Minimal surfaces in minimally convex domains

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Several complex variables and partial differential equations

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References

This is joint work with Antonio Alarcón, Barbara Drinovec Drnovšek, and Francisco J. López. It is based on the following preprints:

A. Alarcón; B. Drinovec Drnovšek; F. F.; F.J. López:

- Every bordered Riemann surface is a complete conformal minimal surface bounded by Jordan curves. (2015)
  arxiv.org/abs/1503.00775
- Minimal surfaces in minimally convex domains. In preparation.

A more complete set of three introductory lectures:

- F. Forstnerič: The Beirut Lectures on Minimal Surfaces.
  http://www.fmf.uni-lj.si/∼forstneric/

Basic facts on minimal surfaces

Let M be an open Riemann surface and let $u = (u_1, \ldots, u_n) : M \to \mathbb{R}^n$ be a smooth conformal (angle preserving) immersion for some $n \geq 3$. Then the following are equivalent:

- u is a minimal immersion, i.e., its mean curvature vector vanishes identically: $\mathbf{H}(u) = 0$.
- u is harmonic: $\triangle u = 0$.
- $\partial u = (\partial u_1, \ldots, \partial u_n)$ is a holomorphic 1-form on M satisfying

$$(\partial u_1)^2 + (\partial u_2)^2 + \cdots + (\partial u_n)^2 = 0.$$

Hence every conformal minimal immersion $u : M \to \mathbb{R}^n$ is of the form

$$u(x) = u(p) + \int_p^x \Re\vartheta \qquad (p, \ x \in M)$$

where $\vartheta = (\vartheta_1, \ldots, \vartheta_n)$ is a $\mathbb{C}^n$-valued holomorphic 1-form on M satisfying $\vartheta_1^2 + \cdots + \vartheta_n^2 = 0$ and $|\vartheta_1|^2 + \cdots + |\vartheta_n|^2 > 0$.

Minimally convex ($\mathfrak{M}$-convex) domains

Definition

Let D be a domain in $\mathbb{R}^n$ for some $n \geq 3$. A function $u \in \mathscr{C}^2(D)$ is said to be minimal (strongly) plurisubharmonic if the restriction $u|_{L \cap D}$ is (strongly) subharmonic on any affine 2-plane $L \subset \mathbb{R}^n$.

This holds if and only if the sum of the smallest two eigenvalues of its Hessian $\mathrm{Hess}\,_u$ is nonnegative (positive) at every point of D.

The restriction of a minimal (strongly) plurisubharmonic function on D to a minimal surface $M \subset D$ is a (strongly) subharmonic function on M.

Definition

A domain $D \subset \mathbb{R}^n$ is said to be minimally convex ($\mathfrak{M}$-convex, also called 2-convex), if it admits a smooth minimal strongly plurisubharmonic exhaustion function $\rho \colon D \to \mathbb{R}$.

Smooth minimally convex domains

Let D be a domain in $\mathbb{R}^3$ (not necessarily bounded) with $\mathscr{C}^2$ boundary. Denote by $\kappa_1(x)$ and $\kappa_2(x)$ the principal curvatures of its boundary bD from the interior side at a point $x \in bD$.

Then D is minimally convex if and only if its boundary bD is mean-convex in the sense that

$$\kappa_1(x) + \kappa_2(x) \geq 0 \quad \forall x \in bD.$$

In particular, if $\Sigma \subset \mathbb{R}^3$ is an embedded minimal surface then every connected component of $\mathbb{R}^3 \setminus \Sigma$ is a minimally convex domain.

The domain D as above is said to be strongly minimally convex if

$$\kappa_1(x) + \kappa_2(x) > 0 \quad \forall x \in bD.$$

Such D admits a strongly 2-convex defining function.

