

Holomorphic Legendrian curves

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Complex contact manifolds

Kobayashi 1959 A **complex contact manifold** is a pair (X, ξ) where:

- X is a complex manifold of odd dimension $2n + 1 \geq 3$,
- ξ is a holomorphic hyperplane subbundle of the tangent bundle TX which is **maximally nonintegrable**, in the sense that the **O'Neill tensor**

$$O : \xi \times \xi \rightarrow TX/\xi = L, \quad (v, w) \mapsto [v, w] \pmod{\xi}$$

(also called the **Frobenius obstruction**) is nondegenerate.

- Equivalently, every point $p \in X$ admits an open neighborhood $U \subset X$ such that

$$\xi|_U = \ker \alpha,$$

where α is a holomorphic 1-form on U satisfying

$$\alpha \wedge (d\alpha)^n \neq 0 \quad \dots \text{ a volume form on } X.$$

Such ξ is a **holomorphic contact structure**, and α is a **contact form**.

On the other hand, $\alpha \wedge d\alpha = 0$ defines a **hypersurface foliation**.

Darboux's theorem and stability results

Contact complex manifolds (X, ζ) and (X', ζ') are **contactomorphic** if there exists a biholomorphism $F: X \rightarrow X'$ satisfying

$$dF_x(\zeta_x) = \zeta'_{F(x)} \quad \text{for all } x \in X.$$

Example (Model complex contact space)

$$(\mathbb{C}^{2n+1}, \zeta_0 = \ker \alpha_0), \quad \alpha_0 = dz + \sum_{j=1}^n x_j dy_j.$$

Darboux 1882; Moser 1965 Every complex contact manifold (X^{2n+1}, ζ) is locally contactomorphic to $(\mathbb{C}^{2n+1}, \zeta_0)$.

Gray 1959 If (X, ζ) is a compact contact manifold then any small contact perturbation ζ' of ζ is contactomorphic to ζ .

LeBrun & Salamon 1994 Any two complex contact structures on a simply connected compact complex manifold are contactomorphic.

The normal bundle of a contact structure

A holomorphic 1-form α with $\zeta = \ker \alpha$ is determined up to a nowhere vanishing multiplier f ; note that $f\alpha \wedge (d(f\alpha))^n = f^{n+1}\alpha \wedge (d\alpha)^n$. Thus, (X, ζ) admits a **complex contact atlas** $\{(U_j, \alpha_j)\}$ with $\alpha_i = f_{i,j}\alpha_j$ on $U_{i,j} = U_i \cap U_j$.

LeBrun & Salamon 1994 The collection (α_j) determines a holomorphic 1-form $\alpha \in \Gamma(X, \Omega^1(L))$ given by the tautological projection

$$TX \xrightarrow{\alpha} L := TX/\zeta \quad \text{the normal bundle.}$$

From $d(f\alpha) = df \wedge \alpha + fd\alpha$ we see that

$$d\alpha \text{ is a section of } \Lambda^2(\zeta^*) \otimes L.$$

Thus, letting $K_X = \Lambda^{2n+1}(T^*X)$ (the canonical bundle), we see that

$$\alpha \wedge (d\alpha)^n \neq 0 \text{ is a section of } K_X \otimes L^{\otimes(n+1)}.$$

This provides a holomorphic line bundle isomorphism

$$L^{\otimes(n+1)} \cong K_X^{-1} = \Lambda^{2n+1}(TX).$$

The space of complex contact structures

Conversely, assume X^{2n+1} is a complex manifold with $H^1(X, \mathbb{Z}_{n+1}) = 0$ and $c_1(TX)$ divisible by $n+1$. Then there exists the line bundle

$$L = K_X^{-1/(n+1)}.$$

Given a holomorphic 1-form $\alpha \in \Gamma(X, \Omega^1(L))$, consider

$$\alpha \wedge (d\alpha)^n \in \Gamma(X, \Omega^{2n+1}(K_X^{-1})) = \Gamma(X, \mathcal{O}).$$

If X is **compact** then $\Gamma(X, \mathcal{O}) = \mathbb{C}$. The map

$$\Gamma(X, \Omega^1(L)) \ni \alpha \mapsto \alpha \wedge (d\alpha)^n \in \mathbb{C}$$

is homogeneous of degree $n+1$. Hence, if X admits a complex contact structure, then the set of all such structures is the complement of a degree $n+1$ hypersurface in $\mathbb{P}(\Gamma(X, \Omega^1(L)))$ (a complex manifold).

