# New developments in nonlinear holomorphic approximation theory

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# Abstract

**I will discuss some developments in holomorphic approximation theory, with emphasis on manifold-valued maps.**

The presentation is based on the following preprints:

**B. Chenoweth**: Carleman Approximation of Maps into Oka manifolds.
https://arxiv.org/abs/1804.10680

**J.-E. Fornæss, F. Forstnerič, E.F. Wold:**
Holomorphic approximation: the legacy of Weierstrass, Runge, Oka-Weil, and Mergelyan. https://arxiv.org/abs/1802.03924

**F. Forstnerič**: Mergeljan's and Arakeljan's theorems for manifold-valued maps. https://arxiv.org/abs/1801.04773

**F. Forstnerič**: Holomorphic embeddings and immersions of Stein manifolds: a survey. In: Geometric Complex Analysis (Proceedings of the KSCV12 Symposium, Gyeong-Ju, South Korea, 2017), Springer Proceedings in Mathematics and Statistics, to appear.
http://www.arxiv.org/abs/1709.05630

# From Runge and Weierstrass to Oka and Weil

- **Weierstrass 1885** Every continuous function on a compact interval in $\mathbb{R}$ is the uniform limit of a sequence of holomorphic polynomials.

- **Runge 1885** If $K$ is a compact set with connected complement in $\mathbb{C}$ then every $f \in \mathcal{O}(K)$ is a uniform limit of holomorphic polynomials. More generally, for every compact $K \subset \mathbb{C}$ we can approximate functions in $\mathcal{O}(K)$ by rational functions with poles in $\mathbb{C} \setminus K$.

- **Behnke and Stein 1949, Koditz and Timmann 1975** If $K$ is a compact set in a Riemann surface $X$ then every $f \in \mathcal{O}(K)$ can be approximated uniformly on $K$ by meromorphic functions on $X$ without poles in $K$, and by functions in $\mathcal{O}(X)$ if $K$ has no holes.

- **K. Oka 1936, A. Weil 1935** If $X \subset \mathbb{C}^n$ is a domain of holomorphy (or a **Stein manifold**, **K. Stein 1951**) and $K \subset X$ is a compact $\mathcal{O}(X)$-convex set, then every $f \in \mathcal{O}(K)$ is a uniform limit of functions in $\mathcal{O}(X)$.

# Oka manifolds

Runge's theorem can be viewed as an approximation result for holomorphic maps $X \to \mathbb{C}$ and $X \to \mathbb{CP}^1$.

*To which extent does it generalize to maps $X \to Y$ from Stein manifolds $X$ in the absence of topological obstructions?*

A natural analytic obstruction is the **Kobayashi hyperbolicity** and **Kobayashi-Eisenmann volume hyperbolicity** of the target manifold $Y$. These mean that $Y$ admits few maps $\mathbb{C}^n \to Y$. The following approximation property eliminates these obstructions.

**Definition (F. 2009)**

A complex manifold $Y$ is an **Oka manifold** if every holomorphic map $K \to Y$ from a compact convex set $K \subset \mathbb{C}^n$ (for any $n \in \mathbb{N}$) is a uniform limit of entire maps $\mathbb{C}^n \to Y$.

It suffices to test this property on compact convex polyhedra $K \subset \mathbb{C}^n$.

# A Runge theorem for manifold-valued maps

**Theorem (F., 2006–2010)**

*Let $X$ be a Stein space and $Y$ be an Oka manifold. Given a compact $\mathcal{O}(X)$-convex subset $K \subset X$ and a continuous map $f_0 \colon X \to Y$ which is holomorphic on $K$, we can approximate $f_0$ uniformly on $K$ by a holomorphic map $f_1 \colon X \to Y$ homotopic to $f_0$.*

*We can include interpolation on a subvariety of $X$, parameters, etc.*

*The analogous result holds for sections of any stratified fibre bundle with Oka fibres, or a stratified elliptic submersion over a Stein base.*

Special cases:

- $Y = \mathbb{C}$: the **Oka-Weil approximation theorem, 1935**
- $Y$ complex homogeneous: the **Oka-Grauert theorem, 1939–1958**.
- $Y$ is an elliptic manifold: **Gromov, 1989**.

**Slapar & F. 2007** Runge's approximation theorem holds for maps to an arbitrary complex manifold if we allow a homotopic deformation of the Stein structure on the source Stein manifold $X$.