F.R. Harvey & H.B. Lawson, Jr.: p-convexity, p-plurisubharmonicity and the Levi problem. Indiana Univ. Math. J. **62** (2013) 149–169
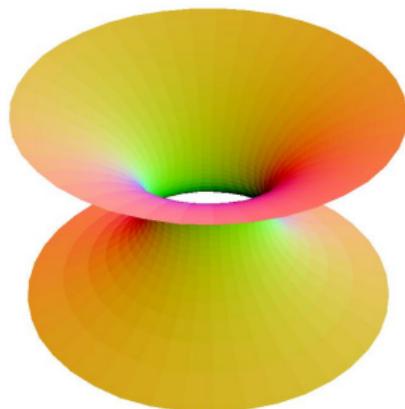
Example: the complement of a catenoid

Example

Let D be the domain

$$D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 > \cosh^2 z\}.$$

Since the boundary of D is a minimal surface (the standard catenoid), D is minimally convex. However, D is clearly not convex and its fundamental group equals $\mathbb{Z}$.

Complete proper minimal surfaces in 𝔐-convex domains

The following is our first main result.

Theorem (1)

Let D be a minimally convex domain in $\mathbb{R}^3$ and let M be a compact bordered Riemann surface with $bM \neq \emptyset$.

(a) *Every conformal minimal immersion $u \colon M \to D$ of class $\mathscr{C}^1(M)$ can be approximated, uniformly on compacts in $\mathring{M} = M \setminus bM$, by conformal complete proper minimal immersions $\widetilde{u} \colon \mathring{M} \to D$.*

(b) *If D is bounded and bD is $\mathscr{C}^2$ strongly mean-convex, then there is a constant $C > 0$ depending only on D such that $\widetilde{u}$ can be chosen continuous on M and*

$$\|\widetilde{u} - u\|_{0,M} \leq C \max_{\zeta \in bM} \sqrt{\mathrm{dist}(u(\zeta), bD)}.$$

Recall that an immersion $u \colon \mathring{M} \to \mathbb{R}^n$ is said to be complete if the pull-back $u^* ds^2$ of the Euclidean metric is a complete metric on $\mathring{M}$.

Optimality of $\mathfrak{M}$-convexity

Minimal convexity is an optimal condition for Theorem 1.

Indeed, if bD fails to be mean-convex at some point $p \in bD$, then

$$\kappa_1(p) + \kappa_2(p) < 0$$

and hence the complement of D is locally strongly $\mathfrak{M}$-convex at p.

Thus there is a strongly 2-convex function $\phi : U \to \mathbb{R}$ on a neighborhood $U \subset \mathbb{R}^3$ of p such that

$$\phi > 0 \text{ on } D \cap U, \quad \phi = 0 \text{ on } bD \cap U, \quad \phi < 0 \text{ on } U \setminus \overline{D}.$$

The maximum principle applied to $\phi \circ u$ shows that there does not exist any proper conformal minimal immersion $u : \mathring{M} \to D \cap U$ with boundary values in $bD \cap U$. Thus Theorem 1-(b) fails.

Proper conformal minimal surfaces in convex domains

We expect that the analogous result holds for $(n-1)$-convex domains $D \subset \mathbb{R}^n$, $n > 3$. For smoothly bounded domains this means that the sum of they principal curvatures at each boundary point is ≥ 0.

However, due to technical difficulties, we can only prove it for minimal surfaces in (linearly) convex domains when $n > 3$.

Theorem (2)

Let D be a convex domain in $\mathbb{R}^n$ for some $n > 3$, and let M be a compact bordered Riemann surface. Then the following hold.

(a) *Every conformal minimal immersion $u \colon M \to D$ of class $\mathscr{C}^1(M)$ can be approximated, uniformly on compacts in $\mathring{M}$, by conformal complete proper minimal immersions $\widetilde{u} \colon \mathring{M} \to D$.*

(b) *If $n \geq 5$ then $\widetilde{u}$ can be chosen an embedding.*

(c) *If D has smooth strongly convex boundary then $\widetilde{u}$ can be chosen continuous on M.*

Holomorphic null curves in $\mathbb{C}^n$

The proofs of these results rely on the connection between conformal minimal surfaces in $\mathbb{R}^n$ and holomorphic null curves in $\mathbb{C}^n$.

