x^\lambda} + [\Gamma_\mu, \Gamma_\lambda] \right) \cdot \sigma$$

This is a scalar $n \times n$ matrix depending on $x \in M$.

Now compare this with the curvature matrix

$$\left\{ \begin{array}{l} \Theta = d\theta + \theta \wedge \theta \\ \\ \theta = \sum\limits_\lambda \Gamma_\lambda \cdot dx^\lambda \end{array} \right\}$$

$$d\theta = \sum_{\lambda,\mu} \frac{\partial \Gamma_\lambda}{\partial x^\mu} dx^\mu \wedge dx^\lambda$$

$$\textcircled{H} = \sum_{\lambda,\mu} \frac{\partial \Gamma_\lambda}{\partial x^\mu} dx^\mu \wedge dx^\lambda + \left(\sum \Gamma_\lambda dx^\lambda\right) \wedge \left(\sum \Gamma_\mu dx^\mu\right)$$

$$= \sum_{\lambda<\mu} \left(\frac{\partial \Gamma_\mu}{\partial x^\lambda} - \frac{\partial \Gamma_\lambda}{\partial x^\mu}\right) dx^\lambda \wedge dx^\mu$$

$$+ \sum_{\lambda<\mu} \left(\Gamma_\lambda \Gamma_\mu - \Gamma_\mu \Gamma_\lambda\right) dx^\lambda \wedge dx^\mu$$

$$-\textcircled{H} = \sum_{\lambda<\mu} \underbrace{\left(\frac{\partial \Gamma_\lambda}{\partial x^\mu} - \frac{\partial \Gamma_\mu}{\partial x^\lambda} + [\Gamma_\mu, \Gamma_\lambda]\right)}_{\text{the matrix of } [\eta_\lambda,\, \eta_\mu].} \cdot dx^\lambda \wedge dx^\mu$$

Conclusion:

$$\boxed{\left\langle \textcircled{H}, \frac{\partial}{\partial x^\lambda} \wedge \frac{\partial}{\partial x^\mu} \right\rangle = \text{the matrix of } [\eta_\mu, \eta_\lambda].}$$

COROLARRY. The following are equivalent:

(i) $\textcircled{H} \equiv 0$

(ii) $[\eta_\lambda, \eta_\mu] = 0$ (horizontal fields commute)

(iii) H is integrable (Frobenius)

(iv) There exist local fiber charts in which $D = d$ = the exterior der.; $\theta \equiv 0$.

We now express the curvature in terms of commutators of vector fields and of covariant derivatives.

<u>Def</u>. Given a connection D on $E \to M$ and a vector field $\xi \in \mathfrak{X}(M)$, we let

$$D_\xi : \mathcal{C}^\infty(M,E) \to \mathcal{C}^\infty(M,E)$$

denote the covariant derivative in direction ξ.

<u>Expression</u> <u>in</u> <u>a</u> <u>local</u> <u>frame</u> $f = (e_1, \dots, e_r)$:

$$s \underset{f}{\simeq} \sigma$$

$$D_\xi \cdot s = Ds \cdot \xi \underset{f}{\simeq} d\sigma(\xi) + \theta(\xi)\,\sigma$$

Here, θ is the connection matrix in frame f.

<u>PROPOSITION</u>. Let $\Theta = \Theta(D)$ denote the curvature tensor of the connection D. For every pair of vector fields $\xi, \eta \in \mathfrak{X}(M)$ we have

$$\boxed{\Theta(\xi, \eta) = [D_\xi, D_\eta] - D_{[\xi,\eta]}} \qquad \square$$

<u>Note</u>: this formula agrees with

<u>Demailly</u>: <u>Complex Analytic Geometry</u>, Proposition 5.3.6. (p. 5)

or: <u>Andrié</u>, <u>Lafontaine</u>: Holomorphic Curves in Symplectic Geom. Chapter III, Def. 2.3.1. on p. 90.

However, in Do Carmo: Differential Geometry, p. 89, we find a formula with the opposite sign!

Proof. Let $x = (x_1, \dots, x_n)$ be local coordinates on M at a point $z_0 \in M$.

Choose a frame such that $\theta|_{z_0} = 0$.

(Such frame exists: from $\theta = g^{-1}\tilde{\theta} g + g^{-1} dg$ it suffices to choose g such that

$$g(z_0) = Id, \quad dg(z_0) = -\tilde{\theta}|_{z_0};$$

then $\theta|_{z_0} = \tilde{\theta}|_{z_0} - \tilde{\theta}|_{z_0} = 0$.)

Let:

$$\begin{cases} \theta = \sum_{j=1}^{n} \theta_j \, dx_j \\ \xi = \sum \xi_j \frac{\partial}{\partial x_j} \; ; \; \eta = \sum \eta_k \cdot \frac{\partial}{\partial x_k} \\ s \simeq_{\ell} \sigma \end{cases}$$

Then: $D_\xi s \simeq_\ell (d\sigma + \theta \cdot \sigma) \cdot \xi$

$$= d\sigma \cdot \xi + \theta(\xi) \cdot \sigma$$

We now calculate the second covariant derivatives at the point z_0, taking into account $\theta|_{z_0} = 0$:

In components:

$$D_\xi s \simeq_\ell \sum_j \left(\xi_j \frac{\partial \sigma}{\partial x_j} + \xi_j \theta_j \sigma \right) = \sum_j \xi_j \left(\frac{\partial \sigma}{\partial x_j} + \theta_j \sigma \right)$$

$$D_\eta \cdot D_\xi \cdot s \simeq_\S \sum_k \eta_k \frac{\partial}{\partial x_k} \cdot \sum_j \xi_j \left(\frac{\partial \sigma}{\partial x_j} + \theta_j \cdot \sigma \right)$$

$$= \sum_{j,k} \eta_k \left[\frac{\partial \xi_j}{\partial x_k} \left(\frac{\partial \sigma}{\partial x_j} + \theta_j \sigma \right) + \xi_j \frac{\partial^2 \sigma}{\partial x_j \partial x_k} + \xi_j \frac{\partial \theta_j}{\partial x_k} \sigma + \xi_j \theta_j \frac{\partial \sigma}{\partial x_k} \right]$$

$$(\text{at } z_0) = \sum_{j,k} \left(\eta_k \frac{\partial \xi_j}{\partial x_k} \cdot \frac{\partial \sigma}{\partial x_j} + \eta_k \xi_j \frac{\partial^2 \sigma}{\partial x_j \partial x_k} + \eta_k \xi_j \frac{\partial \theta_j}{\partial x_k} \cdot \sigma \right)$$

$$D_\xi D_\eta s - D_\eta \cdot D_\xi s = [D_\xi, D_\eta] \cdot s$$

$$\underset{\text{at } z_0}{\simeq} \sum_{j,k} \left(\xi_k \frac{\partial \eta_j}{\partial x_k} - \eta_k \frac{\partial \xi_j}{\partial x_k} \right) \cdot \frac{\partial \sigma}{\partial x_j} + \sum_{j,k} \frac{\partial \theta_j}{\partial x_k} \left(\xi_k \eta_j - \xi_j \eta_k \right) \cdot \sigma$$

$$= d\sigma \cdot [\xi, \eta] + d\theta\, (\xi, \eta) \cdot \sigma$$

<u>Indeed:</u> $[\xi, \eta] = \left[\sum_k \xi_k \frac{\partial}{\partial x_k}, \sum_j \eta_j \frac{\partial}{\partial x_j} \right]$ (commutator)

$$= \sum_{j,k} \xi_k \frac{\partial \eta_j}{\partial x_k} \cdot \frac{\partial}{\partial x_j} - \eta_j \frac{\partial \xi_k}{\partial x_j} \frac{\partial}{\partial x_k}$$

$$= \sum_{j,k} \left(\xi_k \frac{\partial \eta_j}{\partial x_k} - \eta_k \frac{\partial \xi_j}{\partial x_k} \right) \cdot \frac{\partial}{\partial x_j}$$

$$d\theta = \sum_{j,k} \frac{\partial \theta_j}{\partial x_k} dx_k \wedge dx_j$$

$$d\theta\, (\xi, \eta) = \sum_{j,k} \frac{\partial \theta_j}{\partial x_k} (dx_k \wedge dx_j)(\xi, \eta)$$

$$= \sum \frac{\partial \theta_j}{\partial x_k} \cdot \left(\xi_k \eta_j - \xi_j \cdot \eta_k \right)$$

