From Stein manifolds to Oka manifolds: the h-principle in complex analysis

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Flexibility versus rigidity in complex geometry

A central question of complex analysis and geometry is to understand the space $\mathcal{O}(X, Y)$ of holomorphic maps $X \to Y$ between a pair of complex manifolds. Are there many maps, or few maps? Which properties can they have?

There are many holomorphic maps $\mathbb{C} \to \mathbb{C}$ and $\mathbb{C} \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, but there are no nonconstant algebraic maps $\mathbb{C} \to \mathbb{C}^*$ or holomorphic maps $\mathbb{C} \to \mathbb{C} \setminus \{0, 1\}$. Manifolds with the latter property are called **hyperbolic**. Hyperbolicity is an obstruction to solving global complex analytic problems.

On the opposite side, **Oka theory** studies complex manifolds Y which admit many holomorphic maps $X \to Y$ from all **Stein manifolds**, i.e., closed complex submanifolds of affine complex spaces $\mathbb{C}^N$. Oka theory provides solutions to many complex analytic problems in the absence of topological obstructions. It has numerous applications: classification theory for principal fibre bundles on Stein manifolds, the h-principles for holomorphic immersions and submersions, the optimal embedding theorem for Stein manifolds, factorisation theory for holomorphic maps from Stein manifolds to matrix groups, the theory of minimal surfaces in Euclidean spaces, ...

$$\boxed{\textbf{Oka theory = the h-principle in complex geometry}}$$

The first main protagonist: Stein manifolds

Karl Stein, 1951

A complex manifold X is holomorphically complete (Stein) if

1. holomorphic functions on X separate points, and
2. for every discrete sequence $p_j \in X$ there is a holomorphic function $f \in \mathcal{O}(X)$ with $\lim_{j\to\infty} |f(p_j)| = +\infty$.

   Equivalently, for every compact subset $K \subset X$, its holomorphic hull $\widehat{K}_{\mathcal{O}(X)}$ is also compact.

Examples and characterizations of Stein manifolds

- Open Riemann surfaces are Stein (Behnke and Stein, 1949).
- Closed complex submanifolds of Euclidean spaces $\mathbb{C}^N$ are Stein. Conversely, every Stein n-manifold is a closed complex submanifold of $\mathbb{C}^{2n+1}$ (Remmert, Bishop, Narasimhan, 1956–61).
- The category of Stein manifolds is closed under taking holomorphic covering, Cartesian products, and holomorphic vector bundles.
- A complex manifold X is Stein iff it admits a strongly plurisubharmonic exhaustion function $\rho : X \to \mathbb{R}$, $dd^c\rho = \mathrm{i}\partial\bar{\partial}\rho > 0$ (Grauert, 1958).
- Every Stein subvariety of a complex space admits an open Stein neighbourhood (Siu, 1976).

A **Stein space** is a complex space with singularities having similar function theoretic properties as Stein manifolds.

The Oka–Grauert principle

Kiyoshi Oka 1939, Hans Grauert 1958

For principal and their associate fibre bundles (e.g. vector bundles) on Stein spaces, the holomorphic classification agrees with the topological classification.

The gist of the proof is to show that there are many holomorphic maps $X \to Y$ from any Stein manifold X to every complex homogeneous manifold Y, and many holomorphic sections of any holomorphic fibre bundle $Z \to X$ with a homogeneous fibre over a Stein base X.

The second main protagonist: Oka manifolds

A complex manifold Y is called an **Oka manifold** (F. 2009) if maps $X \to Y$ from any Stein space X satisfy all forms of the **Oka principle**:

- (a) Every continuous map $f\colon X \to Y$ is homotopic to a holomorphic map.
- (b) If $f\colon X \to Y$ is holomorphic on (a neighbourhood of) a compact holomorphically convex subset $K = \widehat{K} \subset X$ and on a closed complex subvariety X_0 of X, then there is a homotopy from f to a holomorphic map $F\colon X \to Y$ consisting of maps with the same properties which approximate f on K and agree with f on X_0.
- (c) A similar statement holds for maps $f\colon X \times P \to Y$ depending continuously on a parameter in a compact Hausdorff space P.