# Contributions by many people

Many colleagues have contributed significantly to the development to this subject:

- **Forster & Ramspott, 1966–1968** Oka principle for Oka pairs of sheaves. Applications to problems in Stein geometry.
- **Henkin & Leiterer, 1986–** New proof and extensions of the classical Oka-Grauert principle; applications.
- **Prezelj, F., 2000–** Proof of Gromov's Oka principle.
  **Prezelj 2010, 2016**: Extension to 1-convex manifolds.
- **Lárusson, 2003–** A model structure for the Oka principle. Numerous results about the class of Oka manifolds.
- **Kutzschebauch and collaborators, 2008–**
  Density property and flexibility as sufficient conditions for the Oka property. New applications to problems in Stein geometry.
- **Studer, 2018** An axiomatic point of view on the Oka principle.

The status of the subject and open problems are summarised in my book **Stein Manifolds and Holomorphic Mappings**, Ergebnisse vol. 56, 2nd ed., Springer, 2017.

# What about Mergelyan's theorem?

**Mergelyan 1951** If $K$ is a compact set in $\mathbb{C}$ without holes, then every function in $\mathscr{A}(K) = \mathscr{C}(K) \cap \mathcal{O}(\mathring{K})$ is a uniform limit of entire functions.

In view of Runge's theorem, Mergelyan's theorem is equivalent to

The Mergelyan property (MP): $\mathscr{A}(K) = \overline{\mathcal{O}}(K)$.

**Vitushkin 1966** Characterization of MP in terms of continuous capacity.

**Bishop 1958 (localization theorem)** Let $K$ be a compact set in a Riemann surface $X$. If every point $p \in K$ has a compact neighborhood $D_p \subset X$ such that $K \cap D_p$ has MP, then $K$ has MP. In particular, a compact set without holes in an open Riemann surface has DP.

**Boivin and Jiang 2004 (converse to Bishop's theorem)**
If a compact set $K$ in a Riemann surface $X$ has MP, then for every closed coordinate disc $D_p \subset X$ the set $K \cap D_p$ also has MP.

**Verdera 1986** If $K \subset \mathbb{C}$ is a compact set and $f \in \mathscr{C}^1(\mathbb{C})$ satisfies $\bar{\partial} f = 0$ on $K$, then $f$ can be approximated in the $\mathscr{C}^1(\mathbb{C})$ norm by functions in $\mathscr{C}^1(\mathbb{C})$ holomorphic in small neighborhoods of $K$.

# $\bar{\partial}$-proof of Bishop's localization theorem

**Sakai 1972** Let $f \in \mathscr{A}(K)$, $f$ continuous in a neighborhood of $K$. Cover $K$ by finitely many compact sets $D_1, \ldots, D_m$ as in Bishop's theorem such that $\mathring{D}_j$ is an open cover of $K$. Let $\chi_j$ be a subordinate smooth partition of unity. By the assumption, given $\epsilon > 0$ we have functions $f_j \in \mathscr{C}(D_j) \cap \mathcal{O}(K \cap D_j)$ such that $\|f_j - f\|_{\mathscr{C}(K\cap D_j)} < \epsilon$. Set

$$g = \sum_{j=1}^{m} \chi_j f_j.$$

On some open neighborhood $U$ of $K$ we then have

$$\|g - f\|_{\mathscr{C}(U)} < \epsilon, \quad \bar{\partial} g = \sum_{j=1}^{m} f_j \,\bar{\partial}\chi_j = \sum_{j=1}^{m} (f_j - f)\,\bar{\partial}\chi_j = O(\epsilon).$$

Let $\chi \in \mathscr{C}_0^\infty(U)$ be a cut-off function with $0 \le \phi \le 1$ and $\chi \equiv 1$ near $K$. Then $\|\chi\,\bar{\partial} g\|_{\mathscr{C}(\overline{U})} = O(\epsilon)$ and so $T(\chi\bar{\partial} g) = O(\epsilon)$, where $T$ is a Cauchy-Green operator (**Behnke & Stein 1949**). Hence, the function $\tilde{f} = g - T(\chi\bar{\partial} g) \in \mathcal{O}(K)$ approximates $f$ to a precision $O(\epsilon)$ on $K$.