<u>Note</u>: $\begin{cases} d\sigma \cdot [\xi, \eta] \simeq D_{[\xi,\eta]} \cdot s & (\text{since } \theta|_{z_0} = 0) \\ d\theta|_{z_0} = \Theta|_{z_0} & (\text{since } \theta|_{z_0} = 0). \end{cases}$ ∎

II. HERMITEAN VECTOR BUNDLES AND CHERN CONNECTION

§1. Hermitian metrics, ~~Chern connection & curvature~~

$f = (e_1, \ldots, e_r)$ local frame for $\begin{array}{c} \mathbb{C}^r \hookrightarrow E \\ \downarrow \pi \\ M \end{array}$

h hermitian metric on E =

= a field of h. metrics on fibers E_x, $x \in M$

The matrix of h in frame $f = (e_1, \ldots, e_r)$:

$$\boxed{h(f)_{\rho,\sigma} = \langle e_\sigma, e_\rho \rangle_h \, ; \qquad \rho, \sigma = 1, \ldots, r}$$

$h(f)$: Hermitian matrix valued functions on a chart in M.

$$\xi = \sum_{\sigma=1}^{r} \xi^\sigma e_\sigma \, ; \quad \eta = \sum_{\rho=1}^{r} \eta^\rho e_\rho \qquad \text{sections of } E$$

$$\langle \xi, \eta \rangle_h = \sum \xi^\sigma \cdot \overline{\eta^\rho} \cdot \langle e_\sigma, e_\rho \rangle$$

$$= \sum_{\rho,\sigma=1,\ldots,r} \overline{\eta^\rho} \cdot h_{\rho\sigma} \cdot \xi^\sigma$$

$$\boxed{\langle \xi, \eta \rangle = {}^t\overline{\eta} \cdot h \cdot \xi}$$

${}^t\overline{h} = h$ (hermitian matrix).

CHANGE OF FRAME:

$\mathfrak{f} = (e_1, \ldots, e_r)$ ... frame

$\xi = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^r \end{pmatrix} \in \mathbb{C}^r$ ... components in this frame.

$e = \mathfrak{f} \cdot \xi = \sum_{j=1}^{r} \xi^j e_j$.

Metric : $h(\mathfrak{f})_{(\xi)} = {}^t\bar{\xi} \cdot h(\mathfrak{f}) \cdot \xi$; $h(\mathfrak{f}) = (h_{\rho\sigma})$

Let g be a change of frame (hol. fn. in $GL(r, \mathbb{C})$).

$$\boxed{(\mathfrak{f} \cdot g)\,\xi = \mathfrak{f} \cdot (g\,\xi)}$$

$\Longrightarrow$ $\begin{cases} \text{if } \xi \text{ is the component vector of } e \text{ in frame } \mathfrak{f}\cdot g \\ \text{then } g\xi \text{ is the comp. vector in frame } \mathfrak{f}. \end{cases}$

$$\therefore\ {}^t\bar{\xi} \cdot h(\mathfrak{f} \cdot g) \cdot \xi = {}^t\overline{g \cdot \xi} \cdot h(\mathfrak{f})(g\,\xi) \quad \left(= \langle e, e \rangle_h \right)$$

$$= {}^t\bar{\xi} \cdot {}^t\bar{g} \cdot h(\mathfrak{f}) \cdot g \cdot \xi$$

$$\Longrightarrow \boxed{h(\mathfrak{f}g) = {}^t\bar{g} \cdot h(\mathfrak{f}) \cdot g}$$

Change of frame for metric.

Change of frame for metric.

Equivalently : $h(\mathfrak{f}) = h$, $h(\tilde{\mathfrak{f}}) = \tilde{h}$, $\tilde{\mathfrak{f}} = \mathfrak{f} \cdot g$

$$\Rightarrow \boxed{\tilde{h} = {}^t\bar{g} \cdot h \cdot g}$$

IN COMPONENTS   HC-3   ⊗

Change of frame: g ... $GL(n,\mathbb{C})$-valued fn.,

$f = (e_1, \dots, e_n)$ frame $g = (g_{\rho\sigma})$

$$f \cdot g = \Big(\dots, \underbrace{\sum_\rho e_\rho \cdot g_{\rho\sigma}}_{\sigma\text{-th entry}}, \dots \Big) = \text{new frame.}$$

$$\boxed{h(f \cdot g) = {}^t\bar{g} \cdot h(f) \cdot g}$$

Proof:
$$e = \sum_\sigma \xi^\sigma \cdot (f \cdot g)_\sigma$$
$$= \sum_{\rho,\sigma} \xi^\sigma \cdot g_{\rho\sigma} e_\rho$$
$$= \sum_{\rho,\sigma} g_{\rho\sigma} \cdot \xi^\sigma \cdot e_\rho$$
$$= \sum (g \cdot \xi)^\rho \cdot e_\rho$$

Thus, if a vector $\xi = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^n \end{pmatrix}$ represents the components of an element e in $f \cdot g$-frame then $g \cdot \xi$ (matrix product) represents the same element in f-frame: $(f \cdot g)\xi = f \cdot (g\xi)$

So: $|e|_h^2 = {}^t\bar{\xi} \cdot h(f \cdot g) \cdot \xi =$
$= {}^t\overline{(g\xi)} \cdot h(f)(g\xi)$ } since we are measuring the same e in two ways
$= {}^t\bar{\xi}\left({}^t\bar{g}\, h(f) \cdot g\right) \cdot \xi$

$$\Longrightarrow \boxed{h(f \cdot g) = {}^t\bar{g} \cdot h(f) \cdot g.}$$

□

Calculation in holomorphic frames.

Let f and $\tilde{f} = f \cdot g$ be holomorphic frames, g a holomorphic transition matrix

Set $h(f) = h$, $h(\tilde{f}) = \tilde{h}$.

We have $\boxed{\tilde{h} = {}^t\bar{g} \cdot h \cdot g}$ Differentiate:

$$\partial \tilde{h} = {}^t\bar{g}(\partial h \cdot g + h \cdot \partial g) \qquad (\partial({}^t\bar{g}) = 0)_{g \text{ holomorphic!}}$$

$$\tilde{h}^{-1} \partial \tilde{h} = ({}^t\bar{g} h g)^{-1} \cdot {}^t\bar{g}(\partial h \cdot g + h \partial g)$$

$$= g^{-1} \cdot h^{-1} \cdot (\partial h \cdot g + h \cdot \partial g)$$

$$= g^{-1}(h^{-1}\partial h) g + g^{-1}\partial g \qquad (*)$$

Set: $\begin{cases} \theta = \theta(f) = h(f)^{-1} \cdot \partial h(f) \dots (1,0)\text{-form} \\ \tilde{\theta} = \theta(f \cdot g) = \tilde{h}^{-1} \cdot \partial \tilde{h} \ \dots \text{ in gauge } \tilde{f}. \end{cases}$

The above $(*)$ shows that

$$\boxed{\tilde{\theta} = g^{-1} \theta g + g^{-1} \partial g} \quad (1,0)\text{-form!}$$

Hence θ is a connection matrix (it satisfies the correct change of gauge formula).

