

# OKA MANIFOLDS AND MINIMAL SURFACES

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# Abstract

In this lecture, I will explain a connection between

- **modern Oka theory**, a branch of complex analysis, and
- **the theory of minimal surfaces in  $\mathbb{R}^n$  and null curves in  $\mathbb{C}^n$  for any  $n \geq 3$ .**

Based on collaboration with

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# Who is Oka?

## Kiyoshi Oka, Japanese mathematician, 1901–1978



Kiyoshi Oka was one of the pioneers of complex analysis in several variables. He solved the famous **Levi problem**, characterising **domains of holomorphy** in  $\mathbb{C}^n$  (a global condition concerning the space of holomorphic functions) by **pseudoconvexity** (a geometric condition).

# Milestones in the classical Oka-Grauert theory

**Oka, 1939:** A holomorphic line bundle over a domain of holomorphy is holomorphically trivial if (and only if) it is topologically trivial.

**Stein, 1951:** Studied a class of complex manifolds, **Stein manifolds**, whose properties are similar to those of domains of holomorphy. They have a big algebra of global holomorphic functions (separation of points by holomorphic functions, holomorphic convexity).

**Behnke, Stein, 1949:** Every open Riemann surface is a Stein manifold.

**Remmert, 1956; Bishop, Narasimhan, 1961:** Stein manifolds are the closed complex submanifolds of complex Euclidean spaces  $\mathbb{C}^N$ .

**Grauert, 1958:** For principal fiber bundles with complex Lie group fiber over a Stein base, the topological and the holomorphic classifications coincide. This holds in particular for complex vector bundles.

**The Oka-Grauert Principle:** There are only topological obstructions to solving cohomologically formulated complex analytic problems on Stein manifolds (and Stein spaces).

# Who is Grauert?

## Hans Grauert, German mathematician, 1930–2011



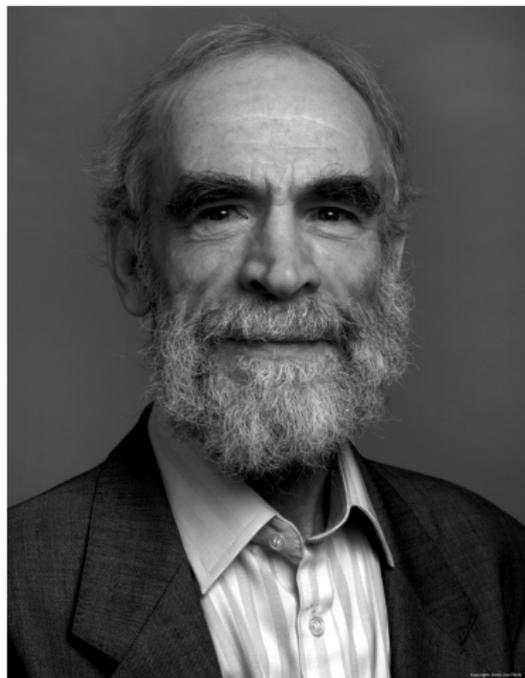
Grauert was one of the most influential mathematicians in complex analysis and analytic geometry during the second half of 20th century.

Besides the Oka-Grauert principle, he and **Reinhold Remmert** developed the theory of **complex analytic spaces** and **coherent analytic sheaves**.

He also gave numerous major contributions to complex geometry.

# Everybody knows who is...

## Mikhail Gromov, 1943–; Abel prize 2009



The Oka-Grauert theory was revitalized by Mikhael Gromov whose paper *Oka's principle for holomorphic sections of elliptic bundles*, *J. Amer. Math. Soc.* 2 (1989) marks the beginning of **modern Oka theory**.

The emphasis shifted to the **homotopy theoretic aspect**, focusing on those analytic properties of a complex manifold  $Y$  which insure that every continuous map  $X \rightarrow Y$  from a Stein manifold  $X$  is homotopic to a holomorphic map, with certain natural additions.

