Complete conformal minimal surfaces with Jordan boundaries in $\mathbb{R}^n$

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References

This is joint work with Antonio Alarcón, Barbara Drinovec Drnovšek, and Francisco J. López.

It is based on the following preprints:

A. Alarcón; B. Drinovec Drnovšek; F. F.; F.J. López:

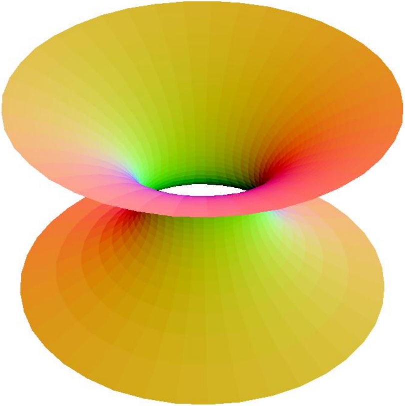
- Every bordered Riemann surface is a complete conformal minimal surface bounded by Jordan curves. (2015) arxiv.org/abs/1503.00775
- Minimal surfaces in minimally convex domains. In preparation.

A more complete set of three introductory lectures:

- F. Forstnerič: The Beirut Lectures on Minimal Surfaces. http://www.fmf.uni-lj.si/∼forstneric/

From Euler's area minimizing surfaces of rotation...

1744 **Euler** The only area minimizing surfaces of rotation in $\mathbb{R}^3$ are planes and catenoids.



. . . to the concept of a minimal surface

Meusnier 1776 A smooth surface $M \subset \mathbb{R}^3$ is *locally area minimizing* (among the surfaces with the same boundary) if and only if its *mean curvature* function H vanishes everywhere.

Definition

A smoothly immersed surface $M \to \mathbb{R}^n$ is said to be a **minimal surface** if its **mean curvature vector** $\mathbf{H} : M \to \mathbb{R}^n$ is identically zero: $\mathbf{H} = 0$.

For $n = 3$ and working in isothermal coordinates we have $\mathbf{H} = H \cdot \mathbf{N}$, where $\mathbf{N}$ is the unit normal vector field to M and

$$\mathrm{H} = \frac{\kappa_1 + \kappa_2}{2}$$

is the **mean curvature** function. Here, κ_1 and κ_2 are the **principal curvatures** of the surface. Their product $\mathrm{K} = \kappa_1 \kappa_2 : M \to \mathbb{R}$ is the **Gauss curvature** function of M. Note that $\mathrm{H} = 0 \Rightarrow \mathrm{K} \leq 0$.

The helicoid (Archimedes' screw)

1776 **Meusnier** The **helicoid** is a minimal surface.

$$x = \rho \cos(\alpha\theta), \quad y = \rho \sin(\alpha\theta), \quad z = \theta$$

1842 **Catalan** The helicoid and the plane are the only **ruled** minimal surfaces in $\mathbb{R}^3$ (unions of straight lines).

Conformal minimal = conformal harmonic

Let M be an open Riemann surface and let $u = (u_1, \ldots, u_n) : M \to \mathbb{R}^n$ be a smooth immersion for some $n \geq 3$. The following are equivalent:

- u is conformal minimal: $\mathbf{H}(u) = 0$.
- u is conformal harmonic: $\triangle u = 0$.
- $\partial u = (\partial u_1, \ldots, \partial u_n)$ is a holomorphic 1-form on M satisfying

$$(\partial u_1)^2 + (\partial u_2)^2 + \cdots + (\partial u_n)^2 = 0.$$

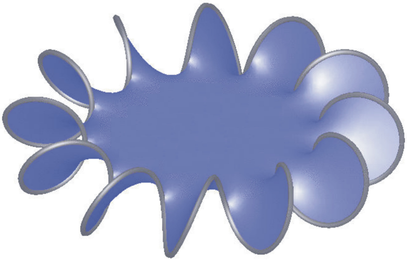
Hence every conformal minimal immersion $u \colon M \to \mathbb{R}^n$ is of the form

$$u(x) = u(p) + \int_p^x \Re\vartheta \qquad (p,\ x \in M)$$

where $\vartheta = (\vartheta_1, \ldots, \vartheta_n)$ is a $\mathbb{C}^n$-valued holomorphic 1-form on M satisfying $\vartheta_1^2 + \cdots + \vartheta_n^2 = 0$ and $|\vartheta_1|^2 + \cdots + |\vartheta_n|^2 > 0$.

