Every meromorphic function is the Gauss map of a conformal minimal surface

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Abstract

Let M be an open Riemann surface.

- We prove that every meromorphic function on M is the complex Gauss map of a conformal minimal immersion $M \to \mathbb{R}^3$ which may furthermore be chosen as the real part of a holomorphic null curve $M \to \mathbb{C}^3$.
- Analogous results are proved for conformal minimal immersions $M \to \mathbb{R}^n$ and null curves $M \to \mathbb{C}^n$ for any $n > 3$.
- We also show that every conformal minimal immersion $M \to \mathbb{R}^n$ is isotopic (through conformal minimal immersions) to a flat one, and we identify the path connected components of the space of all conformal minimal immersions $M \to \mathbb{R}^n$ for any $n \geq 3$.

Based on joint work with **Antonio Alarcón and Francisco J. López, University of Granada**; http://arxiv.org/abs/1604.00514

Robert Osserman, 1926–2011

The connection between complex analysis and minimal surface theory goes back to **Riemann** and **Weierstrass**.

Robert Osserman was a modern pioneer of this field. His book *A survey of minimal surfaces* (Dover, New York,1986) remains a classic.

Conformal minimal surfaces in $\mathbb{R}^n$

Let M be an open Riemann surface and $n \geq 3$. The following are equivalent for a **conformal immersion** $X = (X_1, \ldots, X_n) : M \to \mathbb{R}^n$:

- X parametrizes a minimal surface.
- X has identically vanishing mean curvature vector.
- X is harmonic: $\triangle X = 0$.
- $\Phi = \partial X = (\phi_1, \ldots, \phi_n)$ is a nowhere vanishing holomorphic 1-form satisfying the following nullity condition:

$$(\phi_1)^2 + (\phi_2)^2 + \cdots + (\phi_n)^2 = 0.$$

Conversely, if $\Phi = (\phi_1, \ldots, \phi_n)$ satisfies the nullity condition and

$$\int_\gamma \Re(\Phi) = 0 \quad \text{for all} \quad \gamma \in H_1(M; \mathbb{Z}),$$

then

$$X(p) = X(p_0) + \int_{p_0}^{p} 2\Re\Phi, \quad p_0, p \in M$$

is a conformal minimal immersion $M \to \mathbb{R}^n$.

Weierstrass representation of minimal surfaces

Fix a nowhere vanishing holomorphic 1-form θ on M. The above shows that every conformal minimal immersion $X\colon M \to \mathbb{R}^n$ is of the form

$$X(p) = X(p_0) + \int_{p_0}^{p} \Re(f\theta), \quad p, p_0 \in M,$$

where $f\colon M \to \mathfrak{A}_*^{n-1} = \mathfrak{A}^{n-1} \setminus \{0\}$ is a holomorphic map with values in the **null quadric**

$$\mathfrak{A}^{n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \colon z_1^2 + z_2^2 + \cdots + z_n^2 = 0\}$$

such that the $\mathbb{C}^n$-valued 1-form $f\theta$ has **vanishing real periods**.

The **flux** of a conformal minimal immersion $X\colon M \to \mathbb{R}^n$ is the group homomorphism $\mathrm{Flux}_X\colon H_1(M;\mathbb{Z}) \to \mathbb{R}^n$ given by

$$\mathrm{Flux}_X(\gamma) = \int_\gamma \Im(\partial X) = -\mathrm{i}\int_\gamma \partial X \quad \text{for every closed curve } \gamma \subset M.$$

Construction of null curves and CMI's

We have $\mathrm{Flux}_X = 0$ iff X admits a harmonic conjugate surface $Y\colon M \to \mathbb{R}^n$. In this case, $Z = X + \imath Y\colon M \to \mathbb{C}^n$ is a **holomorphic null curve**, i.e., a holomorphic immersion satisfying the nullity condition

$$(dZ_1)^2 + (dZ_2)^2 + \cdots + (dZ_n)^2 = 0.$$

Thus, we have bijective correspondences (up to constants):

$$\{Z\colon M \to \mathbb{C}^n \text{ null curve}\} \longleftrightarrow \{f\colon M \to \mathfrak{A}_* \text{ holomorphic, } f\theta \text{ exact}\}$$