# Rigidity versus flexibility in complex geometry

In complex analysis and geometry, there is an important dichotomy between **holomorphic rigidity** and **holomorphic flexibility**.

A quintessential rigidity property is **Brody hyperbolicity**.

## Definition

A complex manifold  $Y$  is Brody hyperbolic if every holomorphic map  $\mathbb{C} \rightarrow Y$  is constant.

Another one is **Kobayashi hyperbolicity** with differential-geometric flavour. These two properties coincide on any compact manifold.

A Riemann surface is hyperbolic if and only if it is a quotient of the disc  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . Hence, most Riemann surfaces are hyperbolic.

Most complex manifolds are **almost hyperbolic**. The famous **Green-Griffiths Conjecture** claims that for any compact complex manifold  $Y$  of general type (i.e., of maximal Kodaira dimension), all nonconstant images of  $\mathbb{C}$  lie in a proper complex subvariety  $Y' \subset Y$ .

# The convex approximation property

## What is a suitable notion of holomorphic flexibility of $Y$ ?

A heuristic answer: The existence of ‘many’ entire maps  $\mathbb{C}^n \rightarrow Y$ .

### Definition (F., 2005)

A complex manifold  $Y$  enjoys the **Convex Approximation Property (CAP)** if every holomorphic map  $K \rightarrow Y$  from a neighborhood of a compact convex set  $K$  in some Euclidean space  $\mathbb{C}^n$  can be approximated, uniformly on  $K$ , by entire maps  $\mathbb{C}^n \rightarrow Y$ .

When  $Y = \mathbb{C}$ , this is a very special case of the **Oka-Weil approximation theorem**, also called **Runge’s theorem** in one variable.

It holds for functions  $X \rightarrow \mathbb{C}$  on any Stein manifold  $X$ , with approximation on compact holomorphically convex subsets  $K = \hat{K}_{\theta(X)}$ .

# CAP $\iff$ Oka–Grauert principle; Oka manifolds

## Theorem (F., 2005-2010)

*If a complex manifold  $Y$  enjoys CAP, then maps  $X \rightarrow Y$  from any Stein manifold (or Stein space)  $X$  enjoy the **Oka–Grauert principle**.*

*This means in particular that any continuous map  $f_0: X \rightarrow Y$  is homotopic to a holomorphic map  $f_1: X \rightarrow Y$ . Furthermore, if  $f_0$  is holomorphic on a compact holomorphically convex set  $K \subset X$  and on a closed complex subvariety  $X' \subset X$ , then the homotopy can be chosen fixed on  $X'$  and close to  $f_0$  on  $K$ .*

*Analogous result holds for families of maps. In particular, the inclusion  $\mathcal{O}(X, Y) \hookrightarrow \mathcal{C}(X, Y)$  is a weak homotopy equivalence.*

This answers a question of Gromov from 1989. A complex manifold  $Y$  satisfying these equivalent properties is called an **Oka manifold**.

**F.F.: Stein manifolds and holomorphic mappings. Ergebnisse, 3. Folge, 56. Springer-Verlag, Berlin-Heidelberg (2011)**

# Complex Lie groups are Oka manifolds

**Grauert, 1958:** Every complex Lie group, and every complex homogeneous manifold, is an Oka manifold.

**Sketch of proof:** Let  $Y$  be a complex Lie group with the Lie algebra  $\mathfrak{g} = T_1 Y$ . Denote by  $\exp: \mathfrak{g} \rightarrow Y$  the exponential map.

It suffices to show that CAP holds. Let  $K$  be a compact convex set in some  $\mathbb{C}^n$ . If  $f: K \rightarrow Y$  is a holomorphic map whose range lies close to  $1 \in Y$ , then  $f = \exp h$  for some  $h: K \rightarrow \mathfrak{g}$ . The Oka-Weil theorem gives entire maps  $\tilde{h}: \mathbb{C}^n \rightarrow \mathfrak{g}$  approximating  $h$  on  $K$ . Then,  $\tilde{f} = \exp \tilde{h}: \mathbb{C}^n \rightarrow Y$  is an entire map approximating  $f$  on  $K$ .