The Plateau problem

1873 **Plateau** Minimal surfaces can be obtained as soap films.



1932 **Douglas, Radó** Every continuous injective closed curve in $\mathbb{R}^n$ for $n \geq 3$ spans a minimal surface.

1965 **Calabi's Conjecture:** Bounded minimal surfaces in $\mathbb{R}^3$ can not be complete.

First main theorem

An immersion $u : M \to \mathbb{R}^n$ is said to be **complete** if the pullback $u^* ds^2$ of the Euclidean metric on $\mathbb{R}^n$ is a complete metric on M. Equivalently, the image in $\mathbb{R}^n$ of any divergent curve in M has infinite length.

Theorem (1)

Let M be a compact bordered Riemann surface. Every conformal minimal immersion $u_0 : M \to \mathbb{R}^n$ $(n \geq 3)$ of class $\mathscr{C}^1(M)$ can be approximated arbitrarily closely in the $\mathscr{C}^0(M)$ topology by a continuous map $u : M \to \mathbb{R}^n$ such that

- $u|_{\mathring{M}} : \mathring{M} = M \setminus bM \to \mathbb{R}^n$ *is a conformal complete minimal immersion, and*
- $u|_{bM} : bM \to \mathbb{R}^n$ *is a topological embedding.*

In particular, $u(bM) \subset \mathbb{R}^n$ consists of finitely many (necessarily non rectifiable) Jordan curves.

If $n \geq 5$ there exist embeddings $u : M \hookrightarrow \mathbb{R}^n$ with these properties.

Comments and history

Theorem 1 shows that every finite collection of smooth Jordan curves in $\mathbb{R}^n$ spanning a connected minimal surface can be uniformly approximated by families of Jordan curves spanning complete connected minimal surfaces. Hence it can be viewed as an **approximate solution of the Plateau problem by complete minimal surfaces.** It gives the most decisive known **counterexample to the Calabi Conjecture**.

This is the first result of its kind in the literature which provides the control of both

(a) the conformal type of M (any bordered Riemann surface), and

(b) the topology of the boundary $u(bM) \subset \mathbb{R}^n$ (Jordan curves).

There exist several previous results concerning part (a), but without the control of the conformal structure (Nadirashvili 1996, Martín and Nadirashvili 2007, Alarcón 2010, Alarcón and López 2014).

Earlier attempts to control the topology of the boundary (part (b)) were inconclusive (incomplete proofs).

Holomorphic null curves in $\mathbb{C}^n$

We shall now explain the tools used in the proof.
Let M be an open or a compact bordered Riemann surface.

Definition

A holomorphic immersion

$$F = (F_1, F_2, \ldots, F_n) \colon M \to \mathbb{C}^n$$

is a **null curve** if the derivative $F' = (F'_1, F'_2, \ldots, F'_n)$ with respect to any local holomorphic coordinate $\zeta = x + \imath y$ on M satisfies

$$(F'_1)^2 + (F'_2)^2 + \ldots + (F'_n)^2 = 0.$$

We denote by $\mathfrak{A} \subset \mathbb{C}^n$ the **null quadric**

$$\mathfrak{A} = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon \sum_{j=1}^{n} z_j^2 = 0\}.$$

The nullity condition is equivalent to $F'(\zeta) \in \mathfrak{A}_* = \mathfrak{A} \setminus \{0\}$.

Connection between null curves and minimal surfaces

- If $F = u + \imath v : M \to \mathbb{C}^n$ is a holomorphic null curve, then

$$u = \Re F : M \to \mathbb{R}^n, \quad v = \Im F : M \to \mathbb{R}^n$$

  are conformal harmonic (hence minimal) immersions into $\mathbb{R}^n$.

