Holomorphic immersions and embeddings of Stein manifolds into Stein manifolds with the density property

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Embeddings and immersions into Euclidean spaces

Stein 1951 A Stein manifold is a complex manifold S which admits many holomorphic maps $S \to \mathbb{C}^n$.

Remmert 1956; Narasimhan 1960, Bishop 1961
Every Stein manifold S admits

- a proper holomorphic embedding $S \hookrightarrow \mathbb{C}^{2\dim S+1}$;
- a proper holomorphic immersion $S \to \mathbb{C}^{2\dim S}$;
- a proper holomorphic map $S \to \mathbb{C}^{\dim S+1}$;
- an almost proper holomorphic map $S \to \mathbb{C}^{\dim S}$.

Eliashberg and Gromov 1992; Schürmann 1997
A Stein manifold S^d immerses properly holomorphically into $\mathbb{C}^{\left[\frac{3d+1}{2}\right]}$ and, if $d > 1$, it embeds properly holomorphically into $\mathbb{C}^{\left[\frac{3d}{2}\right]+1}$.

Question: Which Stein manifolds, besides the Euclidean spaces, contain all Stein manifolds of appropriate dimension as closed complex submanifolds?

The (volume) density property

Varolin 2000 A complex manifold X enjoys the **density property (DP)** if every holomorphic vector field on X can be approximated by Lie combinations of $\mathbb{C}$-complete holomorphic vector fields.

If ω is a holomorphic volume form on X, the pair (X, ω) enjoys the **volume density property (VDP)** if the analogous property holds for the holomorphic vector fields with vanishing ω-divergence.

Andersén 1990; Andersén & Lempert 1992 $\mathbb{C}^n$ for $n > 1$ enjoys DP, and also VDP for the volume form $\omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$.

A Stein manifold with DP or VDP is highly symmetric and has a big holomorphic automorphism group. More precisely, we have the following result.

Approximating biholomorphisms by automorphisms

Theorem (Andersén-Lempert, Forstnerič-Rosay, Varolin)

Let X be a Stein manifold with DP. Assume that

$$F_t \colon \Omega_0 \to \Omega_t \subset X \quad (t \in [0,1])$$

is a smooth isotopy of biholomorphic maps between Stein Runge domains in X, with $F_0 = \mathrm{Id}|_{\Omega_0}$. Then, $F_1 \colon \Omega_0 \to \Omega_1$ is a limit of holomorphic automorphisms of X, uniformly on compacts in Ω_0.

There is an analogous result for isotopies of biholomorphic maps preserving the volume form on a Stein manifold with VDP.

Theorem (Kaliman and Kutzschebauch)

A Stein manifold with the (volume) density property is an Oka manifold.

The converse is an open problem.

The main theorem

Theorem (Andrist, F., Ritter, Wold 2016; F. 2017)

Assume that S is a Stein manifold, and X is a Stein manifold with the density property or the volume density property.

(a) *If* $\dim X > 2\dim S$, *then any continuous map* $S \to X$ *is homotopic to a proper holomorphic embedding* $S \hookrightarrow X$.

(b) *If* $\dim X = 2\dim S$, *then any continuous map* $S \to X$ *is homotopic to a proper holomorphic immersion with simple double points.*

In addition, we have the approximation and interpolation results for proper holomorphic immersions and embeddings.

Problem

Is the same true if X is a Stein Oka manifold?

Does every Stein Oka manifold enjoy the (volume) density property?

Examples of manifolds with DP or VDP

- $\mathbb{C}^n$ for $n \geq 1$ satisfies VDP for $dz_1 \wedge \cdots \wedge dz_n$ (Andersén).
- $\mathbb{C}^n$ for any $n > 1$ satisfies DP (Andersén and Lempert).
- $(\mathbb{C}^*)^n$ with the volume form $\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$ satisfies VDP (Varolin). It is not known whether DP holds when $n > 1$.
- If G is a linear algebraic group and $H \subset G$ is a closed proper reductive subgroup, then $X = G/H$ is a Stein manifold with the density property, except when $X = \mathbb{C}$, $(\mathbb{C}^*)^n$, or a $\mathbb{Q}$-homology plane with fundamental group $\mathbb{Z}_2$ (Kaliman, Donzelli & Dvorsky).
- In particular, a linear algebraic group with connected components different from $\mathbb{C}$ or $(\mathbb{C}^*)^n$ has DP (Kaliman and Kutzschebauch).
- If $p : \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function with smooth reduced zero fibre, then $X = \{xy = p(z)\}$ has DP (K&K). The same is true if the source $\mathbb{C}^n$ of p is an arbitrary Stein manifold with DP.
- A Cartesian product $X_1 \times X_2$ of two Stein manifolds X_1, X_2 with DP also has DP. The analogous result holds for VDP (K&K).