# Mergelyan's theorem in higher dimensions

The situation is much more complicated. Here is a brief summary when $K$ is the closure of a pseudoconvex domain $D$ in $\mathbb{C}^n$:

**Henkin 1969, Lieb 1969, Kerzman 1971**: MP holds for strongly pseudoconvex domains $D$ with smooth boundary: every function in $\mathscr{A}^r(D) = \mathscr{C}^r(\overline{D}) \cap \mathcal{O}(D)$ is a $\mathscr{C}^r(\overline{D})$-limit of functions in $\mathcal{O}(\overline{D})$.

**Diederich & Fornæss 1976**: MP fails on a worm domain.

**Fornæss & Nagel 1977** The Mergelyan property holds in the presence of transverse holomorphic vector fields near the set of weakly pseudoconvex boundary points (the degeneracy set). This holds in particular for bounded pseudoconvex domains with real analytic boundaries in $\mathbb{C}^2$.

**Beatrous & Range 1980** MP holds for $f \in \mathscr{A}(D)$ if $f$ can be approximated on a neighborhood of the degeneracy set of $bD$.

**Laurent-Thiebaut & F. 2007**: MP holds if the degeneration set of $bD$ is a Levi flat hypersurface with Levi foliation defined by a closed nowhere vanishing 1-form. (This excludes the Diederich-Fornæss worm.)

# Strongly pseudoconvex handlebodies

Sakai's proof applies to a compact set $K$ with a basis of Stein neighborhoods on which we can solve the $\bar{\partial}$-equation with uniform estimates (independent of the neighborhood). For instance, it applies to strongly pseudoconvex domains.

While approximation on weakly pseudoconvex domains is of interest, it does not play a role in constructions of global holomorphic maps from Stein manifolds. On the other hand, it is essential to understand approximation on sets of the following type.

**Definition**

An **admissible set** in a complex manifold $X$ is a Stein compact of the form $S = K \cup M$, where $M = S \setminus K$ is a totally real submanifold of $X$:

$$T_x M \cap \mathrm{i} T_x M = \{0\} \quad \text{for all} \quad x \in M.$$

$S$ is **strongly admissible** (or a **strongly pseudoconvex handlebody**) if in addition $K$ is the closure of a strongly pseudoconvex domain (possibly disconnected).

# Mergelyan approximation on handlebodies

**Theorem (Hörmander & Wermer 1968; F. 2005; Manne, Øvrelid, Wold 2011; Fornæss, F., Wold 2018)**

*Assume that $X$ is a complex manifold.*

1. *Let $S = K \cup M$ be an admissible set in $X$ with $M$ of class $\mathscr{C}^k$, $k \geq 1$. Then for any $f \in \mathcal{O}(K) \cap \mathscr{C}^k(M)$ there exists a sequence $f_j \in \mathcal{O}(S)$ with $\lim_{j\to\infty} \|f_j - f\|_{\mathscr{C}^k(S)} = 0$.*
2. *Assume in addition that $K = \overline{D}$ is the closure of a strongly pseudoconvex domain $D \Subset X$. Given $f \in \mathscr{A}(S)$ there is a sequence $f_j \in \mathcal{O}(S)$ such that $\lim_{j\to\infty} \|f_j - f\|_{\mathscr{C}(S)} = 0$.*
3. *If in addition $f \in \mathscr{A}^k(S)$ for some $k \in \mathbb{N}$ and $bD \in \mathscr{C}^\ell$ with $\ell \geq \max\{2, k\}$, then $f_j \in \mathcal{O}(S)$ can be chosen such that $\lim_{j\to\infty} \|f_j - f\|_{\mathscr{C}^k(S)} = 0$.*

**These results also hold for manifold-valued maps** as we shall explain. A less precise version of this theorem has been used in the proof of the Runge theorem for maps from Stein manifolds to Oka manifolds.

