Complete bounded submanifolds in different geometries

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Abstract

We survey recent constructions of **complete bounded submanifolds** in several geometries:

- **holomorphic submanifolds** (the problem of **Paul Yang, 1977**)
- **null holomorphic curves** and **conformal minimal surfaces** in **Euclidean spaces** (**the Calabi-Yau problem, 1965 & 2000**)
- **Legendrian curves** in **contact complex manifolds**.

A noncompact submanifold M (immersed or embedded) of a manifold X is said to be **bounded** if it is relatively compact.

Let $\mathfrak{g}$ be a Riemannian metric on X. A submanifold $M \subset X$ is said to be **complete** if the pull-back of $\mathfrak{g}$ to M is a complete metric on M. Equivalently, every divergent curve in M (i.e., one that leaves every compact subset of M) has infinite $\mathfrak{g}$-length in X.

If M is bounded, this notion is independent of the choice of $\mathfrak{g}$.

Part I: Complete bounded complex submanifolds of $\mathbb{C}^n$

Paul Yang 1977 Do there exist complete bounded complex submanifolds of complex Euclidean spaces?

Peter Jones 1979 There is a bounded complete holomorphic immersion $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\} \to \mathbb{C}^2$, embedding $\mathbb{D} \to \mathbb{C}^3$, and proper embedding $\mathbb{D} \to \mathbb{B}^4$. (Based on **C. Fefferman**: Every $\phi \in \mathrm{BMO}_{\mathbb{R}}(\mathbb{T})$ equals $\phi = u + \tilde{v}$ where $u, v \in L^\infty(\mathbb{T})$, $\tilde{v}$ the Hilbert transform of v.)

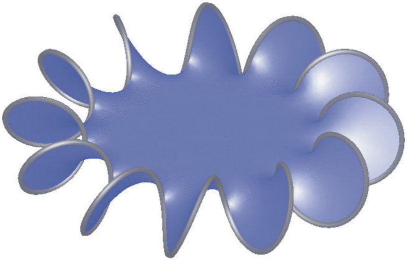
Martin, Umehara and Yamada 2009 There exist complete bounded holomorphic curves in $\mathbb{C}^2$ with arbitrary finite topology.

Theorem

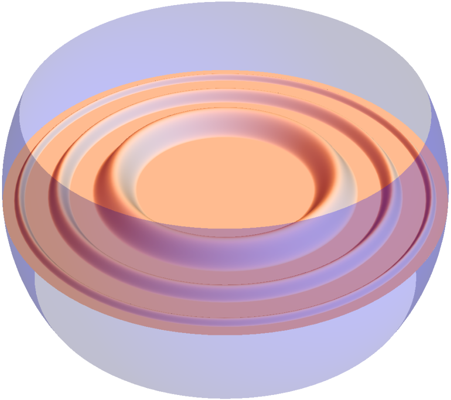
Alarcón and Forstnerič 2013 *Every bordered Riemann surface admits a complete proper holomorphic immersion to $\mathbb{B}^2$ and a complete proper holomorphic embedding to $\mathbb{B}^3$.*

A disc on the way of becoming complete

The illustration shows a **minimal disc** solving a **Plateau problem**. By twisting the boundary curve enough to make it everywhere non-rectifiable, the disc becomes complete. Holomorphic disc are minimal, in fact, absolute area minimizers.



Ripples on a disc increase boundary distance



Complete bounded surfaces abound in nature

Idea of the construction – Pythagora's theorem

Let M be a bordered Riemann surface, and let ds^2 denote the Euclidean metric on $\mathbb{C}^n$.

