Hyperbolic domains in real Euclidean spaces

Franc Forstnerič

Univerza v Ljubljani

Institute of Mathematics, Physics and Mechanics

Complex Analysis and Geometry
Ufa, Russia, 16–19 November 2021

Abstract

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 3$. We introduce an intrinsic Kobayashi-type (Finsler) **minimal pseudometric** $g_\Omega : T\Omega = \Omega \times \mathbb{R}^n \to \mathbb{R}_+$ defined in terms of conformal harmonic discs. Such discs parameterize minimal surfaces in $\mathbb{R}^n$.

Its integrated form is the **minimal pseudodistance** $\rho_\Omega : \Omega \times \Omega \to \mathbb{R}_+$, also defined by chains of conformal harmonic discs.

On the unit ball $\mathbb{B}^n$, $g_{\mathbb{B}^n}$ coincides with the **Cayley–Klein metric**, one of the classical models of hyperbolic geometry.

I shall present several sufficient conditions for a domain Ω to be (complete) hyperbolic, meaning that g_Ω is a (complete) metric; equivalently, ρ_Ω is a (complete) distance function.

F. F. & David Kalaj, Hyperbolicity theory for conformal minimal surfaces in $\mathbb{R}^n$. `https://arxiv.org/abs/2102.12403`**, February 2021**

Barbara Drinovec Drnovšek and F. F., Hyperbolic domains in real Euclidean spaces. `https://arxiv.org/abs/2109.06943`**, Sept 2021**

The minimal pseudodistance on a domain in $\mathbb{R}^n$

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc, and let Ω be a domain in $\mathbb{R}^n$.

Denote by $\mathrm{CH}(\mathbb{D}, \Omega)$ the space of all harmonic discs $f : \mathbb{D} \to \Omega$ which are conformal:

$$f_x \cdot f_y = 0, \quad |f_x| = |f_y|; \qquad z = x + \mathrm{i}y \in \mathbb{D}.$$

Fix a pair of points $\mathbf{x}, \mathbf{y} \in \Omega$ and consider finite chains of conformal harmonic discs $f_i \in \mathrm{CH}(\mathbb{D}, \Omega)$ and points $a_i \in \mathbb{D}$ $(i = 1, \ldots, k)$ such that

$$f_1(0) = \mathbf{x}, \quad f_{i+1}(0) = f_i(a_i) \text{ for } i = 1, \ldots, k-1, \quad f_k(a_k) = \mathbf{y}.$$

To any such chain we associate the number

$$\sum_{i=1}^{k} \frac{1}{2} \log \frac{1 + |a_i|}{1 - |a_i|} \geq 0.$$

The pseudodistance $\rho_\Omega : \Omega \times \Omega \to \mathbb{R}_+$ is the infimum of the numbers obtained in this way. Clearly it satisfies the triangle inequality.

If $\Omega \subset \mathbb{C}^n$ and we use only holomorphic discs, we get the Kobayashi pseudodistance $\mathcal{K}_\Omega$. Hence, $\rho_\Omega \leq \mathcal{K}_\Omega$. These pseudodistances agree on domains in $\mathbb{R}^2 = \mathbb{C}$, but strict inequality holds if $n > 2$.

The minimal pseudometric

Define a Finsler pseudometric $g_\Omega : \Omega \times \mathbb{R}^n \to \mathbb{R}_+$ on $(\mathbf{x}, \mathbf{v}) \in \Omega \times \mathbb{R}^n$ by

$$g_\Omega(\mathbf{x}, \mathbf{v}) = \inf\{1/r > 0 : \exists f \in \mathrm{CH}(\mathbb{D}, \Omega),\ f(0) = \mathbf{x},\ f_x(0) = r\mathbf{v}\}.$$

Clearly, g_Ω is upper-semicontinuous and absolutely homogeneous:

$$g_\Omega(\mathbf{x}, t\mathbf{v}) = |t|\, g_\Omega(\mathbf{x}, \mathbf{v}) \quad \text{for} \quad t \in \mathbb{R}.$$

If $\Omega \subset \mathbb{C}^n$ and using only holomorphic disc gives the Kobayashi pseudometric.