# Approximation of manifold-valued maps

## Lemma

*Assume that $K \subset X$ is a compact set with $\mathscr{A}(K) = \overline{\mathcal{O}}(K)$. Let $Y$ be a complex manifold and $f \in \mathscr{A}(K, Y)$. If $G_f = \{(x, f(x)) : x \in K\}$ has a Stein neighborhood in $X \times Y$, then $f \in \overline{\mathcal{O}}(K, Y)$.*

## Proof.

Let $W \subset X \times Y$ be a Stein neighborhood of the graph $G_f$.
Choose a proper holomorphic embedding $\iota : W \hookrightarrow \mathbb{C}^N$ and a holomorphic retraction $\rho : \Omega \to W$ from a Stein domain $\Omega \subset \mathbb{C}^N$ (**Docquier-Grauert 1960**). Assuming that $\mathscr{A}(K) = \overline{\mathcal{O}}(K)$, we can approximate the map

$$K \ni x \mapsto \iota(x, f(x)) \in \Omega \subset \mathbb{C}^N$$

by holomorphic maps $F\colon U \to \Omega$ on open neighborhoods of $K$. The map

$$pr_Y \circ \iota^{-1} \circ \rho \circ F : U \to Y$$

then approximates $f$ on $K$. □

# A Stein neighborhood theorem of Poletsky

The following result of Poletsky provides Stein neighborhoods of a graph over a Stein compact, assuming local Mergelyan approximability.

**Theorem (E. Poletsky 2013)**

*Let $K$ be a Stein compact in a complex manifold $X$ and let $f \in \mathscr{A}(K, Y)$, where $Y$ is an arbitrary complex manifold. Assume that every point $p \in K$ has a compact neighborhood $D_p \subset X$ such that*

$$f|_{K \cap D_p} \in \mathcal{O}(K \cap D_p).$$

*Then, the graph*

$$G_f = \{(x, f(x)) \in X \times Y : x \in K\}$$

*is a Stein compact.*

The proof is similar in spirit to the proof of **Siu's theorem (1976)** that every Stein subvariety in a complex space admits a basis of open Stein neighborhoods. It relies on **fusion of plurisubharmonic functions**.

# Mergelyan approximation of manifold-valued maps

## Corollary

- *If a compact set $K$ in a Riemann surface $X$ has the Mergelyan property for functions, then it has the Mergelyan property for maps to an arbitrary complex manifold.*
- *The same holds if $K$ is a Stein compact with $\mathscr{C}^1$ boundary in an arbitrary complex manifold $X$.*
- *A compact strongly pseudoconvex handlebody has the Mergelyan property for maps to any complex manifold.*

In particular, most of the known Mergelyan-type theorems for functions on smoothly bounded pseudoconvex domains also hold for manifold-valued maps.

# Unbounded closed sets with bounded exhaustion hulls

Let $X$ be a Stein manifold and $E \subset X$ be a closed subset. Set

$$\widehat{E}_{\mathcal{O}(X)} = \widehat{E} = \bigcup_{j=1}^{\infty} \widehat{E}_j$$

where $E_1 \subset E_2 \subset \cdots \subset \bigcup_j E_j = E$ is a normal exhaustion by compacts. The set $\widehat{E}$ is independent of the choice of the exhaustion.

The closed set $E$ has **bounded exhaustion hulls** if for any compact $K \subset X$ there is a bigger compact $K' \subset X$ such that

$$\widehat{K \cup E} \subset (K \cup E) \cup K'.$$

For closed sets $E \subset \mathbb{C}$ this is the **Carleman-Arakelyan condition** that $\mathbb{CP}^1 \setminus E$ is connected and locally connected at $\infty$. This condition is used in the classical theorems of Carleman and Arakelyan:

**Carleman 1926** Approximation of continuous functions on closed curves in $\mathbb{C}$ by entire functions in the fine $\mathscr{C}^0$-topology;

**Arakelyan 1964** Uniform approximation of functions in $\mathscr{A}(E)$.

# Carleman approximation on totally real submanifolds

**Manne 1993** If $X$ is Stein and $E \subset X$ is a closed $\mathscr{C}^k$ totally real submanifold that is $\mathcal{O}(X)$-convex and has bounded exhaustion hulls, then $E$ admits **$\mathscr{C}^k$-Carleman approximation** (i.e, approximation in the fine $\mathscr{C}^k(E)$ topology) by functions in $\mathcal{O}(X)$.

**Magnusson & Wold 2016** If $X$ is a Stein manifold and $E \subset X$ is a closed $\mathcal{O}(X)$-convex set which admits $\mathscr{C}^0$ Carleman approximation by functions in $\mathcal{O}(X)$, then $E$ has bounded exhaustion hulls.