- Let $F_0 \colon \overline{M} \to \mathbb{C}^n$ be a holomorphic immersion satisfying $|F_0| \geq r_0 > 0$ on bM. We try to increase the boundary distance on M with respect to the induced metric $F_0^* ds^2$ by $\delta > 0$.
- To this end, we approximate F_0 uniformly on a compact set in M by an immersion $F_1 : \overline{M} \to \mathbb{C}^n$ which at a point $p \in bM$ adds a displacement for approximately δ in a direction $V \in \mathbb{C}^n$, $|V| = 1$, approximately orthogonal to the point $F_0(p) \in \mathbb{C}^n$. The boundary distance increases by $\approx \delta$, while the outer radius increases by δ^2:

$$|F_1(p)| \approx \sqrt{|F_0(p)|^2 + \delta^2} \approx |F_0(p)| + \frac{\delta^2}{2|F_0(p)|} \leq |F_0(p)| + \frac{\delta^2}{2r_0}.$$

- Choosing $\delta_j > 0$ such that $\sum_j \delta_j = +\infty$ while $\sum_j \delta_j^2 < \infty$, we obtain by induction a limit immersion $F = \lim_{j\to\infty} F_j \colon M \to \mathbb{C}^n$ with bounded outer radius and with complete metric $F^* ds^2$.

The first main tool – the Riemann-Hilbert problem

This idea can be realized on short arcs $I \subset bM$, on which F_0 does not vary too much, by approximately solving a **Riemann-Hilbert problem**.

Lemma

Let $\mathbb{D} = \{\zeta \in \mathbb{C} \colon |\zeta| < 1\}$ *and* $\mathbb{T} = b\,\mathbb{D} = \{\zeta \in \mathbb{C} \colon |\zeta| = 1\}$.

Let $f \in \mathscr{A}(\mathbb{D}, \mathbb{C}^n)$, *and let* $g \colon \mathbb{T} \times \overline{\mathbb{D}} \to \mathbb{C}^n$ *be a continuous map such that for each* $\zeta \in \mathbb{T}$ *we have* $g(\zeta, \cdot\,) \in \mathscr{A}(\mathbb{D}, \mathbb{C}^n)$ *and* $g(\zeta, 0) = f(\zeta)$.

Given $\epsilon > 0$ *and* $0 < r < 1$, *there are a number* $r' \in [r, 1)$ *and a disc* $h \in \mathscr{A}(\mathbb{D}, \mathbb{C}^n)$ *with* $h(0) = f(0)$ *satisfying the following conditions:*

(i) *for any* $\zeta \in \mathbb{T}$ *we have* $\operatorname{dist}\big(h(\zeta), g(\zeta, \mathbb{T})\big) < \epsilon$,

(ii) *for any* $\zeta \in \mathbb{T}$ *and* $\rho \in [r', 1]$ *we have* $\operatorname{dist}\big(h(\rho\zeta), g(\zeta, \overline{\mathbb{D}})\big) < \epsilon$,

(iii) *for any* $|\zeta| \le r'$ *we have* $|h(\zeta) - f(\zeta)| < \epsilon$, *and*

(iv) *if* $g(\zeta, \cdot\,) = f(\zeta)$ *is the constant disc for all* $\zeta \in \mathbb{T} \setminus J$, *where* $J \subset \mathbb{T}$ *is an arc, then* $|h - f| < \epsilon$ *outside a neighborhood of* J *in* $\overline{\mathbb{D}}$.

Proof of the Riemann-Hilbert lemma

Write

$$g(\zeta, z) = f(\zeta) + \lambda(\zeta, z), \qquad \zeta \in \mathbb{T},\ z \in \overline{\mathbb{D}},$$

where λ is continuous on $\mathbb{T} \times \overline{\mathbb{D}}$ and holomorphic in $z \in \mathbb{D}$, with $\lambda(\zeta, 0) = 0$. Approximate λ by Laurent polynomials

$$\lambda(\zeta, z) = \frac{1}{\zeta^m} \sum_{j=1}^{N} A_j(\zeta) z^j = \frac{z}{\zeta^m} \sum_{j=1}^{N} A_j(\zeta) z^{j-1}$$

with polynomial coefficients $A_j(\zeta)$. Choose an integer $k > m$ and set

$$h_k(\zeta) = f(\zeta) + \lambda(\zeta, \zeta^k) = f(\zeta) + \zeta^{k-m} \sum_{j=1}^{N} A_j(\zeta) \left(\zeta^k\right)^{j-1}, \quad |\zeta| \leq 1.$$