$$Z(p) = Z(p_0) + \int_{p_0}^{p} f\theta; \qquad p \in M.$$

$$\{X\colon M \to \mathbb{R}^n \text{ conformal minimal}\} \longleftrightarrow \{f\colon M \to \mathfrak{A}_* \text{ holo., } \Re(f\theta) \text{ exact}\}$$

$$X(p) = X(p_0) + \int_{p_0}^{p} \Re(f\theta); \qquad p \in M.$$

Example: the catenoid and the helicoid

Example: The **catenoid** and the **helicoid** are **conjugate minimal surfaces** – the real and the imaginary part of the same null curve

$$Z(\zeta) = (\cos\zeta, \sin\zeta, -\mathrm{i}\zeta) \in \mathbb{C}^3, \qquad \zeta = u + \mathrm{i}v \in \mathbb{C}.$$

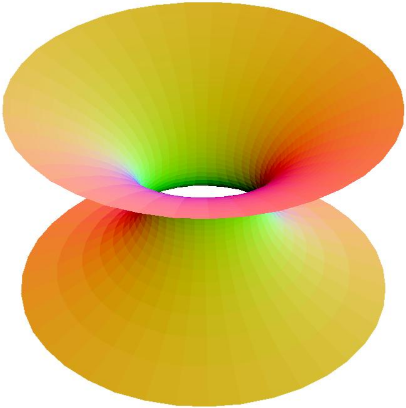
Consider the following family of minimal surfaces in $\mathbb{R}^3$ for $t \in \mathbb{R}$:

$$\begin{aligned} X_t(\zeta) &= \Re\left(e^{\mathrm{i}t} Z(\zeta)\right) \\ &= \cos t \begin{pmatrix} \cos u \cdot \cosh v \\ \sin u \cdot \cosh v \\ v \end{pmatrix} + \sin t \begin{pmatrix} \sin u \cdot \sinh v \\ -\cos u \cdot \sinh v \\ u \end{pmatrix} \end{aligned}$$

At $t = 0$ we have a catenoid and at $t = \pm\pi/2$ a helicoid.

Euler's theorem

1744 **Euler** The only area minimizing surfaces of rotation in $\mathbb{R}^3$ are planes and catenoids.



A Catenoid over Granada

Catenoids appear in nature, sometime in unexpected places. The neck of this cloud over Sierra Nevada seems catenoidal.
© Guido Montañés, http://apod.nasa.gov/apod/ap131126.html

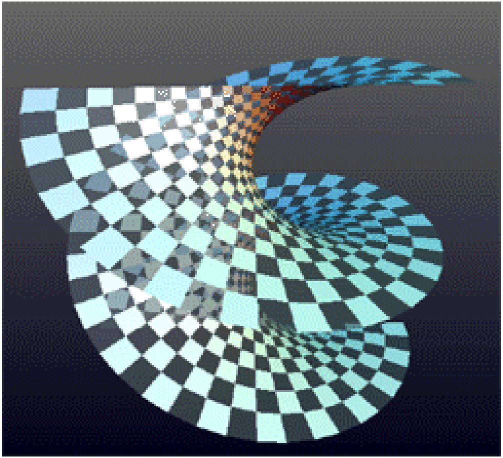
The helicoid (Archimedes' screw)

1776 **Meusnier** The helicoid is a minimal surface.

1842 **Catalan** The helicoid and the plane are the only **ruled** embedded minimal surfaces in $\mathbb{R}^3$ (i.e., unions of straight lines).

The Helicatenoid

The family of minimal surfaces $X_t(\zeta) = \Re\left(e^{\mathrm{i}t} Z(\zeta)\right)$, $\zeta \in \mathbb{C}$, $t \in \mathbb{R}$:



The generalized Gauss map

We now come to the main subject of this talk.

Let $X = (X_1, \ldots, X_n) \colon M \to \mathbb{R}^n$ be a conformal minimal immersion.

Since the 1-form $\partial X = (\partial X_1, \ldots, \partial X_n)$ is holomorphic and nowhere vanishing, it determines the Kodaira type holomorphic map

$$G_X \colon M \to \mathbb{CP}^{n-1}, \quad G_X(p) = [\partial X_1(p) \colon \cdots : \partial X_n(p)] \quad (p \in M).$$

The map G_X is known as the **generalized Gauss map of X** and is of great importance in the theory of minimal surfaces.