Theorem

The minimal pseudodistance ρ_Ω is obtained by integrating the pseudometric g_Ω:

$$\rho_\Omega(\mathbf{x}, \mathbf{y}) = \inf_\gamma \int_0^1 g_\Omega(\gamma(t), \dot\gamma(t))\, dt, \qquad \mathbf{x}, \mathbf{y} \in \Omega,$$

where the infimum is over all piecewise smooth paths $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = \mathbf{x}$ and $\gamma(1) = \mathbf{y}$.

The proof is similar to the one for the Kobayashi pseudometric.

Metric decreasing properties

A conformal surface M is **hyperbolic** if its universal covering space is the disc $\mathbb{D}$. Such a surface carries the **Poincaré metric**, $\mathcal{P}_M$, the unique Riemannian metric such that any conformal covering map $h : \mathbb{D} \to M$ is an isometry from $(\mathbb{D}, \mathcal{P}_{\mathbb{D}})$ onto $(M, \mathcal{P}_M)$. The Poincaré metric on $\mathbb{D}$ is

$$\mathcal{P}_{\mathbb{D}}(z, \xi) = \frac{|\xi|}{1 - |z|^2}, \qquad z \in \mathbb{D},\ \xi \in \mathbb{C}.$$

For every conformal harmonic map $f : \mathbb{D} \to \Omega$ we have that

$$g_\Omega(f(z), df_z(\xi)) \le \mathcal{P}_{\mathbb{D}}(z, \xi), \qquad z \in \mathbb{D},\ \xi \in \mathbb{C},$$

and g_Ω **is the largest pseudometric on Ω with this property.**

For $z = 0$ this is immediate from the definition of g_Ω. For other points, we precompose f by $\phi \in \mathrm{Aut}(\mathbb{D})$ interchanging z and 0.

The same holds for conformal harmonic maps $(M, \mathcal{P}_M) \to (\Omega, g_\Omega)$.

Any rigid map $R : \mathbb{R}^n \to \mathbb{R}^m$ $(n \le m)$ with $R(\Omega) \subset \Omega'$ is metric-decreasing:

$$g_{\Omega'}(R(\mathbf{x}), R(\mathbf{v})) \le g_\Omega(\mathbf{x}, \mathbf{v}), \qquad \mathbf{x} \in \Omega,\ \mathbf{v} \in \mathbb{R}^n.$$

In particular,

$$\Omega \subset \Omega' \implies g_\Omega \geq g_{\Omega'}.$$

We also introduce a Finsler pseudometric on $\Omega \times \mathbb{G}_2(\mathbb{R}^n)$, where $\mathbb{G}_2(\mathbb{R}^n)$ denotes the Grassmann manifold of 2-planes in $\mathbb{R}^n$, by

$$\mathcal{M}_\Omega(\mathbf{x}, \Lambda) = \inf\{1/\|df_0\| : f \in \mathrm{CH}(\mathbb{D}, \Omega),\ f(0) = \mathbf{x},\ df_0(\mathbb{R}^2) = \Lambda\}.$$

Here, $\|df_0\|$ denotes the operator norm of the differential $df_0 : \mathbb{R}^2 \to \mathbb{R}^n$.

It clearly follows that for any vector $\mathbf{v} \in \mathbb{R}^n$ we have

$$g_\Omega(\mathbf{x}, \mathbf{v}) = |\mathbf{v}| \cdot \inf\{\mathcal{M}_\Omega(\mathbf{x}, \Lambda) : \Lambda \in \mathbb{G}_2(\mathbb{R}^n), \mathbf{v} \in \Lambda\}.$$

Note that the 2-planes Λ containing a given vector $v \neq 0$ form an $(n-2)$-sphere. This is an important difference with respect to the Kobayashi metric — a vector $v \in \mathbb{C}^n$ determines a unique complex line Λ.

