

A properly embedded holomorphic disc in the ball
with finite area
and dense boundary curve

Franc Forstnerič

Univerza v Ljubljani



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The main result

Notation:

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} = \mathbb{B}^1$$

$$\mathbb{B}^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z|^2 = \sum_{j=1}^n |z_j|^2 < 1\}$$

We prove the following result which answers a question posed by Filippo Bracci (private communication, June 2017).

Theorem

Given $n > 1$ and $\epsilon > 0$, there exists a proper holomorphic embedding $F: \mathbb{D} \hookrightarrow \mathbb{B}^n$, extending to an injective holomorphic immersion $F: \overline{\mathbb{D}} \setminus \{\pm 1\} \rightarrow \overline{\mathbb{B}^n}$, such that $\text{Area}(F(\mathbb{D})) < \epsilon$ and the boundary curve $F(b\mathbb{D} \setminus \{\pm 1\})$ is everywhere dense in the sphere $b\mathbb{B}^n$.

The map F extends holomorphically to an open neighborhood of $\overline{\mathbb{D}} \setminus \{\pm 1\}$. A similar phenomenon exists on $\overline{\mathbb{D}} \setminus \{1\}$.

An embedded disc in \mathbb{B}^2 with bounded defining function and dense boundary curve

Berndtsson 1980 An embedded holomorphic disc of finite area in the ball \mathbb{B}^2 is the zero set of a bounded holomorphic function on \mathbb{B}^2 .

Corollary

There is a bounded holomorphic function on \mathbb{B}^2 whose zero set is a smooth curve of finite area, biholomorphic to the disc, and whose real analytic boundary curve is dense in $b\mathbb{B}^2$.

Globevnik & Stout 1989 Every strongly pseudoconvex domain $D \subset \mathbb{C}^n$ ($n \geq 2$) with real analytic boundary contains a proper holomorphic disc $F: \mathbb{D} \rightarrow D$ of arbitrarily small area such that $\overline{F(\mathbb{D})} = F(\mathbb{D}) \cup \bar{\omega}$, where ω is a given nonempty connected subset of bD .

The main new point is that we find properly **embedded** holomorphic discs with these properties, even in the lowest dimensional case $n = 2$.

A result from contact geometry

Let $n \geq 2$. The sphere $b\mathbb{B}^n = S^{2n-1}$ carries the contact structure ξ given by the distribution of complex tangent lines. Let $x, y \in \mathbb{R}^{n-1}$, $z \in \mathbb{R}$. Removing a point from $b\mathbb{B}^n$ we obtain the Euclidean space $(\mathbb{R}_{(x,y,z)}^{2n-1}, \xi_0)$ with its standard contact structure

$$\xi_0 = \ker(dz + xdy), \quad xdy = \sum_{j=1}^{n-1} x_j dy_j.$$

A smooth curve $f: \mathbb{R} \rightarrow b\mathbb{B}^n$ is said to be **complex tangential**, or **ξ -Legendrian**, if $\dot{f}(t) \in \xi_{f(t)}$ holds for every $t \in \mathbb{R}$.

Lemma (1)

Let $n \in \mathbb{N}$. Every continuous map $f: \mathbb{R} \rightarrow \mathbb{R}^{2n+1}$ can be approximated as closely as desired in the fine \mathcal{C}^0 topology by a real analytic injective Legendrian immersion $\tilde{f}: \mathbb{R} \hookrightarrow (\mathbb{R}^{2n+1}, \xi_0)$ which extends to an entire Legendrian embedding $F: \mathbb{C} \hookrightarrow \mathbb{C}^{2n+1}$. In particular, we can ensure that the curves $\Lambda_{\pm} = \tilde{f}(\mathbb{R}_{\pm})$ are everywhere dense in $b\mathbb{B}^n$.

Sketch of proof of Lemma 1

The contact form $\alpha_0 = dz + xdy$ on \mathbb{R}^{2n+1} extends to a holomorphic contact form on \mathbb{C}^{2n+1} .

A real analytic Legendrian map $f: I \rightarrow \mathbb{R}^{2n+1}$ on an interval $I \subset \mathbb{R}$ extends by complexification to a holomorphic Legendrian map from a neighborhood of I in \mathbb{C} .