### Theorem (Chenoweth 2018)

*Let $X$ be a Stein manifold and $Y$ be an Oka manifold. Assume that $K \subset X$ is a compact $\mathscr{O}(X)$-convex subset and $E \subset X$ is a closed totally real submanifold of class $\mathscr{C}^k$ $(k \in \mathbb{N})$ which is $\mathscr{O}(X)$-convex, has bounded exhaustion hulls, and $S = K \cup E$ is $\mathscr{O}(X)$-convex.*

*Then, every map $f \in \mathscr{C}^k(X, Y)$ which is holomorphic on a neighbourhood of $K$ (or just on $\mathring{K}$ if $K$ is a compact strongly pseudoconvex domain) can be approximated in the fine $\mathscr{C}^k$ topology on $S$ by holomorphic maps $F \colon X \to Y$.*

# Outline of proof of Chenoweth's theorem

If $E$ has bounded exhaustion hulls, then for every compact set $K \subset X$ the hull $\widehat{K \cup E}$ is closed and $\mathcal{O}(X)$-convex.
If in addition $E = \widehat{E}_{\mathcal{O}(X)}$, then there is a normal exhaustion $(K_j)_{j \in \mathbb{N}}$ of $X$ by $\mathcal{O}(X)$-convex compacts such that

$$K_j \cup E \text{ is } \mathcal{O}(X)\text{-convex for every } j \in \mathbb{N}.$$

The proof proceeds by inductively using Mergelyan's theorem and Runge's theorem for maps to Oka manifolds. Precisely, one finds a sequence of maps $f_j \in \mathscr{C}^k(X, Y) \cap \mathcal{O}(K_j, Y)$ such that

- $f_j$ approximates $f_{j-1}$ uniformly on $K_{j-1}$,
- $f_j$ approximates $f_{j-1}$ in $\mathscr{C}^k(E \cap K_{j+1})$, and
- $f_j = f$ on $E \setminus K_{j+1}$.

If the approximations are close enough, then the sequence $f_j$ converges (uniformly on compacts in $X$ and in the fine $\mathscr{C}^k(E)$-topology) to a map $F \in \mathcal{O}(X, Y)$ with the desired properties.

# Arakelyan's theorem for maps from plane domains to compact homogeneous manifolds

**Arakelyan 1964–1971** The following conditions are equivalent for a closed set $E$ in a domain $X \subset \mathbb{C}$:

(a) Every function in $\mathscr{A}(E)$ is a uniform limit of functions in $\mathcal{O}(X)$.

(b) The complement $X^* \setminus E$ of $E$ in the one point compactification $X^* = X \cup \{*\}$ of $X$ is connected and locally connected. (Equivalently, $E$ has bounded exhaustion hulls.)

**Scheinberg 1978** The same holds if $E$ is a closed set in a Riemann surface $X$ such that the closure of the interior of $E$ admits a covering by a locally finite family of pairwise disjoint open sets, each of finite genus.

**Theorem (F. 2018)**

*Assume that $Y$ is a compact complex homogeneous manifold. If $E$ is an Arakelyan set in a domain $X \subset \mathbb{C}$, then every continuous map $X \to Y$ which is holomorphic in $\mathring{E}$ can be approximated uniformly on $E$ by holomorphic maps $X \to Y$.*

# The difference from Carleman approximation

In Carleman approximation, the subset $E \subset X$ on which we are approximating has no interior near infinity, so it is possible to inductively apply Mergelyan theorem with continuous (or smooth) gluing of a new map with the old one on a suitably chosen compact subset $K \subset E$.

In Arakelyan's theorem, the set $E$ may have nonempty interior also near infinity, and hence we must solve the $\bar{\partial}$-equation with uniform bounds and use this solution in the gluing procedure for manifold-valued maps.

It is classical (see e.g. Ahlfors) that for any compact set $K \subset \mathbb{C}$ and function $g \in L^p(K)$, $p > 2$, the function

$$T_K(g)(z) = \frac{1}{\pi} \int_K \frac{g(\zeta)}{z - \zeta} \, dudv, \quad z \in \mathbb{C}, \ \zeta = u + \mathrm{i}v,$$

vanishes at infinity and satisfies the uniform Hölder condition with exponent $\alpha = 1 - 2/p$ on all of $\mathbb{C}$; moreover, $T_K \colon L^p(K) \to \mathscr{C}^\alpha(\mathbb{C})$ is a continuous linear operator mapping $\mathscr{C}(K)$ boundedly into $\mathscr{C}(\mathbb{C})$.