- Conversely, a conformal minimal immersion $u : \mathbb{D} \to \mathbb{R}^n$ of the disc $\mathbb{D} = \{\zeta \in \mathbb{C} \colon |\zeta| < 1\}$ is the real part of a holomorphic null curve $F : \mathbb{D} \to \mathbb{C}^n$. (This fails on multiply connected Riemann surfaces.)

- If $F = u + \imath v \colon M \to \mathbb{C}^n$ is a null curve then

$$F^* ds^2_{\mathbb{C}^n} = 2u^* ds^2_{\mathbb{R}^n} = 2v^* ds^2_{\mathbb{R}^n}.$$

- **Hence the real and the imaginary part of a complete null curve in $\mathbb{C}^n$ are complete conformal minimal surfaces in $\mathbb{R}^n$.**

Example: catenoid and helicoid

Example: The **catenoid** and the **helicoid** are **conjugate minimal surfaces** – the real and the imaginary part of the same null curve

$$F(\zeta) = (\cos\zeta, \sin\zeta, -i\zeta) \in \mathbb{C}^3, \qquad \zeta = x + iy \in \mathbb{C}.$$

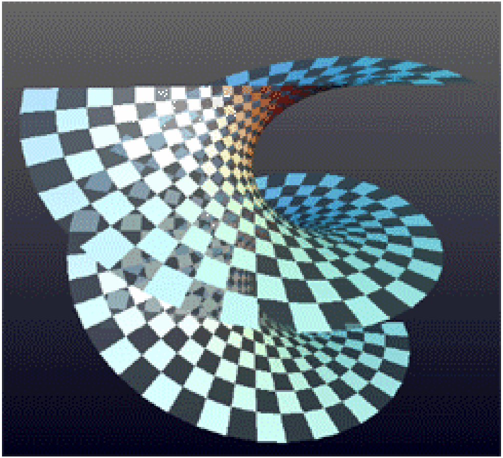
Consider the following family of minimal surfaces in $\mathbb{R}^3$ for $t \in \mathbb{R}$:

$$\begin{aligned} u_t(\zeta) &= \Re\left(e^{it} F(\zeta)\right) \\ &= \cos t \begin{pmatrix} \cos x \cdot \cosh y \\ \sin x \cdot \cosh y \\ y \end{pmatrix} + \sin t \begin{pmatrix} \sin x \cdot \sinh y \\ -\cos x \cdot \sinh y \\ x \end{pmatrix} \end{aligned}$$

At $t = 0$ we have a catenoid and at $t = \pm\pi/2$ a helicoid.

Helicatenoid (Source: Wikipedia)

The family of minimal surfaces $u_t(\zeta) = \Re\left(e^{it}F(\zeta)\right)$, $t \in \mathbb{R}$:



Connection with Oka theory

It is not a big exaggeration to say that the theory of minimal surfaces in $\mathbb{R}^n$, and the companion theory of holomorphic null curves in $\mathbb{C}^n$, is an application of the fact that the punctured null quadric

$$\mathfrak{A}_* = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n_* \colon \sum_{j=1}^{n} z_j^2 = 0\}$$

is an elliptic manifold, and hence an Oka manifold.

In fact, the linear (hence complete) holomorphic vector fields

$$V_{i,j} = z_i \frac{\partial}{\partial z_j} - z_j \frac{\partial}{\partial z_i}, \quad 1 \leq i < j \leq n$$

span the tangent space to $\mathfrak{A}_*$ at every point.

Riemann-Hilbert method for null discs in $\mathbb{C}^3$

Lemma

Let $F\colon \overline{\mathbb{D}} \to \mathbb{C}^3$ *be a null disc of class* $\mathscr{A}^1(\mathbb{D})$*. Assume that*

- I *is a proper closed segment in the circle* $\mathbb{T} = b\mathbb{D}$*,*
- $r\colon \mathbb{T} \to [0,1]$ *is a continuous function supported on* I*, and*
- $\sigma\colon I \times \overline{\mathbb{D}} \to \mathbb{C}^3$ *is a map of class* $\mathscr{C}^1$ *such that for every* $\zeta \in I$ *the map* $\overline{\mathbb{D}} \ni z \mapsto \sigma(\zeta, z)$ *is an immersed null disc with* $\sigma(\zeta, 0) = 0$*.*

