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PROPER HOLOMORPHIC MAPS FROM BALLS

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1. Introduction and statement of the results. If $\Gamma \subset \mathrm{U}(n)$ is a finite unitary group, the quotient $\mathbf{C}^n/\Gamma$ can be realized as a normal algebraic subvariety V in some $\mathbf{C}^s$ according to a theorem of Cartan [4]. In order to do this we choose a finite number of homogeneous Γ-invariant holomorphic polynomials $q_1, \ldots, q_s$ that generate the algebra of all Γ-invariant polynomials [17]; the induced map $Q = (q_1, \ldots, q_s): \mathbf{C}^n \to \mathbf{C}^s$ is proper and induces a homeomorphism of $\mathbf{C}^n/\Gamma$ onto the image $V = Q(\mathbf{C}^n)$. The restriction of Q to the unit ball $\mathbf{B}^n$ maps the ball properly onto a domain G in V.

Rudin proved a partial converse to this [22]: If $f: \mathbf{B}^n \to G$ is a proper holomorphic map from the ball onto a domain in $\mathbf{C}^n$, $n \geqslant 2$, that extends to a $\mathbf{C}^1$ map on $\overline{\mathbf{B}}^n$, then there are a finite unitary group Γ and an automorphism ϕ of $\mathbf{B}^n$ such that $f = \eta \circ Q \circ \phi$, where $Q: \mathbf{B}^n \to \mathbf{B}^n/\Gamma$ is the quotient projection and $\eta: \mathbf{B}^n/\Gamma \to G$ is a biholomorphic map. The group Γ is generated by reflections, i.e., elements of finite order which fix a complex hyperplane. A result of Bedford and Bell [2] implies the same result even when f does not extend to the closure of $\mathbf{B}^n$; moreover, we may replace G by an arbitrary normal complex space of dimension n. See also [19]. The quotient $\mathbf{C}^n/\Gamma$ is nonsingular if and only if the group Γ is generated by reflections, i.e., elements of finite order in $\mathrm{U}(n)$ that fix a complex hyperplane [12, 20, 22]. The boundary of the image G is never smooth in this case [22].

In this paper we shall study the structure of proper maps from balls into strictly pseudoconvex domains G in complex manifolds. A finite unitary group $\Gamma \subset \mathrm{U}(n)$ is call *fixed point free* if 1 is not the eigenvalue of any $\gamma \in \Gamma \backslash \{1\}$. Equivalently, Γ is fixed point free if it acts without fixed points on $\mathbf{C}^n \backslash \{0\}$.

1.1. THEOREM. *Let $f: \mathbf{B}^n \to G$, $n \geqslant 2$, be a proper holomorphic map into a relatively compact, strictly pseudoconvex domain G in a complex manifold. If f extends to a $\mathbf{C}^1$ map on $\overline{\mathbf{B}}^n$, then there exist a finite fixed point free unitary group Γ and an automorphism ϕ of $\mathbf{B}^n$ such that*

$$f = \eta \circ Q \circ \phi, \tag{1.1}$$

where $Q: \mathbf{B}^n \to \mathbf{B}^n/\Gamma$ is the quotient projection and $\eta: \mathbf{B}^n/\Gamma \to f(\mathbf{B}^n)$ is the normalization of the subvariety $f(\mathbf{B}^n)$ of G.

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Conversely, given a finite unitary group $\Gamma \subset \mathrm{U}(n)$, let $q_1, \ldots, q_s$ be a basis of invariants. The associated polynomial map $Q = (q_1, \ldots, q_s): \mathbf{C}^n \to \mathbf{C}^s$ induces a biholomorphism of the quotient $\mathbf{C}^n/\Gamma$ onto the algebraic subvariety $V = Q(\mathbf{C}^n)$ of $\mathbf{C}^s$. If Γ is fixed point free, the point $Q(0) = 0$ is the only possible singularity of V, and therefore the image $Q(\mathbf{B}^n) = \Omega$ of the ball is a strictly pseudoconvex domain with real-analytic boundary in the Stein space V. Hence there is a bounded, strictly convex domain G with real-analytic boundary in some $\mathbf{C}^N$ and a proper holomorphic embedding $\eta: V \to \mathbf{C}^N$ that maps Ω properly into G [10, 11]. The composition $f = \eta \circ Q: \mathbf{C}^n \to \mathbf{C}^N$ maps $\mathbf{B}^n$ properly into G and satisfies (1.1). This shows that every fixed point free unitary group arises in the context of Theorem 1.1.