Definition

Let M be an open or a bordered Riemann surface. A holomorphic immersion

$$F = (F_1, F_2, \ldots, F_n) \colon M \to \mathbb{C}^n, \quad n \geq 3$$

is a **null curve** if the derivative $F' = (F_1', F_2', \ldots, F_n')$ with respect to any local holomorphic coordinate $\zeta = x + iy$ on M satisfies

$$(F_1')^2 + (F_2')^2 + \ldots + (F_n')^2 = 0.$$

The nullity condition is equivalent to $F'(\zeta) \in A_* = A \setminus \{0\}$ where

$$A = \big\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon \sum_{j=1}^{n} z_j^2 = 0 \big\} \ldots \textbf{the null quadric}.$$

A_* is **elliptic** in the sense of Gromov, and hence an **Oka manifold**.

Connection between null curves and minimal surfaces

- If $F = f + \mathrm{i}g \colon M \to \mathbb{C}^n$ is a holomorphic null curve, then

$$f = \Re F \colon M \to \mathbb{R}^n, \quad g = \Im F \colon M \to \mathbb{R}^n$$

are conformal harmonic (hence minimal) immersions into $\mathbb{R}^n$.

- Conversely, a conformal minimal immersion (CMI) $f \colon \mathbb{D} \to \mathbb{R}^n$ of the disc $\mathbb{D} = \{|\zeta| < 1\}$ is the real part of a null disc $F \colon \mathbb{D} \to \mathbb{C}^n$.

- Hence every conformal minimal immersion $M \to \mathbb{R}^n$ is of the form

$$f(x) = f(p) + \Re \int_p^x \phi \qquad (p, \ x \in M)$$

where $\phi = (\phi_1, \ldots, \phi_n)$ is a $\mathbb{C}^n$-valued holomorphic 1-form on M without zeros, satisfying the nullity condition

$$\phi_1^2 + \phi_2^2 + \cdots + \phi_n^2 = 0$$

and such that $\Re\phi = (\Re\phi_1, \ldots, \Re\phi_n)$ has vanishing periods over all closed curves in M.

Local geometry of a strongly $\mathfrak{M}$-psh function

The following lemma gives small conformal minimal discs (Levi discs) on which a strongly $\mathfrak{M}$-psh function increases quadratically.

Lemma (Local minimal discs 'going up')

Let D be a domain in $\mathbb{R}^3$ and let $\rho \colon D \to \mathbb{R}$ be a $\mathscr{C}^2$ minimal strongly plurisubharmonic function with the critical locus P.
For every compact set $L \subset D \setminus P$ there exist a constant $c = c_L > 0$ and families of embedded null holomorphic discs $\sigma_{\mathbf{x}}^j = \alpha_{\mathbf{x}}^j + \mathrm{i}\beta_{\mathbf{x}}^j \colon \overline{\mathbb{D}} \to \mathbb{C}^3$ ($\mathbf{x} \in L$, $j = 1, 2$) depending locally smoothly on the point $\mathbf{x} \in L$ and satisfying the following conditions:

- (a) $\sigma_{\mathbf{x}}^j(0) = 0$.
- (b) $\{\mathbf{x} + \alpha_{\mathbf{x}}^j(\zeta) : \zeta \in \overline{\mathbb{D}}\} \subset D$.
- (c) *The function $\overline{\mathbb{D}} \ni \zeta \mapsto \rho\big(\mathbf{x} + \alpha_{\mathbf{x}}^j(\zeta)\big)$ is strongly convex and satisfies*

$$\rho\big(\mathbf{x} + \alpha_{\mathbf{x}}^j(\zeta)\big) \geq \rho(\mathbf{x}) + c|\zeta|^2, \quad \zeta \in \overline{\mathbb{D}}.$$

Proof of the lemma

Assume that $\rho \colon D \to \mathbb{R}$ is a $\mathscr{C}^2$ function on a domain $D \subset \mathbb{R}^n$. We extend it to a function on the tube $\mathcal{T}_D = D \times i\mathbb{R}^n \subset \mathbb{C}^n$ which is independent of the imaginary variable.

