Runge tubes

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Runge cylinders in $\mathbb{C}^2$

It was an open question for a long time whether it is possible to embed $\mathbb{C}^* \times \mathbb{C}$ as a **Runge domain** $\Omega \subset \mathbb{C}^2$, i.e., such that holomorphic polynomials are dense in $\mathcal{O}(\Omega)$. (Here, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.)

Such hypothetical domains have been called **Runge cylinders** in $\mathbb{C}^2$. The question arose in connection with the classification of Fatou components for Hénon maps by **E. Bedford and J. Smillie (1991)**.

Note that the standard embedding $\mathbb{C}^* \times \mathbb{C} \hookrightarrow \mathbb{C}^2$ is not Runge since, using the coordinates (z, w) on $\mathbb{C}^2$, the holomorphic function $1/z$ on $\mathbb{C}^* \times \mathbb{C}$ cannot be approximated by holomorphic polynomials in (z, w).

A complex manifold X is said to be a **Stein manifold** if X admits a proper holomorphic embedding as a closed complex submanifold of some complex Euclidean space $\mathbb{C}^N$; in this case we can take $N = \left[\frac{3n}{2}\right] + 1$ when $n > 1$, and $N = 3$ when $n = 1$. A Riemann surface X is a Stein manifold if (and only if) it is open (non compact).

Existence and plenitude of Runge tubes

In a recent joint work with **Erlend Fornæss Wold (University of Oslo)**, we gave a simple proof of the following considerably more general result.
`https://arxiv.org/abs/1801.07645`

Theorem (1)

Let X be a Stein manifold and $\theta : X \hookrightarrow \mathbb{C}^n$ be a proper holomorphic embedding. Let $E \to X$ denote the normal bundle associated to θ. Then, θ is approximable uniformly on compacts in X by holomorphic embeddings $\tilde{\theta} \colon E \hookrightarrow \mathbb{C}^n$ whose images $\tilde{\theta}(E)$ are Runge domains in $\mathbb{C}^n$.

To get a Runge embedding of $\mathbb{C}^* \times \mathbb{C}$ into $\mathbb{C}^2$ from Theorem 1, one embeds $X = \mathbb{C}^*$ onto the algebraic curve $A = \{zw = 1\} \subset \mathbb{C}^2$ and notes that any vector bundle over $\mathbb{C}^*$ (and in fact over any open Riemann surface) is trivial by **Oka's theorem (1939)**.

Runge tubes over open Riemann surfaces

It is known that every open Riemann surface, X, embeds properly holomorphically into $\mathbb{C}^3$, and a plenitude of them embed into $\mathbb{C}^2$.

Corollary (Runge tubes over open Riemann surfaces)

If X is an open Riemann surface which admits a proper holomorphic embedding into $\mathbb{C}^2$, then $X \times \mathbb{C}$ admits a Runge embedding into $\mathbb{C}^2$.

For every open Riemann surface X and $k \geq 2$, $X \times \mathbb{C}^k$ embeds as a Runge domain into $\mathbb{C}^{k+1}$.

It is a long standing open problem whether every open Riemann surface embeds as a closed complex curve in $\mathbb{C}^2$. Here are two most general known results; in each case $X \times \mathbb{C}$ embeds as a Runge domain into $\mathbb{C}^2$.

Wold and F. (2009) A bordered Riemann surface which embeds nonproperly holomorphically into $\mathbb{C}^2$ also embeds properly into $\mathbb{C}^2$.

Wold and F. (2013) Every circled domain in $\mathbb{C}$ with at most finitely many punctures (and at most countably many disc holes) embeds properly holomorphically into $\mathbb{C}^2$.

A parabolic basin

The existence of a Runge embedding $\mathbb{C}^* \times \mathbb{C} \hookrightarrow \mathbb{C}^2$ has also been proved recently by Bracci et al.

Theorem (F. Bracci, J. Raissy, and B. Stensønes, 2017)

For every $n \geq 2$ there exists a (non-polynomial) holomorphic automorphism of $\mathbb{C}^n$ with a parabolic fixed point at 0 whose basin of attraction is biholomorphic to $\mathbb{C} \times (\mathbb{C}^*)^{n-1}$.

Note that any attacting basin of an automorphism ϕ of $\mathbb{C}^n$ (or in a Stein manifold) is always a Runge domain in $\mathbb{C}^n$. Indeed, if the iterates ϕ^k $(k \in \mathbb{N})$ converge to a point uniformly on a compact set K, then same holds on its polynomial hull $\widehat{K}$.