A (unique) complex contact structure on $\mathbb{C}\mathbb{P}^{2n+1}$

Let z_1, \dots, z_{2n+2} be complex coordinates on \mathbb{C}^{2n+2} and

$$\theta = z_1 dz_2 - z_2 dz_1 + \cdots + z_{2n+1} dz_{2n+2} - z_{2n+2} dz_{2n+1}.$$

Then, $\ker \theta$ determines a contact structure on $\mathbb{C}\mathbb{P}^{2n+1} = \mathbb{C}_*^{2n+2} / \mathbb{C}^*$.

Let θ_j ($j = 1, \dots, 2n+2$) be the pull-back of θ to the affine hyperplane

$$\mathbb{C}^{2n+1} \cong H_j = \{z_j = 1\} \subset \mathbb{C}^{2n+2}.$$

For example,

$$\theta_1 = dz_2 + z_3 dz_4 - z_4 dz_3 + \cdots.$$

Then (H_j, θ_j) is contactomorphic to $(\mathbb{C}^{2n+1}, \alpha_0)$ for each j , and this collection defines a contact atlas on $X = \mathbb{C}\mathbb{P}^{2n+1}$. We have

$$K_X^{-1} = \mathcal{O}_X(2n+2), \quad L = K_X^{-1/(n+1)} = \mathcal{O}_X(2),$$

$$\alpha \in \Gamma(\mathbb{C}\mathbb{P}^{2n+1}, \Omega^1(2)).$$

Contact hypersurfaces in complex symplectic manifolds

Let (Z, ω) be a **holomorphic symplectic manifold**, $\dim Z = 2n + 2 \geq 4$; ω a holomorphic 2-form on Z ,

$$d\omega = 0, \quad \omega^{n+1} \neq 0 \quad \text{a volume form on } Z.$$

A holomorphic vector field V on Z is a **Liouville vector field** for ω if

$$L_V \omega = \omega \iff d(i_V \omega) = \omega \iff \phi_t^* \omega = e^t \omega.$$

Here ϕ_t is the flow of V . The holomorphic 1-form

$$\theta = i_V \omega = V \lrcorner \omega$$

induces a holomorphic contact form $\alpha = \theta|_{TX}$ on any complex hypersurface $X \subset Z$ transverse to V . Indeed:

$$\theta \wedge (d\theta)^n = i_V \omega \wedge \omega^n = \frac{1}{n+1} i_V (\omega^{n+1})$$

is a volume form on X provided that V is transverse to X .

The converse process is the **symplectization** of a contact manifold (X, α) :

$$Z = \mathbb{C} \times X, \quad \omega = d(e^t \alpha) = e^t (dt \wedge \alpha + d\alpha), \quad i_{\partial_t} \omega|_{t=0} = \alpha.$$

Examples in \mathbb{C}^{2n+2}

Example: \mathbb{C}^{2n+2} , $\omega = \sum_{j=0}^n d\zeta_j \wedge dz_j$ the standard symplectic form,

$$V = \sum_{j=0}^n \zeta_j \partial_{\zeta_j}, \quad \theta = i_V \omega = \sum_{j=0}^n \zeta_j dz_j, \quad d\theta = \omega;$$

$$W = \sum_{j=0}^n \zeta_j \partial_{\zeta_j} + z_j \partial_{z_j}, \quad \theta = i_W \omega = \sum_{j=0}^n \zeta_j dz_j - z_j d\zeta_j, \quad d\theta = 2\omega.$$

Any complex hypersurface $X \subset \mathbb{C}^{2n+2}$ transverse to V or W carries the complex contact structure $\ker \theta \cap TX$.

