Minimal surfaces in minimally convex domains

Franc Forstnerič

University of Ljubljana & IMFM

http://www.fmf.uni-lj.si/∼forstneric/

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References

This is joint work with Antonio Alarcón, Barbara Drinovec Drnovšek, and Francisco J. López. It is based on the following preprints:

A. Alarcón; B. Drinovec Drnovšek; F. F.; F.J. López:

- Every bordered Riemann surface is a complete conformal minimal surface bounded by Jordan curves. (2015) arxiv.org/abs/1503.00775
- Minimal surfaces in minimally convex domains. In preparation.

A set of three introductory lectures:

- F. Forstnerič: The Beirut Lectures on Minimal Surfaces. http://www.fmf.uni-lj.si/∼forstneric/

Basic facts on minimal surfaces

Let M be an open Riemann surface and let $u = (u_1, \ldots, u_n) : M \to \mathbb{R}^n$ be a smooth conformal (angle preserving) immersion for some $n \geq 3$. Then the following are equivalent:

- u is a minimal immersion, i.e., its mean curvature vector vanishes identically: $\mathbf{H}(u) = 0$.
- u is harmonic: $\triangle u = 0$.
- $\partial u = (\partial u_1, \ldots, \partial u_n)$ is a nowhere vanishing holomorphic 1-form and
$$(\partial u_1)^2 + (\partial u_2)^2 + \cdots + (\partial u_n)^2 = 0.$$

Hence every conformal minimal immersion $u\colon M \to \mathbb{R}^n$ is of the form

$$u(x) = u(p) + \int_p^x \Re\vartheta \qquad (p,\ x \in M)$$

where $\vartheta = (\vartheta_1, \ldots, \vartheta_n)$ is a $\mathbb{C}^n$-valued holomorphic 1-form on M satisfying $\vartheta_1^2 + \cdots + \vartheta_n^2 = 0$ and $|\vartheta_1|^2 + \cdots + |\vartheta_n|^2 > 0$.

Minimally convex ($\mathfrak{M}$-convex) domains

Definition

Let D be a domain in $\mathbb{R}^n$ for some $n \geq 3$.

- A function $\rho \in \mathscr{C}^2(D)$ is minimal (strongly) plurisubharmonic, denoted $\rho \in \mathfrak{M}\mathrm{Psh}(D)$, if the restriction $u|_{L \cap D}$ is (strongly) subharmonic on any affine 2-plane $L \subset \mathbb{R}^n$.
- D is minimally convex ($\mathfrak{M}$-convex, also called 2-convex by Harvey and Lawson) if it admits a smooth minimal strongly plurisubharmonic exhaustion function $\rho \colon D \to \mathbb{R}$.

Simple observations:

- A function ρ is minimal (strongly) plurisubharmonic iff the sum of the smallest two eigenvalues of its Hessian Hess_ρ is nonnegative (positive) at every point of D.
- The restriction of a minimal plurisubharmonic function on D to a minimal surface $M \subset D$ is a subharmonic function on M.

Smooth minimally convex domains

Let D be a domain in $\mathbb{R}^n$ (not necessarily bounded) with $\mathscr{C}^2$ boundary. Denote by $\kappa_1(x) \le \kappa_2(x) \le \cdots \le \kappa_{n-1}(x)$ the principal curvatures of its boundary bD from the interior side at a point $x \in bD$. Then:

- D is minimally convex iff

$$\kappa_1(x) + \kappa_2(x) \ge 0 \quad \forall x \in bD.$$

- If $n = 3$ then such bD is said to be mean-convex.
- In particular, if $\Sigma \subset \mathbb{R}^3$ is an embedded minimal surface then every connected component of $\mathbb{R}^3 \setminus \Sigma$ is a minimally convex domain.
- D is said to be strongly minimally convex if

$$\kappa_1(x) + \kappa_2(x) > 0 \quad \forall x \in bD.$$

- A domain is strongly minimally convex iff it admits a minimal strongly plurisubharmonic defining function.