Oka–Weil–Cartan 1936–53: $\mathbb{C}$ is an Oka manifold.

Oka 1939: $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is an Oka manifold.

Grauert 1958: Every complex homogeneous manifold is an Oka manifold.

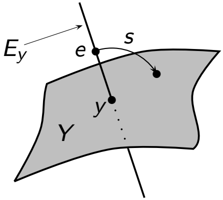
Kobayashi and Ochiai 1977: A compact complex manifold of general type is not dominable by the affine space, so it is not an Oka manifold.

Oka manifolds are rare in the universe of complex manifolds

Elliptic manifolds

Gromov 1989 A complex manifold Y is called **elliptic** if it admits a **dominating holomorphic spray**: A holomorphic map $s\colon E\to Y$, defined on the total space E of a holomorphic vector bundle over Y, such that

$$s(0_y)=y \ \text{ and } \ ds_{0_y}(E_y)=T_yY \quad \text{for all } y\in Y.$$



An ostensibly weaker condition, **subellipticity**, asks for a finite family of sprays $s_j : E_j \to Y$ with

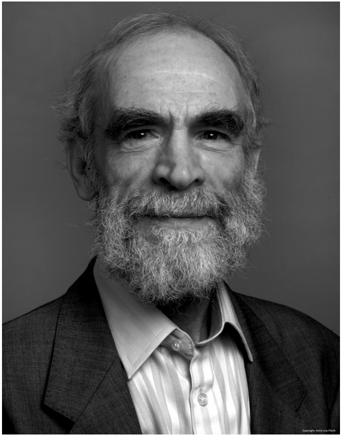
$\sum_j (ds_j)_{0_y}(E_{j,y}) = T_yY$ for all $y\in Y$.

On algebraic manifolds, one defines **algebraic (sub-) ellipticity**.

Ellipticity easily implies the following **relative Oka principle:**

Given a holomorphic map $f_0 : X\to Y$, a compact $\mathcal{O}(X)$-convex subset $K\subset X$, and a homotopy of holomorphic maps $f_t : K\to Y$ $(t\in[0,1])$, we can approximate f_1 on K by holomorphic maps $X\to Y$.

Gromov's Oka principle



Gromov 1989: Every elliptic manifold is an Oka manifold. The Oka principle also holds for sections of elliptic submersions $Z \to X$ over a Stein base.

Prezelj & F., 2000–2: detailed proof.

F. 2003: The same holds for subelliptic manifolds and submersions.

Problems: (1) Can one characterise the class of Oka manifolds by a Runge approximation property from simple domains in Euclidean spaces?

(2) Is every Oka manifold elliptic?

Examples of elliptic manifolds

- **Every $\mathbb{C}$-homogeneous manifold Y is elliptic.** If G is a complex Lie group acting holomorphically transitively on Y and $\mathfrak{g}$ is the Lie algebra of G, then the map $Y \times \mathfrak{g} \xrightarrow{s} Y$, $(y, v) \mapsto \mathrm{e}^v \cdot y$, is a dominating spray.

- **A flexible manifold Y is elliptic.** Such Y admits $\mathbb{C}$-complete holomorphic vector fields $V_1, \ldots, V_k$ spanning the tangent space $T_y Y$ at every point. Let ϕ_t^j denote the flow of V_j for time $t \in \mathbb{C}$. The map $s : Y \times \mathbb{C}^k \to Y$,

$$s(y, t_1, \ldots, t_k) = \phi_{t_1}^1 \circ \cdots \circ \phi_{t_k}^k (y),$$

  is then a dominating spray on Y.

- **An algebraically flexible manifold is algebraically elliptic.**
  The tangent bundle of such a manifold is pointwise spanned by complete algebraic vector fields with algebraic flows.

- **Arzhantsev, Kaliman and Zaidenberg 2024, Banecki 2025**
  Every rational projective manifold is algebraically elliptic.

- **Kaliman and Zaidenberg 2024** Every algebraically subelliptic manifold is also algebraically elliptic.