At this point a natural question is what are the fixed point free unitary groups. It is more appropriate to talk about the unitary representations of finite groups. Recall that a *unitary representation* of an abstract finite group Γ is a homomorphism $\pi: \Gamma \to \mathrm{U}(n)$. The number n is called the *degree* of π. (See [26, Chapter 4] for a concise account of the theory of representations of finite groups.) A representation $\pi: \Gamma \to \mathrm{U}(n)$ is called *fixed point free* if $0 \in \mathbf{C}^n$ is the only fixed point of any $\pi(\gamma)$, $\gamma \in \Gamma \setminus \{1\}$. There exists a complete classification of such groups which was carried out in order to solve the Clifford–Klein *spherical space form problem.* A beautiful exposition of this can be found in Chapters 5–7 of Wolf's book [26]. If $\pi: \Gamma \to \mathrm{O}(n)$ is a fixed point free orthogonal representation, then the quotient of any sphere $S_r = \{x \in \mathbf{R}^n : \|x\| = r\}$ modulo Γ is a *spherical space form*, i.e., a complete connected Riemannian manifold with constant positive Gauss curvature $K = 1/r^2$. Conversely, every spherical space form is obtained in this way [26, Theorem 5.1.2]. Every unitary representation is also orthogonal under the standard identification of $\mathbf{C}^n$ with $\mathbf{R}^{2n}$. Thus, if $\pi: \Gamma \to \mathrm{U}(n)$ is a fixed point free unitary representation, the quotient $\mathbf{B}^n/\pi(\Gamma)$ is a complex space with smooth boundary $b\mathbf{B}^n/\pi(\Gamma)$ that is a spherical space form.

The condition that a finite group Γ admit a fixed point free representation imposes severe restrictions on its structure. In particular, all Sylow p-subgroups of Γ for odd p are cyclic; the Sylow 2-subgroups are either cyclic (groups of type A) or generalized quaternionic (groups of type B) [26, p. 161]. Fixed point free representations of groups of type A can be found in [26, p. 168]. Groups of type B are subdivided into five subtypes, and their fixed point free representations are described in Section 7.2 of [26]. The finite subgroups of SU(2), which are all fixed point free, play a special role in the representation theory of groups of type B. Besides the cyclic groups, SU(2) contains binary dihedral and binary polyhedral groups which arise as the preimages of the rotation groups of regular solids in $\mathbf{R}^3$ under the two-to-one homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Each finite subgroup of SU(2) has a basis of invariants q_1, q_2, q_3 satisfying one relation; these invariants can be found in Klein's book [13, pp. 50–63]. They are also treated in Chapter 4 of [24]. We do not know of any systematic treatment of the invariant theory of other fixed point free groups. In contrast, the problem of invariants is satisfactorily solved for finite reflection groups. (See [24] and the references in [22].)

A fixed point free representation $\pi : \Gamma \to \mathrm{U}(n)$ completely determines the biholomorphic type of $\mathsf{B}^n/\pi(\Gamma)$. Recall that two representations $\pi_1, \pi_2 : \Gamma \to \mathrm{U}(n)$ are *equivalent* if there exists an $A \in \mathrm{GL}(\mathsf{C}^n)$ such that

$$A \circ \pi_1(\gamma) = \pi_2(\gamma) \circ A \qquad \text{for all} \quad \gamma \in \Gamma.$$

A result of Prill [20, p. 382] implies that the quotients $\mathsf{B}^n/\pi_1(\Gamma)$ and $\mathsf{B}^n/\pi_2(\Gamma)$ are biholomorphically equivalent if and only if there exists an automorphism $h : \Gamma \to \Gamma$ such that the representation π_1 is equivalent to $\pi_2 \circ h$.

In view of Theorem 1.1 we may ask further which fixed point free groups arise from proper maps of balls $f : \mathsf{B}^n \to \mathsf{B}^N$, $N > n$. Since every strictly pseudoconvex domain can be embedded holomorphically into a ball [11, 15], every finite fixed point free unitary group Γ arises in this way if we do not require that the map f is smooth on $\bar{\mathsf{B}}^n$. On the other hand, we shall show that in general there is no Γ-invariant proper *rational* map into a ball.

1.2. THEOREM. *Let $\pi : \Gamma \to U(n)$, $n \geqslant 2$, be a fixed point free representation. If there exists a $\pi(\Gamma)$-invariant rational proper map $f : \mathsf{B}^n \to \mathsf{B}^N$ for some integer N, then the group Γ is of type A, i.e., all of its Sylow subgroups are cyclic, and the irreducible fixed point free representations of Γ are of odd degree. If $n = 2^k$ for some integer k, then the group Γ is cyclic.*

Except for some examples involving certain representations of cyclic groups (see Section 3) we do not know which groups of type A actually arise in this connection. It would be of interest to pursue this question at least in the case when the codimension $N - n$ is sufficiently low. Also we do not know what the answer is if we require that the map f be of class $\mathbf{C}^p$ on $\bar{\mathsf{B}}^n$ for some $p > 0$. (The result of Løw [15] implies that f can be made continuous on $\bar{\mathsf{B}}^n$.)