Gizatullin surfaces and the Koras-Russell cubic

- **Andrist 2017** A generic Gizatullin surface has the density property. A smooth affine algebraic surface X is a **Gizatullin surface** if $\mathrm{Aut}_{\mathrm{alg}}(X)$ acts transitively on X up to finitely many points. Every such surface X admits a fibration $\pi\colon X \to \mathbb{C}$ whose generic fiber equals $\mathbb{C}$ and there is only one exceptional fiber. If this exceptional fiber is reduced, then X has the density property.
- **Leuenberger 2016** established DP for a family of hypersurfaces

$$X = \{(x, y, z) \in \mathbb{C}^{n+3} : x^2 y = a(z) + x b(z)\},$$

where $x, y \in \mathbb{C}$ and $a, b \in \mathbb{C}[z]$ are polynomials in $z \in \mathbb{C}^{n+1}$. This family includes the **Koras-Russell cubic threefold**

$$C = \{(x, y, z_0, z_1) \in \mathbb{C}^4 : x^2 y + x + z_0^2 + z_1^3 = 0\}.$$

This threefold is diffeomorphic to $\mathbb{R}^6$, but is not algebraically isomorphic to $\mathbb{C}^3$ (Makar-Limanov, Dubouloz).

It remains an open question whether C is biholomorphic to $\mathbb{C}^3$.

Proper embeddings of strongly pseudoconvex domains

The density property is a holomorphic flexibility (anti-hyperbolicity) property which is needed to accommodate maps from holomorphically 'big' Stein manifolds. No such condition is necessary if the source manifold is holomorphically 'small'. For example:

Theorem (Drinovec Drnovšek & F., 2010)

Let X be an n-dimensional Stein manifold, and let D be a relatively compact, smoothly bounded, strongly pseudoconvex domain in a Stein manifold S of dimension d.

If $2d < n$ then every holomorphic map $f : \overline{D} \to X$ can be approximated uniformly on compacts in D by proper holomorphic embeddings $D \hookrightarrow X$ (immersions with simple double points if $2d = n$).

This result, which is used in the proof of our main theorem, is obtained by using holomorphic peak functions and the Riemann-Hilbert method to push the image of bD to infinity in X. Ealier results were obtained by many authors (**Løw, F., Globevnik, Hakim, Stenønes, Dor,...**).

Preliminaries

A compact set K in a topological space S is said to be **regular** if K is the closure of its interior $\mathring{K}$.

Definition

A pair $K \subset L$ of compact convex sets in $\mathbb{R}^N$ is a **simple convex pair** if there are a linear function $\lambda \colon \mathbb{R}^N \to \mathbb{R}$ and constant $a \in \mathbb{R}$ such that

$$K = \{z \in L : \lambda(z) \leq a\}. \tag{1}$$

The following lemma is obvious.

Lemma

Given regular compact convex sets $C \subset B$ in $\mathbb{R}^N$ and an open set $U \subset \mathbb{R}^N$ containing C, there is a finite sequence of regular compact convex sets $K_1 \subset K_2 \subset \cdots \subset K_{m+1} = B$ such that $C \subset K_1 \subset U$ and (K_i, K_{i+1}) is a simple convex pair for every $i = 1, \ldots, m$.

Slices and slabs

If K is a compact (convex) set in $\mathbb{R}^N$, then a **slice** of K is the intersection of K with a real affine hyperplane:

$$K_a = \{x \in K : \lambda(x) = a\},$$

and a **slab** of K is a subset of the form

$$K_{a,b} = \{x \in K : a \leq \lambda(x) \leq b\}$$

where $a < b$ and $\lambda \colon \mathbb{R}^N \to \mathbb{R}$ is a linear function.