This is an analytic disc satisfying $h_k(0) = f(0)$. For $\zeta = \mathrm{e}^{\mathrm{i}t} \in \mathbb{T}$ we have

$$h_k(\mathrm{e}^{\mathrm{i}t}) = f(\mathrm{e}^{\mathrm{i}t}) + \lambda(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{k\mathrm{i}t}) \approx g(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}kt)}),$$

and hence (i) holds. It is easy to verify the other conditions for big k.

Exposing boundary points on a Riemann surface

The Riemann-Hilbert method could lead to **sliding curtains** (at least in low dimensions), creating shortcuts in the induced metric on M. We **eliminate shortcuts** by the **exposing of points method**.

Erlend F. Wold & F.F. 2009 Construction of proper holomorphic embeddings of certain bordered Riemann surfaces into $\mathbb{C}^2$.

Set $bM = \cup_i C_i$ where C_i is a Jordan curve. Subdivide $C_i = \cup_j I_{i,j}$ such that any two adjacent arcs $I_{i,j-1}$, $I_{i,j}$ meet at a common endpoint $p_{i,j}$.

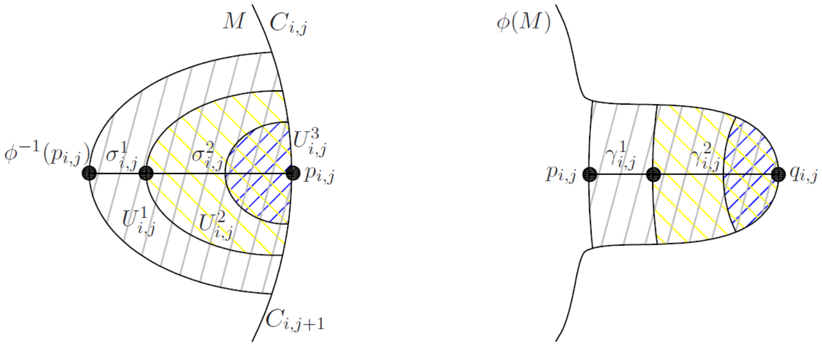
At the point $x_{i,j} = F_0(p_{i,j}) \in \mathbb{C}^n$ we attach to $F_0(\overline{M})$ a smooth real curve $\lambda_{i,j}$ of length $> \delta$ which increases the outer radius by $< \delta^2$. Let $y_{i,j}$ be other endpoint of $\lambda_{i,j}$.

Choose an arc $\gamma_{i,j} \subset R \setminus M$ attached to M at $p_{i,j}$, with the other endpoint $q_{i,j}$. Extend F_0 to a smooth diffeomorphism $\gamma_{i,j} \to \lambda_{i,j}$ mapping $q_{i,j}$ to $y_{i,j}$. Use Mergelyan to approximate F_0 by a holomorphic map from a neighborhood of $\overline{M} \cup \gamma_{i,j}$ to $\mathbb{C}^n$.

Exposing a boundary point

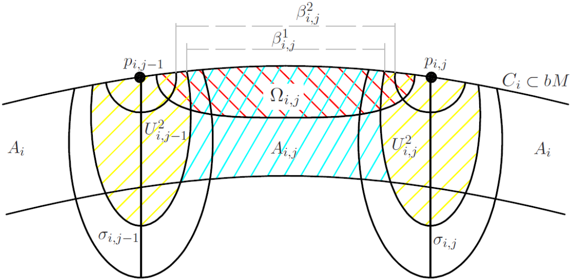
The main point: there is a biholomorphism $\phi\colon \overline{M} \to \phi(\overline{M}) \subset R$ sending each $p_{i,j} \in bM$ to the other endpoint $q_{i,j}$ of the attached arc $\gamma_{i,j} \subset R$, and close to the identity away from the points $p_{i,j}$. Define G by