Since $\sum_{j=1}^n (\partial X_j)^2 = 0$, G_X assumes values in the hyperquadric

$$Q^{n-2} = \{[z_1 : \ldots : z_n] \in \mathbb{CP}^{n-1} : z_1^2 + \cdots + z_n^2 = 0\} = \pi(\mathfrak{A}_*^{n-1}),$$

where $\pi \colon \mathbb{C}_*^n \to \mathbb{CP}^{n-1}$ denotes the canonical projection.

The main theorem

Theorem

Let M be an open Riemann surface and let $n \geq 3$ be an integer.

- *For every holomorphic map $\mathscr{G}\colon M \to Q^{n-2} \subset \mathbb{CP}^{n-1}$ there exists a conformal minimal immersion $X\colon M \to \mathbb{R}^n$ with the generalized Gauss map $G_X = \mathscr{G}$ and with vanishing flux (hence, X is the real part of a holomorphic null curve $Z\colon M \to \mathbb{C}^n$).*
- *If in addition the map $\mathscr{G}$ is full (i.e., its image is not contained in any proper projective subspace of $\mathbb{CP}^{n-1}$), then X can be chosen to have arbitrary flux and to be an embedding if $n \geq 5$ and an immersion with simple double points if $n = 4$.*

Problem

Suppose that M and $\mathscr{G}$ are algebraic (so $M = \overline{M} \setminus \{p_1, \ldots, p_m\}$ with $\overline{M}$ a compact Riemann surface); can X be chosen algebraic?

The Weierstrass formula for $n = 3$

The quadric $Q^1 \subset \mathbb{CP}^2$ is the image of a quadratically embedded Riemann sphere $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$, and the **complex Gauss map** of a conformal minimal immersion $X = (X_1, X_2, X_3)\colon M \to \mathbb{R}^3$ is defined to be the holomorphic map

$$g_X = \frac{\partial X_3}{\partial X_1 - \mathrm{i}\,\partial X_2} = \frac{\partial X_2 - \mathrm{i}\,\partial X_1}{\mathrm{i}\,\partial X_3} : M \longrightarrow \mathbb{CP}^1.$$

The function g_X is the stereographic projection of the real Gauss map $N = (N_1, N_2, N_3)\colon M \to \mathbb{S}^2 \subset \mathbb{R}^3$ to the Riemann sphere $\mathbb{CP}^1$:

$$g_X = \frac{N_1 + \mathrm{i} N_2}{1 - N_3} : M \longrightarrow \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1.$$

We can recover the differential $\partial X = (\partial X_1, \partial X_2, \partial X_3)$ from the pair (g_X, ϕ_3), with $\phi_3 = \partial X_3$, by the classical **Weierstrass formula**:

$$\partial X = \Phi = (\phi_1, \phi_2, \phi_3) = \left(\frac{1}{2}\left(\frac{1}{g_X} - g_X\right), \frac{\mathrm{i}}{2}\left(\frac{1}{g_X} + g_X\right), 1 \right) \phi_3.$$

The case $n = 3$

Conversely, given a pair (g, ϕ_3) consisting of a holomorphic map $g\colon M \to \mathbb{CP}^1$ and a meromorphic 1-form ϕ_3 on M, the meromorphic 1-form $\Phi = (\phi_1, \phi_2, \phi_3)$ defined by the Weierstrass formula satisfies

$$(\phi_1)^2 + (\phi_2)^2 + (\phi_3)^2 = 0.$$

Φ is the differential ∂X of a conformal minimal immersion $X\colon M \to \mathbb{R}^3$ iff it is holomorphic and nowhere vanishing, and **its real periods vanish**.

Example (The helicatenoid)

$$Z(\zeta) = (\cos\zeta, \sin\zeta, -\mathrm{i}\zeta) \in \mathbb{C}^3, \quad \zeta \in \mathbb{C}$$

$$\partial Z = (-\sin\zeta, \cos\zeta, -\mathrm{i})\,d\zeta$$

$$g_Z(\zeta) = \frac{\partial Z_2 - \mathrm{i}\,\partial Z_1}{\mathrm{i}\,\partial Z_3} = \cos\zeta + \mathrm{i}\sin\zeta.$$

Every meromorphic function is the Gauss map

Hence, the main theorem takes the following form in dimension 3.

Corollary

Let M be an open Riemann surface. Every holomorphic map $g\colon M \to \mathbb{CP}^1$ is the complex Gauss map of a holomorphic null curve $Z = X + \imath Y\colon M \to \mathbb{C}^3$, and hence of conformal minimal immersion $X = \Re Z\colon M \to \mathbb{R}^3$.