Let $D \cdot \xi = d\xi + \theta \cdot \xi$

$$= \underbrace{\partial \xi + \theta \cdot \xi}_{D^{1,0} \cdot \xi} + \underbrace{\bar{\partial} \xi}_{D^{0,1}}$$

The Chern connection $D^{0,1} = \bar{\partial}$.

be the associated covariant derivative

§2. CHERN CONNECTION

In components:

$$\xi = \sum_{\sigma=1}^{n} \xi^{\sigma} e_{\sigma}$$

$$\theta = \left(\theta^{\rho}_{\sigma}\right) \quad \text{the connection matrix}$$

$$D e_{\rho} = \sum_{\sigma=1}^{n} \theta^{\sigma}_{\rho} \cdot e_{\sigma} \quad ; \quad \rho = 1, \dots, n.$$

$$D\xi = \sum_{\sigma} d\xi^{\sigma} \cdot e_{\sigma} + \sum_{\rho} \xi^{\rho} \cdot D e_{\rho}$$

$$= \sum_{\sigma} d\xi^{\sigma} \cdot e_{\sigma} + \sum_{\rho,\sigma} \xi^{\rho} \cdot \theta^{\sigma}_{\rho} \cdot e_{\sigma}$$

$$= \sum_{\sigma} \Big(d\xi^{\sigma} + \sum_{\rho} \xi^{\rho} \cdot \theta^{\sigma}_{\rho} \Big) \cdot e_{\sigma}$$

$$= \sum_{\sigma} \Big(d\xi^{\sigma} + \sum_{\rho} \theta^{\sigma}_{\rho} \cdot \xi^{\rho} \Big) \cdot e_{\sigma}$$

$$\boxed{D\xi \approx_{f} d\xi + \theta \cdot \xi} = \text{E-valued 1-form on } M$$

$$\left\{ \begin{array}{l} D^{1,0}\xi = \partial\xi + \theta \cdot \xi \\ D^{0,1}\xi = \bar{\partial}\xi \end{array} \right\} \quad \text{Chern connection \& covariant derivation}$$

D is an h-metric connection : METRIC CONNECTION

This means

$$d\langle e_\sigma, e_\rho\rangle = dh_{\rho\sigma}$$

$$= \langle De_\sigma, e_\rho\rangle + \langle e_\sigma, De_\rho\rangle \quad (\text{Leibnitz rule})$$

$$= \langle \sum_\tau \theta_\sigma^\tau e_\tau, e_\rho\rangle + \langle e_\sigma, \sum_\tau \theta_\rho^\tau, e_\tau\rangle$$

$$= \sum_\tau h_{\rho\tau}\cdot\theta_\sigma^\tau + \sum_\tau {}^t\overline{\theta_\tau^\rho}\cdot h_{\tau\sigma}$$

In matrix notation, we must have

$$\boxed{dh = h\cdot\theta + {}^t\overline{\theta}\cdot h}$$ ← condition for an h-connection.

Verification for $\theta = h^{-1}\partial h$:

$$h\theta + {}^t\overline{\theta}\cdot h = h(h^{-1}\partial h) + {}^t\overline{\partial h}\cdot {}^t\overline{h^{-1}}\cdot h$$

$$= \partial h + \overline{\partial}({}^t\overline{h})\cdot({}^t\overline{h})^{-1}\cdot h \quad ({}^t\overline{h} = h)$$

$$= \partial h + \overline{\partial} h$$

$$= dh \quad \checkmark$$

Proposition. The Chern connection $D = D^{1,0} + \overline{\partial}$ is the unique linear h-metric connection satisfying $D^{0,1} = \overline{\partial}$. (It is only defined on holomorphic Hermitean vector bundles.)

PROPOSITION (SPECIAL FRAME)

For every point $x_0 \in M$ there exists a local holo. frame f such that $h(f)$ satisfies

$$\left\{\begin{array}{l} h(f)(x_0) = Id \\ d\,h(f)(x_0) = 0 \end{array}\right\}$$ SUCH f is SPECIAL at x_0.

Proof. Let f_0 be some frame. Any other frame is given by $f = f_0 \cdot g$, g a local holom. map to $GL(r,\mathbb{C})$.

Then $h(f) = {}^t\bar{g} \cdot h(f_0) \cdot g$.

First condition can be achieved by a constant g. Assume this was done; choose local holo. coordinates

$$z = (z_1, \ldots, z_m) \text{ on } M, \quad z(x_0) = 0.$$

Let $$h_{\rho\sigma} = \delta_{\rho\sigma} - \sum_j \left(A^\rho_{\sigma j}\, z_j + \overline{A^\rho_{\sigma j}\, z_j} \right) + O(|z|^2)$$

Set $$A(z) = \left(\sum_{j=1}^{m} A^\rho_{\sigma j} z^j \right)_{\sigma\rho} = \left(\sum_j - \frac{\partial h_{\rho\sigma}}{\partial z_j}(0) \cdot z_j \right)_{\sigma\rho}$$

$$g(z) \stackrel{\text{def}}{=} I + A(z) \ldots \text{ change of frame.}$$

From ${}^t\bar{h} = h$ we get

$$\overline{{}^t I - (A + \bar{A})} = I - (A + \bar{A}) \Rightarrow {}^t\bar{A} + {}^tA = A + {}^t\bar{A}$$

$\Rightarrow$ (by comparing z and $\bar{z}$ part) $A = {}^tA$.

Then $${}^t\bar{g} \cdot h \cdot g = (I + {}^t\bar{A} + O(2)) \cdot (I - A - \bar{A} + O(2)) \cdot (I + A + O(2))$$

$$= I + {}^t\bar{A} - (A + \bar{A}) + A + O(2)$$

$$= I + (\bar{A} - A - \bar{A} + A) + O(2) = I + O(2).$$

Thus, $f = f_0 \cdot g$ is a special frame at x_0.

HC - 8

CHERN

CONNECTION IN A SPECIAL HOLOMORPHIC FRAME

Recall that for any holomorphic frame $f = (e_1, \dots, e_r)$, the Chern connection matrix is

$$\theta = h^{-1}\partial h, \qquad h_{\alpha\beta} = \langle e_\beta, e_\alpha \rangle .$$

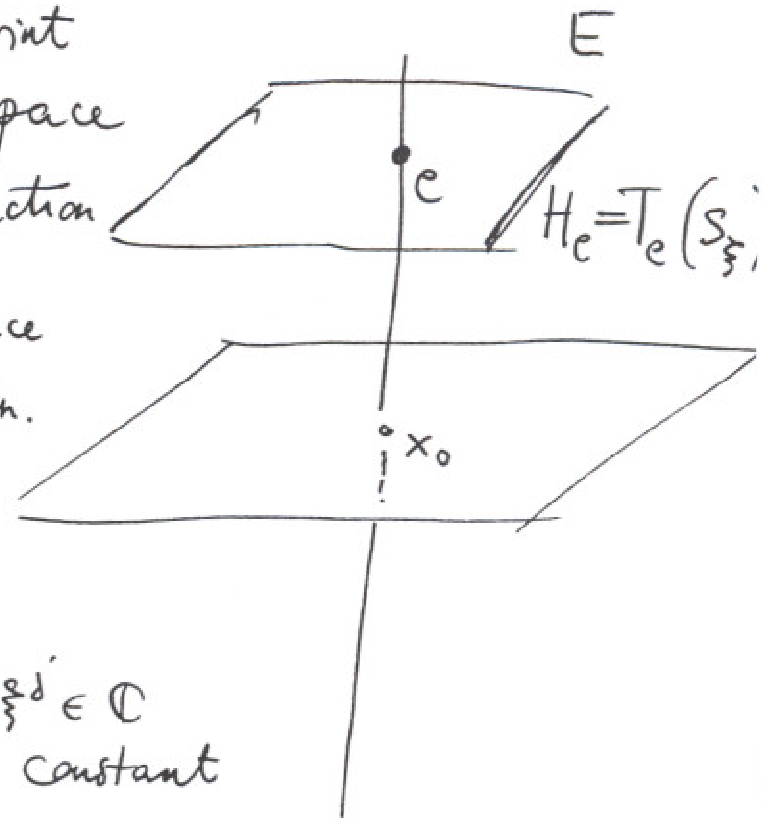
Suppose now that the frame f is special at $x_0 \in M$:

$$h_{\alpha\beta}(x_0) = \delta_{\alpha\beta} \; ; \quad dh_{\alpha\beta}(x_0) = 0 .$$

Then: $\quad \theta(x_0) = 0 .$

Hence: $\quad s = \sum_j \sigma^j \cdot e_j \Rightarrow Ds\Big|_{x_0} = \sum_j d\sigma^j \cdot e_j \Big|_{x_0}$

CONSEQUENCE: At every point $e \in E$, the horizontal space H_e of the Chern connection is just the tangent space at e of the constant section.