$$G = F_0 \circ \phi : M \to \mathbb{C}^n.$$



Increasing the boundary distance

In the metric $G^*(ds^2)$ on M, the distance to the yellow neighborhoods of the points $p_{i,j} \in bM$ increased by the length of $\lambda_{i,j}$ which is $> \delta$.
Apply the Riemann-Hilbert method on the arc $\beta^2_{i,j} \subset bM$ to increase the distance to it by $> \delta$. These two deformations are performed in almost orthogonal directions, so they don't cancel each other.
The boundary distance increased by $> \delta$ and the outer radius by $< \delta^2$.



Embedded complete complex submanifolds

This method works well on any bordered Riemann surface M and allows a complete control of the complex structure (i.e., no part of M needs to be cut away in order to keep its image suitably bounded). This was the main novelty with respect to the previous results in the literature.

Diasadvantages:

- It does not give complete bounded **embeddings into $\mathbb{C}^2$**, and
- it does not work on **higher dimensional manifolds**.

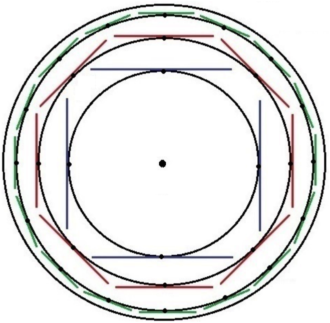
Another idea: start with a closed complex submanifold $X \subset \mathbb{C}^n$.

In the ball $\mathbb{B}^n \subset \mathbb{C}^n$, choose a suitable **labyrinth** $\mathfrak{F} = \cup_j K_j$, where each K_j is a closed ball (or polytope) in an affine real hyperplane $\Lambda_j \subset \mathbb{C}^n$, such that any path in $\mathbb{B}^n \setminus \mathfrak{F}$ terminating on $b\mathbb{B}^n$ has infinite length.

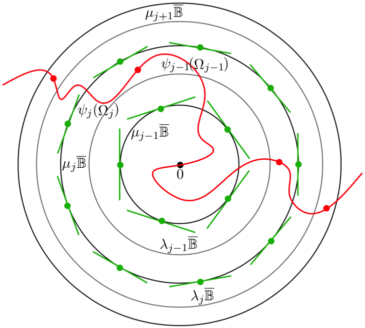
Then, use holomorphic automorphisms of $\mathbb{C}^n$ to push X away from $\mathfrak{F}$.

A complex subvariety avoiding a labyrinth

A labyrinth consisting of tangent balls. Any divergent curve in $\mathbb{B}^n$ avoiding all except finitely many of these balls has infinite length.



The subvariety $X \subset \mathbb{C}^n$ is twisted by holomorphic automorphisms so that it avoids the labyrinth $\mathfrak{F}$. The image is ambiently complete.



A theorem of Globevnik, Alarcón and López

Theorem

For every closed complex submanifold $X \subset \mathbb{C}^n$ and compact set $L \subset X \cap \mathbb{B}^n$ there exists a Runge domain $\Omega \subset X \cap \mathbb{B}^n$ with $L \subset \Omega$ which admits a complete proper holomorphic embedding into $\mathbb{B}^n$.

In particular, every open orientable surface S admits a complex structure J such that the Riemann surface $R = (S, J)$ admits a complete proper holomorphic embedding to $\mathbb{B}^2$.

This gives an affirmative answer to Yang's original question in all dimensions and codimensions. The shortcoming is that **one cannot control the complex structure of the examples**.

Also, this uses that $\mathbb{C}^n$ has a lot of holomorphic automorphisms (**the Andersén-Lempert theory**), which fails in most other interesting geometries that we wish to consider.