If a domain $D \subset \mathbb{C}$ is invariant under the conjugation $\tau(z) = \bar{z}$ and $f: D \rightarrow \mathbb{C}^{2n+1}$ is a holomorphic Legendrian map, then

$$F(z) = \frac{1}{2} \left(f(z) + \overline{f(\bar{z})} \right) \in \mathbb{C}^{2n+1}, \quad z \in D,$$

is a holomorphic Legendrian map satisfying $F(z) = \overline{F(\bar{z})}$; in particular, $F: D \cap \mathbb{R} \rightarrow \mathbb{R}^{2n+1}$ is a real analytic Legendrian map.

Using our recent work **Alarcón, F., López, Compositio Math. 2017** and the above symmetrization trick, we inductively construct an entire Legendrian map $F: \mathbb{C} \hookrightarrow \mathbb{C}^{2n+1}$ such that $\tilde{f} = F|_{\mathbb{R}^{2n+1}}$ is an injective immersion approximating f in the fine \mathcal{C}^0 topology on \mathbb{R}^{2n+1} .

The scheme of proof of the main theorem

A suitably chosen (thin) complexification of the embedded \mathcal{C}^ω Legendrian curve $\Lambda = f(\mathbb{R}) \subset b\mathbb{B}^n$ is an embedded complex disc $\Sigma_0 = f(S) \subset \mathbb{C}^n$ of arbitrarily small area such that $\Sigma_0 \cap \overline{\mathbb{B}^n} = \Lambda$. Here, $S \subset \mathbb{C}$ is a thin strip around \mathbb{R} .

Note that $\sigma \circ f: \mathbb{R} \rightarrow \mathbb{C}\mathbb{P}^{n-1}$ is an immersion, where $\sigma: \mathbb{C}_*^n \rightarrow \mathbb{C}\mathbb{P}^{n-1}$ denotes the canonical projection.

We shall find an embedded disc in \mathbb{B}^n satisfying the theorem by pulling the strip $f(S) \subset \mathbb{C}_*^n$ slightly into the ball along the curve $f(\mathbb{R}) \subset b\mathbb{B}^n$, where the amount of pulling decreases very fast as we go to infinity. When doing so, we shall pay special attention to ensure injectivity; this is a fairly delicate task since the curve $\Lambda = f(S) \cap b\mathbb{B}^n$ is dense in $b\mathbb{B}^n$.

The principal difficulty is that injectivity is not an open condition among immersions of noncompact manifolds in any fine topology. (However, immersions form an open set in the fine \mathcal{C}^1 topology.)

The scheme of proof of the main theorem

To this end, we shall be considering smoothly bounded, simply connected strips $D \subset \mathbb{C}$, symmetric with respect to both coordinate axes, with

$$\mathbb{R} \subset D \subset \bar{D} \subset S,$$

and functions $h = u + iv \in \mathcal{A}^1(D) = \mathcal{C}^1(\bar{D}) \cap \mathcal{O}(D)$ such that

$$u > 0 \text{ on } \mathbb{R} \quad \text{and} \quad e^{2u} < |f|^2 \text{ on } bD.$$

The map of class $\mathcal{A}^1(D)$ defined by

$$F = f_h := e^{-h} f : \bar{D} \rightarrow \mathbb{C}_*$$

then satisfies

$$|F|^2 = e^{-2u} |f|^2 < |f|^2 = 1 \text{ on } \mathbb{R}, \quad |F|^2 = e^{-2u} |f|^2 > 1 \text{ on } bD.$$

Hence,

$$F(\mathbb{R}) \subset \mathbb{B}^n \quad \text{and} \quad F(bD) \cap \bar{\mathbb{B}}^n = \emptyset.$$

The main lemma

Assume in addition that F is transverse to the sphere $b\mathbb{B}^n$. Let

$$(*) \quad \Omega \subset \{z \in D : F(z) \in \mathbb{B}^n\} = F^{-1}(\mathbb{B}^n)$$

denote the connected component of $F^{-1}(\mathbb{B}^n)$ containing \mathbb{R} . It follows that $\overline{\Omega} \subset D$ and $F|_{\Omega}: \Omega \rightarrow \mathbb{B}^n$ is a proper holomorphic map extending holomorphically to $\overline{\Omega}$ and mapping $b\Omega$ to the sphere $b\mathbb{B}^n$. Clearly, Ω is Runge in \mathbb{C} and hence conformally equivalent to the disc.