F.R. Harvey & H.B. Lawson, Jr.: *p*-convexity, *p*-plurisubharmonicity and the Levi problem. Indiana Univ. Math. J. **62** (2013) 149–169

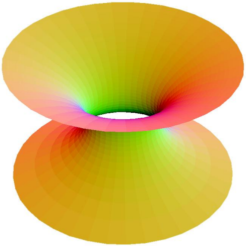
Example: the complement of a catenoid

Example

Let D be the domain

$$D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 > \cosh^2 z\}.$$

Since bD is a minimal surface (the standard catenoid), D is minimally convex. However, D is not convex and its fundamental group equals $\mathbb{Z}$.



A minimal surface which is not a plane has negative Gaussian curvature, hence its complement is never locally convex.

Minimally convex hulls

Definition

Let D be a domain in $\mathbb{R}^n$ and K a compact set in D. The **minimal hull** of K in D is the set

$$\widehat{K}_{\mathfrak{M},D} = \{x \in D \colon \rho(x) \le \sup_K \rho \quad \forall \rho \in \mathfrak{M}\mathrm{Psh}(D)\}.$$

If $D = \mathbb{R}^n$ we write $\widehat{K}_{\mathfrak{M}} = \widehat{K}_{\mathfrak{M},\mathbb{R}^n}$ and call it the minimal hull of K.

- By the maximum principle for subharmonic functions, every compact minimal surface $M \subset D$ with boundary $bM \subset K$ lies in $\widehat{K}_{\mathfrak{M},D}$.
- Clearly $K \subset \widehat{K}_{\mathfrak{M}} \subset \mathrm{Co}(K) =$ the convex hull of K. Both inclusions are proper in general.
- A domain $D \subset \mathbb{R}^n$ is $\mathfrak{M}$-convex iff the minimal hull $\widehat{K}_{\mathfrak{M},D}$ of any compact set $K \subset D$ is compact.

Minimal hulls are described by minimal discs

The following result is analogous to Poletsky's description of the polynomial hull of $K \subset \mathbb{C}^n$ by bounded sequences of holomorphic discs whose boundaries converge to K in the measure theoretic sense.

Theorem

(a) *Let D be a minimally convex domain in $\mathbb{R}^3$ and let K be a compact set in D. A point $p \in D$ belongs to the minimal hull $\widehat{K}_{\mathfrak{M},D}$ iff there exist a relatively compact set $U \Subset D$ containing K and a sequence of conformal minimal discs $f_j \colon \overline{\mathbb{D}} \to U$ such that for all $j = 1, 2, \ldots$ we have $f_j(0) = p$ and*

$$\left|\{t \in [0, 2\pi] : \mathrm{dist}(f_j(e^{\mathrm{i}t}), K) < 1/j\}\right| \geq 2\pi - 1/j.$$

(b) *A similar result holds for compact sets in convex domain $D \subset \mathbb{R}^n$, $n > 3$.*

Complex analysis $\longleftrightarrow$ minimal surfaces

A dictionary between complex analysis and minimal surface theory:

holomorphic curves	$\longleftrightarrow$	minimal surfaces
plurisubharmonic functions	$\longleftrightarrow$	minimal psh functions
pseudoconvex domains	$\longleftrightarrow$	minimally convex domains
polynomial hulls	$\longleftrightarrow$	minimal hulls
Levi-flat boundary	$\longleftrightarrow$	minimal surface boundary
holomorphic functions	$\longleftrightarrow$	?

Complete proper minimal surfaces in $\mathfrak{M}$-convex domains

The following is our first main result.

Theorem (1)

Let D be a minimally convex domain in $\mathbb{R}^3$ and let M be a compact bordered Riemann surface with boundary $bM \neq \emptyset$.

(a) *Every conformal minimal immersion $u\colon M \to D$ of class $\mathscr{C}^1(M)$ can be approximated, uniformly on compacts in $\mathring{M} = M \setminus bM$, by conformal complete proper minimal immersions $\widetilde{u}\colon \mathring{M} \to D$.*

(b) *If D is bounded and strongly minimally convex, then there is a constant $C > 0$ depending only on D such that $\widetilde{u}$ can be chosen continuous on M and satisfying the estimate*

$$\|\widetilde{u} - u\|_{0,M} \leq C \cdot \max_{\zeta \in bM} \sqrt{\mathrm{dist}(u(\zeta), bD)}.$$

Recall that an immersion $u : \mathring{M} \to \mathbb{R}^n$ is said to be complete the pull-back $u^* ds^2$ of the Euclidean metric is a complete metric on $\mathring{M}$.