Part II: Holomorphic Legendrian curves

A **holomorphic directed system** on a complex manifold X is given by a conical complex subvariety $\mathscr{G} \subset TX$ of the tangent bundle. Holomorphic **integral curves** are complex curves tangent to $\mathscr{G}$.

Example (Pfaffian and contact systems)

Let $\xi \subset TX$ be a holomorphic vector subbundle. A complex curve $F\colon M \to X$ is **horizontal**, or **isotropic**, or an **integral curve** if

$$dF_x(T_xM) \subset \xi_{F(x)} \quad \text{for all } x \in M.$$

The case of interest is when ξ is **completely nonintegrable**, in the sense that repeated commutators of vector fields tangent to ξ span TX. When $\dim X = 2k+1$, $\mathrm{rank}\,\xi = 2k$ and first order commutators span, we have $\xi = \ker\alpha$ where α is a holomorphic 1-form satisfying

$$\alpha \wedge \alpha^k \neq 0 \quad \ldots \quad \textbf{a contact form}.$$

Darboux 1882: Locally near each point we have $\xi = \ker\alpha_0$ with

$$\alpha_0 = dz + \sum_{j=1}^{k} x_j dy_j.$$

Standard contact system on $\mathbb{C}^{2k+1}$

Consider the standard contact space $(\mathbb{C}^{2k+1}, \alpha_0)$. Holomorphic integral curves are called **Legendrian curves**. They are plentiful:

Theorem (Alarcón, F., López 2016)

1. *Every immersed Legendrian curve $M \to \mathbb{C}^{2k+1}$ can be approximated uniformly on compacts by properly embedded Legendrian curves.*
2. *Let M be a compact bordered Riemann surface. Every Legendrian curve $M \to \mathbb{B}^{2k+1}$ can be approximated uniformly on compacts in $\mathring{M}$ by complete proper Legendrian embeddings $\mathring{M} \to \mathbb{B}^{2k+1}$.*
3. *Let M be a compact bordered Riemann surface. Every Legendrian curve $M \to \mathbb{C}^{2k+1}$ of class $\mathscr{A}^1(M)$ can be uniformly approximated by topological embeddings $F\colon M \to \mathbb{C}^{2k+1}$ such that $F|_{\mathring{M}}\colon \mathring{M} \to \mathbb{C}^{2k+1}$ is a complete Legendrian embedding.*

Comments about the proof

Consider $\mathbb{C}^3_{(x,y,z)}$ with the contact form $\alpha = dz + xdy$. A Legendrian curve $(x, y, z)\colon M \to \mathbb{C}^3$ is a holomorphic map such that xdy is an exact 1-form and $z = -\int^{\cdot} xdy$.

In an approximation problem on a Runge domain $D \subset M$, first create a **period dominating spray** $(x(\cdot\,,\zeta), y(\cdot\,,\zeta))\colon D \to \mathbb{C}^2$ depending holomorphically on $\zeta \in \mathbb{C}^\ell$, $\ell = \mathrm{rank} H_1(M;\mathbb{Z})$. The approximated spray $(\tilde{x}(\cdot\,,\zeta), \tilde{y}(\cdot\,,\zeta))\colon M \to \mathbb{C}^2$ then contains an element for which $\tilde{x}(\cdot\,,\zeta)d\tilde{y}(\cdot\,,\zeta)$ is exact on D.

Change of topology: extend x, y smoothly to the arc E attached to $D \subset M$ such that $\int_E xdy$ has the correct value. In particular, ensure that $\int_C xdy = 0$ over the new cycle C formed in part by E. Use period dominating sprays and Mergelyan approximation.

The Riemann-Hilbert lemma holds for Legendrian curves: if the central curve $f\colon M \to \mathbb{C}^3$ and all attached boundary discs $g(p, \cdot)\colon \overline{\mathbb{D}} \to \mathbb{C}^3$ $(p \in bM)$ are Legendrian, we can choose a Legendrian approximate solution $h\colon M \to \mathbb{C}^3$ to the Riemann-Hilbert problem.