If g is nonconstant, then we can find X with arbitrary flux.

In dimension 3, the complex Gauss map is fundamental in the theory; it completely determines several important geometric properties of the minimal surface, and there is a huge literature devoted to it.

Stable minimal surfaces in $\mathbb{R}^3$

We give an example illustrating the last assertion.

A minimal surface $S \subset \mathbb{R}^3$ is said to be **stable** if any relatively compact smoothly bounded domain $D \subset S$ has minimal area among all small variations of $\overline{D}$ which keep the boundary bD fixed.

Barbosa and do Carmo, 1976 Let $X \colon M \to \mathbb{R}^3$ be a conformal minimal immersion. The minimal surface $X(M) \subset \mathbb{R}^3$ is stable if the spherical image $g_X(M) \subset \mathbb{CP}^1$ of its Gauss map has area less than 2π.

This holds in particular if $g_X(M)$ lies in the unit disk $\mathbb{D} \subset \mathbb{C}$.

Corollary

If M is an open Riemann surface and $g \colon M \to \mathbb{CP}^1$ is a holomorphic map whose image $g(M)$ has area less than 2π, then there is a stable conformal minimal immersion $M \to \mathbb{R}^3$ with the complex Gauss map g.

Idea of proof of the main theorem

Since an open Riemann surface M is homotopy equivalent to a wedge of circles and the projection $\pi\colon \mathbb{C}_*^n \to \mathbb{CP}^{n-1}$ is a $\mathbb{C}_*$-bundle, every holomorphic map $\mathscr{G}\colon M \to \mathbb{CP}^{n-1}$ lifts to a holomorphic map $f = (f_1, \ldots, f_n)\colon M \to \mathbb{C}_*^n$ such that

$$\mathscr{G} = \pi \circ f = [f_1 : \cdots : f_n]\colon M \to \mathbb{CP}^{n-1}.$$

Clearly, $\mathscr{G}(M) \subset Q^{n-2}$ if and only if $f(M) \subset \mathfrak{A}_*$.

To prove the theorem, we find a **holomorphic multiplier** $h\colon M \to \mathbb{C}_*$ such that the $\mathbb{C}^n$-valued holomorphic 1-form

$$\Phi = hf\theta = h(f_1, \ldots, f_n)\theta$$

has vanishing periods, so it integrates to a holomorphic null curve

$$Z(p) = X(p) + \imath Y(p) = \int_*^p \Phi, \qquad p \in M.$$

Then, $\partial Z = 2\partial X = \Phi$, and hence $G_X = G_Z = \mathscr{G}$.

The main complex analytic result

Theorem

Let M be an open Riemann surface and let $n \in \mathbb{N}$ be an integer. Let

$$\Phi_t = (\phi_{t,1}, \ldots, \phi_{t,n}), \quad t \in [0,1]$$

be a continuous family of full holomorphic 1-forms on M with values in $\mathbb{C}^n$, and let

$$\mathfrak{q}_t \colon H_1(M;\mathbb{Z}) \to \mathbb{C}^n, \quad t \in [0,1]$$

be a continuous family of group homomorphisms.

Then there exists a continuous family of holomorphic multipliers $h_t \colon M \to \mathbb{C}_*$, $t \in [0,1]$, such that

$$\int_\gamma h_t \Phi_t = \mathfrak{q}_t(\gamma) \quad \text{for every closed curve } \gamma \subset M \text{ and } t \in [0,1].$$

If the above condition holds at $t = 0$ with the constant function $h_0 = 1$, then the homotopy $h_t \colon M \to \mathbb{C}_*$ can be chosen with $h_0 = 1$.

Comments on the main theorem

This result shows that we can arbitrarily change the period map in an isotopy of $\mathbb{C}^n_*$-valued 1-forms by a suitable isotopy of multipliers.

For example, if we begin with an arbitrary $\Phi = \Phi_0$ and take $h_0 = 1$, then we can choose h_t such that $h_1\Phi$ is exact, so it integrates to a null curve.

If $\Re\Phi$ is exact, then $\Re\Phi$ integrates to an immersion $M \to \mathbb{R}^n$ with the Gauss map Φ. Hence, we recover and improve our previous result:

Theorem (Alarcón & F., Crelle, in press)

Every conformal minimal immersion $M \to \mathbb{R}^n$ $(n \geq 3)$ is isotopic (through conformal minimal immersions) to the real part of a holomorphic null curve $M \to \mathbb{C}^n$.