The number $b - a > 0$ is called the **thickness** of the slab $K_{a,b}$.

If K is a compact subset of a manifold S contained in a local chart, then a slice or a slab of K will be understood as a subset of the respective type in the given chart.

(Very) special Cartan pairs

Let S be a Stein manifold.

Definition

A pair (A, B) of compact sets in S is a **special Cartan pair** if

(i) the sets A, B, $D = A \cup B$ and $C = A \cap B$ are $\mathscr{O}(S)$-convex,

(ii) A and B are **separated** in the sense that $\overline{A \setminus B} \cap \overline{B \setminus A} = \varnothing$, and

(iii) there is a holomorphic coordinate system on a neighborhood of B in S in which B and $C = A \cap B$ are regular convex sets. The set B is a **convex bump** and C is the **attaching set**.

A special Cartan pair (A, B) with $C = A \cap B$ is **very special** if

(iv) there is a holomorphic coordinate system on a neighborhood of B in S in which (C, B) is a **simple convex pair**.

Replacing a special Cartan pair by finitely many very special Cartan pairs

Grauert; Henkin & Leiterer A Stein manifold S can be exhausted by an increasing sequence of compact sets $A_1 \subset A_2 \subset \cdots \subset \bigcup_j A_j = X$ such that $A_{j+1} = A_j \cup B_j$, where (A_j, B_j) is a special Cartan pair or B_j is a handle attached to A_j. A special Cartan pair can be replaced by finitely many very special Cartan pairs.

Lemma

Assume that (A, B) is a special Cartan pair in S. Given an open set $W \subset S$ containing A, there is a finite increasing sequence of compact $\mathscr{O}(S)$-convex sets

$$A \subset A_1 \subset A_2 \subset \cdots A_{m+1} = A \cup B =: D$$

such that $A_1 \subset W$, and for every $i = 1, \ldots, m$ we have $A_{i+1} = A_i \cup B_i$ where (A_i, B_i) is a very special Cartan pair and $C_i = A_i \cap B_i$ is an arbitrarily thin slab of B_i.

Proof of the lemma

Proof. Let $C = A \cap B$. By the assumption, there are an open neighborhood $V_0 \subset S$ of B and a biholomorphic map

$$\theta \colon V_0 \to \widetilde{V}_0 \subset \mathbb{C}^d$$

onto an open convex subset $\widetilde{V}_0$ of $\mathbb{C}^d$ such that $\theta(C) \subset \theta(B)$ are regular compact convex set in $\mathbb{C}^d$. We use the chart θ to define the notion of convexity, slices, slabs, and simple convex pairs in V_0.

Pick an open neighborhood U of $C = A \cap B$, with $U \subset W$, and a compact convex set $\widetilde{C} \subset U$ containing C in its interior.

There is a sequence $K_1 \subset K_2 \subset \cdots \subset K_{m+1} = B$ of regular compact convex sets such that

$$B \cap \widetilde{C} \subset K_1 \subset U$$

and (K_i, K_{i+1}) is a simple convex pair for every $i = 1, \ldots, m$, i.e., there are $\mathbb{R}$-linear functions $\lambda_i \colon \mathbb{C}^d \to \mathbb{R}$ and numbers $b_i \in \mathbb{R}$ such that

$$K_i = \{x \in K_{i+1} : \lambda_i(\theta(x)) \leq b_i\}.$$

Proof of the lemma, 2

Choose numbers $a_i < b_i$ close to b_i. For every $i = 1, \ldots, m$ let

$$A_i = A \cup K_i, \quad B_i = \{x \in K_{i+1} : a_i \leq \lambda_i(\theta(x))\}.$$

Assuming that $b_i - a_i > 0$ is sufficiently small for each i, it follows that

$$A \cap B_i = \varnothing \quad \text{for} \quad i = 1, \ldots, m.$$

Then

$$A_i \cup B_i = A \cup K_{i+1} = A_{i+1} \subset A \cup B = D$$

and

$$C_i = A_i \cap B_i = \{x \in K_{i+1} : a_i \leq \lambda_i(\theta(x)) \leq b_i\}.$$

Thus, C_i is a slab of the compact convex set K_{i+1}.