CAP $\iff$ OKA

Definition (F. 2005) A complex manifold Y enjoys the **convex approximation property (CAP)** if every holomorphic map $K \to Y$ from a neighbourhood of a compact convex set $K \subset \mathbb{C}^n$ is a uniform limit of holomorphic maps $\mathbb{C}^n \to Y$.

The implication **Elliptic $\Longrightarrow$ CAP** is immediate from the relative Oka principle.

Theorem (F. 2005–2010, 2017)

- *CAP $\iff$ OKA. (This answers Gromov's first question.)*
- *The individual Oka properties are pairwise equivalent.*
- *If $Y \to Z$ is a holomorphic fibre bundle with an Oka fibre, then Y is Oka iff Z is Oka. This leads to the notion of an* **Oka map**.
- *Every Oka manifold Y is the image of a strongly dominating holomorphic map $\mathbb{C}^{\dim Y} \to Y$.*

Lárusson 2005: Model category for Oka theory.

MSC 2020: New subfield **32Q56 Oka principle and Oka manifolds**.

A few examples of Oka manifolds

- A Riemann surface is Oka iff it is not hyperbolic.
- **Grauert 1958** Complex Lie groups and homogeneous manifolds.
- **Gromov 1989** Elliptic complex manifolds.
- Hirzebruch surfaces ($\mathbb{P}^1$ bundles over $\mathbb{P}^1$).
- Hopf manifolds (quotients of $\mathbb{C}^n \setminus \{0\}$ by cyclic groups).
- **Banecki 2025** Rational projective manifolds.
- Complex tori of dimension > 1 with finitely many points removed, or blown up in finitely many points.
- **Lárusson 2013** Smooth toric varieties are Oka. These are quotients $(\mathbb{C}^m \setminus Z)/G$, where Z is a union of coordinate subspaces of $\mathbb{C}^m$ and G is a subgroup of $(\mathbb{C}^*)^m$ acting on $\mathbb{C}^m \setminus Z$ by diagonal matrices.
- **Mok 1988** Every compact Kähler manifold with nonnegative holomorphic bisectional curvature is an Oka manifold.
- **Campana & Peternell 1991** Every compact projective manifold with dim ≤ 3 and with nef tangent bundle is an Oka manifold.

Kusakabe's characterisation of Oka manifolds

Since 2020, major contributions have been made by
Yuta Kusakabe (2026 Math. Soc. Japan Spring Prize).

Kusakabe 2021 A complex manifold Y is said to enjoy **convex relative ellipticity, CRE,** if for every holomorphic map $f\colon U \to Y$ from a bounded convex set $U \subset \mathbb{C}^m$ there is a holomorphic spray $F\colon U \times \mathbb{C}^N \to Y$ with $F(\cdot\,, 0) = f$ satisfying

$$\frac{\partial}{\partial \zeta}\Big|_{\zeta=0} F(z,\zeta) : \mathbb{C}^N \to T_{f(z)}Y \text{ is surjective for every } z \in U.$$

Kusakabe 2021: CRE $\Longrightarrow$ CAP. It follows that

$$\boxed{\text{CRE} \iff \text{CAP} \iff \text{OKA}}$$

Corollary (Kusakabe's localization theorem, 2021)

If a complex manifold Y is a union of Zariski open Oka domains, then Y is Oka.

Complements of polynomially convex sets are Oka

Theorem (Kusakabe, Ann. Math. 2024)

If K is a compact polynomially convex set in $\mathbb{C}^n$ $(n > 1)$ then $\mathbb{C}^n \setminus K$ satisfies CRE, and hence is an Oka manifold.

The analogous result holds if $\mathbb{C}^n$ is replaced by any Stein manifold X having many complete holomorphic vector fields (Varolin's density property).

In fact, CRE of $\mathbb{C}^n \setminus K$ holds in a stronger form, which gives a holomorphically varying family of Fatou–Bieberbach domains with given centres in $\mathbb{C}^n \setminus K$.