A hyperbolic contact system on $\mathbb{C}^{2k+1}$

Theorem (F., 2016)

For any $k \geq 1$ there exists a holomorphic contact system ξ on $\mathbb{C}^{2k+1}$ which is **Kobayashi hyperbolic***; in particular, every holomorphic Legendrian curve $\mathbb{C} \to (\mathbb{C}^{2k+1}, \xi)$ is constant.*

Idea of proof: We take $\alpha = \Phi^*\alpha_0$ where $\alpha_0 = dz + \sum_{j=1}^k x_j dy_j$ is the standard contact form on $\mathbb{C}^{2k+1}$ and $\Phi\colon \mathbb{C}^{2k+1} \to \Omega \subset \mathbb{C}^{2k+1}$ is a **Fatou-Bieberbach map** whose image Ω avoids the union of cylinders

$$K = \bigcup_{N=1}^{\infty} 2^{N-1} b\mathbb{D}^{2k}_{(x,y)} \times C_N \overline{\mathbb{D}}_z.$$

If $C_N \geq k2^{3N+1}$ for all $N \in \mathbb{N}$, then $\mathbb{C}^{2k+1} \setminus K$ is α_0-hyperbolic; hence $(\mathbb{C}^{2k+1}, \alpha)$ is hyperbolic.

Darboux charts around immersed Legendrian curves

Let (X, ξ) be an arbitrary contact complex manifold.

Theorem (Alarcón & F. 2017)

Let R be an open Riemann surface with a nowhere vanishing holomorphic 1-form θ, and let $f \colon R \to (X, \xi)$ be a holomorphic Legendrian immersion. Then, every compact set in R has a neighborhood $U \subset R$ and a holomorphic immersion $F \colon U \times \mathbb{B}^{2k} \to X$ such that the contact structure $F^*\xi$ is given by ($x \in U$, the other coordinates Euclidean)

$$\alpha = dz - y\theta(x) - \sum_{j=2}^{k} y_j dx_j. \qquad \textbf{Darboux chart}$$

Corollary

Let $M \subset R$ be a compact bordered Riemann surface. Then $f|_M$ can be uniformly approximated by topological embeddings $F \colon M \to X$ such that $F|_{\mathring{M}} \colon \mathring{M} \to X$ is a complete Legendrian embedding.

Part III: Null holomorphic curves and minimal surfaces

Another classical example of a directed system are **null holomorphic curves** in $\mathbb{C}^n$ and **minimal surfaces**, in $\mathbb{R}^n$.

Let M be an open Riemann surface.

A null holomorphic curve is a holomorphic immersion $Z = (Z_1, \ldots, Z_n) \colon M \to \mathbb{C}^n$ $(n \geq 3)$ whose derivative satisfies

$$(dZ_1)^2 + \cdots + (dZ_n)^2 = 0.$$

An immersion $X = (X_1, \ldots, X_n) \colon M \to \mathbb{R}^n$ is a **conformal minimal (=harmonic) immersion**, abbreviated **CMI**, iff $\partial X = (\partial X_1, \ldots, \partial X_n)$ is a **holomorphic 1-form** on M satisfying the same equation:

$$(\partial X_1)^2 + \cdots + (\partial X_n)^2 = 0.$$

The real part $X = \Re Z$ of a null curve is a CMI; converse holds on simply connected domains.

Weierstrass representation of minimal surfaces

Fix a nowhere vanishing holomorphic 1-form θ on M. Then every conformal minimal immersion $X\colon M \to \mathbb{R}^n$ is of the form

$$X(p) = X(p_0) + \int_{p_0}^{p} \Re(f\theta), \quad p, p_0 \in M,$$

where $f\colon M \to A^{n-1} \setminus \{0\}$ is a holomorphic map into the **null quadric**

$$A^{n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \colon z_1^2 + z_2^2 + \cdots + z_n^2 = 0\}$$

such that the $\mathbb{C}^n$-valued 1-form $f\theta$ has **vanishing real periods**.