It is easily verified that each (A_i, B_i) is a very special Cartan pair.

The Main Lemma

Lemma

Assume that S is a complex manifold of dimension d, and X is a Stein manifold of dimension $n \geq 2d$ with the (volume) density property.
Let (A, B) be a special Cartan pair in S, $C := A \cap B$, and $D := A \cup B$.
Assume that

(a) *L is a compact $\mathscr{O}(X)$-convex set in X,*

(b) *K is a compact set in $\mathring{A} \setminus C$ such that $K \cup B$ is $\mathscr{O}(S)$-convex, and*

(c) *$f : A \to X$ is a holomorphic map such that*

$$f(A \setminus \mathring{K}) \subset X \setminus L \quad (\iff \quad f^{-1}(L) \subset \mathring{K}).$$

Then we can approximate f uniformly on A by a holomorphic immersion $\tilde{f} : D = A \cup B \to X$ (embedding if $n > 2d$) such that

$$\tilde{f}(D \setminus \mathring{K}) \subset X \setminus L \quad (\iff \quad \tilde{f}^{-1}(L) \subset \mathring{K}).$$

Proof, 1: reductions

By the first lemma, we assume that (A, B) is a very special Cartan pair.

Let $W \subset S$ be a neighborhood of A and $f\colon W \to X$ be a holomorphic map such that

$$f(W \setminus \mathring{K}) \subset X \setminus L.$$

By the theorem by Drinovec Drnovšek and myself we may assume, after shrinking $W \supset A$ and by approximation, that $f\colon W \to X$ is a **proper holomorphic immersion** whose image is a closed immersed complex submanifold of X with simple double points (embedded if $n > 2d$).

Hence, the sets $f(A)$, $f(B)$, $f(C)$, $f(D)$, $f(K \cup C)$ are $\mathscr{O}(X)$-convex.

Furthermore, we have $L \cap f(W) \subset f(K)$, so the sets $L' := L \cup f(K)$ and $L' \cup f(C)$ are also $\mathscr{O}(X)$-convex.

Proof, 2: constructing a spray over C

Recall that the attaching set $C = A \cap B$ can be chosen an arbitrarily thin slab of the convex set B.

A suitable choice of C ensures that f (a proper immersion with simple double points) is an embedding on a small convex neighborhood $\overline{U} \Subset W$ of C.

The normal bundle of the immersion $f\colon W \to X$ is holomorphically trivial over $\overline{U}$ by the Oka-Grauert principle.

Hence, there are a neighborhood $W_1 \subset W$ of A and a holomorphic map

$$F\colon W_1 \times \mathbb{D}^{n-d} \to X$$

which is injective (an embedding) on $U \times \mathbb{D}^{n-d}$ and satisfies

$$F(z,0) = f(z) \quad \text{for all} \quad z \in W_1.$$

Here, $\mathbb{D}^{n-d}$ denotes the unit polydisc in $\mathbb{C}^{n-d}$.

Proof, 3: an isotopy $r_t : V \to V$

Recall that the sets $L' = L \cup f(K)$ and $L' \cup f(C)$ are $\mathscr{O}(X)$-convex.

By shrinking $U \supset C$ and rescaling the variable $w \in \mathbb{D}^{n-d}$, we may assume that the Stein domain

$$\Omega := F(U \times \mathbb{D}^{n-d}) \subset X \setminus L'$$

is sufficiently small around $f(C)$ such that $L' \cup \overline{\Omega}$ is $\mathscr{O}(X)$-convex.

Hence, there is a Stein neighborhood $\Omega' \subset X$ of L' such that

$$\overline{\Omega} \cap \overline{\Omega}' = \varnothing, \text{ and } \Omega_0 := \Omega \cup \Omega' \text{ is a Stein Runge domain in } X.$$

Choose a convex neighborhood $V \supset B \cup \overline{U}$ and an isotopy $r_t : V \to V$ ($t \in [0,1]$) of injective holomorphic maps (dilations) such that

1. $r_0 = \mathrm{Id}_V$,
2. $r_t(U) \subset U$ for all $t \in [0,1]$, and
3. $r_1(V) \subset U$.