Fix $\mathbf{x} \in D$ and $\mathbf{u} \in \mathbb{R}^n$. The Hessian $\operatorname{Hess}_\rho(\mathbf{x})$ has coefficients

$$b_{j,k} := \frac{\partial^2 \rho}{\partial x_j \partial x_k}(\mathbf{x}) = 4 \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(\mathbf{x}) \in \mathbb{R}.$$

Elementary calculation gives for $\mathbf{w} = \mathbf{u} + i\mathbf{v} \in \mathbb{C}^n$:

$$\frac{1}{2} \operatorname{Hess}_\rho(\mathbf{x}; \mathbf{u}) = \frac{1}{4} \Re \left(\sum_{j,k=1}^n b_{j,k} w_j w_k \right) + \mathcal{L}_\rho(\mathbf{x}; \mathbf{w}).$$

Applying this to $-i\mathbf{w} = \mathbf{v} - i\mathbf{u}$ and noting that $\mathcal{L}_\rho(\mathbf{x}; \pm i\mathbf{w}) = \mathcal{L}_\rho(\mathbf{x}; \mathbf{w})$ while the first term on the right hand side changes sign, we obtain

$$\operatorname{Hess}_\rho(\mathbf{x}; \mathbf{u}) + \operatorname{Hess}_\rho(\mathbf{x}; \mathbf{v}) = 4\mathcal{L}_\rho(\mathbf{x}; \mathbf{u} + i\mathbf{v}).$$

Proof of the lemma, 2

Set $a_j = \frac{\partial \rho}{\partial x_j}(\mathbf{x}) \in \mathbb{R}$ for $j = 1, \ldots, n$. For every point $\mathbf{z} = \mathbf{x} + i\mathbf{y} \in \mathcal{T}_D$ and vector $\mathbf{w} = \mathbf{u} + i\mathbf{v} \in \mathbb{C}^n$ near $0 \in \mathbb{C}^n$ we have

$$
\begin{aligned}
\rho(\mathbf{z} + \mathbf{w}) &= \rho(\mathbf{x}) + \sum_{j=1}^{n} a_j u_j + \frac{1}{2} \operatorname{Hess}_\rho(\mathbf{x}; \mathbf{u}) + o(|\mathbf{u}|^2) \\
&= \rho(\mathbf{x}) + \Re\left(\sum_{j=1}^{n} a_j w_j + \frac{1}{4} \sum_{j,k=1}^{n} b_{j,k} w_j w_k\right) + \mathcal{L}_\rho(\mathbf{x}; \mathbf{w}) + o(|\mathbf{w}|^2).
\end{aligned}
$$

Denote by $\Sigma_{\mathbf{x}} \subset \mathbb{C}^n$ the local complex hypersurface near $0 \in \mathbb{C}^n$:

$$
\Sigma_{\mathbf{x}} = \big\{\mathbf{w} : \sum_{j=1}^{n} a_j w_j + \frac{1}{4} \sum_{j,k=1}^{n} b_{j,k} w_j w_k = 0\big\}.
$$

It follows that for every $\mathbf{z} = \mathbf{x} + i\mathbf{y} \in \mathcal{T}_D$ and $\mathbf{w} \in \Sigma_{\mathbf{x}}$ we have

$$
\rho(\mathbf{z} + \mathbf{w}) = \rho(\mathbf{z}) + \mathcal{L}_\rho(\mathbf{x}; \mathbf{w}) + o(|\mathbf{w}|^2).
$$

Proof of the lemma, 3

Assume now that $n = 3$ and $\rho : D \to \mathbb{R}$ is strongly $\mathfrak{M}$-psh. The formula

$$0 < \text{Hess}_\rho(\mathbf{x}; \mathbf{u}) + \text{Hess}_\rho(\mathbf{x}; \mathbf{v}) = 4\mathcal{L}_\rho(\mathbf{x}; \mathbf{u} + i\mathbf{v})$$

shows that $\mathcal{L}_\rho(\mathbf{x}; \mathbf{w}) > 0$ on every null vector $\mathbf{w} \in A_*$.

Indeed, the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal and have the same length when $\mathbf{w} = \mathbf{u} + i\mathbf{v} \in A_*$, so the above expression is proportional to the Laplacian $\triangle_L(\rho|_L)$ of ρ on the affine 2-plane $L = \mathbf{x} + \text{Span}\{\mathbf{u}, \mathbf{v}\}$. Since ρ is strongly $\mathfrak{M}$-psh, this expression is positive.