Optimality of $\mathfrak{M}$-convexity

Minimal convexity is an optimal condition for Theorem 1.

Indeed, if bD fails to be mean-convex at some point $p \in bD$ then

$$\kappa_1(p) + \kappa_2(p) < 0$$

and hence the complement of D is locally strongly $\mathfrak{M}$-convex at p.

Thus there is a strongly 2-convex function $\phi : U \to \mathbb{R}$ on a neighborhood $U \subset \mathbb{R}^3$ of p such that

$$\phi > 0 \text{ on } D \cap U, \quad \phi = 0 \text{ on } bD \cap U, \quad \phi < 0 \text{ on } U \setminus \overline{D}.$$

The maximum principle applied to $\phi \circ u$ shows that there does not exist any proper conformal minimal immersion $u : \mathring{M} \to D \cap U$ with boundary values in $bD \cap U$. Thus Theorem 1-(b) fails.

Proper conformal minimal surfaces in convex domains

We expect that the analogous result holds for $(n-1)$-convex domains $D \subset \mathbb{R}^n$, $n > 3$. For smoothly bounded domains this means that the sum of their principal curvatures at each boundary point is ≥ 0.

However, due to technical difficulties we can only prove it for minimal surfaces in convex domains when $n > 3$.

Theorem (2)

Let D be a convex domain in $\mathbb{R}^n$ for some $n > 3$, and let M be a compact bordered Riemann surface. Then the following hold.

(a) *Every conformal minimal immersion $u \colon M \to D$ of class $\mathscr{C}^1(M)$ can be approximated, uniformly on compacts in $\mathring{M}$, by conformal complete proper minimal immersions $\widetilde{u} \colon \mathring{M} \to D$.*

(b) *If $n \geq 5$ then $\widetilde{u}$ can be chosen an embedding.*

(c) *If D has $\mathscr{C}^2$ strongly convex boundary then $\widetilde{u}$ can be chosen continuous on M and $\|\widetilde{u} - u\|_{0,M} \leq C \cdot \max_{\zeta \in bM} \sqrt{\mathrm{dist}(u(\zeta), bD)}$.*

Holomorphic null curves in $\mathbb{C}^n$

The proofs of these results rely on the connection between conformal minimal surfaces in $\mathbb{R}^n$ and holomorphic null curves in $\mathbb{C}^n$.

Definition

Let M be an open or a bordered Riemann surface. A holomorphic immersion

$$F = (F_1, F_2, \ldots, F_n) \colon M \to \mathbb{C}^n, \quad n \geq 3$$

is a **null curve** if the derivative $F' = (F_1', F_2', \ldots, F_n')$ with respect to any local holomorphic coordinate $\zeta = x + \imath y$ on M satisfies

$$(F_1')^2 + (F_2')^2 + \ldots + (F_n')^2 = 0.$$

The nullity condition is equivalent to $F'(\zeta) \in A_* = A \setminus \{0\}$ where

$$A = \Big\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon \sum_{j=1}^{n} z_j^2 = 0 \Big\} \ldots$$ **the null quadric**.

A_* is **elliptic** in the sense of Gromov, and hence an **Oka manifold**.

Connection between null curves and minimal surfaces

- If $F = f + \imath g \colon M \to \mathbb{C}^n$ is a holomorphic null curve, then

$$f = \Re F \colon M \to \mathbb{R}^n, \quad g = \Im F \colon M \to \mathbb{R}^n$$

are conformal harmonic (hence minimal) immersions into $\mathbb{R}^n$.

- Conversely, a conformal minimal immersion (CMI) $f : \mathbb{D} \to \mathbb{R}^n$ of the disc $\mathbb{D} = \{|\zeta| < 1\}$ is the real part of a null disc $F \colon \mathbb{D} \to \mathbb{C}^n$.