Example: the special complex linear group

$$SL_2(\mathbb{C}) = \left\{ \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} : z_{11}z_{22} - z_{21}z_{12} = 1 \right\} \subset \mathbb{C}^4$$

with the holomorphic contact form

$$\theta = z_{11} dz_{22} - z_{21} dz_{12}.$$

Contact structure on $\mathbb{P}(T^*Z)$

Let Z^{n+1} be a complex manifold. The holomorphic cotangent bundle T^*Z carries the tautological 1-form θ given in any local coordinates z_0, \dots, z_n on Z and the induced fiber coordinates ζ_0, \dots, ζ_n by

$$\theta = \zeta_0 dz_0 + \dots + \zeta_n dz_n \quad (= \mathbf{pdq} \text{ in classical notation}).$$

Then, $\ker \theta$ determines a contact structure ζ on the projectivised cotangent bundle $X = \mathbb{P}(T^*Z)$. On the affine chart $\{\zeta_j = 1\}$ we have

$$\zeta = \ker\left(dz_j + \sum_{i \neq j} \zeta_i dz_i\right).$$

Note that $d\theta = d\zeta \wedge dz = \omega$ is the **canonical symplectic form** on T^*Z , and $\theta = i_V \omega$ where $V = \sum_{j=0}^n \zeta_j \partial_{\zeta_j}$ is the **Euler vector field** (i.e., radial vector field on the fibres of $T^*Z \rightarrow Z$).

Isotropic and Legendrian submanifolds

A smooth map $F: M \rightarrow (X, \zeta)$ is said to be **isotropic** if

$$dF_p(T_p M) \subset \zeta_{F(p)}, \quad p \in M.$$

An isotropic immersion is **Legendrian** if $\dim_{\mathbb{R}} M = 2n$ is maximal.

If $\zeta = \ker \alpha$ then $F: M \rightarrow X$ is isotropic iff $F^* \alpha = 0$.

Lemma

*If $\dim X = 2n + 1$ and F is an isotropic immersion, then $\dim_{\mathbb{R}} M \leq 2n$; if $\dim_{\mathbb{R}} M = 2n$ then $F(M)$ is an immersed **complex submanifold** of X .*

Proof.

Note that $\omega := d\alpha|_{\zeta}$ is a holomorphic symplectic form on $\zeta = \ker \alpha$.

Isotropic subspaces $U \subset \zeta_x$ ($x \in X$) are characterized by the condition

$U \subset U_{\omega}^{\perp}$. Note that U_{ω}^{\perp} is a complex subspace of ζ_x , so we also have

$U^{\mathbb{C}} := \text{Span}_{\mathbb{C}}(U) \subset U_{\omega}^{\perp}$. From $\dim_{\mathbb{C}} U^{\mathbb{C}} + \dim_{\mathbb{C}} U_{\omega}^{\perp} = 2n$ we get the

result. Note that $\dim_{\mathbb{R}} U = 2n$ iff $U = U^{\mathbb{C}} = U_{\omega}^{\perp}$, so U is complex.

Hence, if $\dim_{\mathbb{R}} M = 2n$ then $dF_p(T_p M) \subset \zeta_{F(p)}$ is a complex subspace

for every $p \in M$, so $F(M)$ is a complex submanifold. □

How many Legendrian submanifolds are there?

What can be said about the existence of (proper) complex isotropic and Legendrian submanifolds of a complex contact manifold (X, ξ) ?

Example

Let $(\mathbb{C}^{2n+1}, \xi_0 = \ker \alpha_0)$ with $\alpha_0 = dz + \sum_{j=1}^n x_j dy_j$. Given a holomorphic function $z = z(y_1, \dots, y_n)$, the formula

$$dz - \sum_{j=1}^n \frac{\partial z}{\partial y_j} dy_j = 0$$

shows that $y \mapsto (-\partial z / \partial y, y, z(y))$ is a Legendrian submanifold.