Proof

$$e = \sum \xi^j e_j(x_0)$$

$$S_\xi(z) \stackrel{\text{def}}{=} \sum_j \xi^j \cdot e_j(z) \; ; \qquad \xi^j \in \mathbb{C} \text{ constant}$$

$$DS_\xi \Big|_{x_0} = \sum_j d\xi^j \cdot e_j(x_0) = 0 .$$

– calculate in a fixed holomorphic fr

$$D^2\xi = D(d\xi + \theta\cdot\xi)$$
$$= d(d\xi + \theta\cdot\xi) + \theta\wedge(d\xi + \theta\cdot\xi)$$
$$= d\theta\wedge\xi - \theta\wedge d\xi + \theta\wedge d\xi + \theta\wedge$$
$$= (d\theta + \theta\wedge\theta)\cdot\xi$$

$$\Theta(g) = d\theta + \theta\wedge\theta$$ = the Chern cu matrix

= $(r\times r)$-matrix of 2-f

Intrinsic meaning: $\Theta = D^2$ is a 2-form on M with values in $Hom(E,E)$.

$$\Theta \in \mathcal{C}_2^\infty(M, Hom(E,E)) = \mathcal{C}_2^\infty(M, E^*\otimes E)$$

$$\theta = h^{-1}\partial h$$
$$d\theta = \partial\theta + \bar\partial\theta$$
$$= \partial(h^{-1}\partial h) + \bar\partial(h^{-1}\partial h)$$
$$= [\partial h^{-1}\wedge\partial h] + [\bar\partial h^{-1}\wedge\partial h + h^{-1}\bar\partial\partial h]$$
$$= \underbrace{[-h^{-1}\partial h\cdot h^{-1}\wedge\partial h]}_{\partial\theta = -\theta\wedge\theta} + \underbrace{[-h^{-1}\bar\partial h\cdot h^{-1}\wedge\partial h + h^{-1}\partial\bar\partial h]}_{\bar\partial\theta}$$
$$= \bar\partial\theta = -h^{-1}\partial\bar\partial h - h^{-1}\bar\partial l$$

CHERN CURVATURE: $\boxed{\Theta = -h^{-1}\cdot\partial\bar\partial h + h^{-1}\partial h \wedge h^{-1}\bar\partial h}$

Intrinsically: $\Theta \in \mathcal{C}^{\infty}_{(1,1)}(M, \mathrm{Hom}(E,E))$

(1,1)-form with values in $\mathrm{Hom}(E,E) = E^*\otimes E$.

The above expression is a matrix-valued (1,1)-form, corresponding to the given holomorphic frame.

THE CHERN CURVATURE TENSOR:

$$\Theta = \sum_{\rho,\sigma=1,\dots,r} \Theta^{\rho}_{\sigma}\, e^*_\sigma \otimes e_\rho$$

$$\boxed{\Theta = \sum_{\substack{\rho,\sigma=1,\dots,r\\ i,j=1,\dots,m}} \Theta^{\rho}_{\sigma\, ij}\cdot e^*_\sigma \otimes e_\rho \cdot dz^i \wedge d\bar z^j}$$

where $z=(z^1,\dots,z^m)$ are local holomorphic coordinates on M.

IN A SPECIAL FRAME: $x_0 \in M$, $h(x_0) = \mathrm{Id}$, $dh(x_0)=0$

$$\Theta_{x_0} = -\partial\bar\partial h(x_0) \qquad \text{(matrix)}$$

$$= -\sum_{i,j=1}^{m} \frac{\partial^2 h}{\partial z^i \partial \bar z^j}\cdot dz^i \wedge d\bar z^j$$

$$\Theta^{\rho}_{\sigma}\Big|_{x_0} = -\sum_{i,j=1}^{m} \frac{\partial^2 h_{\rho,\sigma}}{\partial z^i \partial \bar z^j}\cdot dz^i \wedge d\bar z^j.$$

TENSOR: $\boxed{\Theta_{x_0} = -\sum_{\rho,\sigma} \partial\bar\partial h_{\rho,\sigma}\Big|_{x_0}\cdot e^*_\sigma \otimes e_\rho}$.

III. CURVATURE AND CONVEXITY: LINE BUNDLES

CC 1 — CC 10

$\square$

III. CURVATURE AND CONVEXITY : LINE BUNDLES

Let $\pi : E \to M$ be a Hermitian hol line bundle.

Any non vanishing holomorphic section $e(z)$ (over an open set in M) is a holomorphic frame for E over that set $U \subset M$.

It induces a trivialization

$$\begin{cases} U \times \mathbb{C} \xrightarrow{\approx} E|_U \\ (z, \xi) \longrightarrow \xi \cdot e(z) \end{cases}$$

Let

$$\begin{cases} h(z) = \|e(z)\|^2; \\ \varphi(z, \xi) = \|\xi \cdot e(z)\|^2 = |\xi|^2 \cdot h(z). \end{cases}$$

Usually we write $h(z) = e^{-\psi(z)}$;

$$\boxed{\varphi(z, \xi) = |\xi|^2 \cdot e^{-\psi(z)}}$$

From the general theory we have:

$$\begin{cases} \theta = \frac{1}{h} \partial h = \partial(\log h) \\ \quad = -\partial \psi \quad \text{... the connection form of } E \\ \Theta = \bar\partial \theta = \bar\partial \partial \log h \\ \quad = \partial \bar\partial \psi \quad \text{... the curvature form of } E. \end{cases}$$

Note: $i\Theta = i\partial\bar\partial\psi = i \cdot \sum_{j,k} \frac{\partial^2 \psi}{\partial z^j \partial \bar z^k} \cdot dz^j \wedge d\bar z^k$

is the Levi form of ψ (a real (1,1)-form).

DEFINITION.

(a) $\Theta > 0$ (in the sense of Griffiths)

iff $i\Theta$ is a positive form

(ie, it is positive on all complex lines in $T_z M$)

iff ψ is strongly plurisubharmonic

iff $\left(\dfrac{\partial^2 \psi}{\partial z^j \partial \bar{z}^k}\right) > 0$ is a positive definite Hermitian matrix.

(b) $\Theta < 0 \iff -\psi$ is strongly plurisubharmonic

$$\iff \left(\frac{\partial^2 \psi}{\partial z^j \partial \bar{z}^k}\right) < 0 .$$

(c) Θ has signature (p, q) at a point $z_0 \in M$

if $\left(\dfrac{\partial^2 \psi(z_0)}{\partial z^j \partial \bar{z}^k}\right)$ has signature (p, q)

(p positive, q negative eigenvalues)

Remark. The definition is independent of the choice of e (= holo. frame): Given another e', we have $e' = g \cdot e$, $g \neq 0$ holomorphic. Hence

$$\begin{cases} \|e'\|^2 = |g|^2 \cdot \|e\|^2 \\ \log \|e'\|^2 = \log |g|^2 + \log \|e\|^2 \\ \partial\bar{\partial} \log \|e'\|^2 = \partial\bar{\partial} \log \|e\|^2 \quad (\text{since } \partial\bar{\partial} \log |g|^2 = 0). \end{cases}$$

PROPOSITION. Let $E \to M$ be a hermitian holomorphic line bundle with the Chern curvature tensor Θ and the hermitian squared length $\varphi : E \to \mathbb{R}_+$.
Then the following hold:

(a) $\Theta > 0 \iff \{\varphi > c\}$ is strongly pseudoconvex along $\Sigma = \{\varphi = c\}$ for all $c > 0$

$\iff \frac{1}{\varphi}$ is strongly psh on $E \setminus M$.