# Cousin-I problem with bounds on unbounded sets in $\mathbb{C}$

A pair of closed subsets $(A, B)$ in a domain $X \subset \mathbb{C}$ is a *Cartan pair* if

$$K = A \cap B \text{ is compact and } \overline{A \setminus B} \cap \overline{B \setminus A} = \varnothing.$$

Pick a smooth function $\chi \colon X \to [0, 1]$ such that $\chi = 0$ in a neighborhood of $\overline{A \setminus B}$ and $\chi = 1$ in a neighborhood of $\overline{B \setminus A}$. Then,

$$\mathcal{A}(g) = \chi g - T_K(g\bar{\partial}\chi), \qquad \mathcal{B}(g) = (\chi - 1)g - T_K(g\bar{\partial}\chi),$$

are sup-norm bounded linear operators $\mathcal{A} \colon \mathscr{A}(K) \to \mathscr{A}(A)$, $\mathcal{B} \colon \mathscr{A}(K) \to \mathscr{A}(B)$, satisfying

$$g = \mathcal{A}(g) - \mathcal{B}(g), \qquad g \in \mathscr{A}(K).$$

# A nonlinear gluing lemma

By using the above and the implicit function theorem, we prove the following lemma.

**Lemma**

*Let $(A, B, K = A \cap B)$ be as above. Given a ball $0 \in W \subset \mathbb{C}^n$ and numbers $r \in (0, 1)$ and $\epsilon > 0$, there is a $\delta > 0$ satisfying the following property. For every map $\gamma\colon K \times W \to K \times \mathbb{C}^n$ of the form*

$$\gamma(z, w) = \big(z, g(z, w)\big), \quad z \in K, \ w \in W,$$

*and of class $\mathscr{A}(K \times W)$, with $\mathrm{dist}_{K\times W}(\gamma, \mathrm{Id}) < \delta$, there exist maps*

$$\alpha_\gamma : A \times rW \to A \times \mathbb{C}^n, \qquad \beta_\gamma : B \times rW \to B \times \mathbb{C}^n$$

*of the same form and of class $\mathscr{A}(A \times rW)$ and $\mathscr{A}(B \times rW)$, respectively, $\epsilon$-close to the identity and depending smoothly on $\gamma$, such that $\alpha_{\mathrm{Id}} = \mathrm{Id}$, $\beta_{\mathrm{Id}} = \mathrm{Id}$, and*

$$\gamma \circ \alpha_\gamma = \beta_\gamma \quad \text{holds on } K \times rW.$$

# A dominating spray on $Y$

Since the manifold $Y$ is compact homogeneous, the tangent bundle $TY$ is spanned by complete holomorphic vector fields $V_1, \dots, V_n$, $n = \dim Y$. Let $\phi^j_t$ $(t \in \mathbb{C})$ denote the flow of $V_j$. The holomorphic map

$$Y \times \mathbb{C}^n \ni (y,t) = (y, t_1, \dots, t_n) \mapsto s(y,t) = \phi^1_{t_1} \circ \cdots \circ \phi^n_{t_n}(y) \in Y$$

clearly satisfies

$$\partial s(y,t)/\partial t_j|_{t=0} = V_j(y), \qquad j = 1, \dots, n,$$

so it is a *dominating spray* on the trivial bundle $\mathcal{E} = Y \times \mathbb{C}^n \to Y$.

By compactness of $Y$ there are constants $c_0, c_1 > 0$ such that

$$\mathrm{dist}_Y(y, s(y,\xi)) \le c_1|\xi|, \qquad y \in Y,\ \xi \in \mathbb{C}^n,\ |\xi| \le c_0. \qquad (1)$$

# Outline of proof, 1

We follow the scheme of proof of Arakelyan's theorem given by **Rosay and Rudin 1989**, adapting it to manifold-valued maps. We take $X = \mathbb{C}$.