The Cayley–Klein metric on the ball $\mathbb{B}^n$ of $\mathbb{R}^n$, $n \geq 3$

Theorem (F.–Kalaj 2021)

The minimal metric $g^2_{\mathbb{B}^n}$ on the unit ball $\mathbb{B}^n$ equals the **Cayley–Klein metric***:*

$$g_{\mathbb{B}^n}(\mathbf{x},\mathbf{v})^2 = \frac{(1-|\mathbf{x}|^2)|\mathbf{v}|^2 + |\mathbf{x}\cdot\mathbf{v}|^2}{(1-|\mathbf{x}|^2)^2} = \frac{|\mathbf{v}|^2}{1-|\mathbf{x}|^2} + \frac{|\mathbf{x}\cdot\mathbf{v}|^2}{(1-|\mathbf{x}|^2)^2}.$$

We also have

$$g_{\mathbb{B}^n}(\mathbf{x},\mathbf{v}) = \frac{\sqrt{1-|\mathbf{x}|^2\sin^2\phi}}{1-|\mathbf{x}|^2}\,|\mathbf{v}|, \qquad \mathbf{x}\in\mathbb{B}^n,\ \mathbf{v}\in\mathbb{R}^n,$$

where $\phi \in [0, \pi/2]$ is the angle between the vector $\mathbf{v}$ and the line $\mathbb{R}\mathbf{x} \subset \mathbb{R}^n$.

The **Beltrami–Cayley–Klein model of hyperbolic geometry** was introduced by **Arthur Cayley (1859)**, **Eugenio Beltrami (1868)**, and **Felix Klein (1871–73)**. The underlying space is the unit ball, geodesics are straight line segments with endpoints on the boundary sphere, and the distance between points on a geodesic is given by cross ratio. This metric is the restriction of the Kobayashi metric on the complex ball $\mathbb{B}^n_{\mathbb{C}} \subset \mathbb{C}^n$ to points in $\mathbb{B}^n = \mathbb{B}^n_{\mathbb{C}} \cap \mathbb{R}^n$ and vectors in $\mathbb{R}^n$. It is a special case of the metric on convex domains in $\mathbb{R}^n$ introduced by **David Hilbert** in 1885.

Schwarz–Pick lemma for conformal harmonic discs in the ball

This theorem is a corollary to the following Schwarz–Pick lemma.

Theorem (F.F. – D. Kalaj 2021)

Let $f:\mathbb{D}\to\mathbb{B}^n$ is a harmonic map for some $n\geq 2$ which is a conformal immersion at a point $z\in\mathbb{D}$. Denote by $\theta\in[0,\pi/2]$ the angle between the vector $f(z)$ and the plane $df_z(\mathbb{R}^2)$. Then at this point we have that

$$(*)\qquad \|df_z\| \;\leq\; \frac{1}{R}\cdot\frac{1-|f(z)|^2}{1-|z|^2},$$

where $R=\sqrt{1-|f(z)|^2\sin^2\theta}$ is the radius of the affine disc

$$\Sigma=(f(z)+df_z(\mathbb{R}^2))\cap\mathbb{B}^n.$$

Equality holds if and only if f is a conformal diffeomorphism onto Σ.

The proof reduces to $z=0$ by precomposing f with $\psi\in\mathrm{Aut}\,\mathbb{D}$ interchanging 0 and z. On the other hand, we cannot interchange $f(0)$ and 0 by an automorphism of the ball preserving the class of conformal harmonic maps.