Similarly, every **holomorphic null curve** is of the form

$$Z(p) = Z(p_0) + \int_{p_0}^{p} f\theta, \qquad p \in M$$

where f is as above and $f\theta$ has **vanishing periods**.

Example: the catenoid and the helicoid

Example

Consider the null curve

$$Z(\zeta) = (\cos\zeta, \sin\zeta, -\mathrm{i}\zeta) \in \mathbb{C}^3, \qquad \zeta = u + \mathrm{i}v \in \mathbb{C},$$

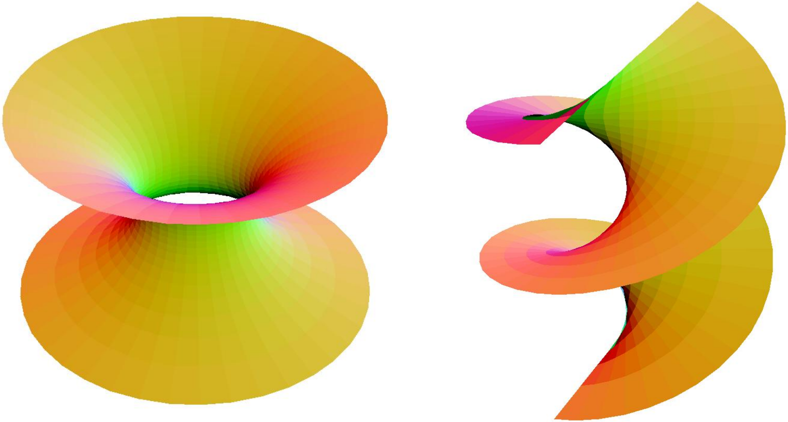
$$\partial Z = (-\sin\zeta, \cos\zeta, -\mathrm{i})d\zeta, \quad \sin^2\zeta + \cos^2\zeta + (-\mathrm{i})^2 = 0,$$

and the associated family of minimal surfaces in $\mathbb{R}^3$ for $t \in \mathbb{R}$:

$$\begin{aligned} X_t(\zeta) &= \Re\left(e^{\mathrm{i}t} Z(\zeta)\right) \\ &= \cos t \begin{pmatrix} \cos u \cdot \cosh v \\ \sin u \cdot \cosh v \\ v \end{pmatrix} + \sin t \begin{pmatrix} \sin u \cdot \sinh v \\ -\cos u \cdot \sinh v \\ u \end{pmatrix} \end{aligned}$$

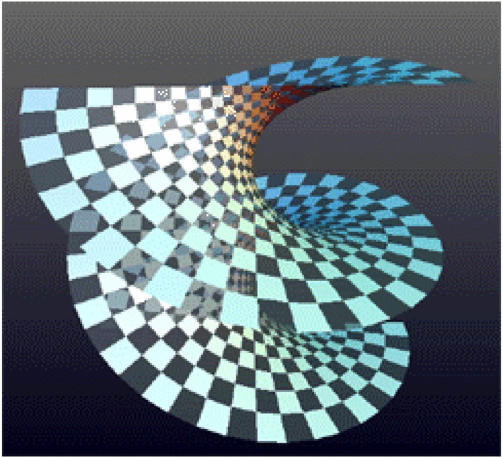
At $t = 0$ we have a **catenoid** and at $t = \pm\pi/2$ a **helicoid**.

The catenoid and the helicoid



The Helicatenoid (Source: Wikipedia)

The family of minimal surfaces $X_t(\zeta) = \Re\left(e^{it} Z(\zeta)\right)$, $\zeta \in \mathbb{C}$, $t \in \mathbb{R}$:



The Calabi-Yau problem for minimal surfaces

Calabi 1965 Conjecture: every complete minimal surface in $\mathbb{R}^3$ is unbounded.