Since the closed set $E \subset \mathbb{C}$ has the BEH property, there are closed discs $\Delta_1 \subset \Delta_2 \subset \cdots \subset \bigcup_{i=1}^{\infty} \Delta_i = \mathbb{C}$ such that, letting $H_i = H_{E \cup \Delta_i}$ denote the union of holes of $E \cup \Delta_i$, we have that

$$\Delta_i \cup \overline{H}_i \subset \mathring{\Delta}_{i+1}, \qquad i = 1, 2, \ldots .$$

Set

$$E_0 = E, \qquad E_i = E \cup \Delta_i \cup \overline{H}_i \ \text{ for } i \in \mathbb{N}.$$

Note that $E_i$ is a closed set with connected complement in $\mathbb{C}$,

$$E_i \subset E_{i+1}, \quad \bigcup_{i=0}^{\infty} E_i = \mathbb{C}, \quad \text{and} \quad E \setminus \Delta_{i+1} = E_i \setminus \Delta_{i+1}.$$

In particular, every $E_i$ is an Arakelyan set.

# Outline of proof, 2

Let $f = f_0 \colon \mathbb{C} \to Y$ be a continuous map such that $f_0|_E \in \mathscr{A}(E)$. Fix a Riemannian distance function $\mathrm{dist}_Y$ on $Y$.

We wish to show that for every $\epsilon > 0$ there is a holomorphic map $F \colon \mathbb{C} \to Y$ such that

$$\sup_{z \in E} \mathrm{dist}_Y(f(z), F(z)) < \epsilon. \tag{2}$$

To this aim, we inductively construct a sequence of continuous maps $f_i \colon \mathbb{C} \to Y$ $(i = 1, 2, \ldots)$ such that

$$f_i|_{E_i} \in \mathscr{A}(E_i, Y)$$

and

$$\mathrm{dist}_Y(f_i(z), f_{i-1}(z)) < 2^{-i}\epsilon, \quad z \in E_{i-1}. \tag{3}$$

Since the sets $E_i$ exhaust $\mathbb{C}$, the sequence $f_i$ converges uniformly on compacts to a holomorphic map $F = \lim_{i \to \infty} f_i \colon \mathbb{C} \to Y$ satisfying (2).

# Outline of proof, 3

Assume inductively that $f_{i-1} \colon \mathbb{C} \to Y$ is a continuous map with $f_{i-1}|_{E_{i-1}} \in \mathscr{A}(E_{i-1})$. Pick a closed disc $\Delta \subset \mathbb{C}$ such that

$$\Delta_i \cup \overline{H}_i \subset \Delta \subset \mathring{\Delta}_{i+1}. \tag{4}$$

Note that $E_i \setminus \Delta = E_{i-1} \setminus \Delta = E \setminus \Delta$. Since $E_{i-1}$ has no holes, $E_{i-1} \cap \Delta_{i+1}$ has no holes either. As $Y$ is an Oka manifold, Mergelyan's theorem furnishes for any $c > 0$ a holomorphic map $h \colon \mathbb{C} \to Y$ satisfying

$$\mathrm{dist}_Y(f_{i-1}(z), h(z)) < c, \quad z \in E_{i-1} \cap \Delta_{i+1}. \tag{5}$$

To get the next map $f_j$, we glue $f_{j-1}$ and $h$ over the Cartan pair decomposition $(A, B)$ of $E_i = A \cup B$ defined by

$$A = \overline{E_i \setminus \Delta}, \quad B = E_i \cap \Delta_{i+1},$$
$$A \cap B = K = E_i \cap \overline{\Delta_{i+1} \setminus \Delta}. \tag{6}$$

Since $E_i$ has no holes, $K$ has no holes either.

# Outline of proof, 4

By Poletsky's Theorem, the graph

$$\Gamma = \{(z, f_{i-1}(z) : z \in K\} \subset \mathbb{C} \times Y$$

admits an open Stein neighborhood $\Omega \subset \mathbb{C} \times Y$.

Recall that $s : \mathcal{E} = Y \times \mathbb{C}^n \to Y$ is a dominating spray on $Y$. We identify $Y$ with the zero section $\mathcal{E}$ and set

$$\mathcal{E}' = \ker ds|_Y = \{(y, \xi) : y \in Y, \ \xi \in \mathbb{C}^n, \ ds_{0_y}(\xi) = 0\}.$$

We consider $\mathcal{E}' \subset \mathcal{E}$ as holomorphic vector bundles over $\mathbb{C} \times Y$ by pulling them back by the projection $\mathbb{C} \times Y \to Y$. The spray map $s$ defines the dominating fibre-spray

$$\sigma \colon \mathcal{E} \to \mathbb{C} \times Y, \qquad \sigma(z, y, \xi) = (z, s(y, \xi)), \tag{7}$$

where $z \in \mathbb{C}$, $y \in Y$, and $\xi \in \mathbb{C}^n$.