- Hence every conformal minimal immersion $M \to \mathbb{R}^n$ is of the form

$$f(x) = f(p) + \Re \int_p^x \phi \qquad (p,\ x \in M)$$

where $\phi = (\phi_1, \ldots, \phi_n)$ is a $\mathbb{C}^n$-valued holomorphic 1-form on M without zeros, satisfying the nullity condition

$$\phi_1^2 + \phi_2^2 + \cdots + \phi_n^2 = 0$$

and such that $\Re\phi = (\Re\phi_1, \ldots, \Re\phi_n)$ has vanishing periods over all closed curves in M.

Local geometry of a strongly $\mathfrak{M}$-psh function

The following lemma gives small conformal minimal discs on which a strongly $\mathfrak{M}$-psh function increases quadratically.

Lemma (minimal discs 'going up')

Let D be a domain in $\mathbb{R}^3$ and let $\rho\colon D \to \mathbb{R}$ be a $\mathscr{C}^2$ minimal strongly plurisubharmonic function with the critical locus P.

For every compact set $L \subset D \setminus P$ there exist a constant $c = c_L > 0$ and families of embedded null holomorphic discs $\sigma^j_{\mathbf{x}} = \alpha^j_{\mathbf{x}} + \mathrm{i}\beta^j_{\mathbf{x}} \colon \overline{\mathbb{D}} \to \mathbb{C}^3$ $(\mathbf{x} \in L,\ j = 1, 2)$ depending locally smoothly on the point $\mathbf{x} \in L$ and satisfying the following conditions:

(a) $\sigma^j_{\mathbf{x}}(0) = 0$.

(b) $\{\mathbf{x} + \alpha^j_{\mathbf{x}}(\zeta) : \zeta \in \overline{\mathbb{D}}\} \subset D$.

(c) *The function $\overline{\mathbb{D}} \ni \zeta \mapsto \rho\big(\mathbf{x} + \alpha^j_{\mathbf{x}}(\zeta)\big)$ is strongly convex and satisfies*

$$\rho\big(\mathbf{x} + \alpha^j_{\mathbf{x}}(\zeta)\big) \geq \rho(\mathbf{x}) + c|\zeta|^2, \quad \zeta \in \overline{\mathbb{D}}.$$

Proof of the lemma

Assume that $\rho\colon D \to \mathbb{R}$ is a $\mathscr{C}^2$ function on a domain $D \subset \mathbb{R}^n$. We extend it to a function on the tube $\mathcal{T}_D = D \times \mathrm{i}\mathbb{R}^n \subset \mathbb{C}^n$ which is independent of the imaginary variable.

Fix $\mathbf{x} \in D$ and $\mathbf{u} \in \mathbb{R}^n$. The Hessian $\mathrm{Hess}_\rho(\mathbf{x})$ has coefficients

$$b_{j,k} := \frac{\partial^2 \rho}{\partial x_j \partial x_k}(\mathbf{x}) = 4\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(\mathbf{x}) \in \mathbb{R}.$$

Elementary calculation gives for $\mathbf{w} = \mathbf{u} + \mathrm{i}\mathbf{v} \in \mathbb{C}^n$:

$$\mathrm{Hess}_\rho(\mathbf{x};\mathbf{u}) = \frac{1}{2}\,\Re\left(\sum_{j,k=1}^{n} b_{j,k} w_j w_k\right) + 2\mathcal{L}_\rho(\mathbf{x};\mathbf{w})$$

where $\mathcal{L}_\rho(\mathbf{x};\mathbf{w}) = 1/4\sum_{j,k=1}^n b_{j,k} w_j \overline{w}_k$ is the Levi form.
Applying this to $-\mathrm{i}\mathbf{w} = \mathbf{v} - \mathrm{i}\mathbf{u}$ and noting that $\mathcal{L}_\rho(\mathbf{x};\pm\mathrm{i}\mathbf{w}) = \mathcal{L}_\rho(\mathbf{x};\mathbf{w})$ while the first term on the right hand side changes sign, we obtain

$$\mathrm{Hess}_\rho(\mathbf{x};\mathbf{u}) + \mathrm{Hess}_\rho(\mathbf{x};\mathbf{v}) = 4\mathcal{L}_\rho(\mathbf{x};\mathbf{u}+\mathrm{i}\mathbf{v}).$$