Let $\varkappa\colon \mathbb{T} \times \overline{\mathbb{D}} \to \mathbb{C}^3$ *be given by*

$$\varkappa(\zeta, z) = F(\zeta) + \sigma\big(\zeta, r(\zeta)z\big).$$

Given numbers $\epsilon > 0$, $0 < \rho_0 < 1$ *and an open neighborhood* U *of* I *in* $\overline{\mathbb{D}}$*, there exist a number* $\rho' \in [\rho_0, 1)$ *and a null holomorphic immersion* $G\colon \overline{\mathbb{D}} \to \mathbb{C}^3$ *such that* $G(0) = F(0)$ *and*

i) $\operatorname{dist}(G(\zeta), \varkappa(\zeta, \mathbb{T})) < \epsilon$ *for all* $\zeta \in \mathbb{T}$*,*

ii) $\operatorname{dist}(G(\rho\zeta), \varkappa(\zeta, \overline{\mathbb{D}})) < \epsilon$ *for all* $\zeta \in \mathbb{T}$ *and all* $\rho \in [\rho', 1)$*, and*

iii) G *is* ϵ*-close to* F *in the* $\mathscr{C}^1$ *topology on* $(\overline{\mathbb{D}} \setminus U) \cup \rho'\overline{\mathbb{D}}$*.*

Riemann-Hilbert lemma for null discs in $\mathbb{C}^n$, $n > 3$

Let Θ denote the holomorphic bilinear form on $\mathbb{C}^n$ given by

$$\Theta(z,w) = \sum_{j=1}^{n} z_j w_j,$$

so $\mathfrak{A}^{n-1} = \{z \in \mathbb{C}^n \colon \Theta(z,z) = z_1^2 + \ldots + z_n^2 = 0\}$.

We prove the same lemma also for $n > 3$ provided that the null discs, attached to $F(\zeta)$ at boundary points $\zeta \in \mathbb{T}$, have constant null direction:

$$\varkappa(\zeta, z) = F(\zeta) + \sigma\big(\zeta, r(\zeta) z\big)\, \mathbf{u}$$

and the direction null vector $\mathbf{u} \in \mathfrak{A}_*^{n-1}$ satisfies

$$\Theta(\mathbf{u}, F'(\zeta)) \neq 0 \qquad \forall \zeta \in \overline{\mathbb{D}}.$$

Riemann-Hilbert method for minimal surfaces

Two further generalizations are used in our constructions:

1. The central null disc $\overline{\mathbb{D}} \to \mathbb{C}^n$ can be replaced by a null curve $F : M \to \mathbb{C}^n$, where M is any compact bordered Riemann surface.

   The stated lemmas are used on small discs abutting boundary arcs in bM, and the deformations furnished by these lemmas are glued with the identity map on the rest of M by the gluing-of-sprays technique.

2. By considering real parts of null curves we also get the corresponding Riemann-Hilbert lemma for conformal minimal immersions $u : M \to \mathbb{R}^n$, $n \geq 3$.

Outline of proof of Theorem 1

Theorem 1 is obtained by a recursive application of the following lemma.

Lemma (1)

Let M be a compact bordered Riemann surface, let $n \geq 3$ be a natural number, and let $u : M \to \mathbb{R}^n$ be a conformal minimal immersion.

Given a point $p_0 \in \mathring{M}$ and numbers $\epsilon > 0$ (small), $\lambda > 0$ (big), there is a conformal minimal immersion $\widehat{u} : M \to \mathbb{R}^n$ such that

$$||\widehat{u} - u||_{0,M} < \epsilon, \quad \mathrm{dist}_{\widehat{u}}(p_0, bM) > \lambda.$$

Furthermore, we use that for a generic conformal minimal immersion $u : M \to \mathbb{R}^n$ $(n \geq 3)$ the boundary $u : bM \to \mathbb{R}^n$ is embedded; if $n \geq 5$ then a generic such u is an embedding $M \hookrightarrow \mathbb{R}^n$.