Since  $K$  is convex, there is a homotopy  $f_t: K \rightarrow Y$  from  $f_1 = f$  to a constant map  $f_0: K \rightarrow p \in Y$ . Choose a big integer  $k \in \mathbb{N}$  and write

$$f_1 = f_{1/k} \cdot (f_{1/k}^{-1} \cdot f_{2/k}) \cdots (f_{(k-1)/k}^{-1} \cdot f_1).$$

Each map  $g_j = f_{(j-1)/k}^{-1} \cdot f_{j/k}: K \rightarrow Y$  is close to 1, hence the previous argument applies. Thus,  $Y$  enjoys CAP, and hence is Oka.

# Gromov's theorem: Elliptic manifolds

Gromov replaced the use of the exponential map by dominating sprays.

## Definition

A **spray** on a complex manifold  $Y$  is a triple  $(E, \pi, s)$ , where  $\pi: E \rightarrow Y$  is a holomorphic vector bundle and  $s: E \rightarrow Y$  is a holomorphic map satisfying  $s(0_y) = y$  for every point  $y \in Y$ . The spray is **dominating** if  $s|_{E_y} \rightarrow Y$  is a submersion at  $0_y \in E_y$  for every point  $y \in Y$ . The manifold  $Y$  is **elliptic** if it admits a dominating spray.

## Theorem (Gromov, 1989)

*Every elliptic manifold is an Oka manifold. (The converse is not known.)*

In view of the main theorem, it suffices to prove that an elliptic manifold satisfies CAP. This is done by using the dominating spray to linearize the approximation problem, thereby reducing it to the Oka-Weil theorem for sections of holomorphic vector bundles over Stein manifolds.

# Manifolds generated by complete vector fields

## Corollary

If  $Y$  admits  $\mathbb{C}$ -complete holomorphic vector fields  $V_1, \dots, V_k$  spanning the tangent space  $T_Y Y$  at every point, then  $Y$  is elliptic, and hence Oka.

**Proof:** Let  $\phi_t^j$  denote the flow of  $V_j$  for time  $t \in \mathbb{C}$ . Then, the map  $s: E = Y \times \mathbb{C}^k \rightarrow Y$ ,  $s(y, t_1, \dots, t_k) = \phi_{t_1}^1 \circ \dots \circ \phi_{t_k}^k(y)$  is a dominating spray on  $Y$ . Indeed,  $\frac{\partial}{\partial t_j} \Big|_{t=0} s(y, t) = V_j(y)$ .

## Example

A spray of this type exists on  $Y = \mathbb{C}^n \setminus A$ , where  $A$  is algebraic subvariety with  $\dim A \leq n - 2$  (i.e.,  $A$  contains no divisors). Indeed, we can use finitely many **shear vector fields** of the form

$$V(z) = f(\pi(z))v, \quad z \in \mathbb{C}^n,$$

where  $v \in \mathbb{C}^n$ ,  $\pi: \mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$  is linear projection such that  $\pi(v) = 0$ , and  $f \in \mathcal{O}(\mathbb{C}^{n-1})$  vanishes on the subvariety  $\pi(A) \subset \mathbb{C}^{n-1}$ .

# Properties of Oka manifolds

Here are a few properties which can be proved by using the characterization of the class of Oka manifolds by CAP.