Proof of the lemma, 2

Set $a_j = \frac{\partial \rho}{\partial x_j}(\mathbf{x}) \in \mathbb{R}$ for $j = 1, \ldots, n$. For every point $\mathbf{z} = \mathbf{x} + \imath \mathbf{y} \in \mathcal{T}_D$ and vector $\mathbf{w} = \mathbf{u} + \imath \mathbf{v} \in \mathbb{C}^n$ near $0 \in \mathbb{C}^n$ we have

$$\begin{aligned} \rho(\mathbf{z} + \mathbf{w}) &= \rho(\mathbf{x}) + \sum_{j=1}^{n} a_j u_j + \frac{1}{2} \mathrm{Hess}_{\rho}(\mathbf{x}; \mathbf{u}) + o(|\mathbf{u}|^2) \\ &= \rho(\mathbf{x}) + \Re \left(\sum_{j=1}^{n} a_j w_j + \frac{1}{4} \sum_{j,k=1}^{n} b_{j,k} w_j w_k \right) + \mathcal{L}_{\rho}(\mathbf{x}; \mathbf{w}) + o(|\mathbf{w}|^2). \end{aligned}$$

Denote by $\Sigma_{\mathbf{x}} \subset \mathbb{C}^n$ the local complex hypersurface near $0 \in \mathbb{C}^n$ given by

$$\Sigma_{\mathbf{x}} = \left\{ \mathbf{w} : \sum_{j=1}^{n} a_j w_j + \frac{1}{4} \sum_{j,k=1}^{n} b_{j,k} w_j w_k = 0 \right\}.$$

It follows that for every $\mathbf{z} = \mathbf{x} + \imath \mathbf{y} \in \mathcal{T}_D$ and $\mathbf{w} \in \Sigma_{\mathbf{x}}$ we have

$$\rho(\mathbf{z} + \mathbf{w}) = \rho(\mathbf{x}) + \mathcal{L}_{\rho}(\mathbf{x}; \mathbf{w}) + o(|\mathbf{w}|^2).$$

Proof of the lemma, 3

Assume now that $n = 3$ and $\rho : D \to \mathbb{R}$ is strongly $\mathfrak{M}$-psh. The formula

$$\mathrm{Hess}_\rho(\mathbf{x};\mathbf{u}) + \mathrm{Hess}_\rho(\mathbf{x};\mathbf{v}) = 4\mathcal{L}_\rho(\mathbf{x};\mathbf{u}+\mathrm{i}\mathbf{v})$$

shows that $\mathcal{L}_\rho(\mathbf{x};\mathbf{w}) > 0$ on every null vector $\mathbf{w} \in A_*$.

Indeed, if $\mathbf{w} = \mathbf{u} + \mathrm{i}\mathbf{v} \in A_*$ then the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are orthogonal and have the same length, so the left hand side is proportional to the Laplacian $\triangle_L(\rho|_L)$ on the affine 2-plane $L = \mathbf{x} + \mathrm{Span}\{\mathbf{u},\mathbf{v}\}$. Since ρ is strongly $\mathfrak{M}$-psh, $\triangle_L(\rho|_L) > 0$ whence $\mathcal{L}_\rho(\mathbf{x};\mathbf{w}) > 0$.

The complex hypersurface $\Sigma_\mathbf{x} \subset \mathbb{C}^3$ contains a pair of small holomorphic null discs $\mathfrak{N}^1_\mathbf{x}, \mathfrak{N}^2_\mathbf{x}$ obtained by integrating the pair of holomorphic 1-dimensional distributions $T_\mathbf{z}\Sigma_\mathbf{x} \cap (\mathbf{z} + A)$, $\mathbf{z} \in \Sigma_\mathbf{x}$.

The projections of $\mathfrak{N}^1_\mathbf{x}, \mathfrak{N}^2_\mathbf{x}$ to $\mathbb{R}^3$ are conformal minimal discs $\mathfrak{M}^1_\mathbf{x}, \mathfrak{M}^2_\mathbf{x}$ as in the lemma. We have a consistent choice of ordering of the two discs on every simply connected set of points $\mathbf{x} \in D$, as well as uniform estimates on any compact set where ρ has no critical points.