Proof, 4: an isotopy $\psi_t\colon \Omega_0 \to X$

Define the isotopy $\phi_t\colon V \times \mathbb{D}^{n-d} \to V \times \mathbb{D}^{n-d}$ $(t \in [0,1])$ by

$$\phi_t(z,w) = (r_t(z), w), \quad z \in V,\ w \in \mathbb{D}^{n-d}.$$

Since $r_1(V) \subset U$, we have that

$$\phi_1(V \times \mathbb{D}^{n-d}) \subset U \times \mathbb{D}^{n-d}.$$

Consider the isotopy $\psi_t\colon \Omega_0 \to X$ $\ (t \in [0,1])$ defined by

$$\psi_t = F \circ \phi_t \circ F^{-1} \ \text{ on } \ \Omega = F(U \times \mathbb{D}^{n-d}); \quad \psi_t = \mathrm{Id} \ \text{ on } \Omega'.$$

Note that $\psi_0 = \mathrm{Id}$ on $\Omega_0 = \Omega \cup \Omega'$ and $\psi_t(\Omega_0)$ is Runge in X for all t.

The crucial point: If X enjoys the density property, we can approximate the biholomorphism $\psi_1\colon \Omega_0 \to \psi_1(\Omega_0) \subset X$ uniformly on compacts in Ω_0 by holomorphic automorphisms $\Psi \in \mathrm{Aut}(X)$.

Proof, 5: the map $G : V \times \mathbb{D}^{n-d} \to X$

Recall that $V \supset B$. Consider the injective holomorphic map

$$G := \Psi^{-1} \circ F \circ \phi_1 : V \times \mathbb{D}^{n-d} \to X.$$

Since $\phi_1(V \times \mathbb{D}^{n-d}) \subset U \times \mathbb{D}^{n-d}$, we have

$$(F \circ \phi_1)(V \times \mathbb{D}^{n-d}) \subset F(U \times \mathbb{D}^{n-d}) = \Omega \subset X \setminus L'.$$

Since $\psi_1 = \mathrm{Id}$ on $\Omega' \supset L'$, Ψ approximates the identity on a neighborhood of L', and hence

$$G(V \times \mathbb{D}^{n-d}) \subset \Psi^{-1}(\Omega) \subset X \setminus L'.$$

We also have that $F \circ \phi_1 = \psi_1 \circ F$ on $U \times \mathbb{D}^{n-d}$ and hence

$$G = \Psi^{-1} \circ F \circ \phi_1 = \Psi^{-1} \circ \psi_1 \circ F \quad \text{on} \quad U \times \mathbb{D}^{n-d}.$$

Since $\Psi^{-1} \circ \psi_1 \approx \mathrm{Id}$ on $\Omega = F(U \times \mathbb{D}^{n-d})$, we see that

$$G \approx F \text{ uniformly on compacts in } U \times \mathbb{D}^{n-d} \supset C \times \mathbb{D}^{n-d}.$$

Proof, 6: gluing F and G

Hence, we can glue F and G into a holomorphic map

$$\widetilde{F} \colon (A \cup B) \times \frac{1}{2}\mathbb{D}^{n-d} \to X$$

such that

$$\widetilde{F} \approx F \text{ on } A \times \frac{1}{2}\mathbb{D}^{n-d}, \quad F \approx G \text{ on } B \times \frac{1}{2}\mathbb{D}^{n-d}.$$

The holomorphic map $\tilde{f} := \widetilde{F}(\cdot\,, 0) \colon A \cup B = D \to X$ then satisfies the conclusion of Lemma (after a generic perturbation). Indeed, if the approximations are sufficiently close then

$$\tilde{f}(B) \cap L = \varnothing \;\text{ and }\; \tilde{f}(D \setminus \mathring{K} \subset X \setminus L.$$

A similar argument applies if X enjoys the volume density property.

Conclusion: The noncritical case of the theorem follows from the lemma by a standard induction procedure. The analysis near an attached handle (in S) reduces to the noncritical case (as in Oka theory).

Exotic embeddings into $\mathbb{C}^n$

A submanifold $S \subset \mathbb{C}^n$ is said to be **complete** if the Euclidean metric induces a complete metric on S.