The main theorem is an immediate consequence of the following lemma.

Lemma (2)

Let $n > 1$. Given $\epsilon > 0$, there exist a smoothly bounded, simply connected, symmetric domain $D \subset \mathbb{C}$ satisfying $\mathbb{R} \subset D \subset \overline{D} \subset S$ and a function $h = u + iv \in \mathcal{A}^1(D)$ such that the map $F = e^{-h}f: \overline{D} \rightarrow \mathbb{C}_^n$ is an injective immersion transverse to $b\mathbb{B}^n$ and satisfying*

(α) $\text{Area}(F(\Omega)) < \epsilon$ where Ω is given by (), and*

(β) the curve $F(b\Omega) \subset b\mathbb{B}^n$ is everywhere dense in the sphere $b\mathbb{B}^n$.

The function space $\mathcal{H}(\mathbb{D})$

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}, \quad \mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\},$$

$$\mathcal{H} = \{h = u + iv \in \mathcal{A}^\infty(\mathbb{D}) : h(\bar{z}) = \overline{h(z)}, \quad u|_{\mathbb{T}} = 0 \text{ near } \pm 1\}.$$

Note that every function $h = u + iv \in \mathcal{H}$ is determined by a smooth real function $u \in \mathcal{C}^\infty(\mathbb{T})$, supported away from the points ± 1 and satisfying $u(e^{it}) = u(e^{-it})$ for all $t \in \mathbb{R}$, by the formula

$$h(z) = T[u](z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} u(e^{i\theta}) d\theta, \quad z \in \mathbb{D}.$$

We shall write

$$\mathcal{H}^\pm = \{h = u + iv \in \mathcal{H} : \pm u \geq 0\}, \quad \mathcal{H}_*^\pm = \mathcal{H}^\pm \setminus \{0\}.$$

The sets \mathcal{H}^\pm and \mathcal{H}_*^\pm are real cones, $\mathcal{H}^+ \cap \mathcal{H}^- = \{0\}$, and for every $h = u + iv \in \mathcal{H}_*^+$ we have that

$$u > 0 \text{ on } \mathbb{D}, \quad \frac{\partial u}{\partial x}(-1) > 0, \quad \frac{\partial u}{\partial x}(1) < 0, \quad v = 0 \text{ on } \mathbb{R} \cap \mathbb{D}.$$

A general position lemma

Lemma (3)

Let $f : \overline{\mathbb{D}} \rightarrow \mathbb{C}_*^n$ ($n > 1$) be a map of class $\mathcal{A}^\infty(\mathbb{D})$ such that

(a) $|f|^2 := |f_1|^2 + \cdots + |f_n|^2 \leq 1$ on $(-1, 1) = \mathbb{D} \cap \mathbb{R}$,

(b) $|f| > 1$ on $\mathbb{T} \setminus \{\pm 1\}$, and

(c) $\sigma \circ f = [f_1 : f_2 : \cdots : f_n] : \overline{\mathbb{D}} \rightarrow \mathbb{C}\mathbb{P}^{n-1}$ is an immersion.

Assume that $E \subset b\mathbb{B}^n$ is a compact smoothly embedded curve such that $f(\pm 1) \notin E$. Given a number $\eta \in (0, 1)$, there is a function $h \in \mathcal{H}_*^+$ arbitrarily close to 0 in $\mathcal{C}^1(\overline{\mathbb{D}})$ such that the $\mathcal{A}^\infty(\mathbb{D})$ immersion

$$f_h := e^{-h} f : \overline{\mathbb{D}} \rightarrow \mathbb{C}_*^n$$

satisfies:

- 1 $|f_h| < 1$ on $(-1, 1)$ and $|f_h| > (1 - \eta)|f| + \eta > 1$ on $\mathbb{T} \setminus \{\pm 1\}$,
- 2 f_h is transverse to the sphere $b\mathbb{B}^n$, and
- 3 $f_h(\overline{\mathbb{D}}) \cap E = \emptyset$.