- **Up-and-down:** If  $\pi: E \rightarrow B$  is a holomorphic fiber bundle whose fiber  $\pi^{-1}(b)$  ( $b \in B$ ) is Oka, then  $B$  is Oka if and only if  $E$  is Oka.
- In particular, if  $\pi: E \rightarrow B$  is a holomorphic covering projection then  $B$  is Oka if and only if  $E$  is Oka. Therefore, every holomorphic quotient of  $\mathbb{C}^n$  and of  $\mathbb{C}^n \setminus \{0\}$  is Oka (tori, Hopf manifolds).
- A Riemann surface is Oka precisely when it is not hyperbolic; these are the surfaces  $\mathbb{C}P^1$ ,  $\mathbb{C}$ ,  $\mathbb{C} \setminus \{0\}$ , and complex tori.
- No (volume) hyperbolic manifold  $X$  is Oka.
- No manifold of Kodaira general type is Oka.

# Examples of Oka manifolds

- Complex Lie groups and homogeneous spaces ( $\mathbb{C}^n$ ,  $\mathbb{C}\mathbb{P}^n, \dots$ );
- $\mathbb{C}^n \setminus A$  and  $\mathbb{P}^n \setminus A$ , where  $A$  is tame subvariety of codimension  $\geq 2$ ;
- Hirzebruch surfaces ( $\mathbb{P}^1$  bundles over  $\mathbb{P}^1$ );
- Hopf manifolds (quotients of  $\mathbb{C}^n \setminus \{0\}$  by a cyclic group);
- algebraic manifolds that are Zariski locally affine ( $\cong \mathbb{C}^n$ );
- certain modifications of such (blowing up points, removing subvarieties of codimension  $\geq 2$ );
- $\mathbb{C}^n$  blown up at all points of a tame discrete sequence;
- complex torus of dimension  $> 1$  with finitely many points removed, or blown up at finitely many points;
- smooth toric varieties:  $(\mathbb{C}^m \setminus Z)/G$ , where  $Z$  is a union of coordinate subspaces of  $\mathbb{C}^m$  and  $G$  is a subgroup of  $(\mathbb{C}^*)^m$  acting on  $\mathbb{C}^m \setminus Z$  by diagonal matrices.

# Connection with the classical minimal surface theory

## Example

The punctured null quadric in  $\mathbb{C}^n$  for  $n \geq 3$ ,

$$\mathfrak{A}_* = \{z = (z_1, \dots, z_n) : z_1^2 + \dots + z_n^2 = 0\},$$

is elliptic, and hence an Oka manifold.

**Proof:** The holomorphic vector fields on  $\mathbb{C}^n$  given by

$$V_{j,k} = z_j \frac{\partial}{\partial z_k} - z_k \frac{\partial}{\partial z_j}, \quad 1 \leq j, k \leq n$$

are linear and hence complete, tangential to the null quadric  $\mathfrak{A}$ , and they generate  $T_z \mathfrak{A}_*$  at each point  $z \in \mathfrak{A}_*$ .

Hence, the composition of their flows is a dominating spray on  $\mathfrak{A}_*$ .

# Conformal minimal immersions $M \rightarrow \mathbb{R}^n$

## Theorem (Classical)

Let  $M$  be a surface endowed with a conformal structure. The following are equivalent for a **conformal** immersion  $\mathbf{X} : M \rightarrow \mathbb{R}^n$  ( $n \geq 3$ ):

- $\mathbf{X}$  is minimal (a stationary point of the area functional).
- $\mathbf{X}$  has identically vanishing mean curvature vector:  $\mathbf{H} = 0$ .
- $\mathbf{X}$  is harmonic:  $\Delta \mathbf{X} = 0$ .

Indeed, we have  $\Delta \mathbf{X} = 2\zeta \mathbf{H}$  where  $\zeta = |\mathbf{X}_u|^2 = |\mathbf{X}_v|^2$  and  $\zeta = u + iv$  be a local holomorphic coordinate on  $M$ .