Bryant 1981 Every compact Riemann surface embeds as a complex Legendrian curve in $\mathbb{C}P^3$. **Main idea:** consider rational maps

$$M \rightarrow \mathbb{C}_{(x,y,z)}^3 \subset \mathbb{C}P^3, \quad t \mapsto (-\dot{z}(t)/\dot{y}(t), y(t), z(t)).$$

Find (y, z) such that this is an (Legendrian) embedding $M \hookrightarrow \mathbb{C}P^3$.

Proper Legendrian curves in $(\mathbb{C}^{2n+1}, \zeta_0)$

Theorem (Alarcón, F., López, Compositio Math., in press)

Let M be an open Riemann surface and $K \subset M$ be a compact set in M whose complement has no relatively compact connected components.

Then every holomorphic Legendrian curve $F: K \rightarrow \mathbb{C}^{2n+1}$ ($n \in \mathbb{N}$) on an open neighborhood of K can be approximated uniformly on K by proper holomorphic Legendrian embeddings $\tilde{F}: M \hookrightarrow \mathbb{C}^{2n+1}$.

Furthermore, given a pair of indices $\{i, j\} \subset \{1, 2, \dots, 2n+1\}$ with $i \neq j$, we may choose $\tilde{F} = (\tilde{F}_1, \tilde{F}_2, \dots, \tilde{F}_{2n+1})$ as above such that $(\tilde{F}_i, \tilde{F}_j): M \rightarrow \mathbb{C}^2$ is a proper map.

Example: If $(x, y): \mathbb{C} \rightarrow \mathbb{C}^2$ is a proper holomorphic immersion and

$$z(\zeta) := z_0 - \int_0^\zeta x(t) dy(t), \quad \zeta \in \mathbb{C},$$

then $F = (x, y, z): \mathbb{C} \rightarrow \mathbb{C}^3$ is a proper Legendrian immersion.

Proof, 1: The basic scheme

Consider $\mathbb{C}^3_{(x,y,z)}$ with the contact form $\alpha_0 = dz + xdy$.

A holomorphic map $(x, y, z): M \rightarrow \mathbb{C}^3$ is Legendrian if and only if xdy is an **exact 1-form** (with vanishing periods over all closed curves in M) and

$$z = - \int xdy.$$

The construction proceeds by inductively enlarging the domain of the Legendrian curve. Let $\rho: M \rightarrow \mathbb{R}_+$ be a strongly subharmonic Morse exhaustion function. We must consider two cases:

The noncritical case: Let $D \subset D'$ be Runge domains in M of the form

$$D = \{\rho < c\}, \quad D' = \{\rho < c'\}, \quad d\rho \neq 0 \text{ on } \overline{D'} \setminus D.$$

The critical case: ρ has a single critical point $p \in D' \setminus \overline{D}$.

The (only) nontrivial case is when the Morse index of p is equals one (critical points of Morse index zero are local minima of ρ).

Proof, 2: The period map

The noncritical case: Let $C_1, \dots, C_\ell \subset D$ be closed curves forming a basis of the homology group $H_1(D; \mathbb{Z}) \cong H_1(D'; \mathbb{Z}) = \mathbb{Z}^\ell$ such that $\bigcup_{j=1}^\ell C_j$ is Runge in M . Consider the **period map**

$$\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_\ell) : \mathcal{A}^1(D)^2 \rightarrow \mathbb{C}^\ell$$

$$\mathcal{P}_j(x, y) = \int_{C_j} x \, dy, \quad x, y \in \mathcal{A}^1(D), \quad j = 1, \dots, \ell.$$