The proof of this theorem is much more involved than our construction. It is not clear whether there exist more general parabolic basins.

Manifolds with density property

Varolin 2000 A complex manifold Y enjoys the **density property (DP)** if every holomorphic vector field on Y can be approximated by Lie combinations of $\mathbb{C}$-complete holomorphic vector fields.

A Lie algebra $\mathfrak{g}$ of holomorphic vector fields on Y enjoys DP if it is densely generated by the complete vector fields that it contains. If Y carries a holomorphic volume form ω, then the density property for the Lie algebra $\mathfrak{g}(\omega)$ of all holomorphic vector fields with vanishing ω-divergence is called the **volume density property (VDP)** of (Y, ω).

Andersén 1990; Andersén & Lempert 1992 $\mathbb{C}^n$ enjoys DP for $n > 1$, and VDP for the volume form $\omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$ for $n \geq 1$.

In fact, **every polynomial holomorphic vector field on $\mathbb{C}^n$ is a finite sum of polynomial shear vector fields** of the form

$$V(z) = V(z', z_n) = f(z') \frac{\partial}{\partial z_n}, \qquad W(z) = f(z') z_n \frac{\partial}{\partial z_n},$$

where $f \in \mathbb{C}[z_1, \ldots, z_{n-1}]$, and their $GL_n(\mathbb{C})$ conjugates.

Approximating biholomorphisms by automorphisms

A Stein manifold Y with DP or VDP is highly symmetric and has a very big holomorphic automorphism group $\mathrm{Aut}(Y)$. In particular:

Theorem (Andersén-Lempert, Forstnerič-Rosay, Varolin)

Let Y be a Stein manifold with DP. Assume that

$$F_t \colon \Omega_0 \to \Omega_t \subset Y, \qquad t \in [0,1],$$

is a smooth isotopy of biholomorphic maps between Stein Runge domains in Y, with $F_0 = \mathrm{Id}|_{\Omega_0}$. Then, $F_1 \colon \Omega_0 \to \Omega_1$ is a limit of holomorphic automorphisms of Y, uniformly on compacts in Ω_0.

The analogous result holds for isotopies of biholomorphic maps preserving a holomorphic volume form on a Stein manifold with VDP.

The theorem also holds if $\Omega_t = F_t(\Omega_0)$ is a neighborhood of a compact $\mathcal{O}(Y)$-convex set $K_t = F_t(K_0)$, with uniform approximation on K_0.

Runge tubes in Stein manifolds with DP

The following is our main result with E.F. Wold.

Theorem (2)

Let X and Y be Stein manifolds with $\dim X < \dim Y$, and assume that Y has the density property.

Suppose that $\theta : X \hookrightarrow Y$ is a holomorphic embedding with $\mathcal{O}(Y)$-convex image (this holds in particular if θ is proper), and let $E \to X$ denote the normal bundle associated to θ.

Then, θ is approximable uniformly on compacts in X by holomorphic embeddings $\tilde{\theta}\colon E \hookrightarrow Y$ whose images $\tilde{\theta}(E)$ are Runge domains in Y.

A locally closed subset Z of a complex manifold Y is said to be $\mathcal{O}(Y)$-*convex* if for every compact set $K \subset Z$, its $\mathcal{O}(Y)$-convex hull

$$\widehat{K}_{\mathcal{O}(Y)} = \{y \in Y : |f(y)| \leq \sup_K |f| \quad \forall f \in \mathcal{O}(Y)\}$$

is compact and contained in Z.

Flexibility properties Stein manifolds with DP

Every Stein manifold Y with (V)DP enjoys a number of holomorphic flexibility properties, similar to those of Euclidean spaces:

- It is an **Oka manifold**: every continuous map $X \to Y$ from a Stein manifold X which is holomorphic on a compact $\mathcal{O}(X)$-convex set $K \subset X$ can be approximated on K by holomorphic maps $X \to Y$;
- Y is **infinitely transitive**: every finite set of points in Y can be simultaneously moved to any other set with the same number of points by an automorphism of Y;
- **Andrist, F., Ritter, Wold 2016; F. 2017** If X is a Stein manifold and $\dim Y > 2\dim X$, then any continuous map $X \to Y$ is homotopic to a proper holomorphic embedding $X \hookrightarrow Y$ (and to a proper holomorphic immersion if $\dim Y = 2\dim X$).

Corollary

Every Stein manifold Y with DP contains a Runge domain biholomorphic to the total space E of a holomorphic vector bundle over an arbitrary Stein manifold X with $2\dim X < \dim Y$.