The new addition is that **all immersions in the isotopy can have the same Gauss map**.

Proof

A desired family of holomorphic multipliers $h_t \colon M \to \mathbb{C}_* \ (t \in I = [0,1])$ is constructed by induction with respect to an exhaustion of M by an increasing sequence of smoothly bounded Runge domains

$$D_1 \subset D_2 \subset \cdots \subset \cup_{j=1}^{\infty} D_j = M.$$

Three main ingredients are employed at every step, combining complex analysis with Gromov's convex integration theory:

- construction of multipliers on an arc (or a loop) in M that give approximately correct values of periods;
- construction of a period dominating spray of multipliers (this device is used to make arbitrary small corrections of the periods);
- Mergelyan's approximation theorem is used to approximate continuous families of continuous multipliers on curves by continuous families of holomorphic multipliers on Runge domains $D \Subset M$.

Main Lemma 1

The following lemma provides homotopies of continuous multiplier functions which enable us to prescribe the periods. Set $I = [0, 1]$.

Lemma (1)

Let $f\colon I^2 = I \times I \to \mathbb{C}^n$ and $\alpha\colon I \to \mathbb{C}^n$ be continuous maps. Assume that the path $f_t := f(t, \cdot)\colon I \to \mathbb{C}^n$ is nowhere flat for every $t \in I$. Then there exists a continuous function $h\colon I^2 \to \mathbb{C}_*$ such that

$$h(t, s) = 1 \quad \text{for } t \in I \text{ and } s = 0, 1$$

and

$$\int_0^1 h(t, s) f(t, s)\, ds = \alpha(t), \quad t \in I.$$

If in addition $\int_0^1 f(0, s)\, ds = \alpha(0)$, then h can be chosen such that $h(0, s) = 1$ for $s \in [0, 1]$.

Proof of Lemma 1, part 1

It suffices to prove that for any $\epsilon > 0$ there exists $h\colon I^2 \to \mathbb{C}_*$ such that

$$\left| \int_0^1 h(t,s) f(t,s)\, ds - \alpha(t) \right| < \epsilon, \quad t \in I.$$

The exact result is obtained by splitting $I = I_1 \cup I_2 = [0, 1/2] \cup [1/2, 1]$, applying the approximate result with a small $\epsilon > 0$ on I_1, and a period dominating argument on I_2 (see Lemma 2) to correct the error.

Since f_t is nowhere flat and hence full for each fixed $t \in [0,1]$, there is a division $0 = s_0 < s_1 < \cdots < s_N = 1$ of I such that

$$\mathrm{span}\{f_t(s_1), \ldots, f_t(s_N)\} = \mathbb{C}^n \quad \text{for all } t \in I.$$

Set

$$V_j(t) = \int_{s_{j-1}}^{s_j} f_t(s)\, ds \approx f_t(s_j)(s_j - s_{j-1}), \quad j = 1, \ldots, N.$$

By passing to a finer division, we may therefore assume that

$$\mathrm{span}\{V_1(t), \ldots, V_N(t)\} = \mathbb{C}^n, \quad t \in I.$$

Proof of Lemma 1, part 2

For each $t \in I$ we let $\Sigma_t \subset \mathbb{C}^N$ denote the affine complex hyperplane

$$\Sigma_t = \Big\{(g_1, \ldots, g_N) \in \mathbb{C}^N : \sum_{j=1}^N g_j V_j(t) = \alpha(t)\Big\}.$$

Clearly, there exists a continuous map $g = (g_1, \ldots, g_N) \colon I \to \mathbb{C}^N$ such that $g(t) \in \Sigma_t$ for every $t \in I$. (We may view g as a section of the affine bundle over I whose fiber over the point $t \in I$ equals Σ_t.) Hence

$$\sum_{j=1}^N \int_{s_{j-1}}^{s_j} g_j(t) f_t(s)\, ds = \sum_{j=1}^N g_j(t) V_j(t) = \alpha(t), \quad t \in I.$$

Note that

$$\sum_{j=1}^N V_j(t) = \sum_{j=1}^N \int_{s_{j-1}}^{s_j} f_t(s)\, ds = \int_0^1 f_t(s)\, ds.$$

Hence, if $\int_0^1 f(0,s)\, ds = \alpha(0)$ then g can be chosen such that $g(0) = (1, \ldots, 1) \in \mathbb{C}^N$. We assume in the sequel that this holds.