We may assume that $y \in \mathcal{A}^1(D)$ is nonconstant. We find a holomorphic spray $X(\cdot, \zeta) : \bar{D} \rightarrow \mathbb{C}$ ($\zeta \in \mathbb{C}^\ell$) of class $\mathcal{A}^1(D)$ and of the form

$$X(u, \zeta) = x(u) + \sum_{k=1}^{\ell} \zeta_k g_k(u), \quad u \in \bar{D}, \quad \zeta \in \mathbb{C}^\ell.$$

such that $X(\cdot, 0) = x$ and

$$\frac{\partial}{\partial \zeta} \Big|_{\zeta=0} \mathcal{P}(X(\cdot, \zeta), y) : \mathbb{C}^\ell \longrightarrow \mathbb{C}^\ell \text{ is an isomorphism.}$$

Proof, 3: Sprays and Runge's theorem

By Runge's theorem we can find holomorphic maps

$$\tilde{x}(\cdot, \cdot): M \times \mathbb{C}^\ell \rightarrow \mathbb{C}, \quad \tilde{y}: M \rightarrow \mathbb{C}$$

approximating X, y (respectively) in $\mathcal{C}^1(\overline{D})$.

Since $\mathcal{P}(X(\cdot, 0), y) = 0$, the period domination condition implies (by the implicit function theorem) that there is $\zeta_0 \in \mathbb{C}^\ell$ close to 0 such that

$$\mathcal{P}(\tilde{x}(\cdot, \zeta_0), \tilde{y}) = 0.$$

Hence, the 1-form $\tilde{x}(\cdot, \zeta_0)d\tilde{y}$ is **exact** on \overline{D}' . Fix a point $p_0 \in D$ and set

$$\tilde{z}(p) = z(p_0) - \int_{p_0}^p \tilde{x}(\cdot, \zeta_0)d\tilde{y}, \quad p \in D'.$$

The Legendrian curve

$$(\tilde{x}(\cdot, \zeta_0), \tilde{y}, \tilde{z}) : \overline{D}' \rightarrow \mathbb{C}^3$$

approximates (x, y, z) in $\mathcal{C}^1(\overline{D})$. This establishes the noncritical case.

Proof, 4: The critical case

The critical case: This amounts to a change of topology of the sublevel set. The new bigger domain $D' \subset M$ deformation retracts onto $\bar{D} \cup E$, where E is a smooth arc attached to \bar{D} with its endpoints $a, b \in bD$.

Let $(x, y, z): \bar{D} \rightarrow \mathbb{C}^3$ be a Legendrian curve. We extend the functions x, y to smooth functions $\tilde{x}, \tilde{y}: \bar{D} \cup E \rightarrow \mathbb{C}$ such that

$$\int_E \tilde{x} d\tilde{y} = z(b) - z(a).$$

This ensures that the extended function

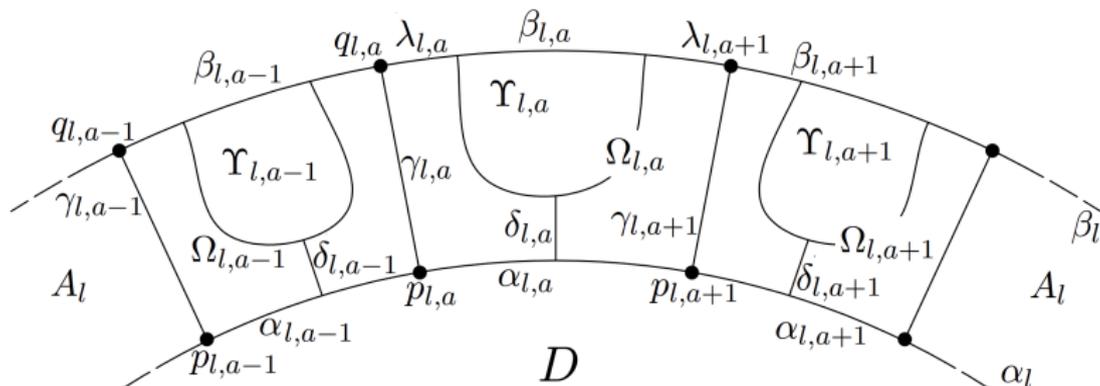
$$\tilde{z}(p) = z(a) + \int_a^p \tilde{x} d\tilde{y}, \quad p \in \bar{D} \cup E \subset M$$

is well defined and matches the function z on \bar{D} .