(b) $\Theta < 0 \iff \{\varphi < c\}$ is strongly pseudoconvex along $\Sigma = \{\varphi = c\}$, $\forall c > 0$

$\iff \varphi$ is strongly psh. on $E \setminus M$.

Proof. Fix a point $z_0 \in M$ and a local holomorphic section $e(z)$ for z near z_0 such that

$h(z) = \|e(z)\|^2$ satisfies $h(z_0) = 1$, $dh_{z_0} = 0$.

Then $\varphi(z, \xi) = \|\xi e(z)\|^2 = |\xi|^2 \cdot h(z)$.

Write $h(z) = e^{-\psi(z)}$; then $\begin{cases} \psi(z_0) = 0 \\ d\psi_{z_0} = 0. \end{cases}$

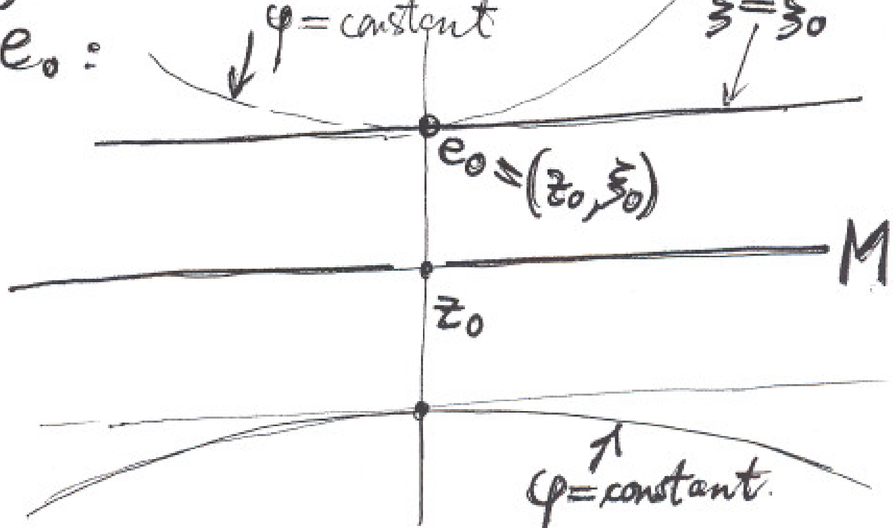
It follows that

$\Theta_{z_0} = -\partial\bar\partial h(z_0) = \partial\bar\partial \psi(z_0)$.

Hence $\Theta_{z_0} > 0 \iff \psi$ is strongly psh at z_0.

Note also that the horizontal distribution along the fiber E_{z_0} is the actual horizontal:

At every point $e_0 = (z_0, \xi_0)$, $\xi_0 \neq 0$, the horizontal space is just (the tangent to) the hyperplane $\xi = \xi_0$, and at the same time this is the complex tangent space to the level set of φ through e_0:



Hence: $\textcircled{H}_{z_0} > 0 \iff \psi$ spsh at z_0

$\iff \dfrac{1}{\varphi} = \dfrac{e^{\psi}}{|\xi|^2}$ is strongly psh along $E_{z_0} \setminus \{0\}$.

$\iff \{\varphi = c\}$ is strongly psc. from the side $\{\varphi > c\}$ for all $c > 0$ along E_{z_0}.

We get similar equivalences when $\textcircled{H}_{z_0} < 0$:

$\iff -\psi$ spsh at $z_0 \iff e^{-\psi}$ spsh at z_0

$\iff \varphi = |\xi|^2 e^{-\psi}$ spsh on $E_{z_0} \setminus \{0\}$

This finishes the proof.

<u>more generally</u> :

(A) positive eigendirections for Ⓗ correspond to positive eigendirections of the exterior $\{\varphi > c\}$ in the complex tangential directions;

(B) negative eigendirections for Ⓗ correspond to positive eigendirections of the interior tube $\{\varphi < c\}$ in the complex tangential directions.

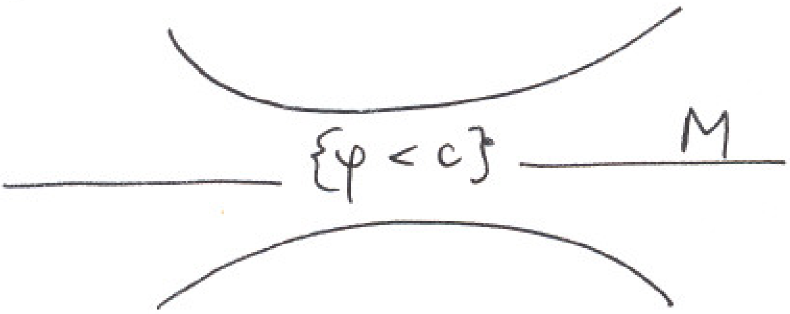
(The radial eigenvalue is positive for φ from inside, and for $1/\varphi$ from outside.)

<u>Remark</u>. In local coordinates on E, the tube $\{\varphi < c\}$ is the Hartogs domain.

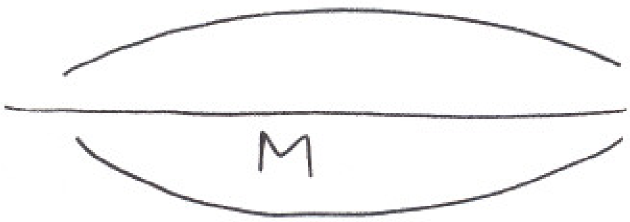
$$\{(z,\xi) \mid |\xi|^2 e^{-\psi(z)} < c\} = \{(z,\xi) \mid |\xi|^2 < c\, e^{\psi(z)}\}.$$

By the classical theory, this tube is strongly psc. along $\{\varphi = c\}$ if and only if the function $+\psi$ is strongly psh ($\Leftrightarrow$ $H < 0$).

Likewise, $\{\varphi > c\} = \{|\xi|^2 e^{-\psi(z)} > c\}$ is strongly psc. iff $-\psi$ is spsh $\Leftrightarrow$ $H > 0$.



POSITIVELY CURVED



NEGATIVELY CURVED

A fundamental example: the tautological line bundle over $\mathbb{P}^m$.

Let $M = \mathbb{P}^m = \mathbb{P}(\mathbb{C}^{m+1})$

= complex lines through 0 in $\mathbb{C}^{m+1}$.