Theorem (Wold & F. 2020)

Every holomorphic map $f\colon U \to \mathbb{C}^n \setminus K$ from a convex domain $U \subset \mathbb{C}^m$ extends to a holomorphic map $F\colon U \times \mathbb{C}^n \to \mathbb{C}^n \setminus K$ such that for all $\zeta \in U$,

- *$F(\zeta, \cdot\,) : \mathbb{C}^n \to \mathbb{C}^n \setminus K$ is injective (a Fatou–Bieberbach map), and*
- *$F(\zeta, 0) = f(\zeta)$ for all $\zeta \in U$.*

What is the shape of Oka domains?

Conjecture: Every Oka domain in a complex manifold is pseudoconcave.

Theorem (Wold & F., 2023)

If $E \subset \mathbb{C}^n$, $n > 1$, is a closed convex set which does not contain any affine real line, then the concave domain $\mathbb{C}^n \setminus E$ is an Oka manifold.

Example

If $\phi : \mathbb{C}^{n-1} \times \mathbb{R} \to \mathbb{R}_+$ is a strictly convex function then the concave domain

$$\{(z, x + \mathrm{i}y) \in \mathbb{C}^n : y < \phi(z, x)\} \quad \text{is Oka}$$

while $\{y > \phi(z, x)\}$ is hyperbolic. The halfspace $\{y > 0\}$ has neither property.

Theorem (Kusakabe & F. 2024)

Let $X \subset \mathbb{CP}^n$ be a flag manifold and $L \to X$ an ample holomorphic line bundle. For any semipositive hermitian metric h on L, the tube $\{h < 1\}$ is Oka.

Grauert 1961: If (L, h) is a negative holomorphic hermitian line bundle on a compact complex manifold X then the tube $\{0 < h < 1\}$ is hyperbolic.

Every projective Oka manifold is elliptic

Gromov 1989 Is every Oka manifold elliptic?

Andrist, Shcherbina & Wold 2016 If K is a compact set in $\mathbb{C}^n$ ($n \geq 3$) with infinitely many limit points then $\mathbb{C}^n \setminus K$ is not elliptic or subelliptic.

Kusakabe 2024 If K is a compact polynomially convex set in $\mathbb{C}^n$, $n > 1$, then $\mathbb{C}^n \setminus K$ is an Oka domain.

These results give examples of open (noncompact) nonelliptic Oka manifolds, thereby answering negatively the question of Gromov.
However, Gromov's conjecture holds true for projective manifolds:

Theorem (Lárusson & F. 2026)

Every projective Oka manifold is elliptic.

Our construction of dominating sprays on such manifolds strongly depends on the existence of negative line bundles, so it only applies to projective manifolds.

Problem

Is there a compact Oka manifold which fails to be elliptic?

Scheme of proof

Let $Y \subset \mathbb{CP}^n$ be a projective manifold and $\pi : E \to Y$ a holomorphic vector bundle. Along the zero section $E(0) \cong Y$ of E, we have a natural splitting

$$TE|E(0) \cong E \oplus TY.$$

Let V be a holomorphic vector field on E that vanishes on $E(0)$. There is a neighbourhood $\Omega \subset E$ of $E(0)$ such that for any $e \in \Omega$, the flow $\phi_\tau(e)$ of V with $\phi_0(e) = e$ exists for all $0 \le \tau \le 1$. The holomorphic map

$$s = \pi \circ \phi_1 : \Omega \to Y$$

is then a local spray on Y. Taking $E = N \cdot \mathbb{U}^k|_Y$, where $\mathbb{U} \to \mathbb{CP}^n$ is the universal line bundle and $N, k >> 0$, we construct s whose vertical derivative

$$Vds_y : T_{0_y}E_y \cong E_y \to T_yY$$

is surjective for every $y \in Y$. This means that s is dominating.

Since E is a 1-convex manifold with the exceptional subvariety $E(0)$, the Oka principle (**Prezelj 2010, 2016; Stopar 2013**) gives a global holomorphic spray $E \to Y$ which agrees with s to the second order along $E(0)$. Hence, Y is elliptic.

A new direction: The Oka principle for families

Let X be a smooth manifold. A continuous family $\mathscr{J} = \{J_t\}_{t\in T}$ of smooth integrable Stein structures on X is said to be **tame** if for every compact set $K \subset X$, the family of J_t-convex hulls $\widehat{K}_{J_t}$ $(t \in T)$ is locally bounded (equivalently, upper semicontinuous). This always holds if X is a surface, since the hull $\widehat{K}_J$ is then independent of the complex structure J.