The Schwarz–Pick lemma implies the theorem

Let us see how this Schwarz–Pick lemma implies the theorem. Take $z = 0$. Let $\mathbf{v}$ be a unit vector in a 2-plane Λ, and let $f \in \mathrm{CH}(\mathbb{D}, \mathbb{B}^n)$ be such that

$$\mathbf{x} = f(0) \in \mathbb{B}^n, \quad f_x(0) = \|df_0\|\mathbf{v} \in \mathbb{R}^n, \quad df_0(\mathbb{R}^2) = \Lambda.$$

Let θ denote the angle between $\mathbf{x}$ and Λ. The inequality (*) is equivalent to

$$\frac{\sqrt{1 - |\mathbf{x}|^2 \sin^2\theta}}{1 - |\mathbf{x}|^2} \le \frac{1}{\|df_0\|}.$$

The infimum of the right hand side over all discs f with the given data equals $\mathcal{M}_{\mathbb{B}^n}(\mathbf{x}, \Lambda)$. Since equality holds for some f, we obtain

$$\mathcal{M}_{\mathbb{B}^n}(\mathbf{x}, \Lambda) = \frac{\sqrt{1 - |\mathbf{x}|^2 \sin^2\theta}}{1 - |\mathbf{x}|^2}.$$

Note that $0 \le \theta \le \phi \le \pi/2$ where ϕ is the angle between $\mathbf{x}$ and the line $\mathbb{R}\mathbf{v} \subset \Lambda$. Taking the infimum over all planes Λ containing the vector $\mathbf{v}$ gives

$$g_{\mathbb{B}^n}(\mathbf{x}, \mathbf{v}) = \frac{\sqrt{1 - |\mathbf{x}|^2 \sin^2\phi}}{1 - |\mathbf{x}|^2} = \mathcal{CK}(\mathbf{x}, \mathbf{v}).$$

Schwarz–Pick lemma for harmonic self-map maps of the disc which are conformal at a point

If $n = 2$ then $\theta = 0$ and $R = 1$, so the theorem implies the following corollary generalizing the classical Schwarz–Pick lemma for holomorphic maps $\mathbb{D} \to \mathbb{D}$ due to **Karl Hermann Amandus Schwarz (1869)**, **Henri Poincaré (1884)**, and **Georg Alexander Pick (1915)**.

Corollary

Let $f : \mathbb{D} \to \mathbb{D}$ be a harmonic map. If f is conformal at a point $z \in \mathbb{D}$, then at this point we have that

$$|f'(z)| = \|df_z\| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

Equality holds if and only if f is a conformal diffeomorphism of the disc $\mathbb{D}$.

The estimate fails for some nonconformal harmonic diffeomorphisms of $\mathbb{D}$.

It also fails for harmonic maps which are conformal at a point from the disc to some other domains in $\mathbb{C}$; for example, to the square $\{x + \imath y : |x| < 1, |y| < 1\}$.

Definition

A domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is **hyperbolic** if ρ_Ω is a distance function, and is **complete hyperbolic** if (Ω, ρ_Ω) is a complete metric space.

Example

(A) The ball $\mathbb{B}^n \subset \mathbb{R}^n$, $n \geq 3$, is complete hyperbolic.

(B) Every bounded domain $\Omega \subset \mathbb{R}^n$ is hyperbolic since it is contained in a ball. However, it need not be complete hyperbolic. For example, if $b\Omega$ is smooth and contains a strongly concave boundary point $\mathbf{p} \in b\Omega$, there is a conformal linear disc $\Sigma \subset \Omega \cup \{\mathbf{p}\}$ containing $\mathbf{p}$. Then, $\mathbf{p}$ is at finite ρ_Ω-distance from Ω.

(C) Every bounded strongly convex domain in $\mathbb{R}^n$ is complete hyperbolic.

(D) The half-space $\mathbb{H}^n = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 > 0\}$ is not hyperbolic since the pseudodistance $\rho_{\mathbb{H}^n}$ vanishes on planes $x_1 = \mathit{const}$. However, the minimal distance to $b\mathbb{H} = \{x_1 = 0\}$ is infinite.

Basic properties of hyperbolic domains

Theorem

The following conditions are equivalent for a domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$.