Proof of Lemma 3, #1

Given $h = u + iv \in \mathcal{H}$, we define the functions $\rho = \rho_0$ and ρ_h by

$$\rho = \log |f|: \overline{\mathbb{D}} \rightarrow \mathbb{R}, \quad \rho_h := \log |e^{-h}f| = -u + \rho: \overline{\mathbb{D}} \rightarrow \mathbb{R}.$$

Conditions (a) and (b) on f imply that

$$\rho \leq 0 \text{ on } [-1, 1] = \overline{\mathbb{D}} \cap \mathbb{R}, \quad \rho(\pm 1) = 0, \quad \rho > 0 \text{ on } \mathbb{T} \setminus \{\pm 1\}.$$

It follows that

$$\frac{\partial \rho}{\partial x}(-1) \leq 0 \quad \text{and} \quad \frac{\partial \rho}{\partial x}(1) \geq 0.$$

It is obvious that for any $h \in \mathcal{H}_*^+$ with sufficiently small $\mathcal{C}^0(\overline{\mathbb{D}})$ norm the map f_h satisfies condition (1) of the lemma.

Proof of Lemma 3, #2

Note that the map $f_h = e^{-h}f$ intersects the sphere $b\mathbb{B}^n$ transversely if and only if 0 is a regular value of the function ρ_h .

Since $\frac{\partial \rho}{\partial x}(-1) \leq 0$ and $\frac{\partial \rho}{\partial x}(1) \geq 0$, we have for every $h \in \mathcal{H}_*^+$ that

$$\frac{\partial \rho_h}{\partial x}(-1) = \frac{\partial \rho}{\partial x}(-1) - \frac{\partial u}{\partial x}(-1) < 0, \quad \frac{\partial \rho_h}{\partial x}(1) > 0.$$

Replacing f by $e^{-h}f$ for some such h close to 0, we may assume that $\rho = \log |f|$ satisfies these conditions. Hence, there are discs $U^\pm \subset \mathbb{C}$ around the points ± 1 such that $d\rho \neq 0$ on $\overline{\mathbb{D}} \cap (\overline{U}^+ \cup \overline{U}^-)$. It follows that for all $h \in \mathcal{H}$ with sufficiently small $\mathcal{C}^1(\overline{\mathbb{D}})$ norm we have

$$d\rho_h \neq 0 \text{ on } \overline{\mathbb{D}} \cap (\overline{U}^+ \cup \overline{U}^-).$$

Since $f(\pm 1) \notin E$, we may choose U^\pm small enough such that

$$f_h(\overline{\mathbb{D}} \cap (\overline{U}^+ \cup \overline{U}^-)) \cap E = \emptyset$$

holds for all $h \in \mathcal{H}$ sufficiently close to 0 in $\mathcal{C}^1(\overline{\mathbb{D}})$.

Proof of Lemma 3, #3

Recall that $\rho = \log |f| < 0$ on $(-1, 1) = \mathbb{D} \cap \mathbb{R}$. Hence, there is an open set $U_0 \in \mathbb{D}$ containing $(-1, 1) \setminus (U^+ \cup U^-) \subset \mathbb{R}$ such that $\rho \leq -c < 0$ on \overline{U}_0 for some $c > 0$.

Since $\rho > 0$ on $\mathbb{T} \setminus \{\pm 1\}$, there is an open set $U_1 \in \mathbb{C}$ containing $\mathbb{T} \setminus (U^+ \cup U^-)$ such that $\rho \geq c' > 0$ on $\overline{U}_1 \cap \overline{\mathbb{D}}$ for some $c' > 0$. It follows that for all $h \in \mathcal{H}^+$ sufficiently close to 0 in $\mathcal{C}^1(\overline{\mathbb{D}})$ we have that $\rho_h < 0$ on \overline{U}_0 , $\rho_h > 0$ on $\overline{U}_1 \cap \overline{\mathbb{D}}$, and hence

$$f_h(\overline{\mathbb{D}} \cap (\overline{U}_0 \cup \overline{U}_1)) \cap b\mathbb{B}^n = \emptyset.$$

For such h it follows that

$$\{z \in \overline{\mathbb{D}} : f_h(z) \in b\mathbb{B}^n, d\rho_h(z) = 0\} \subset K := \overline{\mathbb{D}} \setminus (U_0 \cup U_1 \cup U^+ \cup U^-).$$

Consider the family of functions $\rho_{th} = -tu + \rho : \overline{\mathbb{D}} \rightarrow \mathbb{R}$ for $t \in \mathbb{R}$. Since $\partial\rho_{th}/\partial t = -u < 0$ on \mathbb{D} , transversality theorem implies that for a generic choice of t , 0 is a regular value of $\rho_{th}|_{\mathbb{D}}$. By choosing $t > 0$ small enough, the map $f_{th} = e^{-th}f : \overline{\mathbb{D}} \rightarrow \mathbb{C}_*^2$ satisfies conditions (1) and (2). Replace f by f_h .