**Weierstrass representation:** Let  $M$  be a Riemann surface. Fix a nonvanishing holomorphic 1-form  $\theta$  on  $M$ . Then, every conformal minimal immersion  $M \rightarrow \mathbb{R}^n$  ( $n \geq 3$ ) is of the form

$$\mathbf{X}(p) = \mathbf{X}(p_0) + \int_{p_0}^p \Re(\mathbf{f}\theta), \quad p_0, p \in M,$$

where  $\mathbf{f} : M \rightarrow \mathcal{A}_*$  is holomorphic and the real periods of  $\mathbf{f}\theta$  vanish.

# What can be done with these methods?

The **Weierstrass representation** in connection with **Oka theory** (i.e.,  $\mathfrak{A}_*$  is an Oka manifold) allows new constructions of conformal minimal surfaces in  $\mathbb{R}^n$  and holomorphic null curves in  $\mathbb{C}^n$  for any  $n \geq 3$ .

Another useful tool coming from complex analysis is the existence of approximate solutions of the **Riemann-Hilbert problem** for null curves and minimal surfaces (next slide).

The combination of these methods allows us to improve the existing results of many authors, and to obtain new results in the following areas:

- **desingularization and structure theorems;**
- construction of **proper complete conformal minimal surfaces** in certain classes of domains in  $\mathbb{R}^n$ ;
- the **Calabi-Yau problem** for minimal surfaces and null curves.

An important advantage of these methods is that **one can work with a fixed conformal structure of the source Riemann surface.**

# The Riemann-Hilbert problem for minimal surfaces

Assume that  $M$  is a compact bordered Riemann surface and  $\mathbf{X}: M \rightarrow \mathbb{R}^n$  ( $n \geq 3$ ) is a conformal minimal immersion. Let  $I \subset bM$  be an arc.

Let  $\mathbf{Y}: bM \times \overline{\mathbb{D}} \rightarrow \mathbb{R}^n$  be a continuous map of the form

$$\mathbf{Y}(p, \zeta) = \mathbf{X}(p) + f(p, \zeta)\mathbf{u} + g(p, \zeta)\mathbf{v},$$

where  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$  is an orthonormal pair,  $F(p, \cdot) = f(p, \cdot) + ig(p, \cdot)$  is a holomorphic immersion for each  $p \in I$ , and  $F(p, \cdot) = 0$  for  $p \in bM \setminus I$ .

Then, we can find conformal minimal immersions  $\tilde{\mathbf{X}}: M \rightarrow \mathbb{R}^n$  such that

- $\tilde{\mathbf{X}}$  approximates  $\mathbf{X}$  outside a small neighbourhood of  $I$  in  $M$ ;
- $\tilde{\mathbf{X}}(M)$  lies close to  $\mathbf{X}(M) \cup \bigcup_{p \in I} \mathbf{Y}(p, \overline{\mathbb{D}})$ ;
- $\tilde{\mathbf{X}}(p)$  lies close to the curve  $\mathbf{Y}(p, b\mathbb{D})$  for every  $p \in I$ ;
- $\text{Flux}(\tilde{\mathbf{X}}) = \text{Flux}(\mathbf{X})$ .

**A. Alarcón, B. Drinovec Drnovšek, F. Forstnerič, F.J. López,**  
**Proc. London Math. Soc. (3) 111 (2015) 851–886**

# A survey of our main results, 2013–2015

Let  $M$  be an open Riemann surface, and let  $\theta$  be a nonvanishing holomorphic 1-form on  $M$  (such exists by the Oka-Grauert Principle).

- **Oka principle:** Every continuous map  $f_0: M \rightarrow \mathfrak{A}_*$  is homotopic to a holomorphic map  $f: M \rightarrow \mathfrak{A}_*$  such that  $f\theta$  has vanishing periods, and hence it defines a holomorphic null curve  $\int f\theta: M \rightarrow \mathbb{C}^n$ .

**A. Alarcón, F. Forstnerič, Inventiones Math. 196 (2014)**

- **Runge approximation theorem:** If  $K$  is a compact Runge subset of  $M$ , then every conformal minimal immersion  $K \rightarrow \mathbb{R}^n$  on a neighbourhood of  $K$  can be approximated by proper conformal minimal immersions  $M \rightarrow \mathbb{R}^n$ . The analogous result for null curves.