(i) *The family $\mathrm{CH}(\mathbb{D}, \Omega)$ of conformal harmonic discs $\mathbb{D} \to \Omega$ is pointwise equicontinuous for some metric ρ on Ω inducing its natural topology.*

(ii) *Every point $\mathbf{p} \in \Omega$ has a neighbourhood $U \subset \Omega$ and $c > 0$ such that*

$$g_\Omega(\mathbf{x}, \mathbf{u}) \geq c|\mathbf{u}|, \qquad \mathbf{x} \in U, \ \mathbf{u} \in \mathbb{R}^n.$$

(iii) *Ω is hyperbolic.*

(iv) *The minimal distance ρ_Ω induces the standard topology of Ω.*

A domain $\Omega \subset \mathbb{R}^n$ is called **taut** if $\mathrm{CH}(\mathbb{D}, \Omega)$ is a normal family.

Theorem

The following hold for any domain Ω in $\mathbb{R}^n$, $n \geq 3$:

$$\textit{complete hyperbolic} \implies \textit{taut} \implies \textit{hyperbolic}$$

Theorem (B. Drinovec Drnovšek & F. F.)

The following are equivalent for a convex domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$.

1. Ω *is complete hyperbolic.*
2. Ω *is hyperbolic.*
3. Ω *does not contain any 2-dimensional affine subspaces.*
4. Ω *is contained in the intersection of* $n-1$ *halfspaces determined by linearly independent linear functionals.*

For comparison: A convex domain in $\mathbb{C}^n$ is Kobayashi hyperbolic if and only if it does not contain any affine complex line (Barth (1980), Harris (1979)).

The first main step is to show that **the minimal distance to an affine hyperplane is infinite.** This follows from the Schwarz lemma for positive harmonic functions $f : \mathbb{D} \to (0, +\infty) : \ |\nabla f(0)| \leq 2f(0)$. This implies for $\mathbb{H}^n = \{x_1 > 0\}$:

$$g_{\mathbb{H}^n}((x_1, \ldots, x_n), (v_1, \ldots, v_n)) \geq \frac{|v_1|}{2x_1}.$$

For any path $\gamma(t) = (\gamma_1(t), \ldots) \in \mathbb{H}^n$, $t \in [0, 1)$, it follows that

$$\int_0^1 g_{\mathbb{H}^n}(\gamma(t), \dot\gamma(t))\, dt \geq \int_0^1 \frac{|\dot\gamma_1(t)|}{2\gamma_1(t)}\, dt.$$

If $\gamma(t) \to 0$ or $\gamma(t) \to +\infty$ as $t \to 1$ then the integral is $+\infty$.

Let Ω be a domain in $\mathbb{R}^n$. An upper-semicontinuous function $u : \Omega \to [-\infty, +\infty)$ is said to be **minimal plurisubharmonic, MPSH**, if for every affine 2-plane $L \subset \mathbb{R}^n$ the restriction $u : L \cap \Omega \to [-\infty, +\infty)$ is subharmonic (in conformal affine coordinates on L). This class of functions was studied by **Harvey and Lawson** in a series of papers.

Every MPSH function on a domain $\Omega \subset \mathbb{R}^{2n} = \mathbb{C}^n$ is also plurisubharmonic in the usual sense.

A function $u \in \mathscr{C}^2(\Omega)$ is MPSH if and only if

$$\Delta(u|_{\mathbf{x}+\Lambda})(\mathbf{x}) = \mathrm{tr}_\Lambda \mathrm{Hess}_u(\mathbf{x}) \geq 0 \text{ for every } (\mathbf{x}, \Lambda) \in \Omega \times \mathbb{G}_2(\mathbb{R}^n),$$

and this holds if and only if

$$(*) \qquad \lambda_1(\mathbf{x}) + \lambda_2(\mathbf{x}) \geq 0 \text{ for all } \mathbf{x} \in \Omega,$$

where $\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x})$ denote the smallest eigenvalues of the Hessian $\mathrm{Hess}_u(\mathbf{x})$.

We say that $u \in \mathscr{C}^2(\Omega)$ is **strongly minimal plurisubharmonic** if strong inequality holds in (*).