$E = H \to \mathbb{P}^m$ ... the complex line bundle whose fiber over a point $z = [z_0 : \dots : z_m] \in \mathbb{P}^m$ is the complex line $\{\lambda \cdot z = (\lambda z_0, \dots, \lambda z_m) \mid \lambda \in \mathbb{C}\}$

$$\begin{array}{l} \mathbb{P}^m \times \mathbb{C}^{m+1} \supset H \xrightarrow{\tau} \mathbb{C}^{m+1} \\ \downarrow \pi \\ \mathbb{P}^m \end{array}$$

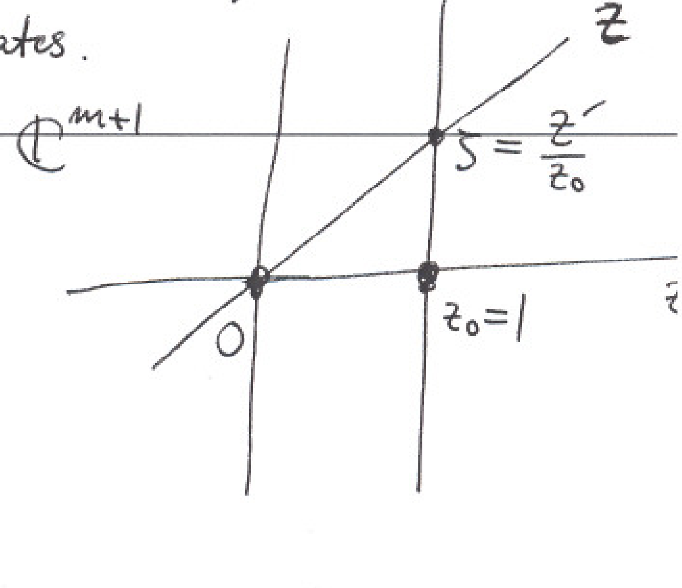
$$H = \{([z], v) \mid z = [z_0 : \dots : z_m] \in \mathbb{P}^m, \; v = \lambda z, \; \lambda \in \mathbb{C}\}$$

Note: $\tau : H \setminus \mathbb{P}^m \to \mathbb{C}^{m+1}_*$ is biholomorphic.

The total space H is obtained by blowing up the origin $0 \in \mathbb{C}^{m+1}$.

$U_j = \{ z = [z_0 : \dots : z_m] \mid z_j \neq 0 \} \approx \mathbb{C}^m$.

On U_0, use $\zeta = \frac{1}{z_0} \cdot z' = \left(\frac{z_1}{z_0}, \dots, \frac{z_m}{z_0}\right) \in \mathbb{C}^m$ as affine coordinates.



Let $\begin{cases} \varphi : H \to \mathbb{R}_+ \\ \varphi([z], v) = |v|^2. \end{cases}$

This is the squared length function for the Hermitean metric on H, obtained from the standard metric on $\mathbb{C}^{m+1}$.

A holomorphic nonvanishing section of $H|_{U_0}$ is given by

$$e([z_*]) = \left(1, \frac{z_1}{z_0}, \dots, \frac{z_m}{z_0}\right) = (1, \zeta_1, \dots, \zeta_m).$$

Hence $$h = \|e\|^2 = 1 + |\zeta_1|^2 + \dots + |\zeta_m|^2 = 1 + |\zeta|^2$$

The Chern curvature of this metric on H is

$$\boxed{i\,\Theta_H = -i\,\partial\bar\partial \log\left(1 + |\zeta|^2\right)}$$

We get the same expression in the affine coordinates over any chart $U_j \subset \mathbb{P}^m$.

It is easily seen that the function $\log(1+|\zeta|^2)$ is strongly plurisubharmonic on $\mathbb{C}^m$; hence $i\,\Theta_H < 0$.

Note that $\omega = -\frac{i}{2\pi}\Theta_H = \frac{i}{2\pi}\partial\bar\partial \log(1+|\zeta|^2) > 0$ is the Fubini–Study Kähler form on $\mathbb{P}^m$.

$$\int_{\mathbb{P}^m} \omega = 1/m! \quad \dots \text{ the volume of } \mathbb{P}^m.$$

Expo sheaf sequence: CC-9

$$\begin{array}{ccccccccc} \to & H^1(\mathbb{P}^m, \mathcal{O}) & \to & H^1(\mathbb{P}^m, \mathcal{O}^*) & \xrightarrow{c_1} & H^2(\mathbb{P}^m, \mathbb{Z}) & \to & H^2(\mathbb{P}^m, \mathcal{O}) \\ & \| & & \| & & \| & & \| \\ & 0 & & \mathrm{Pic}(\mathbb{P}^m) & & \mathbb{Z} & & 0 \end{array}$$

c_1 ... the 1^{st} chern class map ~~induces~~.

$$c_1 : \underbrace{H^1(\mathbb{P}^m, \mathcal{O}^*)}_{\substack{\text{holo}^{\text{"}} \text{ line bdles} \\ \text{on } \mathbb{P}^m}} \xrightarrow{\approx} H^2(\mathbb{P}^m, \mathbb{Z}) = \mathbb{Z}$$

$$c_1(H) = -1$$

$$c_1(H^{-1}) = +1.$$

$$E = H^{-1} = [\Lambda], \quad \Lambda \approx \mathbb{P}^{m-1} \subset \mathbb{P}^m$$

$\uparrow$ projective hyperplane

$\uparrow$ the hyperplane section bundle determined by Λ. $\square$

Every holo. line bundle is isomorphic to $E^{\otimes k}$ for some $k \in \mathbb{Z}_+$, or to $H^{\otimes k}$ for some $k \in \mathbb{N}$.

$$E = \mathcal{O}_{\mathbb{P}^m}(+1), \quad H = \mathcal{O}_{\mathbb{P}^m}(-1); \quad H^{\otimes m} = \mathcal{O}_{\mathbb{P}^m}(-m).$$

The Lelong – Poincaré equation and first Chern class.

Theorem. Let $E \to M$ be a Hermitian holomorphic line bundle over a compact complex manifold, with the Chern curvature form $\Theta(E) = \Theta$.

For any meromorphic section s of E which does not vanish identically on any connected component of M we have

$$\underset{\substack{\Cap \\ H^2_{\mathrm{deRham}}(M)}}{\left[\frac{i}{2\pi}\Theta\right]} = c_1(E)_{\mathbb{R}} = \left\{\sum_j m_j [Z_j]\right\} \in H^2(M, \mathbb{R})$$

where $\sum m_j Z_j$ is the divisor of s.

(Demailly, Sec. 5.13.)

IV.

GRIFFITHS POSITIVITY AND CONVEXITY OF HERMITIAN HOLOMORPHIC VECTOR BUNDLES

BARBARA DRINOVEC DRNOVŠEK & FRANC FORSTNERIČ

ABSTRACT. This is §6 and part of §7 of the paper by the authors, entitled *Strongly pseudoconvex Stein domains as subvarieties of complex manifolds*, arXiv: math/0708.2155.

We recall some basics of complex Hermitian geometry, in particular the notions of *positivity* and *signature* of a Hermitian holomorphic vector bundle (see Definition 0.1). For more complete treatments we refer to the papers of Griffiths [7, 8] and the monographs of Demailly [4, Chapter 5] and Wells [11, Chapter III].

Let M be a compact complex manifold of dimension m and $\pi\colon E \to M$ a holomorphic Hermitian vector bundle with fiber $\mathbb{C}^r$. A Hermitian metric on E is given in each local frame $f = (e_1, \cdots, e_r)$ by a Hermitian $r \times r$ matrix-valued function $h = (h_{\rho\sigma})$ with entries

$$h_{\rho\sigma}(x) = \langle e_\sigma(x), e_\rho(x)\rangle, \quad \rho, \sigma = 1, \ldots, r.$$

(In certain sources the transpose matrix ${}^t h = \bar h$ is used instead, which changes the formulas below accordingly.) Any local section of E is written in this frame as $e(x) = \sum_{\sigma=1}^r \xi^\sigma(x) e_\sigma(x)$ ($= f\cdot\xi$ in matrix notation, thinking of $\xi = {}^t(\xi^1, \ldots, \xi^r)$ as a column vector), and

$$||e(x)||^2 = \langle e(x), e(x)\rangle = \sum_{\rho,\sigma=1}^r h_{\rho\sigma}(x)\bar\xi^\rho(x)\xi^\sigma(x) = {}^t\bar\xi(x)h(x)\xi(x).$$

For any point $x_0 \in M$ there exists a local holomorphic frame $(e_1, \ldots, e_r)$ for E near x_0 whose associated Hermitian matrix satisfies

$$h(x_0) = I, \qquad dh(x_0) = 0.$$

The first condition means $\langle e_\sigma, e_\rho\rangle = \delta_{\rho\sigma}$ and hence the frame is unitary at x_0. Such frame is said to be *special* at x_0 [8, p. 195].