Examples of Stein manifolds with (V)DP

- **Andersén (1990)** $\mathbb{C}^n$ for $n \geq 1$ satisfies VDP for $dz_1 \wedge \cdots \wedge dz_n$.
- **Andersén and Lempert (1992)** $\mathbb{C}^n$ for any $n > 1$ satisfies DP.
- **Varolin (2000)** $(\mathbb{C}^*)^n$ with the volume form $\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$ satisfies VDP It is not known whether DP holds when $n > 1$.
- **Kaliman, Donzelli & Dvorsky (2010)** If G is a linear algebraic group and $H \subset G$ is a closed proper reductive subgroup, then $Y = G/H$ is a Stein manifold with DP, except when $Y = \mathbb{C}, (\mathbb{C}^*)^n$, or a $\mathbb{Q}$-homology plane with fundamental group $\mathbb{Z}_2$.
- **Kaliman and Kutzschebauch (2008)** In particular, a linear algebraic group with connected components different from $\mathbb{C}$ or $(\mathbb{C}^*)^n$ has DP.
- **K& K (2008)** If $p : \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function with smooth reduced zero fibre, then $Y = \{xy = p(z)\}$ has DP. The same is true if the source $\mathbb{C}^n$ of p is an arbitrary Stein manifold with DP.
- **K& K (2008)** A Cartesian product $Y_1 \times Y_2$ of two Stein manifolds Y_1, Y_2 with DP also has DP. The analogous result holds for VDP.

Gizatullin surfaces and the Koras-Russell cubic

- **Andrist 2017** A smooth affine algebraic surface Y is a **Gizatullin surface** if $\mathrm{Aut}_{\mathrm{alg}}(Y)$ acts transitively on Y up to finitely many points. Every such surface admits a fibration $\pi\colon Y \to \mathbb{C}$ whose generic fiber equals $\mathbb{C}$ and there is only one exceptional fiber. **If this exceptional fiber is reduced, then Y has the density property.**

- **Leuenberger 2016** DP holds for a family of hypersurfaces

$$Y = \{(x, y, z) \in \mathbb{C}^{n+3} : x^2 y = a(z) + x b(z)\},$$

where $x, y \in \mathbb{C}$ and $a, b \in \mathbb{C}[z]$ are polynomials in $z \in \mathbb{C}^{n+1}$. This family includes the **Koras-Russell cubic threefold**

$$C = \{(x, y, z_0, z_1) \in \mathbb{C}^4 : x^2 y + x + z_0^2 + z_1^3 = 0\}.$$

This threefold is diffeomorphic to $\mathbb{R}^6$, but is not algebraically isomorphic to $\mathbb{C}^3$; in particular, $\mathrm{Aut}_{\mathrm{alg}}(C)$ does not act transitively on C (**Makar-Limanov, Dubouloz**).

It remains an open question whether C is biholomorphic to $\mathbb{C}^3$.

Preliminaries

Assume that $\pi\colon E \to X$ is a **holomorphic vector bundle** over a Stein manifold X. The total space E is then also a Stein manifold. We write elements of E as $e = (x, v)$, identifying X with the zero section of E:

$$X \cong \{(x, 0) : x \in X\} \subset E.$$

Consider the holomorphic automorphisms $\psi_t \in \mathrm{Aut}(E)$ given by

$$\psi_t(x, v) = (x, tv), \quad t \in \mathbb{C}^*.$$

Note that

$$\psi_t|_X = \mathrm{Id}_X \quad \text{for all } t.$$

A subset $Z \subset E$ is called **radial** if

$$\psi_t(Z) \subset Z \text{ holds for every } t \in [0, 1].$$

Proof of Theorem 2: The inductive step

Lemma

Assume that:

- X *is a Stein manifold,*
- $\pi \colon E \to X$ *is a holomorphic vector bundle,*
- $K \subset L$ *are compact radial* $\mathcal{O}(E)$*-convex subsets of* E*,*
- $\Omega \subset E$ *is an open set containing* $X \cup K$*,*
- Y *is a Stein manifold with DP such that* $\dim Y = \dim E$*, and*
- $\theta : \Omega \hookrightarrow Y$ *is a holomorphic embedding such that* $\theta|_X \colon X \hookrightarrow Y$ *is a Runge embedding and* $\theta(K)$ *is* $\mathcal{O}(Y)$*-convex.*