A. Alarcón, F. F., F.J. López, arxiv.org/abs/1409.6901.

The main lemma

Lemma 1 follows by a recursive application of the following result.

Lemma (2)

Consider a conformal minimal immersion $u : M \to \mathbb{R}^n$ $(n \geq 3)$, *a smooth map* $g \colon bM \to \mathbb{R}^n$, *and a number* $\delta > 0$ *such that*

$$\|u - g\|_{0,bM} < \delta.$$

Fix a point $p_0 \in \mathring{M}$. *For each* $d > 0$ *there is a conformal minimal immersion* $\widehat{u} : M \to \mathbb{R}^n$ *satisfying the following properties:*

(a) $\|\widehat{u} - g\|_{0,bM} < \sqrt{\delta^2 + d^2}$.

(b) $\mathrm{dist}_{\widehat{u}}(p_0, bM) > \mathrm{dist}_u(p_0, bM) + d$.

(c) $\widehat{u}$ *is arbitrarilly close to* u *on a given compact* $K \subset \mathring{M}$.

Lemma 2 $\Longrightarrow$ Lemma 1

Let $\epsilon > 0$ be as in Lemma 1. Pick δ_0 with $0 < \delta_0 < \epsilon$ and set

$$d_0 = \mathrm{dist}_u(p_0, bM), \quad c = \frac{\sqrt{6(\epsilon^2 - \delta_0^2)}}{\pi} > 0.$$

Consider the following sequences defined recursively:

$$d_j := d_{j-1} + \frac{c}{j} > 0, \qquad \delta_j := \sqrt{\delta_{j-1}^2 + \frac{c^2}{j^2}} > 0, \quad j \in \mathbb{N}.$$

Then $\lim_{j\to\infty} d_j = +\infty$ and $\lim_{j\to\infty} \delta_j = \epsilon$.

Lemma 2 furnishes a sequence $u_j : M \to \mathbb{R}^n$ of conformal minimal immersions satisfying

$$\|u_j - u\|_{0,bM} < \delta_j < \epsilon, \qquad \mathrm{dist}_{u_j}(p_0, bM) > d_j.$$

Thus $\mathrm{dist}_{u_j}(p_0, bM) > d_j > \lambda$ for any large enough $j \in \mathbb{Z}_+$.

Outline of proof of Lemma 2, part 1

- By general position we may assume that

$$u(p) - g(p) \neq 0 \text{ for all } p \in bM.$$

- The key idea is to push the u-image of each point $p \in bM$ a distance approximately d in a direction approximately orthogonal to the vector $u(p) - g(p) \in \mathbb{R}^n$. Conditions (a) and (b) will then follow from Pythagoras' Theorem.
- However, this procedure by itself will likely create shortcuts in the new induced metric. Hence we divide bM to finitely many very short arcs $I_1, \ldots, I_k$ so that both g and u vary very little on each I_j when compared to the size d of the desired displacement.
- At each of the endpoints $x_j = u(p_j) \in \mathbb{R}^n$ of these arcs we attach to $u(M) \subset \mathbb{R}^n$ a smooth arc λ_j which remains near x_j, but is spinning fast and has projection of length $> d$ on each line spanned by one of the vectors $u(p_i) - g(p_i)$, $i = 1, \ldots, k$.

Outline of proof, part 2

- By the method of **exposing boundary points** we modify u locally near p_j so that the new immersion $\tilde{u}$ follows the arc λ_j very closely up to the other endpoint $\tilde{u}(p_j) = q_j$ of λ_j. Hence any curve in M terminating on bM near p_j is elongated by approximately d.
- To this new $u = \tilde{u}$ we apply the Riemann-Hilbert method to find a conformal minimal immersion $\widehat{u}\colon M \to \mathbb{R}^n$ which at a point $x \in I_j$ adds a displacement for approximately d in a direction approximately orthogonal to the vector $u(p_j) - g(p_j) \in \mathbb{R}^n$.
- The intrinsic boundary distance with respect to the metric $\widehat{u}^* ds^2$ increases by approximately d (Pythagoras), while

$$|\widehat{u}(x) - g(x)| \approx \sqrt{|u(x) - g(x)|^2 + d^2} \leq \sqrt{\delta^2 + d^2}.$$

A few remarks

1. In our construction the vector $\widehat{u}(x) - u(x)$ is spiralling very fast as x traces the arc I_j, due to the nature of solutions of Riemann-Hilbert boundary value problems.