Proof of Lemma 3, #4

It remains to achieve also condition (3) in the lemma. We have

$$\{z \in \overline{\mathbb{D}} : f(z) \in E\} \subset K := \overline{\mathbb{D}} \setminus (U_0 \cup U_1 \cup U^+ \cup U^-).$$

It is elementary to see that every $z \in \mathbb{D} \setminus (-1, 1)$ there exists $h_1, h_2 \in \mathcal{H}^+$ such that $h_1(z), h_2(z) \in \mathbb{C}$ are \mathbb{R} -linearly independent.

Since the set $K \subset \mathbb{D} \setminus (-1, 1)$ is compact, there are functions $h_1, \dots, h_N \in \mathcal{H}_*^+$ such that for every point $z \in K$ the vectors $h_j(z) \in \mathbb{C}$ ($j = 1, \dots, N$) span \mathbb{C} over \mathbb{R} . The corresponding family of maps

$$f_t(z) = \exp\left(-\sum_{j=1}^N t_j h_j(z)\right) f(z) \in \mathbb{C}^n, \quad z \in \overline{\mathbb{D}}, \quad t \in \mathbb{R}^N,$$

satisfies

$$\left. \frac{\partial f_t(z)}{\partial t_j} \right|_{t=0} = -h_j(z) f(z), \quad j = 1, \dots, N.$$

Proof of Lemma 3, #5

This means that the spray of maps $f_t(z)$ for $z \in K \subset \mathbb{D}$ and $t \in \mathbb{R}^N$ is **dominating in the radial direction** at $t = 0$.

Recall also that the natural projection $\sigma: \mathbb{C}_*^n \rightarrow \mathbb{C}\mathbb{P}^{n-1}$ satisfies

$$\sigma \circ f_t = \sigma \circ f : \overline{\mathbb{D}} \rightarrow \mathbb{C}\mathbb{P}^1 \text{ is an immersion.}$$

It follows that the map

$$\mathbb{R}^N \times K \ni (t, z) \mapsto f_t(z) \in \mathbb{C}^n$$

is a submersion near $t = 0$ onto a 2-dimensional submanifold $W \subset \mathbb{C}^n$. (If $n = 2$ then this map is a submersion to \mathbb{C}^2 .)

Since E is an arc, $E \cap W$ is a set of finite linear measure in W . Since $\dim K = 2$, the transversality theorem implies that for a generic $t \in \mathbb{R}^N$ near 0 the map $f_t: K \rightarrow \mathbb{C}^n$ misses E by dimension reasons, and hence

$$f_t(\overline{\mathbb{D}}) \cap E = \emptyset.$$

This proves Lemma 3.

Special domains in \mathbb{C}

Let $z = x + iy \in \mathbb{C}$. The antiholomorphic involutions

$$\tau_x(x + iy) = -x + iy, \quad \tau_y(x + iy) = x - iy$$

generate an abelian group $\Gamma = \langle \tau_x, \tau_y \rangle \cong \mathbb{Z}_2^2$.

A set $D \subset \mathbb{C}$ is said to be **Γ -invariant** if $\gamma(D) = D$ holds for all $\gamma \in \Gamma$.

A map $\phi: D \rightarrow \mathbb{C}$ on a Γ -invariant set is said to be **Γ -equivariant** if

$$\phi = \gamma \circ \phi \circ \gamma \quad \text{holds for all } \gamma \in \Gamma.$$

Definition (special domains)

A nonempty connected domain $D \subset \mathbb{C}$ is **special** if it is bounded with \mathcal{C}^∞ smooth boundary, simply connected, and Γ -invariant.

If D is a special domain and $\phi: D \rightarrow D'$ is a biholomorphic map onto a bounded domain $D' = \phi(D)$ with smooth boundary satisfying $\phi(0) = 0$ and $\phi'(0) > 0$, then ϕ is Γ -equivariant if and only if D' is special.