Proposition

An upper-semicontinuous function $u : \Omega \to [-\infty, +\infty)$ is MPSH if and only if for each conformal harmonic map $f : M \to \Omega$ from a conformal surface the function $u \circ f : M \to \mathbb{R}$ is subharmonic. If $u \in \mathscr{C}^2(\Omega)$ is strongly MPSH and f is an immersion, then $u \circ f$ is strongly subharmonic on M.

For functions $u \in \mathscr{C}^2(\Omega)$ this follows from the following formula, which holds for every conformal harmonic map $f : \mathbb{D} \to \Omega$:

$$\Delta(u \circ f)(z) = \mathrm{tr}_{df_z(\mathbb{R}^2)} \mathrm{Hess}_u(f(z)) \cdot \|df_z\|^2, \quad z \in \mathbb{D}.$$

Lemma

Let $\mathbf{x}$ be the Euclidean coordinate on $\mathbb{R}^n$, $n \geq 3$.

(a) *The function $\log |\mathbf{x}|$ is MPSH on $\mathbb{R}^n$.*

(b) *If u is MPSH on $\Omega \subset \mathbb{R}^n$ then for any $\mathbf{p} \in \Omega$ the function $\mathbf{x} \mapsto |\mathbf{x} - \mathbf{p}|^2 e^{u(\mathbf{x})}$ and its logarithm are MPSH on Ω.*

Minimally convex domains

A domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, with smooth boundary is **minimally convex** if admits a defining function ρ such that

$$(*) \qquad \mathrm{tr}_\Lambda \mathrm{Hess}_\rho(\mathbf{p}) \geq 0 \text{ for every } \mathbf{p} \in b\Omega \text{ and 2-plane } \Lambda \subset T_\mathbf{p} b\Omega.$$

Ω is **strongly minimally convex** if strict inequality holds.

Condition (*) says that $b\Omega$ has nonnegative (resp. positive) mean sectional curvature on every tangent 2-plane. This holds if and only if the principal normal curvatures $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_{n-1}$ of $b\Omega$ at $\mathbf{p} \in b\Omega$ satisfy

$$\nu_1 + \nu_2 \geq 0 \qquad (\text{resp. } \nu_1 + \nu_2 > 0).$$

Note: A domain $\Omega \subset \mathbb{C}^n$ is Levi pseudoconvex if (*) holds for all complex lines $\Lambda \subset T_\mathbf{p} b\Omega$. Hence, every minimally convex domain in $\mathbb{C}^n$ is also Levi pseudoconvex, but the converse is not true.

Fact: A bounded (strongly) minimally convex domain $\Omega \subset \mathbb{R}^n$ admits a defining function which is (strongly) MPSH on $\overline{\Omega}$.

Theorem (B. Drinovec Drnovšek & F. F.)

Every bounded strongly minimally convex domain is complete hyperbolic.

This is an analogue of **Graham's theorem** (1975) that bounded strongly pseudoconvex domains in $\mathbb{C}^n$ are complete Kobayashi hyperbolic.

Conversely: if $\nu_1 + \nu_2 < 0$ at some point $\mathbf{p} \in b\Omega$ then $\mathbf{p}$ is at finite minimal distance from the interior. In this case there exists an embedded conformal harmonic disc $f : \mathbb{D} \to \Omega \cup \{\mathbf{p}\}$ with $f(0) = \mathbf{p}$ and $f(\mathbb{D}^*) \subset \Omega$.

Corollary

If M is an embedded surface in $\mathbb{R}^3$ such that the minimal distance to any point $\mathbf{p} \in M$ is infinite, then M is a minimal surface.

Problem

Is the minimal distance to an embedded minimal surface $M \subset \mathbb{R}^3$ infinite?

Our proof uses the existence of a strongly minimally plurisubharmonic defining function and an **analogue of the Sibony metric** in this category.