Since $\Omega$ is Stein, we have $\mathcal{E}|_{\Omega} = \mathcal{E}'|_{\Omega} \oplus \mathcal{E}''$ for some holomorphic vector subbundle $\mathcal{E}''$ of $\mathcal{E}|_{\Omega}$. The restricted map $\sigma \colon \mathcal{E}''|_{\Omega} \to \mathbb{C} \times Y$ is then injective holomorphic in a neighborhood $\Theta \subset \mathcal{E}''$ of the zero section of $\mathcal{E}''|_{\Omega}$ (which we identify with $\Omega$) and maps $\Theta$ biholomorphically onto a neighborhood of $\Gamma$ (the graph of $f_{i-1}$ over $K$) in $\mathbb{C} \times Y$.

Hence, by the implicit function theorem, there are a neighborhood $\Omega' \subset \Omega$ of $\Gamma$ in $\mathbb{C} \times Y$, a neighborhood $0 \in W \subset \mathbb{C}^n$, and a holomorphic *transition map*

$$G : Z = \big\{(z, y_1, y_2, \xi) : (z, y_1), (z, y_2) \in \Omega',\ \xi \in W\big\} \to \mathbb{C}^n$$

such that

$$\sigma(z, y_1, \xi) = \sigma\big(z, y_2, G(z, y_1, y_2, \xi)\big), \quad (z, y_1, y_2, \xi) \in Z,$$

and

$$G(z, y, y, \xi) = \xi, \quad (z, y) \in \Omega', \quad \xi \in W. \tag{8}$$

# Outline of proof, 6

By taking $y_1 = f_{i-1}(z)$ and $y_2 = h(z)$ for $z \in K$, we obtain a map

$$g(z,\xi) := G(z, f_{i-1}(z), h(z), \xi) \in \mathbb{C}^n, \quad z \in K,\ \xi \in W,$$

of class $\mathscr{A}(K \times W)$ such that

$$\sigma(z, f_{i-1}(z), \xi) = \sigma(z, h(z), g(z,\xi)), \quad z \in K,\ \xi \in W. \tag{9}$$

Consider the Cartan pair decomposition $(A, B)$ of $E_i = A \cup B$ defined by

$$A = \overline{E_i \setminus \Delta}, \quad B = E_i \cap \Delta_{i+1}, \quad A \cap B = K.$$

Pick a number $0 < r_0 < 1$. Assuming that $h$ is sufficiently uniformly close to $f_{i-1}$ on $K$, the map

$$\gamma(z,\xi) = (z, g(z,\xi)), \quad z \in K,\ \xi \in W, \tag{10}$$

is close to the identity map on $K \times W$ in view of (8), with $\mathrm{dist}_Y(\gamma, \mathrm{Id})$ depending on the constant $c > 0$ in (5).

# Outline of proof, 7

Assuming that $c > 0$ is small enough, the gluing lemma furnishes maps

$$\begin{aligned} \alpha(z,\xi) &= (z, a(z,\xi)), \quad z \in A,\ \xi \in r_0 W, \\ \beta(z,\xi) &= (z, b(z,\xi)), \quad z \in B,\ \xi \in r_0 W, \end{aligned} \tag{11}$$

of class $\mathscr{A}(A \times r_0 W)$ and $\mathscr{A}(B \times r_0 W)$, respectively, uniformly close to the identity on their respective domains (depending on the constant $c > 0$ in (5)) and satisfying $\gamma \circ \alpha = \beta$ on $K \times r_0 W$; equivalently,

$$g(z, a(z,\xi)) = b(z,\xi), \quad z \in K,\ \xi \in r_0 W. \tag{12}$$

From (7), (9), (10), (11), and (12) it follows that

$$s\big(f_{i-1}(z), a(z,\xi)\big) = s\big(h(z), b(z,\xi)\big) \in Y, \quad z \in K, \quad \xi \in r_0 W.$$

Setting $\xi = 0$, the two sides of the above equation define a map $f_i \colon E_i \to Y$ of class $\mathscr{A}(E_i)$. It follows from (1) and the construction of $f_i$ that the estimate (3) holds provided the constant $c > 0$ in (5) is chosen small enough. This concludes the induction step.

THANK YOU

FOR YOUR ATTENTION