Proof of Lemma 1, part 3

By a small perturbation, we may assume that $g_j(t) \in \mathbb{C}_*$ for every $t \in I$ and $j = 1, \ldots, N$. This changes the exact condition to the approximate condition

$$\left| \sum_{j=1}^{N} \int_{s_{j-1}}^{s_j} g_j(t) f_t(s)\, ds - \alpha(t) \right| < \frac{\epsilon}{2}, \quad t \in I.$$

View the vector $g(t) = (g_j(t))_j \in \mathbb{C}^N$ for every fixed $t \in I$ as a step function of the variable $s \in I$ which equals the constant $g_j(t) \in \mathbb{C}_*$ on the j-segment $s \in [s_{j-1}, s_j]$ for every $j = 1, \ldots, N$.

Next, approximate this step function by a continuous function $h_t = h(t, \cdot) \colon I \to \mathbb{C}_*$ $(t \in I)$ which agrees with the step function, except near the division points $s_0, s_1, \ldots, s_N$. This causes an error of size $< \epsilon/2$ provided the modification is supported on sufficiently short segments around the division points, and it yields the desired estimate

$$\left| \int_0^1 h(t,s) f(t,s)\, ds - \alpha(t) \right| < \epsilon, \quad t \in I.$$

This proves Lemma 1, subject to Lemma 2.

Period dominating sprays of multipliers

Lemma (2)

Let I' be a nontrivial closed subinterval of $I = [0, 1]$, let Q be a compact Hausdorff space (the parameter space), and let $n \in \mathbb{N}$.

Given a continuous map $f\colon Q \times I \to \mathbb{C}^n$ such that $f(q, \cdot)$ is full on I' for every $q \in Q$, there exist finitely many continuous functions $g_1, \ldots, g_N\colon I \to \mathbb{C}$ $(N \geq n)$, supported on I', such that the function $h\colon \mathbb{C}^N \times I \to \mathbb{C}$ given by

$$h(\zeta, s) = 1 + \sum_{i=1}^{N} \zeta_i g_i(s), \quad \zeta = (\zeta_1, \ldots, \zeta_N) \in \mathbb{C}^N,\ s \in I$$

is a **period dominating multiplier of** f, *meaning that the map*

$$\frac{\partial}{\partial \zeta}\bigg|_{\zeta=0} \int_0^1 h(\zeta, s) f(q, s)\, ds\colon T_0\mathbb{C}^N \cong \mathbb{C}^N \to \mathbb{C}^n$$

is surjective for every $q \in Q$.

Proof of Lemma 2

Assume that

$$h(\zeta, s) := 1 + \sum_{i=1}^{N} \zeta_i g_i(s), \quad (\zeta, s) \in \mathbb{C}^N \times I.$$

Let $\mathcal{P}\colon Q \times \mathbb{C}^N \to \mathbb{C}^n$ be the map given by

$$\mathcal{P}(q, \zeta) = \int_0^1 h(\zeta, s) f(q, s)\, ds, \quad (q, \zeta) \in Q \times \mathbb{C}^N.$$

Then,

$$\left.\frac{\partial \mathcal{P}(q,\zeta)}{\partial \zeta_i}\right|_{\zeta=0} = \int_0^1 \left.\frac{\partial h(\zeta, s)}{\partial \zeta_i}\right|_{\zeta=0} f(q, s)\, ds = \int_0^1 g_i(s) f(q, s)\, ds.$$

Since $f(q, \cdot)$ is full on I' for every $q \in Q$, there are distinct points $s_1, \ldots, s_N$ in the interior of I' for a big $N \in \mathbb{N}$ such that

$$\mathrm{span}\{f(q, s_1), \ldots, f(q, s_N)\} = \mathbb{C}^n \quad \text{for all } q \in Q.$$