A lemma on conformal maps

The following is a more precise version of a lemma proved by **Erlend Wold** and myself in **J. Math. Pures Appl. 2009**.

Lemma (4)

Assume that $D \subset \mathbb{C}$ is a special domain with the base $D \cap \mathbb{R} = (-a, a)$. Fix a number $b > a$ and set $I^+ = [a, b]$, $I^- = [-b, -a]$, $I = I^+ \cup I^-$.

Given an open neighborhood $V \subset \mathbb{C}$ of I , $\epsilon > 0$, and $\ell \in \mathbb{Z}_+$, there exists a Γ -equivariant biholomorphism $\phi: D \rightarrow \phi(D) = D'$ onto a special domain D' with the base $(-b, b)$ satisfying the following conditions:

- (a) $\phi(0) = 0$, $\phi'(0) > 0$, $\phi(\pm a) = \pm b$,
- (b) $D \subset D' \subset D \cup V$ (hence $D \setminus V = D' \setminus V$), and
- (c) $\|\phi - \text{Id}\|_{\mathcal{C}^\ell(\bar{D} \setminus V)} < \epsilon$.

Proof of Lemma 4, #1

It suffices to prove the result when $D = \mathbb{D}$ and hence $a = 1$. Choose a smaller open neighborhood $V_0 \subset \mathbb{C}$ of $I = I^+ \cup I^-$ such that $\overline{V_0} \subset V$. Pick $\epsilon_0 \in (0, \epsilon)$. Fix a pair of small discs $U_0^+ \Subset U_1^+$ centered at the point $1 \in \mathbb{C}$, let $U_0^- \Subset U_1^-$ be the corresponding discs centered at the point -1 , and set $U_j = U_j^+ \cup U_j^-$ for $j = 0, 1$. We may assume that

$$U_1 \subset V_0 \Subset V, \quad \text{dist}(U_0, \mathbb{C} \setminus U_1) > \frac{\epsilon_0}{2}.$$

Fix an integer $k \in \mathbb{N}$ with $k > 1/(b-1)$ and let

$$I_k = [1, 1 + 1/k] \cup [-1 - 1/k, -1].$$

Choose a smooth map

$$\theta_k: K_k = \left(1 + \frac{1}{2k}\right)\overline{\mathbb{D}} \cup I_k \rightarrow \mathbb{C}$$

which equals the identity on $\left(1 + \frac{1}{2k}\right)\overline{\mathbb{D}}$, maps $[1 + \frac{1}{2k}, 1 + \frac{1}{k}] \subset \mathbb{R}$ diffeomorphically onto $[1 + \frac{1}{2k}, b] \subset \mathbb{R}$, and satisfies

$$\tau_x \circ \theta_k \circ \tau_x = \theta_k \quad \text{where } \tau_x(x + iy) = -x + iy.$$

Proof of Lemma 4, #2

By **Mergelyan's theorem** we can approximate θ_k as closely as desired in $\mathcal{C}^1(K_k)$ by a polynomial map $\vartheta_k: \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\vartheta_k(\pm 1) = \pm 1, \quad \vartheta_k(1 + 1/k) = b, \quad \vartheta_k(-1 - 1/k) = -b.$$

Replacing ϑ_k by $\frac{1}{4} \sum_{\gamma \in \Gamma} \gamma \circ \vartheta_k \circ \gamma$ we ensure that ϑ_k is Γ -equivariant.

Assuming that ϑ_k is sufficiently close to θ_k in $\mathcal{C}^1(K_k)$, it follows that ϑ_k is biholomorphic in an open neighborhood of K_k . Since ϑ_k is Γ -equivariant, it maps $[1, 1 + 1/k]$ diffeomorphically onto $[1, b]$,

$$|\vartheta_k(z) - z| < \frac{\epsilon_0}{2} \quad \text{for all } z \in \left(1 + \frac{1}{2k}\right)\overline{\mathbb{D}} \text{ and } k > \frac{1}{b-1},$$

and that the Γ -invariant domain

$$\Theta_k := \vartheta_k^{-1}(\mathbb{D}) \Subset \left(1 + \frac{1}{4k}\right)\mathbb{D}$$

is an arbitrarily small smooth perturbation of the disc \mathbb{D} , with $\pm 1 \in b\Theta_k$.