**A. Alarcón, F. Forstnerič, F.J. López, Embedded minimal surfaces in  $\mathbb{R}^n$ . Math. Z., in press.**

- **Isotopies:** Every conformal minimal immersion  $M \rightarrow \mathbb{R}^n$  is isotopic to the real part of a holomorphic null curve  $M \rightarrow \mathbb{C}^n$ .

**A. Alarcón, F. Forstnerič, J. reine angew. Math., in press.**

# General position theorems

- Every holomorphic null curve  $M \rightarrow \mathbb{C}^n$  ( $n \geq 3$ ) can be approximated uniformly on compacts by **embedded** null curves  $M \hookrightarrow \mathbb{C}^n$ .

**A. Alarcón, F. Forstnerič: Inventiones Math. 196 (2014)**

- Every conformal minimal immersion  $M \rightarrow \mathbb{R}^n$  for  $n \geq 5$  can be approximated uniformly on compacts by **proper conformal minimal embeddings**  $M \hookrightarrow \mathbb{R}^n$ .

**A. Alarcón, F. Forstnerič & F.J. López: Embedded minimal surfaces in  $\mathbb{R}^n$ . Math. Z., in press.**

- **Open Problems:** Does every open Riemann surface admit a proper conformal minimal embedding into  $\mathbb{R}^4$ ?

Does it admit a proper holomorphic embedding into  $\mathbb{C}^2$ ?

# Complete proper minimal surfaces

## Theorem

Assume that  $M$  is a **compact bordered Riemann surface**. Every conformal minimal immersion  $\mathbf{X}_0: M \rightarrow \mathbb{R}^n$  can be approximated, uniformly on  $M$ , by continuous maps  $\mathbf{X}: M \rightarrow \mathbb{R}^n$  such that  $\mathbf{X}: \mathring{M} \rightarrow \mathbb{R}^n$  is a **complete conformal minimal immersion** and  $\mathbf{X}: bM \rightarrow \mathbb{R}^n$  is a **topological embedding**.  
If  $n \geq 5$  then  $\mathbf{X}: M \rightarrow \mathbb{R}^n$  can be chosen a topological embedding.

This result is a contribution to the **Calabi–Yau problem**. Pioneering results on this topic were obtained by several authors, in particular:

**Martín, F.; Nadirashvili, N.: A Jordan curve spanned by a complete minimal surface. Arch. Ration. Mech. Anal. 184 (2007)**

**T.H. Colding; , W.P. Minicozzi II: The Calabi-Yau conjectures for embedded surfaces. Ann. of Math. (2) 167 (2008)**

**W.H. Meeks III; J. Pérez; A. Ros: The embedded Calabi-Yau conjectures for finite genus. In preparation.**

# Complete proper minimal surfaces

## Theorem

Assume that  $M$  is a **compact bordered Riemann surface**. Let  $D$  be a convex domain in  $\mathbb{R}^n$ . Every conformal minimal immersion  $\mathbf{X}_0: M \rightarrow D$  can be approximated, uniformly on compacts in  $\overset{\circ}{M}$ , by *proper complete conformal minimal immersions*  $\mathbf{X}: \overset{\circ}{M} \rightarrow D$ . If  $D$  is bounded and strongly convex, then  $\mathbf{X}$  can be chosen continuous on  $M$ .