The hypersurface $\Sigma_{\mathbf{x}}$ contains a pair of small holomorphic null discs $\mathfrak{N}_{\mathbf{x}}^1 \cup \mathfrak{N}_{\mathbf{x}}^2$, obtained by integrating the 1-dimensional distributions obtained by intersecting the tangent space to $\Sigma_{\mathbf{x}}$ with the null quadric.

Their projections on $\mathbb{R}^3$ are conformal minimal discs $\mathfrak{M}_{\mathbf{x}}^1 \cup \mathfrak{M}_{\mathbf{x}}^2$ which can be used in the lemma. We have a consistent choice of the ordering of the two discs on every simply connected set of points $\mathbf{x} \in D$, as well as uniform estimates on any compact set where ρ has no critical points.

Riemann-Hilbert method for null discs in $\mathbb{C}^3$

Lemma

Let $F\colon \overline{\mathbb{D}} \to \mathbb{C}^3$ be a null disc of class $\mathscr{A}^1(\mathbb{D})$. Assume that

- I is a proper closed segment in the circle $\mathbb{T} = b\mathbb{D}$,
- $r\colon \mathbb{T} \to [0,1]$ is a continuous function supported on I, and
- $\sigma\colon I \times \overline{\mathbb{D}} \to \mathbb{C}^3$ is a map of class $\mathscr{C}^1$ such that for every $\zeta \in I$ the map $\overline{\mathbb{D}} \ni z \mapsto \sigma(\zeta, z)$ is an immersed null disc with $\sigma(\zeta, 0) = 0$.

Let $\varkappa\colon \mathbb{T} \times \overline{\mathbb{D}} \to \mathbb{C}^3$ be given by

$$\varkappa(\zeta, z) = F(\zeta) + \sigma\big(\zeta, r(\zeta)z\big).$$

Given numbers $\epsilon > 0$, $0 < \rho_0 < 1$ and an open neighborhood U of I in $\overline{\mathbb{D}}$, there exist a number $\rho' \in [\rho_0, 1)$ and a null holomorphic immersion $G\colon \overline{\mathbb{D}} \to \mathbb{C}^3$ such that $G(0) = F(0)$ and

  i) $\mathrm{dist}(G(\zeta), \varkappa(\zeta, \mathbb{T})) < \epsilon$ for all $\zeta \in \mathbb{T}$,
 ii) $\mathrm{dist}(G(\rho\zeta), \varkappa(\zeta, \overline{\mathbb{D}})) < \epsilon$ for all $\zeta \in \mathbb{T}$ and all $\rho \in [\rho', 1)$, and
iii) G is ϵ-close to F in the $\mathscr{C}^1$ topology on $(\overline{\mathbb{D}} \setminus U) \cup \rho'\overline{\mathbb{D}}$.

Riemann-Hilbert lemma for null discs in $\mathbb{C}^n$, $n > 3$

Let Θ denote the holomorphic bilinear form on $\mathbb{C}^n$ given by

$$\Theta(z, w) = \sum_{j=1}^{n} z_j w_j,$$

so $A^{n-1} = \{z \in \mathbb{C}^n \colon \Theta(z, z) = z_1^2 + \ldots + z_n^2 = 0\}$.

We prove the same lemma also for $n > 3$ provided that the null discs, attached to $F(\zeta)$ at boundary points $\zeta \in \mathbb{T}$, have constant null direction:

$$\varkappa(\zeta, z) = F(\zeta) + \sigma(\zeta, r(\zeta)z) \, \mathbf{u}$$

and the direction null vector $\mathbf{u} \in A_*^{n-1}$ satisfies

$$\Theta(\mathbf{u}, F'(\zeta)) \neq 0 \qquad \forall \zeta \in \overline{\mathbb{D}}.$$

Riemann-Hilbert method for minimal surfaces

Two further generalizations are used in our constructions:

1. The central null disc $\overline{\mathbb{D}} \to \mathbb{C}^n$ can be replaced by any null curve $F : M \to \mathbb{C}^n$, where M is any compact bordered Riemann surface.