Riemann-Hilbert method for null discs in $\mathbb{C}^3$

Lemma

Let $F\colon \overline{\mathbb{D}} \to \mathbb{C}^3$ be a null disc of class $\mathscr{A}^1(\mathbb{D})$. Assume that

- *I is a proper closed segment in the circle $\mathbb{T} = b\mathbb{D}$,*
- *$r\colon \mathbb{T} \to [0,1]$ is a continuous function supported on I, and*
- *$\sigma\colon I \times \overline{\mathbb{D}} \to \mathbb{C}^3$ is a map of class $\mathscr{C}^1$ such that for every $\zeta \in I$ the map $\overline{\mathbb{D}} \ni z \mapsto \sigma(\zeta, z)$ is an immersed null disc with $\sigma(\zeta, 0) = 0$.*

Let $\varkappa\colon \mathbb{T} \times \overline{\mathbb{D}} \to \mathbb{C}^3$ be given by

$$\varkappa(\zeta, z) = F(\zeta) + \sigma\big(\zeta, r(\zeta)z\big).$$

Given numbers $\epsilon > 0$, $0 < \rho_0 < 1$ and an open neighborhood U of I in $\overline{\mathbb{D}}$, there exist a number $\rho' \in [\rho_0, 1)$ and a null holomorphic immersion $G\colon \overline{\mathbb{D}} \to \mathbb{C}^3$ such that $G(0) = F(0)$ and

i) $\mathrm{dist}(G(\zeta), \varkappa(\zeta, \mathbb{T})) < \epsilon$ *for all $\zeta \in \mathbb{T}$,*

ii) $\mathrm{dist}(G(\rho\zeta), \varkappa(\zeta, \overline{\mathbb{D}})) < \epsilon$ *for all $\zeta \in \mathbb{T}$ and all $\rho \in [\rho', 1)$, and*

iii) G is ϵ-close to F in the $\mathscr{C}^1$ topology on $(\overline{\mathbb{D}} \setminus U) \cup \rho'\overline{\mathbb{D}}$.

Riemann-Hilbert lemma for null discs in $\mathbb{C}^n$, $n > 3$

Let Θ denote the holomorphic bilinear form on $\mathbb{C}^n$ given by

$$\Theta(z, w) = \sum_{j=1}^{n} z_j w_j.$$

Note that $A^{n-1} = \{z \in \mathbb{C}^n \colon \Theta(z, z) = z_1^2 + \ldots + z_n^2 = 0\}$.

We prove the same lemma also for $n > 3$ provided that the null discs, attached to $F(\zeta)$ at boundary points $\zeta \in \mathbb{T}$, have constant null direction:

$$\varkappa(\zeta, z) = F(\zeta) + \sigma\big(\zeta, r(\zeta)z\big)\, \mathbf{u}$$

and the direction null vector $\mathbf{u} \in A_*^{n-1}$ satisfies

$$\Theta(\mathbf{u}, F'(\zeta)) \neq 0 \qquad \forall \zeta \in \overline{\mathbb{D}}.$$

Riemann-Hilbert method for minimal surfaces

Two further generalizations are used in our constructions:

1. The central null disc $\overline{\mathbb{D}} \to \mathbb{C}^n$ can be replaced by any null curve $F : M \to \mathbb{C}^n$, where M is any compact bordered Riemann surface.

   The stated lemmas are used on small discs abutting boundary arcs in bM, and the deformations furnished by these lemmas are glued with the identity map on the rest of M by the gluing-of-sprays technique.

2. By considering real parts of null curves we also get the corresponding Riemann-Hilbert lemma for conformal minimal immersions $u : M \to \mathbb{R}^n$, $n \geq 3$.

These lemmas are used to inductively push the boundary $u(bM) \subset D$ of a conformal minimal immersion $u \colon M \to D$ closer to bD.

A rigidity theorem for minimal surfaces with FTC

Theorem (3)

Let $S \subset \mathbb{R}^3$ be a connected properly immersed minimal surface with finite total curvature in $\mathbb{R}^3$. If $D \subset \mathbb{R}^3$ is a minimally convex domain containing S then $D = \mathbb{R}^3$ or S is a plane; in the latter case, the connected component of D containing S is a slab, a halfspace or $\mathbb{R}^3$.