Hence, $(\tilde{x}, \tilde{y}, \tilde{z}): \bar{D} \cup E \rightarrow \mathbb{C}^3$ is a **generalized Legendrian curve**.

Now, use period dominating sprays and Mergelyan approximation theorem to conclude the proof similarly as before.

Proof, 5: How to ensure properness of $(x, y): M \rightarrow \mathbb{C}^2$



Assume $\max\{|x|, |y|\} > m$ on bD . Subdivide bD into arcs $\alpha_{l,a}$ such that on each of them, one of the functions $|x|, |y|$ is $> m$. Assume that $|x| > m$ on $\alpha_{l,a}$. Extend x smoothly to the arcs $\gamma_{l,a}$ and $\gamma_{l,a+1}$ such that $|x| > m$, and $|x| > m + 1$ at the outer endpoints of these two arcs. Apply Mergelyan to approximate x on $\overline{D} \cup \gamma_{l,a} \cup \gamma_{l,a+1}$ by $\tilde{x} \in \mathcal{O}(M)$. Choose the disc $Y_{l,a} \subset \Omega_{l,a}$ such that $|\tilde{x}| > m$ on $\overline{\Omega_{l,a}} \setminus Y_{l,a}$. Use Mergelyan to approximate y on \overline{D} by $\tilde{y} \in \mathcal{O}(M)$ such that $|\tilde{y}| > m + 1$ on $Y_{l,a}$. Apply the analogous procedure on every $\Omega_{l,a}$. Then, $\max\{|\tilde{x}|, |\tilde{y}|\} > m + 1$ on bD' and $> m$ on $\overline{D'} \setminus D$.

Rough shape of the space of Legendrian curves in \mathbb{C}^{2n+1}

Let $\mathcal{L}(M, \mathbb{C}^{2n+1})$ be the space of all holomorphic Legendrian immersions $M \rightarrow (\mathbb{C}^{2n+1}, \xi_0)$. We have $(x, y, z) \in \mathcal{L}(M, \mathbb{C}^{2n+1})$ iff $(x, y): M \rightarrow \mathbb{C}^{2n}$ is an exact holomorphic immersion, i.e., $x dy$ is an exact 1-form on M . Thus we have a sequence of maps

$$\mathcal{L}(M, \mathbb{C}^{2n+1}) \xrightarrow{pr} \mathfrak{I}_*(M, \mathbb{C}^{2n}) \xrightarrow{i} \mathfrak{I}(M, \mathbb{C}^{2n}) \xrightarrow{\phi} \mathcal{O}(M, \mathbb{C}_*^{2n}) \xrightarrow{\psi} \mathcal{C}(M, S^{4n-1})$$

where $\mathfrak{I}_*(M, \mathbb{C}^{2n})$ denotes the space of exact holomorphic immersions $M \rightarrow \mathbb{C}^{2n}$, $\mathfrak{I}(M, \mathbb{C}^{2n})$ is the set of all holomorphic immersions,

$$\phi(x, y) = (dx/\theta, dy/\theta): M \rightarrow \mathbb{C}_*^{2n}, \quad (x, y) \in \mathfrak{I}(M, \mathbb{C}^{2n}),$$

and ψ is induced by the retraction $\mathbb{C}_*^{2n} \rightarrow S^{4n-1}$.

Theorem (Lárusson and F., CAG and Math. Z., in press)

All maps in the above sequence are weak homotopy equivalences, and are homotopy equivalence when M has finite topological type.