Theorem (Alárcon & F., 2017)

Given a closed complex submanifold S of $\mathbb{C}^n$ $(n > 1)$, there exists a complete holomorphic embedding $f\colon S \hookrightarrow \mathbb{C}^n$ such that $f(S)$ contains any given countable subset of $\mathbb{C}^n$. In particular, $f(S)$ can be dense in $\mathbb{C}^n$.

The idea is to twist the image of S by holomorphic automorphisms of $\mathbb{C}^n$ so that it avoids a certain sequence of labyrinths, thereby making it complete, while at the same time hitting the given countable set.

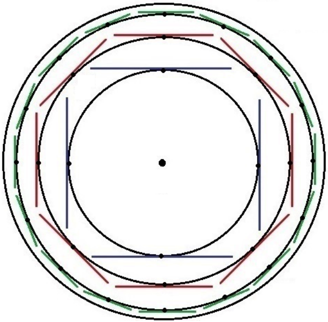
A **labyrinth** is a set of the form

$$\mathfrak{F} = \cup_j K_j \subset r\mathbb{B}^n \ (r > 0),$$

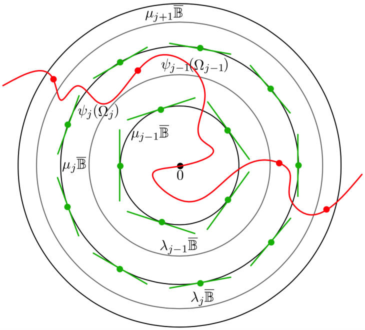
where each K_j is a closed ball (or polytope) in an affine real hyperplane $\Lambda_j \subset \mathbb{C}^n$, $K_i \cap K_j = \varnothing$ for $i \neq j$, such that any path in $\mathbb{C}^n \setminus \mathfrak{F}$ starting inside $r\mathbb{B}^n$ and terminating outside $r\mathbb{B}^n$ has infinite length.

A complex subvariety avoiding a labyrinth

Alárcon, Globevnik, López 2016: A labyrinth $\mathfrak{F}$ consisting of tangent balls. Any divergent curve in $\mathbb{B}^n$ avoiding all except finitely many of these balls has infinite length.



The submanifold $S \subset \mathbb{C}^n$ is twisted by holomorphic automorphisms so that it avoids the labyrinth $\mathfrak{F}$. The image is ambiently complete.



A complete densely embedded disc in the ball

A variation of this technique gives the following result.

Theorem (Alárcon & F., 2017)

Let S be a closed complex submanifold of $\mathbb{C}^n$ $(n > 1)$ intersecting $\mathbb{B}^n$. Given a connected compact subset $K \subset S \cap \mathbb{B}^n$, there are a pseudoconvex Runge domain $\Omega \subset S$ containing K and a complete holomorphic embedding $f : \Omega \hookrightarrow \mathbb{B}^n$ whose image $f(\Omega)$ contains any given countable subset of $\mathbb{B}^n$. In particular, $f(\Omega)$ can be dense in $\mathbb{B}^n$.

Taking $n = 2$ and $S = \mathbb{D} := \{\zeta \in \mathbb{C} \colon |\zeta| < 1\} \hookrightarrow \mathbb{C}^2$ we obtain:

Corollary

There is a complete embedded holomorphic disc $\mathbb{D} \hookrightarrow \mathbb{B}^2$ with a dense image.

A conjecture

We believe that the analogous results hold with $\mathbb{C}^n$ replaced by any Stein manifold with the (volume) density property. More precisely:

Conjecture: Assume that X is a Stein manifold with the density property. Choose a complete Riemannian metric $\mathfrak{g}$ on X.

If S is a Stein manifold, $\dim S < \dim X$, which admits a proper holomorphic embedding $S \hookrightarrow X$, then it also admits a $\mathfrak{g}$-complete holomorphic embedding $S \hookrightarrow X$ with dense image.

Recall that, by our theorem, a proper holomorphic embedding $S \hookrightarrow X$ exists if $\dim X \geq 2 \dim S + 1$.

THANK YOU

FOR YOUR ATTENTION

AND

HAPPY BIRTHDAY TO KANG-TAE