Osserman, Jorge and Meeks 1983 A complete conformal minimal surface in $\mathbb{R}^3$ of finite total Gauss curvature (FTC) is proper; its conformal type is a finitely punctured compact Riemann surface.

On the other hand, omitting FTC leads to counterexamples:

Jorge & Xavier 1980 There exists a complete minimal surface in $\mathbb{R}^3$ with a bounded coordinate function. (**Calabi was somewhat wrong**.)

Nadirashvili 1996 The disc is a complete bounded immersed minimal surface in $\mathbb{R}^3$. **Ferrer, Martin, Meeks 2012** There exist complete bounded immersed minimal surfaces in $\mathbb{R}^3$ with arbitrary topology. (**Calabi was completely wrong**.)

S.T. Yau 2000: Review of geometry and analysis (the Millenium Lecture). **Calabi-Yau problem**: When is Calabi's conjecture true?

Embedded minimal surfaces in $\mathbb{R}^3$

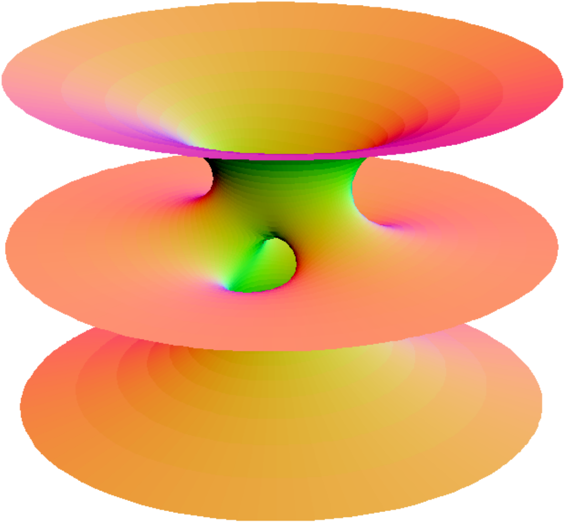
Colding & Minicozzi 2008 A **complete embedded** minimal surface M with **finite topology** in $\mathbb{R}^3$ is **proper**.
(Calabi was right for embedded surfaces with finite topology.)

Meeks-Rosenberg 2005 The helicoid (conformal type $\mathbb{C}$) is the only nonflat, properly embedded, simply connected minimal surface in $\mathbb{R}^3$.

Costa 1984 Besides the plane, the helicoid, and the catenoid, **Costa's surface** was the first example of a complete, properly embedded parabolic minimal surface in $\mathbb{R}^3$.

It is of finite total curvature and has three ends, two catenoidal ones at the top and the bottom (as all FTC properly embedded minimal surfaces besides the plane have) and a planar end in the middle.

Costa's surface



Complete minimal surfaces with Jordan boundaries

Theorem (Alarcón, F., 2015)

Every bordered Riemann surface M admits a complete proper conformal minimal immersion into the ball of $\mathbb{R}^3$.

Theorem (Alarcón, Drinovec, F., López, 2016)

Let M be a compact bordered Riemann surface, and let $n \geq 3$. Every conformal minimal immersion $f\colon M \to \mathbb{R}^n$ can be approximated, uniformly on M, by continuous maps $F\colon M \to \mathbb{R}^n$ such that $F|_{bM}\colon bM \to \mathbb{R}^n$ is a topological embedding and $F|_{\mathring{M}}\colon \mathring{M} \to \mathbb{R}^n$ is a complete immersed conformal minimal surface (embedded if $n \geq 5$).

Our surfaces don't have FTC, but we have a complete control of both the conformal structure (any bordered Riemann surface) and of the boundary (a union of Jordan curves).

Catenoidal cloud over the Sierra Nevada (Granada)

∼ Thank you for your attention ∼

Image Credit & Copyright: Guido Montañés