Proof of Lemma 2

Let $\epsilon > 0$ be small enough such that the intervals $[s_i - \epsilon, s_i + \epsilon]$ $(i = 1, \ldots, N)$ are pairwise disjoint and contained in I'. Let $g_i \colon I \to \mathbb{C}$ be continuous function supported on $(s_i - \epsilon, s_i + \epsilon) \subset I'$ and satisfying

$$\int_0^1 g_i(s)\, ds = \int_{s_i-\epsilon}^{s_i+\epsilon} g_i(s)\, ds = 1.$$

We have that

$$\left.\frac{\partial \mathcal{P}(q,\zeta)}{\partial \zeta_i}\right|_{\zeta=0} = \int_0^1 g_i(s) f(q,s)\, ds \approx f(q,s_i)$$

for all $q \in Q$ and $i \in \{1, \ldots, N\}$. Therefore, if $\epsilon > 0$ is small enough,

$$\mathrm{span}\left\{ \left.\frac{\partial \mathcal{P}(q,\zeta)}{\partial \zeta_1}\right|_{\zeta=0}, \ldots, \left.\frac{\partial \mathcal{P}(q,\zeta)}{\partial \zeta_N}\right|_{\zeta=0} \right\} = \mathbb{C}^n \quad \text{for all } q \in Q.$$

This proves Lemma 2, and hence the Main Theorem.

Spaces of conformal minimal immersions $M \to \mathbb{R}^n$

Assume that M is an open Riemann surface and $n \geq 3$ is an integer.

A conformal minimal immersion $X\colon M \to \mathbb{R}^n$ is said to be **flat** if its image $X(M)$ lies in an affine 2-plane of $\mathbb{R}^n$; otherwise it is **nonflat**. Let

$$\mathfrak{M}(M, \mathbb{R}^n)$$

denote the space of all conformal minimal immersions $M \to \mathbb{R}^n$ endowed with the compact-open topology, and let $\mathfrak{M}_*(M, \mathbb{R}^n)$ denote the open subset of $\mathfrak{M}(M, \mathbb{R}^n)$ consisting of all nonflat immersions.

Fix a nowhere vanishing holomorphic 1-form θ on M and consider the maps

$$\mathfrak{M}(M, \mathbb{R}^n) \longrightarrow \mathscr{O}(M, \mathfrak{A}_*) \hookrightarrow \mathscr{C}(M, \mathfrak{A}_*),$$

where $\mathfrak{A}_* = \mathfrak{A}_*^{n-1} \subset \mathbb{C}^n$ is the punctured null quadric.

The first map is given by $X \mapsto \partial X/\theta$, and the second map is the natural inclusion of the space of all holomorphic maps $M \to \mathfrak{A}_*$ into the space of continuous maps.

Path components of the space $\mathfrak{M}_*(M, \mathbb{R}^n)$

Since $\mathfrak{A}_*$ is an Oka manifold, the inclusion $\mathscr{O}(M, \mathfrak{A}_*) \hookrightarrow \mathscr{C}(M, \mathfrak{A}_*)$ is a weak homotopy equivalence by the main result of Oka theory.

Lárusson and myself recently proved that the restricted map

$$\mathfrak{M}_*(M, \mathbb{R}^n) \to \mathscr{O}(M, \mathfrak{A}_*), \quad X \mapsto \partial X/\theta$$

is also a weak homotopy equivalence. If $H_1(M; \mathbb{Z})$ is finitely generated then both these maps are actually homotopy equivalences.

It follows that the path components of $\mathfrak{M}_*(M, \mathbb{R}^n)$ are in bijective correspondence with the path components of the space $\mathscr{C}(M, \mathfrak{A}_*^{n-1})$. Since M is homotopy equivalent to a bouquet of circles, we have

$$H_1(M; \mathbb{Z}) \cong \mathbb{Z}^\ell, \quad \ell \in \mathbb{Z}_+ \cup \{\infty\},$$

$$\pi_1(\mathfrak{A}_*^2) = H_1(\mathfrak{A}_*; \mathbb{Z}) = \mathbb{Z}_2, \quad \pi_1(\mathfrak{A}_*^{n-1}) = 0 \text{ if } n > 3.$$

Thus, the path components of $\mathfrak{M}_*(M, \mathbb{R}^3)$ are in bijective correspondence with homomorphisms $H_1(M; \mathbb{Z}) \cong \mathbb{Z}^\ell \to \mathbb{Z}_2$, hence with elements of $(\mathbb{Z}_2)^\ell$, and $\mathfrak{M}_*(M, \mathbb{R}^n)$ is path connected if $n > 3$.

Path components of the space $\mathfrak{M}(M, \mathbb{R}^n)$

Theorem (Alarcón, López and F.)