The following immediate corollary is a particular case of the Strong Halfspace Theorem for minimal surfaces due to Hoffman and Meeks.

Corollary

Let S_1 and S_2 be properly immersed minimal surfaces with finite total curvature in $\mathbb{R}^3$. Then S_1 and S_2 can not be disjoint unless they are parallel planes.

A few analogous problems for Levi-flat hypersurfaces

Problem 1: Assume that S is a properly embedded Levi-flat hypersurface in $\mathbb{C}^2$ and $D \subset \mathbb{C}^2$ is a pseudo convex domain containing S. What can be said about the pair (S, D)?

In particular, are there conditions on S which force $D = \mathbb{C}^2$?

Problem 2: Which Levi-flat S are contained in a strongly pseudoconvex domain? (For an example see Hartz, Shcherbina, Tomassini, Problem 2, p. 25 in http://arxiv.org/pdf/1006.1059.pdf)

Problem 3: Assume that $S_1, S_2 \subset \mathbb{C}^2$ are disjoint properly embedded Levi-flat surfaces. What can be said about them?

G. Della Sala, N. Shcherbina: If S_1 is a hyperplane then S_2 is foliated by complex lines parallel to those in S_1.

Sketch of proof of Theorem 3 for the catenoid, I

Proposition. If D is a minimally convex domain in $\mathbb{R}^3$ which contains the catenoid $S = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = \cosh^2 x\}$, then $D = \mathbb{R}^3$.

Proof: Consider the family of catenoidal curves

$$C_t = \{y = t\cosh(x/t)\} \subset \mathbb{R}^2, \quad t > 0$$

and the corresponding catenoidal (minimal) surfaces of rotation:

$$S_t = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = t^2\cosh^2(x/t)\} \subset \mathbb{R}^3.$$

Note that S_t is obtained from $S = S_1$ by a homotety.

If $t > 1$ then $C_t \cap C_1$ consists of two points $(\pm x_t, y_t)$.

Let Σ_t be the part of S_t over $|x| \leq x_t$. Then $\Sigma_1 \subset S$ and $b\Sigma_t \subset S$ for all $t \geq 1$.

Since D is minimally convex and contains S, Kontinuitätssatz gives $\Sigma_t \subset D$ for all $t \geq 1$. These surfaces fill the domain $\overline{\Omega} = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 \geq \cosh^2 x\}$ which therefore belongs to D.

Sketch of proof of Theorem 3 for the catenoid, II

Pick numbers $x_0 > 0, y_0 > 1$ such that the tangent line to the curve $C_1 = \{y = \cosh x\}$ at (x_0, y_0) passes through $(0, 0) \in \mathbb{R}^2$.

Fix a number $r \geq y_0$ and let $\tau = \cosh^{-1}(r) > 0$. Consider the cylinder

$$T = \{(x, y, z) \in \mathbb{R}^3 : |x| \leq \tau, \ y^2 + z^2 = r^2\}.$$

Note that $T \subset D$ and $bT \subset S$. For every $0 < t \leq 1$ let Σ_t denote the part of S_t lying over the segment $|x| \leq x_t := t \cosh^{-1}(r/t) \leq \tau$. Then $b\Sigma_t \subset T$ consists of the two circles in T over the points $\pm x_t$. Hence

$$\bigcup_{0<t\leq 1} b\Sigma_t \subset T \Subset D \quad \text{and} \quad \Sigma_1 \subset D.$$

The Kontinuitätssatz implies that $\bigcup_{0<t\leq 1} \Sigma_t \Subset D$. Since Σ_t contains the circle $\{x = 0, \ y^2 + z^2 = t^2\}$ and these circles converge to $0 \in \mathbb{R}^3$ as $t \to 0$, it follows that $0 \in D$. It is now easily seen (by sliding around concentric circles in the (y, z)-plane with boundaries in D and applying again the Kontinuitätssatz) that $D = \mathbb{R}^3$.

THANK YOU

FOR

YOUR ATTENTION