Homotopy groups of $\mathcal{L}(M, \mathbb{C}^{2n+1})$

The proof combines methods explained above and the parametric version of Gromov's **convex integration lemma**. Since M has the homotopy type of a bouquet of ℓ circles, where $H_1(M; \mathbb{Z}) = \mathbb{Z}^\ell$, we get

Corollary (Lárusson and F., Math. Z., in press)

The space $\mathcal{L}(M, \mathbb{C}^{2n+1})$ is weakly homotopy equivalent to the free ℓ -loop space $\mathcal{L}_\ell S^{4n-1}$, and is homotopy equivalent to it if $\ell < \infty$.

It follows that $\mathcal{L}(M, \mathbb{C}^{2n+1})$ is path connected and simply connected, and for each $k \geq 2$ we have

$$\pi_k(\mathcal{L}(M, \mathbb{C}^{2n+1})) = \pi_k(S^{4n-1}) \times \pi_{k+1}(S^{4n-1})^\ell.$$

In particular, $\mathcal{L}(M, \mathbb{C}^{2n+1})$ is $(4n - 3)$ -connected.

A hyperbolic contact structure on \mathbb{C}^{2n+1}

The situation may be radically different for nonstandard contact structures on \mathbb{C}^{2n+1} . The **Kobayashi pseudometric** associated to a contact structure is defined by using holomorphic Legendrian discs.

Theorem (F., J. Geom. Anal. 2017)

For any $n \geq 1$ there exists a holomorphic contact structure ξ on \mathbb{C}^{2n+1} which is **Kobayashi hyperbolic** and isotopic to ξ_0 . In particular, every holomorphic Legendrian curve from \mathbb{C} or \mathbb{C}^* to (\mathbb{C}^{2n+1}, ξ) is constant.

Idea of proof: We take $\alpha = \Phi^* \alpha_0$ where $\alpha_0 = dz + \sum_{j=1}^n x_j dy_j$ and $\Phi: \mathbb{C}^{2n+1} \rightarrow \Omega \subset \mathbb{C}^{2n+1}$ is a **Fatou-Bieberbach map** whose image Ω avoids the union of countably many cylinders

$$K = \bigcup_{N=1}^{\infty} 2^{N-1} b \mathbb{D}_{(x,y)}^{2n} \times C_N \overline{\mathbb{D}}_z.$$

Assuming that $C_N \geq n 2^{3N+1}$ for all $N \in \mathbb{N}$,

$\mathbb{C}^{2n+1} \setminus K$ is α_0 -hyperbolic; hence, $(\mathbb{C}^{2n+1}, \alpha = \Phi^* \alpha_0)$ is hyperbolic.

On α_0 -hyperbolicity of $\mathbb{C}^3 \setminus K$

Let $\alpha_0 = dz + xdy$ in \mathbb{C}^3 .

Lemma

Assume that $C_N \geq 2^{3N+1}$ for every $N \in \mathbb{N}$ and let

$$K = \bigcup_{N=1}^{\infty} 2^{N-1} b\mathbb{D}_{(x,y)}^2 \times C_N \bar{\mathbb{D}}_z \subset \mathbb{C}^3$$

For every holomorphic α_0 -horizontal disk

$$f(\zeta) = (x(\zeta), y(\zeta), z(\zeta)) \in \mathbb{C}^3 \setminus K, \quad \zeta \in \mathbb{D}$$

with $f(0) \in 2^{N_0} \mathbb{D}^3$ for some $N_0 \in \mathbb{N}$ we have the estimates

$$|x'(0)| < 2^{N_0+1}, \quad |y'(0)| < 2^{N_0+1}, \quad |z'(0)| < 2^{2N_0+1}.$$

Complete bounded Legendrian curves in $(\mathbb{C}^{2n+1}, \xi_0)$

Martín-Umehara-Yamada 2014 Do there exist complete bounded holomorphic Legendrian curves in \mathbb{C}^3 ? Can they have Jordan boundaries? (Analogue of the **Calabi-Yau problem** in the theory of minimal surface.)