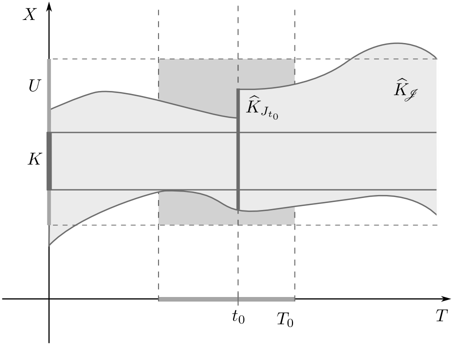


Figure: A tame family of hulls

The Oka principle for families

A continuous map $f\colon T \times X \to Y$ to a complex manifold (Y, J_Y) is said to be **$\mathscr{J}$-holomorphic** if $f_t = f(t,\cdot\,) : X \to Y$ is (J_t, J_Y)-holomorphic for every $t \in T$.

Theorem (Sigurðardóttir & F. 2025)

If $\mathscr{J}$ is a tame family of smooth Stein structures on X for a suitable parameter space T (including all finite CW complexes) and Y is an Oka manifold, then

$\mathscr{J}$-holomorphic maps $T \times X \to Y$ satisfy the Oka principle.

Conversely, a local one-fibre extension property for $\mathscr{J}$-holomorphic functions implies that the family $\mathscr{J}$ is tame.

Example

There is a smooth nontame family $\mathscr{J} = \{J_t\}_{t\in\mathbb{R}}$ of Stein structures on $\mathbb{R}^4$ such that $(\mathbb{R}^4, J_t)$ is biholomorphic to $\mathbb{C}^2$ for every $t \in \mathbb{R}$, and the J_t-convex hull of the closed ball explodes as $t \to 0$.

Universal families of punctured Riemann surfaces

The Teichmüller space $T(M)$ of a Riemann surface M is finite dimensional iff

$$M = \overline{M} \setminus \{p_1, \ldots, p_n\}$$

is a compact Riemann surface of some genus $g \geq$ with $n \neq 0$ punctures. Such M is said to be of **finite conformal type**, and we write $T(M) = T(g, n)$.

Bers 1960 For $2g + n \geq 3$, $T(g, n)$ is a contractible Stein domain in $\mathbb{C}^{3g-3+n}$.

Marković 2018 If $g \geq 2$ then $T(g, n)$ is not a convex domain.

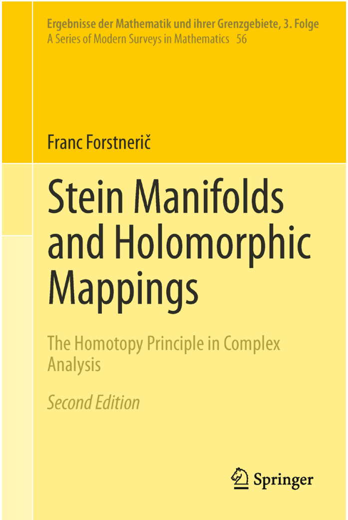
The universal Teichmüller family $\pi : V(g, n) \to T(g, n)$ is a holomorphic submersion with holomorphically varying punctures.

Theorem (F. 2026)

For every $g \geq 0$ and $n \geq 1$, $V(g, n)$ is a Stein manifold. There is a proper holomorphic embedding $V(g, n) \hookrightarrow \mathbb{C}^N$ which is algebraic on every fibre.

It follows that every holomorphic function on $V(g, n)$ is a limit of holomorphic functions which are algebraic on fibres.

Also, $V(g, n)$ admits nontrivial holomorphic maps to any Oka manifold.



The main results on this subject up to 2017 are presented in my Ergebnisse monograph.

Developments after 2017 are summarised in my surveys

Recent developments on Oka manifolds.
Indag. Math., 34(2) (2023) 367–417.

From Stein manifolds to Oka manifolds: the h-principle in complex analysis. Proc. ICM 2026.

Thank you for your attention.