   The stated lemmas are used on small discs abutting boundary arcs in bM, and the deformations furnished by these lemmas are glued with the identity map on the rest of M by the gluing-of-sprays technique.

2. By considering real parts of null curves we also get the corresponding Riemann-Hilbert lemma for conformal minimal immersions $u : M \to \mathbb{R}^n$, $n \geq 3$.

These lemmas are used to inductively push the boundary $u(bM) \subset D$ of a conformal minimal immersion $M \to D$ closer to bD.

A global rigidity theorem for minimal surfaces with finite total curvature

Theorem (3)

Let $S \subset \mathbb{R}^3$ be a complete connected properly immersed minimal surface with finite total curvature in $\mathbb{R}^3$, and let $D \subset \mathbb{R}^3$ be a minimally convex domain containing S.
Then either $D = \mathbb{R}^3$ or S is a plane; in the latter case, the connected component of D containing S is either a slab, or a halfspace, or $\mathbb{R}^3$.

The following immediate corollary is a particular case of the Strong Halfspace Theorem for minimal surfaces due to Hoffman and Meeks.

Corollary

Let S_1 and S_2 be complete immersed minimal surfaces with finite total curvature in $\mathbb{R}^3$. Then S_1 and S_2 can not be disjoint unless they are parallel planes.

Sketch of proof for the catenoid, I

Proposition. If D is a minimally convex domain in $\mathbb{R}^3$ which contains the catenoid $S = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = \cosh^2 x\}$, then $D = \mathbb{R}^3$.

Proof: Consider the family of catenoidal curves

$$C_t = \{y = t \cosh(x/t)\} \subset \mathbb{R}^2, \quad t > 0$$

and the corresponding catenoidal (minimal) surfaces of rotation:

$$S_t = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = t^2 \cosh^2(x/t)\} \subset \mathbb{R}^3.$$

Note that $S = S_1$, and S_t is obtained from S by a homothety.

If $t > 1$ then $C_t \cap C_1$ consists of two points $(\pm x_t, y_t)$. Let Σ_t be the part of S_t over $|x| \leq x_t$. Then $\Sigma_1 \subset S$ and $b\Sigma_t \subset S$ for all $t \geq 1$. Since D is minimally convex and contains S, it follows from the Kontinuitätssatz that $\Sigma_t \subset D$ for all $t \geq 1$. These surfaces fill the domain $\overline{\Omega} = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 \geq \cosh^2 x\}$ which therefore belongs to D.

Sketch of proof for the catenoid, II

Pick numbers $x_0 > 0, y_0 > 1$ such that the tangent line to the curve $C_1 = \{y = \cosh x\}$ at (x_0, y_0) passes through $(0,0) \in \mathbb{R}^2$. Fix a number $r \geq y_0$ and let $\tau = \cosh^{-1}(r) > 0$. Consider the compact cylinder

$$T = \{(x, y, z) \in \mathbb{R}^3 : |x| \leq \tau, \ y^2 + z^2 = r^2\}.$$

Note that $T \subset D$ and $bT \subset S$.

For every $0 < t \leq 1$ let Σ_t denote the part of S_t lying over the segment $|x| \leq x_t := t \cosh^{-1}(r/t) \leq \tau$.

Then $b\Sigma_t \subset T$ consists of the two circles in T over the points $\pm x_t$.

Hence

$$\bigcup_{0 < t \leq 1} b\Sigma_t \subset T \Subset D \ \text{ and } \ \Sigma_1 \subset D.$$

The Kontinuitätssatz implies that $\bigcup_{0 < t \leq 1} \Sigma_t \Subset D$. Since Σ_t contains the circle $\{x = 0, \ y^2 + z^2 = t^2\}$ and these circles converge to $0 \in \mathbb{R}^3$ as $t \to 0$, it follows that $0 \in D$. It is now easily seen (by sliding around concentric circles in the (y, z)-plane with boundaries in D and applying the Kontinuitätssatz) that $D = \mathbb{R}^3$.