Theorem (Alarcón, F., López, Compositio Math., in press)

Let M be a compact bordered Riemann surface. Every Legendrian curve $M \rightarrow \mathbb{C}^{2n+1}$ of class $\mathcal{A}^1(M)$ can be uniformly approximated by topological embeddings $F: M \rightarrow \mathbb{C}^{2n+1}$ such that $F|_{\mathring{M}}: \mathring{M} \rightarrow \mathbb{C}^{2n+1}$ is a complete Legendrian embedding.

Besides the methods explained above, we use the following

Riemann-Hilbert lemma for Legendrian curves: given a Legendrian immersion $f: M \rightarrow \mathbb{C}^{2n+1}$ and a continuous family of Legendrian discs $F(u, \cdot): \bar{\mathbb{D}} \rightarrow \mathbb{C}^{2n+1}$ with $F(u, 0) = f(u)$ for all $u \in bM$, there is a Legendrian approximate solution $H: M \rightarrow \mathbb{C}^{2n+1}$ to the Riemann-Hilbert boundary value problem.

Darboux charts around immersed Legendrian curves

Theorem (Alarcón & F., 2017)

Assume that

- (X, ξ) is a complex contact manifold,
- R is an open Riemann surface,
- $x_1: R \rightarrow \mathbb{C}$ is a holomorphic immersion, and
- $f: R \rightarrow (X, \xi)$ is a holomorphic Legendrian immersion.

Then there exist a Stein neighborhood $\Omega \subset R \times \mathbb{C}^{2n}$ of $R \times \{0\}^{2n}$ and a holomorphic immersion $F: \Omega \rightarrow X$ such that $F(\cdot, 0) = f$ and

$$F^*\xi = \ker\left(dz + \sum_{j=1}^n x_j dy_j\right) \quad \text{on } U \times \mathbb{B}^{2n},$$

where $x_2, \dots, x_n, y_1, \dots, y_n, z$ are Euclidean coordinates on \mathbb{C}^{2n} .

This is proved by standardizing the contact structure $\xi \subset TX$ along the Legendrian curve $f(R)$ and applying **Moser's method**.

Two consequences of the existence of Darboux charts

Corollary (Alarcón & F. 2017)

Assume that

- (X, ξ) is a complex contact manifold,
- R is an open Riemann surface,
- $f: R \rightarrow (X, \xi)$ is a holomorphic Legendrian immersion, and
- $M \subset R$ is a smoothly bounded compact domain.

Then $f|_M$ can be uniformly approximated by topological embeddings $\tilde{f}: M \rightarrow X$ such that $\tilde{f}|_{\overset{\circ}{M}}: \overset{\circ}{M} \rightarrow X$ is a complete Legendrian embedding.

Corollary (Deformation theory for Legendrian curves)

(Assumptions as above.) The space of all small Legendrian deformations of $f|_M: M \rightarrow (X, \xi)$ can be identified with an open set in a complex Banach space which can be explicitly described (as in the standard case when (X, ξ) is the model contact space $(\mathbb{C}^{2n+1}, \xi_0)$).

A few open problems

- 1 How many contact structures are there on \mathbb{C}^3 ? On \mathbb{C}^{2n+1} ?
How could one distinguish them?
- 2 Study complex contact structures (and their isotropic submanifolds) on projectivized cotangent bundles, and on closed adjoint orbits.
- 3 Does every Stein manifold X^{2n+1} satisfying LeBrun-Salamon condition (i.e., such that the canonical bundle K_X has $(n+1)$ -st root) admit a contact structure?

Note that a generic holomorphic 1-form on a Stein manifold is contact on the complement of a complex hypersurface.

- 4 Does the Runge approximation theorem hold for holomorphic contact structures?
Is it possible to approximate a holomorphic contact form on a convex set in \mathbb{C}^{2n+1} by a contact form on all of \mathbb{C}^{2n+1} ?
- 5 Does every Stein contact manifold (X, ξ) admit proper holomorphic Legendrian curves normalized by any bordered Riemann surface?
We have a positive answer for pseudoconvex domains in the model contact space $(\mathbb{C}^{2n+1}, \alpha_0)$.