Proof of Lemma 4, #3

Pick an open neighborhood $W_k^+ \subset \mathbb{C}$ of $[1, 1 + 1/k] \subset \mathbb{R}$ and set $W_k = W_k^+ \cup \tau_x(W_k^+)$. By choosing k big and W_k^+ small, we have

$$W_k \subset U_0 \quad \text{and} \quad \vartheta_k(W_k) \subset V_0.$$

Let

$$\Omega_k = \Theta_k \cup R_k$$

be a special domain with the base $(-1 - 1/k, 1 + 1/k)$, where $R_k \subset W_k$ is a thin Γ -invariant strip around $[1, 1 + 1/k] \cup (-1 - 1/k, -1]$. Hence

$$\vartheta_k(R_k) \subset \vartheta_k(W_k) \subset V_0.$$

The (unique) biholomorphic map

$$\psi_k: \mathbb{D} \rightarrow \Omega_k, \quad \psi_k(0) = 0, \quad \psi_k'(0) > 0$$

is Γ -equivariant and satisfies $\psi_k(\pm 1) = \pm(1 + 1/k)$. As $k \rightarrow \infty$, the domains $\Theta_k \subset \Omega_k$ converge to the disc \mathbb{D} in the sense of **Carathéodory**, and their closures $\overline{\Theta}_k \subset \overline{\Omega}_k$ converge to $\overline{\mathbb{D}}$.

Proof of Lemma 4, #4

It follows by **Rado's theorem** that the sequence of conformal diffeomorphisms $\psi_k: \overline{\mathbb{D}} \rightarrow \overline{\Omega}_k$ converges to the identity map uniformly on $\overline{\mathbb{D}}$ as $k \rightarrow \infty$. In particular, we have

$$|\psi_k(z) - z| < \frac{\epsilon_0}{2}, \quad z \in \overline{\mathbb{D}}$$

for all big $k \in \mathbb{N}$. For such k , the Γ -equivariant biholomorphic map

$$\phi := \vartheta_k \circ \psi_k: \mathbb{D} \rightarrow D' := \vartheta_k(\Omega_k)$$

satisfies the lemma provided that $\epsilon_0 > 0$ is chosen small enough.

The estimate

$$\|\phi - \text{Id}\|_{\mathcal{C}^\ell(\overline{\mathbb{D}} \setminus V)} < \epsilon \quad (\text{condition (c)})$$

follows from the uniform estimate on the bigger set $\overline{\mathbb{D}} \setminus V_0$ in view of the Cauchy estimates and the reflection principle. (Note that ϕ maps the circle $b\mathbb{D}$ to itself outside of V_0 .)

Proof of the Main Lemma, #1

Recall that $S = \{x + iy \in \mathbb{C} : x \in \mathbb{R}, |y| < g(x)\}$ for some $g: \mathbb{R} \rightarrow (0, +\infty)$ and $f: S \hookrightarrow \mathbb{C}^n$ is a holomorphic embedding such that $f(S) \cap \overline{b\mathbb{B}^n} = f(\mathbb{R})$ is a dense Legendrian curve in $b\mathbb{B}^n$. Hence

$$|f(x + iy)| \geq 1 + c(x)|y|^2$$

for a positive smooth function $c: \mathbb{R} \rightarrow (0, \infty)$. If the strip S is chosen thin enough, then $\text{Area}(f(S)) = \int_S |f'|^2 dx dy$ is as small as desired.

Lemma (2)

Let $n > 1$. Given $\epsilon > 0$, there exist a smoothly bounded, simply connected, symmetric domain $D \subset \mathbb{C}$ satisfying $\mathbb{R} \subset D \subset \overline{D} \subset S$ and a function $h = u + iv \in \mathcal{A}^1(D)$ such that the map $F = e^{-h}f: \overline{D} \rightarrow \mathbb{C}_*^n$ is an injective immersion transverse to $b\mathbb{B}^n$, satisfying

(α) $\text{Area}(F(\Omega)) < \epsilon$ where $\Omega = F^{-1}(\mathbb{B}^n) \subset D$, and

(β) the curve $F(b\Omega) \subset b\mathbb{B}^n$ is everywhere dense in the sphere $b\mathbb{B}^n$.