Let $\mathrm{D} = \mathrm{D}^{1,0} + \mathrm{D}^{0,1}$ be the covariant derivative associated to the *Chern connection* on E, i.e., the unique Hermitian connection whose $(0,1)$-part equals $\mathrm{D}^{0,1} = \bar\partial$. The connection matrix θ, and the Chern curvature form Θ, are given in any holomorphic frame by

$$\theta = h^{-1}\partial h, \qquad \Theta = \bar\partial\,\theta = -h^{-1}\partial\bar\partial h + h^{-1}\partial h \wedge h^{-1}\bar\partial h.$$

If the frame is special at x_0, these expressions simplify to

$$\theta(x_0) = 0, \qquad \Theta(x_0) = -\partial\bar\partial h(x_0).$$

For a line bundle ($r = 1$) we have

$$\theta = h^{-1}\partial h = \partial \log h, \qquad \Theta = -\partial\bar\partial \log h.$$

Writing the metric locally as $h = e^{-\psi}$, we get

$$\theta = e^{\psi}\partial(e^{-\psi}) = -\partial\,\psi, \quad \Theta = -\bar\partial\partial\,\psi = \partial\bar\partial\,\psi.$$

Choosing a local holomorphic coordinate system $z = (z^1, \ldots, z^m)$ at x_0, the Chern curvature tensor can be written as

$$\Theta = \sum_{\substack{\rho,\sigma=1,\ldots,r\\ i,j=1,\ldots,m}} \Theta^{\rho}_{\sigma ij} e^*_\sigma \otimes e_\rho \cdot dz^i \wedge d\bar z^j.$$

Here (e^*_σ) is the dual (to (e_ρ)) coframe for the dual bundle E^*.

In the sequel we assume that $(e_\rho)_{\rho=1}^r$ is a local holomorphic frame that is special at x_0. Then $\overline{\Theta^{\rho}_{\sigma ij}}(x_0) = \Theta^{\sigma}_{\rho ji}(x_0)$ and

$$\Theta^{\rho}_{\sigma ij}(x_0) = -\frac{\partial^2 h_{\rho\sigma}}{\partial z^i \partial \bar z^j}(x_0).$$

Choose an element $e = \sum_{\rho=1}^r \xi^\rho e_\rho(x_0) \in E_{x_0} = \pi^{-1}(x_0)$. We associate to Θ the following covector of type $(1,1)$ at x_0:

$$\Theta\{e\} = \frac{\mathrm{i}}{2}\langle \Theta\, e, e\rangle = \frac{\mathrm{i}}{2} \sum_{\substack{\rho,\sigma=1,\ldots,r\\ i,j=1,\ldots,m}} \Theta^{\rho}_{\sigma ij}(x_0)\xi^\sigma \bar\xi^\rho\, dz^i \wedge d\bar z^j.$$

Its coefficients

$$A_{ij}(x_0,\xi) = \sum_{\rho,\sigma=1,\ldots,r} \Theta^{\rho}_{\sigma ij}(x_0)\xi^\sigma \bar\xi^\rho$$

form a Hermitian matrix, and hence $\Theta\{e\}$ is a real valued $(1,1)$-form on M at x_0. Denote by $s(e)$ (resp. $t(e)$) the number of positive (resp. negative) eigenvalues of $\Theta\{e\}$; that is, $(s(e), t(e))$ is the signature of the Hermitian quadratic form

$$\mathbb{C}^m \ni \eta \to \sum_{i,j} A_{ij}(x_0,\xi)\eta^i\bar\eta^j = \sum_{\substack{\rho,\sigma=1,\ldots,r\\ i,j=1,\ldots,m}} \Theta^{\rho}_{\sigma ij}(x_0)\xi^\sigma\bar\xi^\rho\eta^i\bar\eta^j. \tag{0.1}$$

Thus $s(e) + t(e) \le m = \dim M$. The numbers $s(e), t(e)$ only depend on the Hermitian metric on E, and not on the particular choices of frames and coordinates (granted the stated conditions).

We now recall the notion of positivity (resp. negativity) of E in the sense of Griffiths [8]; compare also with *metric q-convexity* [10, p. 222].

Definition 0.1. The pair of numbers $(s(e), t(e))$ defined above is the *signature* of the Hermitian holomorphic vector bundle $E \to M$ at the point $e \in E$, $e \neq 0$. The signature of E is (s, t) where

$$s = \min\{s(e) \colon 0 \neq e \in E\}, \quad t = \min\{t(e) \colon 0 \neq e \in E\}.$$

The bundle E is of *pure signature* (s, t) is $s = s(e)$ and $t = t(e)$ for all $e \in E$, $e \neq 0$. The bundle E is *positive* (resp. *negative*) in the sense of Griffiths if it has signature $(m, 0)$ (resp. $(0, m)$), where $m = \dim M$.

Thus E is positive if the Hermitian quadratic form in (0.1) is positive definite jointly in both variables $\xi \in \mathbb{C}^r, \eta \in \mathbb{C}^m$. (In the earlier paper [7] such bundle was called *weakly positive*. A comparison of different notions of positivity can be found in [8] and in [4, Chapter 7].)

For the dual bundle E^* we have $\Theta^* = -\Theta$; hence E is positive if and only if E^* is negative.

Let $\phi \colon E \to \mathbb{R}_+$ denote the function $\phi(e) = ||e||^2$. In a local frame $(e_1, \ldots, e_r)$ near the point $x = \pi(e) \in M$, we have $e = \sum \xi^\rho e_\rho(x)$ and

$$\phi(e) = \phi(x, \xi) = \sum_{\rho,\sigma=1,\ldots,r} h_{\rho\sigma}(x) \xi^\sigma \bar{\xi}^\rho. \tag{0.2}$$

For a positive number $c \in (0, \infty)$ set

$$W_c = \{e \in E \colon \phi(e) < c\}, \quad \Sigma_c = bW_c = \{e \in E \colon \phi(e) = c\}.$$

The following proposition, essentially due to Andreotti and Grauert (see [1, §23]), explains the connection between the curvature properties of a Hermitian metric and Levi convexity properties of the corresponding norm function ϕ (0.2) on a Hermitian holomorphic vector bundle.

Proposition 0.2. *Let $E \to M$ be a Hermitian holomorphic vector bundle with fiber $\mathbb{C}^r$ over an m-dimensional complex manifold M. Set $n = m + r = \dim E$. Then the following hold:*

(i) *If E has signature $(s(e), t(e))$ at a point $e \in E$ $(e \neq 0)$ then the Levi form of the hypersurface $\Sigma_{\phi(e)}$ has Levi signature $(t(e) + r - 1, s(e))$ at e from the side $\{\phi < \phi(e)\}$.*

(ii) *If E has signature (s, t) then the Levi form of ϕ has signature $(t + r, s)$ (and hence ϕ is $(m - t + 1)$-convex) on $E \backslash M$, and the Levi form of $\frac{1}{\phi}$ has signature $(s + 1, t + r - 1)$ (and hence $\frac{1}{\phi}$ is $(n - s)$-convex) on $E \backslash M$.*

(iii) *In particular, if E is positive then $\frac{1}{\phi}$ is r-convex on $E \backslash M$, and if E is negative then ϕ is strongly plurisubharmonic on $E \backslash M$.*