We define the pseudometric $\mathcal{F}_\Omega : \Omega \times G_2(\mathbb{R}^n) \to \mathbb{R}_+$ by

$$\mathcal{F}_\Omega(\mathbf{x}, \Lambda) = \frac{1}{2} \sup_u \sqrt{\mathrm{tr}_\Lambda \mathrm{Hess}_u(\mathbf{x})}, \quad \mathbf{x} \in \Omega,\ \Lambda \in G_2(\mathbb{R}^n),$$

where the supremum is over all MPSH functions $u : \Omega \to [0, 1]$ such that u is of class $\mathscr{C}^2$ near $\mathbf{x}$, $u(\mathbf{x}) = 0$, and $\log u$ is MPSH on Ω.

The **Sibony metric** is defined in the same way, using log-plurisubharmonic functions on domains in $\mathbb{C}^n$ and complex lines $\Lambda \subset \mathbb{C}^n$.

The main point is that $\mathcal{F}_\Omega$ gives a lower bound for the minimal pseudometric:

Proposition

For any domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, we have that $\mathcal{F}_\Omega \leq \mathcal{M}_\Omega$.

Proof of the proposition

Fix $(\mathbf{x}, \Lambda) \in \Omega \times \mathbb{G}_2(\mathbb{R}^n)$. Let $f \in \mathrm{CH}(\mathbb{D}, \Omega)$ be such that $f(0) = \mathbf{x}$ and $df_0(\mathbb{R}^2) = \Lambda$. Let $u : \Omega \to [0, 1]$ be as in the definition of $\mathcal{F}_\Omega$.

The function $v := u \circ f : \mathbb{D} \to [0, 1]$ is then subharmonic, of class $\mathscr{C}^2$ near the origin, $v(0) = 0$, and $\log v = \log u \circ f : \mathbb{D} \to [-\infty, 0)$ is also subharmonic.

By **Sibony (1981)** we have that

$$\Delta v(0) \leq 4.$$

(The unique extremal function with $\Delta v(0) = 4$ is $v(x + \imath y) = x^2 + y^2$.) Hence,

$$\mathrm{tr}_\Lambda \mathrm{Hess}_u(\mathbf{x}) \cdot \|df_0\|^2 = \Delta v(0) \leq 4.$$

Equivalently,

$$\frac{1}{2}\sqrt{\mathrm{tr}_\Lambda \mathrm{Hess}_u(\mathbf{x})} \leq \frac{1}{\|df_0\|}.$$

The supremum of the left hand side over all admissible functions u equals $\mathcal{F}_\Omega(\mathbf{x}, \Lambda)$, while the infimum of the right hand side over all conformal harmonic discs f as above equals $\mathcal{M}_\Omega(\mathbf{x}, \Lambda)$. Hence, $\mathcal{F}_\Omega \leq \mathcal{M}_\Omega$.

Sketch of proof of the theorem on complete hyperbolicity

We use the above proposition with MPSH function of the form

$$\Psi(\mathbf{y}) = \theta\left(r^{-2}|\mathbf{y}-\mathbf{x}|^2\right) e^{\lambda u(\mathbf{y})}, \quad \mathbf{y} \in \Omega,$$

where $\theta : [0,\infty) \to [0,1]$ is a smooth increasing function such that

$$\theta(t) = t \ \text{ for } 0 \le t \le \frac{1}{2}, \qquad \theta(t) = 1 \ \text{ for } t \ge 1,$$

u is a strongly MPSH defining functions for Ω, $\mathbf{x} \in \Omega$, and $r > 0$ and $\lambda > 0$ are suitably chosen constants. In this way, we show that

$$g_\Omega(\mathbf{x},\mathbf{v}) \ge C\frac{|\mathbf{v}|}{\sqrt{\mathrm{dist}(\mathbf{x}, b\Omega)}}, \quad \mathbf{x} \in \Omega, \ \mathbf{v} \in \mathbb{R}^n.$$

However, to show completeness of g_Ω we need a stronger estimate

$$g_\Omega(\mathbf{x},\mathbf{v}) \ge C\frac{|\mathbf{v}|}{\mathrm{dist}(\mathbf{x}, b\Omega)} \tag{1}$$

for vectors $\mathbf{v}$ which are normal to $b\Omega$ at the closest point $\mathbf{p} \in b\Omega$ to $\mathbf{x}$.