2. The idea of **enlarging the intrinsic boundary distance by spiraling** is reminiscent of Nash's 1956 construction of $\mathscr{C}^1$ isometric immersion of Riemannian manifolds M^n into $\mathbb{R}^{n+2}$. Due to curvature obstructions, his immersions can not be $\mathscr{C}^2$ unless the target dimension is sufficiently big.

3. In our case, complete immersions can be continuous but not $\mathscr{C}^1$ (not even Lipschitz) up the boundary of M. Furthermore, if $u : M \to \mathbb{R}^n$ is a complete conformal minimal immersion then the curves in $u(bM)$ are non rectifiable due to the isoperimetric inequality.

4. Analogous results can be proved for complex curves in $\mathbb{C}^n$, $n \geq 2$, and for holomorphic null curves in $\mathbb{C}^n$, $n \geq 3$.

Proper conformal minimal surfaces in convex domains

The same tools are used to prove the following results on proper complete conformal minimal immersions.

Theorem (2)

Let D be a convex domain in $\mathbb{R}^n$ for some $n \geq 3$, and let M is a compact bordered Riemann surface.

(a) *Every conformal minimal immersion $u\colon M \to D$ of class $\mathscr{C}^1(M)$ can be approximated, uniformly on compacts in $\mathring{M}$, by conformal complete proper minimal immersions $\widetilde{u}\colon \mathring{M} \to D$.*

(b) *If $n \geq 5$ then $\widetilde{u}$ can be chosen an embedding.*

(c) *If D has smooth strongly convex boundary then $\widetilde{u}$ can be chosen continuous on M.*

In the proof we alternately apply Lemma 1 above (to enlarge the intrinsic boundary distance) and the Riemann-Hilbert method.

Minimally convex domains in $\mathbb{R}^3$

Definition

A domain $D \subset \mathbb{R}^3$ is minimally convex (or mean convex) if it admits a smooth exhaustion function $u : D \to \mathbb{R}$ which is 2-plurisubharmonic, in the sense that the sum of the smallest two eigenvalues of $\mathrm{Hess}\,_u$ is positive at every point of $\overline{D}$.

Let D be a domain in $\mathbb{R}^3$ with $\mathscr{C}^2$ boundary. Denote by $\kappa_1(x)$ and $\kappa_2(x)$ the principal curvatures of bD from the interior side at $x \in bD$. Such D is minimally convex if and only if bD is mean-convex in the sense that

$$\kappa_1(x) + \kappa_2(x) \geq 0 \quad \forall x \in bD.$$

The domain D is said to be strongly minimally convex if $\kappa_1(x) + \kappa_2(x) > 0$ for every $x \in bD$.

A domain bounded by a minimal surface $\Sigma \subset \mathbb{R}^3$ is minimally convex since in this case $\kappa_1(x) + \kappa_2(x) = 0$ for every $x \in \Sigma$.

Conformal minimal surfaces in minimally convex domains

Theorem (3)

Let D be a minimally convex domain in $\mathbb{R}^3$ and let M be a compact bordered Riemann surface.

Every conformal minimal immersion $u\colon M \to D$ of class $\mathscr{C}^1(M)$ can be approximated, uniformly on compacts in $\mathring{M} = M \setminus bM$, by conformal complete proper minimal immersions $\widetilde{u}\colon \mathring{M} \to D$.

If D is bounded and bD is $\mathscr{C}^2$ strongly mean-convex, then $\widetilde{u}$ can be chosen continuous on M.

Minimal convexity is an optimal condition for this result:
if bD is strongly mean-concave near some point $p \in bD$ then, by maximum principle, there exist no proper conformal minimal surfaces $M \to D$ in a small neighbourhood of p in D.