**A. Alarcón, B. Drinovec Drnovšek, F. Forstnerič, F.J. López, Proc. London Math. Soc. (3) 111 (2015) 851–886**

Pioneering results in this subject were obtained by several authors:

**F. Martín; S. Morales: Complete proper minimal surfaces in convex bodies of  $\mathbb{R}^3$ . Duke Math. J. 128 (2005)**

**L. Ferrer; F. Martín; , W.H. Meeks III: Existence of proper minimal surfaces of arbitrary topological type. Adv. Math. 231 (2012)**

**A. Alarcón, A.; F.J. López: Null curves in  $\mathbb{C}^3$  and Calabi-Yau conjectures. Math. Ann. 355 (2013)**

# Minimal surfaces in minimally convex domains

A domain  $D \subset \mathbb{R}^3$  is **minimally convex** if it admits a smooth exhaustion function  $\rho: D \rightarrow \mathbb{R}$  such that for every point  $\mathbf{x} \in D$ , the sum of the smallest two eigenvalues of  $\text{Hess}_\rho(\mathbf{x})$  is positive.

A domain  $D$  with  $\mathcal{C}^2$  boundary is minimally convex (also called **mean-convex**) if and only if  $\kappa_1(\mathbf{x}) + \kappa_2(\mathbf{x}) \geq 0$  for each point  $\mathbf{x} \in bD$ , where  $\kappa_1(\mathbf{x}), \kappa_2(\mathbf{x})$  are the principal curvatures of  $bD$  at  $\mathbf{x}$ .

## Theorem

*If  $D \subset \mathbb{R}^3$  is a minimally convex domain and  $M$  is a compact bordered Riemann surface, then every conformal minimal immersion  $\mathbf{X}: M \rightarrow D$  can be approximated, uniformly on compacts in  $\overset{\circ}{M}$ , by proper complete conformal minimal immersions  $\tilde{\mathbf{X}}: \overset{\circ}{M} \rightarrow D$ .*

*If  $bD$  is smooth and  $\kappa_1(\mathbf{x}) + \kappa_2(\mathbf{x}) > 0$  for each  $\mathbf{x} \in bD$ , then  $\tilde{\mathbf{X}}$  can be chosen continuous on  $M$ .*

**A. Alarcón, B. Drinovec Drnovšek, F. Forstnerič, F.J. López,**  
**Minimal surfaces in minimally convex domains, arxiv:1510.04006**

# What about non-orientable minimal surfaces?

Assume that  $N$  is a non-orientable surface with a conformal structure.

There is a 2-sheeted covering  $\pi: M \rightarrow N$  by a Riemann surface  $M$  and a fixed-point-free antiholomorphic involution  $\mathfrak{J}: M \rightarrow M$  (the deck transformation of  $\pi$ ) such that  $N = M/\mathfrak{J}$ .

Every conformal minimal immersion  $\mathbf{Y}: N \rightarrow \mathbb{R}^n$  lifts to a  $\mathfrak{J}$ -invariant conformal minimal immersion  $\mathbf{X}: M \rightarrow \mathbb{R}^n$ , i.e.,

$$\mathbf{X} \circ \mathfrak{J} = \mathbf{X}.$$

Conversely, a  $\mathfrak{J}$ -invariant conformal minimal immersion  $\mathbf{X}: M \rightarrow \mathbb{R}^n$  descends to a conformal minimal immersion  $\mathbf{Y}: N \rightarrow \mathbb{R}^n$ .

# Everything is still true...

## Theorem

Let  $M$  be an open Riemann surface (or a bordered Riemann surface) with a fixed-point-free antiholomorphic involution  $\mathfrak{J}$ .

Then, all results mentioned above hold for  $\mathfrak{J}$ -invariant conformal minimal immersions  $M \rightarrow \mathbb{R}^n$ . The same methods apply.

Hence, all mentioned results also hold for conformal minimal immersions  $N \rightarrow \mathbb{R}^n$  from any non-orientable surface  $N$  endowed with a conformal structure, **without having to change the conformal structure.**

$$\begin{array}{ccc} M & & \\ \pi \downarrow & \searrow \mathbf{x} & \\ N & \xrightarrow{\mathbf{Y}} & \mathbb{R}^n \end{array}$$

**A. Alarcón, F. Forstnerič, F.J. López, In preparation.**