Then there is a domain $\widetilde{\Omega}$*, with* $X \cup L \subset \widetilde{\Omega} \subset E$*, such that* θ *can be approximated uniformly on* K *by holomorphic embeddings*

$$\tilde{\theta} : \widetilde{\Omega} \hookrightarrow Y$$

so that $\tilde{\theta}|_X \colon X \hookrightarrow Y$ *is a Runge embedding and* $\tilde{\theta}(L)$ *is* $\mathcal{O}(Y)$*-convex.*

Proof of the lemma

- Choose a compact $\mathcal{O}(X)$-convex subset $X_0 \subset X$ with $\pi(L) \subset X_0$. Since the embedding $\theta|_X : X \hookrightarrow Y$ is Runge, the image $Y_0 := \theta(X_0) \subset \theta(X)$ is $\mathcal{O}(Y)$-convex.
- Pick a compact $\mathcal{O}(Y)$-convex neighborhood $N \subset \theta(\Omega)$ of Y_0. Thus, $N = \theta(N_0)$ for a compact set $N_0 \subset \Omega$ with $X_0 \subset \mathring{N}_0$.
- Since $\pi(L) \subset X_0$, there exists $\epsilon > 0$ such that $\psi_\epsilon(L) \subset N_0$.
- Since L is $\mathcal{O}(E)$-convex and $\psi_\epsilon \in \mathrm{Aut}(E)$, the set $\psi_\epsilon(L) \subset N_0$ is also $\mathcal{O}(E)$-convex, and hence $\mathcal{O}(N_0)$-convex.
- Since $\theta\colon \Omega \to \theta(\Omega)$ is a biholomorphism and $\psi_\epsilon(L)$ is $\mathcal{O}(N_0)$-convex, the image $\theta(\psi_\epsilon(L))$ is $\mathcal{O}(N)$-convex, and hence also $\mathcal{O}(Y)$-convex (since N is $\mathcal{O}(Y)$-convex).

Proof of the lemma, 2

After shrinking Ω around $X \cup K$, we may assume that it is radial, $\psi_t(\Omega) \subset \Omega$ for all $t \in [0,1]$. Consider the isotopy of injective holomorphic maps

$$\sigma_t : \theta(\Omega) \to \theta(\Omega), \quad t \in [\epsilon, 1],$$

defined by the conjugation condition

$$\theta \circ \psi_t = \sigma_t \circ \theta.$$

Note that $\sigma_1 = \mathrm{Id}$ on $\theta(\Omega)$, and

$(*)$ the compact set $\sigma_t(\theta(K)) \subset Y$ is $\mathcal{O}(Y)$-convex for every $t \in [\epsilon, 1]$.

Indeed, since $\psi_t(K) \subset K$ is clearly $\mathcal{O}(K)$-convex and $\theta : \Omega \to \theta(\Omega)$ is a biholomorphism, we have that

$$\sigma_t(\theta(K)) = \theta(\psi_t(K)) \text{ is } \mathcal{O}(\theta(K))\text{-convex.}$$

Since $\theta(K)$ is $\mathcal{O}(Y)$-convex, the claim follows.

Proof of the lemma, 3

Since $\sigma_t(\theta(K))$ is $\mathcal{O}(Y)$-convex for every $t \in [\epsilon, 1]$ and Y has DP,

σ_ϵ can be approximated uniformly on $\theta(K)$ by $\phi \in \mathrm{Aut}(Y)$.

Since $\psi_\epsilon(L \cup X) = \psi_\epsilon(L) \cup X \subset \Omega$ by the choice of $\epsilon > 0$, there is an open neighborhood $\widetilde{\Omega} \subset E$ of $L \cup X$ such that $\psi_\epsilon(\widetilde{\Omega}) \subset \Omega$.

We claim that the holomorphic embedding

$$\tilde{\theta} = \phi^{-1} \circ \theta \circ \psi_\epsilon : \widetilde{\Omega} \hookrightarrow Y$$

satisfies the lemma. Indeed:

- The sets $\tilde{\theta}(L)$ and $\tilde{\theta}(K)$ are $\mathcal{O}(Y)$-convex (since the sets $\theta(\psi_\epsilon(L))$ and $\theta(\psi_\epsilon(K))$ are $\mathcal{O}(Y)$-convex and $\phi \in \mathrm{Aut}(Y)$).
- $\tilde{\theta}|_X = \phi^{-1} \circ \theta|_X : X \hookrightarrow Y$ is a Runge embedding since $\theta|_X$ is.
- On the set K we have that $\theta \circ \psi_\epsilon = \sigma_\epsilon \circ \theta$ and hence

$$\tilde{\theta} = \phi^{-1} \circ \theta \circ \psi_\epsilon = \phi^{-1} \circ \sigma_\epsilon \circ \theta.$$

  Since $\phi^{-1} \circ \sigma_\epsilon$ is close to the identity on $\theta(K)$ by the choice of ϕ, it follows that $\tilde{\theta}$ is close to θ on K.