Sketch of proof, 2

We follow Ivashkovich and Rosay (2004). The existence of a local negative strongly MPSH peak function, and also of the MPSH anti-peak functions $\log|\mathbf{x} - \mathbf{p}|$, at points $\mathbf{p} \in b\Omega$ implies that for some $c > 0$ we have

$$|\nabla f(z)| \le c\sqrt{|u(f(0))|} \approx \sqrt{\mathrm{dist}(f(0), b\Omega)}, \quad |z| \le \frac{1}{2} \tag{2}$$

for every $f \in \mathrm{CH}(\mathbb{D}, \Omega)$ whose centre $f(0)$ is close enough to $b\Omega$. (This amounts to a **localization argument**, showing that most of the disc is mapped by f close to $f(0)$, and then applying the Schwarz lemma for bounded harmonic functions.) This gives

$$\begin{aligned} |\Delta(u \circ f)(z)| &= |\mathrm{tr}_{df_z(\mathbb{R}^2)}\mathrm{Hess}_u(f(z))| \cdot \|df_z\|^2 \\ &\le c_1|\nabla f(z)|^2 \le C_1|u(f(0))|, \quad |z| \le \frac{1}{2} \end{aligned}$$

for some constant $c_1 > 0$ and $C_1 = c^2 c_1 > 0$. We claim that this gives

$$|\nabla(u \circ f)(0)| \le C_2|u(f(0))|, \quad f \in \mathrm{CH}(\mathbb{D}, \Omega), \tag{3}$$

which implies (1) and hence establishes complete hyperbolicity of Ω.

Proof of (3)

By rescaling we may assume that (2) holds for all $z \in \mathbb{D}$.
Set $v = u \circ f : \mathbb{D} \to (-\infty, 0)$, so we have that

$$|\Delta v(z)| \leq C_1 |v(0)|, \qquad z \in \mathbb{D}.$$

We extend Δv to $\mathbb{C}$ by setting it equal to 0 on $\mathbb{C} \setminus \overline{\mathbb{D}}$. The function

$$g(z) = v(z) - \Big(\frac{1}{2\pi} \log|\cdot| * \Delta v\Big)(z) - C_1|v(0)|, \quad z \in \mathbb{D}$$

is then harmonic on $\mathbb{D}$. Note that

$$\left|\frac{1}{2\pi} \log|\cdot| * \Delta v\right| \leq C_1|v(0)|.$$

Hence, $g \leq v < 0$ on $\mathbb{D}$ and $|g(0)| < (2C_1 + 1)|v(0)|$. Schwarz lemma for negative harmonic functions gives $|\nabla g(0)| \leq 2|g(0)|$, and hence

$$|\nabla v(0)| \leq |\nabla g(0)| + \sup_{\mathbb{D}} |\Delta v| \leq 2|g(0)| + C_1|v(0)| \leq (5C_1 + 2)|v(0)|.$$

This is the estimate (3) with $C_2 = 5C_1 + 2$.

What about unbounded domains?

Proposition

If $\Omega \subset \Omega' \subset \mathbb{R}^n$ are (not necessarily bounded) domains such that Ω' is complete hyperbolic and Ω is strongly minimally convex, then Ω is also complete hyperbolic.

Corollary

For any $a < 1$ and $b \in \mathbb{R}$ the domain

$$\Omega_{a,b} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < az^2 + b\}, \quad 0 \leq a < 1, \ b \in \mathbb{R}$$

is complete hyperbolic. Hence, every strongly minimally convex domain contained in $\Omega_{a,b}$ is complete hyperbolic.

∼ Thank you for your attention ∼