Proof. (See [1, 5, 8, p. 426].) We identify M with the zero section $\{\phi = 0\}$ of E. Fix $e_0 \in E\backslash M$ and let $\Sigma = \Sigma_{\phi(e_0)}$. Choose local holomorphic coordinates $z = (z^1, \ldots, z^m)$ at $x_0 = \pi(e_0)$ and a local holomorphic frame (e_ρ) that is special at x_0. Write $e_0 = \sum_{\rho=1}^r \xi_\rho^0 e_\rho(x_0)$. We have

$$h_{\rho\sigma}(x_0) = \delta_{\rho\sigma}, \quad dh(x_0) = 0, \quad \frac{\partial^2 h_{\rho\sigma}}{\partial z^i \partial \bar z^j}(x_0) = -\Theta^\rho_{\sigma ij}(x_0).$$

This gives

$$\begin{aligned}\partial\bar\partial\phi(e_0) &= \partial_\xi \bar\partial_\xi\big|_{\xi=\xi_0} \sum_{\rho=1}^r \xi^\rho \bar\xi^\rho + \sum_{\rho,\sigma=1,\ldots,r} \partial_z \bar\partial_z h_{\rho\sigma}(x_0) \xi^\sigma \bar\xi^\rho \\ &= \sum_{\rho=1}^r d\xi^\rho \wedge d\bar\xi^\rho - \sum_{\substack{\rho,\sigma=1,\ldots,r \\ i,j=1,\ldots,m}} \Theta^\rho_{\sigma ij}(x_0)\xi^\sigma\bar\xi^\rho dz^i \wedge d\bar z^j \\ &= \sum_{\rho=1}^r d\xi^\rho \wedge d\bar\xi^\rho - \sum_{i,j=1,\ldots,m} A_{ij}(x_0,\xi)\, dz^i \wedge d\bar z^j.\end{aligned}$$

The maximal complex tangent space to Σ at e_0 consists of the vectors $\gamma = (\zeta^1, \ldots, \zeta^r; \eta^1, \ldots, \eta^m)$ with $\sum_{\rho=1}^r \xi_0^\rho \bar\zeta^\rho = 0$. In the ζ-direction (tangential to E_{x_0}) we thus get $r-1$ positive (and no negative or zero) Levi eigenvalues for Σ; in the η-direction (the horizontal direction in $T_{e_0}E$ with respect to the Chern connection) we get $s(e_0)$ negative and $t(e_0)$ positive eigenvalues. Hence the Levi signature of Σ at e_0 is $(t(e_0)+r-1, s(e_0))$. The remaining Levi eigenvalue of ϕ in the radial direction is positive.

If E has (pure) signature (s,t), it follows that the Levi form $\mathcal{L}_\phi$ has (pure) signature $(t+r, s)$ on $E\backslash M$. In particular, if E is negative then ϕ is strongly plurisubharmonic on $E\backslash M$. When replacing ϕ by $-\log\phi$, the eigenvalues of the Levi form in directions tangential to the level set of ϕ change sign, and hence $\mathcal{L}_{-\log\phi}$ has tangential signature $(s, t+r-1)$ on $E\backslash M$. Passing to $e^{-\log\phi} = \frac{1}{\phi}$, the tangential eigenvalues preserve signs while the radial eigenvalue becomes positive, so $\mathcal{L}_{1/\phi}$ has signature $(s+1, t+r-1)$ on $E\backslash M$. In particular, if E is positive then $\frac{1}{\phi}$ is r-convex on $E\backslash M$. ☐

The following result is due to M. Schneider [10].

Theorem 0.3. *Let A be a compact complex submanifold of codimension r in a complex manifold M whose normal bundle $N_{A|M}$ has signature (s,t) with respect to some Hermitian metric. Then there is an open tubular neighborhood $V \subset M$ of A and a smooth function $\rho\colon V\backslash A \to \mathbb{R}$ without critical points that tends to $+\infty$ along A, whose Levi form $\mathcal{L}_\rho$ has at least $s+1$ positive eigenvalues at every point of $V\backslash A$.*

In particular, if the normal bundle $N_{A|M}$ is positive then the Levi form of ρ has $r+1$ positive eigenvalues at every point in $V\backslash A$.

Proof. Assume first that A is a smooth complex hypersurface in M. Let $E \to M$ denote the hyperplane section bundle of the divisor determined by A. Then $E|_A \simeq N_{A|M}$, and there is a holomorphic section $\sigma\colon M \to E$ such that $A = \{x \in M\colon \sigma(x) = 0\}$. Such σ is given by a collection (g_i) of holomorphic functions $g_i\colon U_i \to \mathbb{C}$ on an open covering $\{U_i\}$ of M such that $\{g_i = 0\} = A \cap U_i$ and $dg_i \neq 0$ on $A \cap U_i$. The associated 1-cocycle $g_{ij} = \frac{g_i}{g_j}$ defines the line bundle $E \to M$.

The Hermitian metric of signature (s,t) on the normal bundle $E|_A = N_{A|M}$ extends to a Hermitian metric h on E. On $E|_{U_i} \simeq U_i \times \mathbb{C}$ the metric is given by a positive function $h_i\colon U_i \to (0,\infty)$. Let $||\sigma||_h^2\colon M \to [0,\infty)$ be the squared length of the section $\sigma\colon M \to E$. (On U_i we have $||\sigma||_h^2 = h_i|g_i|^2$.) Schneider showed that for a sufficiently large constant $C > 0$ the metric ϕ on E, defined over U_i by

$$\phi_i = \frac{h_i}{1 + Ch_i|g_i|^2},$$

has signature $(s+1,t)$ over a neighborhood of A (see [10, p. 225]). This means that the $(1,1)$-form $-i\partial\bar\partial \log \phi_i$ has at least $s+1$ positive and t negative eigenvalues at every point. Set $g = ||\sigma||_\phi^2\colon M \to [0,\infty)$, so

$$g|_{U_i} = \phi_i|g_i|^2 = \frac{h_i|g_i|^2}{1 + Ch_i|g_i|^2} = \frac{||\sigma||_h^2}{1 + C||\sigma||_h^2}.$$

It follows that

$$-i\partial\bar\partial \log g|_{U_i} = -i\partial\bar\partial \log \phi_i$$

and hence the Levi form of $-\log g = -\log ||\sigma||_\phi^2$ has at least $s+1$ positive eigenvalues in a deleted neighborhood of A in M (see bottom of page 225 in [10]). Clearly the same holds for $e^{-\log g} = \frac{1}{g} = \frac{1 + C||\sigma||_h^2}{||\sigma||_h^2}$ and hence for $\rho = \frac{1}{||\sigma||_h^2}$. The latter function is noncritical near A and it blows up along A. This settles the hypersurface case.

The general case reduces to the hypersurface case by blowing up M along A [10, §3]. Assume that A has complex dimension m and codimension r in M. Let $\hat{A} = \mathbb{P}(N)$ denote the total space of the fiber bundle over A whose fiber over a point $x \in A$ is $\mathbb{P}(N_x) \simeq \mathbb{P}^{r-1}$, the projective space of complex lines in $N_x \simeq \mathbb{C}^r$. Replacing A by $\hat{A}$ changes M to a new manifold $\hat{M}$ such that $\hat{M}\backslash\hat{A}$ is biholomorphic to $M\backslash A$, and $\hat{A}$ is a smooth complex hypersurface in $\hat{M}$. The restriction of the normal bundle $N_{\hat{A}|\hat{M}}$ to the submanifold $\mathbb{P}(N_x) \subset \hat{A}$ is the universal bundle over $\mathbb{P}(N_x) \simeq \mathbb{P}^{r-1}$ (the inverse of the hyperplane section bundle). This bundle is negative with respect to the Fubini-Study metric on $\mathbb{P}^{r-1}$, and a simple calculation shows that $N_{\hat{A}|\hat{M}}$ has signature $(s, t+r-1)$ if $N_{A|M}$ has signature (s,t). It remains to apply the previous argument (in the hypersurface case) to a deleted neighborhood of $\hat{A}$ in $\hat{M}$ (that is the same as a deleted neighborhood of A in M). $\square$

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