Proof of the Main Lemma, #2

We construct an increasing sequence of special domains

$$\mathbb{D} = D_1 \subset D_2 \subset D_3 \subset \cdots \subset D = \bigcup_{j=1}^{\infty} D_j \subset S$$

whose union D is a simply connected, smoothly bounded, Γ -invariant domain satisfying $\mathbb{R} \subset D \subset \bar{D} \subset S$.

For every $k \in \mathbb{N}$ we let $D_{k+1} = D_k \cup R_k$ be a special domain with the base $(-k-1, k+1) \subset \mathbb{R}$, furnished by Lemma 4. (Recall that R_k is a thin Γ -invariant strip around the interval $(-k-1, k+1)$.)

For each $k \geq 2$ let ψ_k be the biholomorphism

$$\psi_k: D_k \rightarrow D_{k-1}, \quad \psi_k(0) = 0, \quad \psi_k'(0) > 0.$$

Then ψ_k extends to a smooth Γ -equivariant diffeomorphism $\psi_k: \bar{D}_k \rightarrow \bar{D}_{k-1}$ satisfying $\psi_k([-k, k]) = [-k+1, k-1]$ and $\psi_k(\pm k) = \pm(k-1)$. Set

$$\Psi_1 = \text{Id}|_{\bar{D}_1}, \quad \Psi_k = \psi_2 \circ \cdots \circ \psi_k: \bar{D}_k \rightarrow \bar{D}_1 \quad \forall k = 2, 3, \dots$$

Proof of the Main Lemma, #3

We also find a sequence of multipliers

$$h_k = \tilde{h}_k \circ \Psi_k \in \mathcal{A}^\infty(D_k), \quad \tilde{h}_k \in \mathcal{H}_*^+, \quad k \in \mathbb{N}$$

such that the map

$$F_k = e^{-h_k} f: \overline{D}_k \hookrightarrow \mathbb{C}_*^n$$

is an embedding of class $\mathcal{A}^\infty(D_k)$ that is transverse to $b\mathbb{B}^n$, and F_{k+1} approximates F_k as closely as desired uniformly on \overline{D}_k and in $\mathcal{C}^1(\overline{D}_k \setminus U_k)$, where $U_k = U_k^+ \cup U_k^-$ is a small neighborhood of $\{-k, k\}$.

In the induction step, we first apply Lemma 3 to find a small perturbation of F_k such that $F_k(\overline{D}_k)$ intersects the pair of arcs $E_k^+ = f([k, k+1]) \subset b\mathbb{B}^n$ and $E_k^- = f([-k-1, -k]) \subset b\mathbb{B}^n$ only at the points $f(\pm k)$. Lemma 4 then furnishes the next map

$$F_{k+1} = e^{-h_{k+1}} f: \overline{D}_{k+1} \hookrightarrow \mathbb{C}_*^n$$

which is an embedding mapping the strip $\overline{D_{k+1}} \setminus D_k$ into a small neighborhood of the arcs $E_k^+ \cup E_k^-$.

Proof of the Main Lemma, #4

The sequence h_k is chosen such that it converges on $\bar{D} = \bigcup_{j=1}^{\infty} \bar{D}_j \subset S$ to a multiplier

$$h = u + iv = \tilde{h} \circ \Psi \in \mathcal{A}^1(D),$$

with $\tilde{h} = \lim_{k \rightarrow \infty} \tilde{h}_k \in \mathcal{A}^{\infty}(\mathbb{D})$, satisfying

$$u > 0 \text{ on } \mathbb{R} \quad \text{and} \quad e^{2u} < |f|^2 \text{ on } bD,$$

and the limit map

$$F = \lim_{k \rightarrow \infty} F_k = e^{-h} f: \bar{D} \rightarrow \mathbb{C}_*^n$$

is an injective immersion of class $\mathcal{A}^1(D)$ which is transverse to the sphere $b\mathbb{B}^n$. The connected component

$$\Omega \subset \{z \in D : F(z) \in \mathbb{B}^n\} = F^{-1}(\mathbb{B}^n)$$

containing \mathbb{R} then satisfies the main lemma, and $F(\Omega) \subset \mathbb{B}^n$ is a properly embedded holomorphic disc satisfying the main theorem.

THANK YOU

FOR YOUR ATTENTION