Let M be an open connected Riemann surface. The natural inclusion

$$\mathfrak{M}_*(M, \mathbb{R}^n) \hookrightarrow \mathfrak{M}(M, \mathbb{R}^n)$$

induces a bijection of path components of the two spaces. In particular, the set $\pi_0(\mathfrak{M}(M, \mathbb{R}^3))$ of path components of $\mathfrak{M}(M, \mathbb{R}^3)$ is in bijective correspondence with the elements of the free abelian group $(\mathbb{Z}_2)^\ell$, where $H_1(M; \mathbb{Z}) = \mathbb{Z}^\ell$, and $\mathfrak{M}(M, \mathbb{R}^n)$ is path connected if $n > 3$.

The case $n > 3$ trivially follows from the following deformation result.

Theorem (Alarcón, López and F.)

Let M be a connected open Riemann surface. Given a flat conformal minimal immersion $X : M \to \mathbb{R}^n$ $(n \geq 3)$, there exists an isotopy $X_t : M \to \mathbb{R}^n$ $(t \in [0, 1])$ of conformal minimal immersions such that $X_0 = X$ and X_1 is nonflat.

Proof of the deformation theorem

Clearly it suffices to prove the theorem for $n = 3$. Let $X\colon M \to \mathbb{R}^3$ be a flat conformal minimal immersion. We may assume that $\partial X = (1, \imath, 0)\phi_3$ where ϕ_3 is an exact holomorphic 1-form without zeros on M.

The gist of the proof is show that there is nonconstant holomorphic function $g\colon M \to \mathbb{C}_*$ such that **$g\phi_3$ and $g^2\phi_3$ are exact 1-forms**. This is similar to the proof of Lemma 1. Then,

$$\Phi_\lambda = \Big(1 - \lambda^2 g^2, \imath(1 + \lambda^2 g^2), 2\lambda g\Big)\,\phi_3, \quad \lambda \in \mathbb{C}$$

is an exact holomorphic 1-form and the map Φ_λ/ϕ_3 assumes values in $\mathfrak{A}_* \subset \mathbb{C}^3$ for every $\lambda \in \mathbb{C}$. Thus, Φ_λ provides a conformal minimal immersion $X_\lambda\colon M \to \mathbb{R}^3$ by

$$X_\lambda(p) = X(p_0) + 2\int_{p_0}^{p} \Re(\Phi_\lambda), \quad p \in M.$$

Since $\Phi_0 = \partial X$, we have that $X_0 = X$. Furthermore, since g is nonconstant, X_1 is nonflat, and we are done.

Path components of the space $\mathfrak{M}(M, \mathbb{R}^3)$

In dimension $n = 3$, we obtain the above theorem to the effect that

$$\pi_0(\mathfrak{M}(M, \mathbb{R}^3)) = \mathbb{Z}_2^\ell$$

by using the corresponding result for nonflat immersions,

$$\pi_0(\mathfrak{M}_*(M, \mathbb{R}^3)) = \mathbb{Z}_2^\ell,$$

along with the deformation theorem and the following result.

Theorem

Let M be a connected open Riemann surface and let θ be a nowhere vanishing holomorphic 1-form on M. For every group homomorphism

$$\mathfrak{p} \colon H_1(M; \mathbb{Z}) \cong \mathbb{Z}^\ell \to \mathbb{Z}_2$$

there exists a **flat conformal minimal immersion** *$X \colon M \to \mathbb{R}^3$ satisfying $H_1(\partial X / \theta) = \mathfrak{p}$.*

The Gauss map can omit two points

We also prove the following result concerning isotopies of conformal minimal immersions into $\mathbb{R}^3$.

Theorem

Given an open Riemann surface M and a conformal minimal immersion $X\colon M \to \mathbb{R}^3$, there exists an isotopy

$$X_t\colon M \to \mathbb{R}^3, \quad t \in [0,1]$$

of conformal minimal immersions such that $X_0 = X$ and the complex Gauss map of X_1 is nonconstant and avoids any two given points of the Riemann sphere.

There also exists an isotopy X_t as above such that $X_0 = X$ and X_1 is flat.

If M is covered by $\mathbb{C}$, then the Gauss map cannot omit three point of $\mathbb{CP}^1$ by Picard's theorem, unless it is constant and the immersion is flat.

THANK YOU FOR YOUR ATTENTION