Proof of Theorem 1

Pick an exhaustion $K_1 \subset K_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} K_j = E$ by compact radial $\mathcal{O}(E)$-convex sets.

Let $\theta \colon X \hookrightarrow Y$ be a holomorphic Runge embedding. By a theorem of **Docquier and Grauert (1960)** there is a neighbourhood $\Omega_0 \subset E$ of the zero section $X \subset E$ such that θ extends to a holomorphic embedding

$$\theta_0 : \Omega_0 \hookrightarrow Y.$$

Set $K_0 = \varnothing$. Applying the main lemma inductively, we find

open neighbourhoods $\Omega_j \subset E$ of $K_j \cup X$, and

holomorphic embeddings $\theta_j \colon \Omega_j \hookrightarrow Y$,

satisfying the following conditions for every $j \in \mathbb{N}$:

(a) the compact sets $\theta_j(K_j)$ and $\theta_j(K_{j-1})$ are $\mathcal{O}(Y)$-convex,
(b) the embedding $\theta_j|_X \colon X \hookrightarrow Y$ is Runge, and
(c) θ_j approximates θ_{j-1} as closely as desired on K_{j-1}.

Proof of Theorem 1

If the approximations are close enough, the sequence θ_j converges uniformly on compacts in E to a holomorphic embedding $\tilde{\theta}\colon E \hookrightarrow Y$.

Since $\mathcal{O}(Y)$-convexity of a compact set in a Stein manifold Y is a stable property for compact strongly pseudoconvex domains and every compact $\mathcal{O}(Y)$-convex set can be approximated from the outside by such domains, it follows that the image of each K_j remains $\mathcal{O}(Y)$-convex in the limit provided that all approximations were close enough.

Hence, $\tilde{\theta}(E)$ is a Runge domain in Y. This proves the theorem.

Runge tubes around algebraic submanifolds of $\mathbb{C}^n$

The Runge embeddings $E \hookrightarrow Y$ of the normal bundle in Theorems 1 and 2 need not agree with the embedding $\theta : X \hookrightarrow Y$ on the zero section X of E. However, we can ensure this additional condition for algebraic embeddings of codimension at least 2 into $\mathbb{C}^n$.

Theorem (2)

Let $\theta\colon X \hookrightarrow \mathbb{C}^n$ be proper holomorphic embedding onto an algebraic submanifold $A = \theta(X) \subset \mathbb{C}^n$.
If $n \geq \dim A + 2$, then θ extends to a holomorphic Runge embedding $\tilde{\theta}\colon E \hookrightarrow Y$ of the normal bundle E of the embedding θ.

Since every vector bundle over an open Riemann surface is trivial, we get

Corollary

Let X be an affine algebraic curve. Every proper algebraic embedding $\theta\colon X \hookrightarrow \mathbb{C}^{n+1}$ for $n \geq 2$ extends to a holomorphic embedding $\tilde{\theta} : X \times \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ onto a Runge domain in $\mathbb{C}^{n+1}$.

Runge tubes around algebraic submanifolds of $\mathbb{C}^n$

The proof requires the following

Addendum to the main lemma:
If $Y = \mathbb{C}^n$ with $n \geq \dim X + 2$, $\theta\colon \Omega \hookrightarrow Y$ is a holomorphic embedding (where $\Omega \subset E$ is an open neighborhood of $K \cup X$), and $A = \theta(X) \subset \mathbb{C}^n$ is a **closed algebraic submanifold** of $\mathbb{C}^n$, then the approximating holomorphic embedding $\tilde{\theta} : \widetilde{\Omega} \hookrightarrow \mathbb{C}^n$ can be chosen to agree with θ on X.

The proof uses the following result.

Theorem (Kaliman and Kutzschebauch, 2008)

If $A \subset \mathbb{C}^n$ is an algebraic submanifold with $n \geq \dim A + 2$, then every polynomial vector field on $\mathbb{C}^n$ that vanishes on A is a Lie combination of complete polynomial shear vector fields vanishing on A.

By using this result and Serre's Theorem A and B, we can approximate the biholomorphism σ_ϵ (in the proof of Theorem 2) by an automorphism $\phi \in \mathrm{Aut}(\mathbb{C}^n)$ such that $\phi(z) = z$ for all $z \in A$.

THANK YOU

FOR YOUR ATTENTION