Let $f : \mathsf{B}^n \to \mathsf{B}^N$, $n \geqslant 2$, be a proper holomorphic map. Two such maps f_1, f_2 will be called *equivalent* if there exist automorphisms ϕ (resp. ψ) of B^n (resp. B^N) such that $f_1 \circ \phi = \psi \circ f_2$. If f extends to a $\mathbf{C}^2$ map on $\bar{\mathsf{B}}^n$ and $N = n + 1$, then f is rational; If $n \geqslant 3$, f is equivalent to the embedding $z \to (z, 0)$ [25, 6]. There are precisely four nonequivalent rational proper maps of B^2 into B^3 [8, p. 441]. Two of these maps have nontrivial structure groups: $f(z, w) = (z^2, \sqrt{2}\, zw, w^2)$ is invariant with respect to the cyclic two-group generated by the central inversion $(z, w) \to -(z, w)$, and $g(z, w) = (z^3, \sqrt{3}\, zw, w^3)$ is invariant with respect to the cyclic three-group generated by the linear map with the matrix $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^2 \end{pmatrix}$, where $\epsilon = \exp(2\pi i/3)$. We shall give some other examples of such embeddings in Section 3 below.

We shall use the idea of Cima and Suffridge [6] and apply induction to prove

1.3. THEOREM. *Let U be a neighborhood of a point $p \in b\mathsf{B}^n$ in C^n such that $\mathsf{B}^n \cap U$ is connected, and let $f : U \to \mathsf{C}^N$ be a $\mathbf{C}^{N-n+1}$ map that is holomorphic and* *nonconstant on* $\mathsf{B}^n \cap U$, $f(\mathsf{B}^n \cap U) \subset \mathsf{B}^N$, *and* $f(b\mathsf{B}^n \cap U) \subset b\mathsf{B}^N$. *If* $n \geqslant 2$ *and* $N \leqslant 2n - 1$, *then* f *extends to a rational map on* $\mathbf{C}^n$ *that is holomorphic on* B^n *and maps* B^n *into* B^N.

This theorem, together with a result of Faran [9], implies

1.4. COROLLARY. *Let* $f: \mathsf{B}^n \to \mathsf{B}^N$ *be a proper holomorphic map that is of class* $\mathbf{C}^{N-n+1}$ *on* $\overline{\mathsf{B}}^n$. *If* $n \geqslant 2$ *and* $N \leqslant 2n - 2$, *then there are automorphisms* ϕ *of* B^n *and* ψ *of* B^N *such that for all* $(z_1, \ldots, z_n) \in \mathsf{B}^n$

$$\psi \circ f \circ \phi(z_1, \ldots, z_n) = (z_1, \ldots, z_n, 0, \ldots, 0).$$

An example shows that Corollary 1.4 is false for $N = 2n - 1$ [6, p. 499]. It would be interesting to know whether Corollary 1.4 holds under weaker smoothness assumption on f, e.g., when f is of class $\mathbf{C}^2$ on $\overline{\mathsf{B}}^n$.

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2. Proof of Theorem 1.1. Although the existence of a group Γ satisfying (1.1) follows from the work of Bedford and Bell [2], there is a much simpler proof in our case. Since f is of class $\mathbf{C}^1$ on $\overline{\mathsf{B}}^n$ and both $b\mathsf{B}^n$ and bG are strictly pseudoconvex, the Hopf lemma implies that f has maximal rank n at every point of $b\mathsf{B}^n$ [10, p. 549; 18, p. 378]. Thus the branching locus L of f is a compact subvariety of B^n and therefore finite [21, p. 294]. The image $f(\mathsf{B}^n)$ is a subvariety of D according to a theorem of Remmert [16, p. 129]. Let $\eta: V \to f(\mathsf{B}^n)$ be its normalization [16, p. 114], and let $F: \mathsf{B}^n \to V$ be the induced map such that $f = \eta \circ F$. The branching locus L of F is also finite. Denote by W the subvariety $F(L) \cup V_{\text{sing}}$ of V. The restriction

$$F: \mathsf{B}^n \backslash F^{-1}(W) \to V \backslash W \tag{2.1}$$

is a nondegenerate proper map of complex manifolds and therefore a finite covering projection. Since L is finite and the complex codimension of V_{sing} in V is at least two [16, p. 115], the preimage $F^{-1}(W)$ also has codimension at least two in B^n. Thus $\mathsf{B}^n \backslash W$ is simply connected [1, p. 355], and hence (2.1) is the universal covering projection. Therefore the group Γ of deck transformations of (2.1) acts transitively on each fiber. Each $\gamma \in \Gamma$ extends holomorphically across W to an automorphism of B^n, and these extensions form a group $\Gamma \subset \operatorname{Aut}(\mathsf{B}^n)$. The map F factors as $h \circ Q$, where $Q: \mathsf{B}^n \to \mathsf{B}^n/\Gamma$ is the quotient projection. Outside a proper subvariety of B^n/Γ the map h is one-to-one and onto; since both spaces B^n/Γ and V are normal, it follows that h is biholomorphic. Thus we can replace η by $\eta \circ h$ and take $\eta: \mathsf{B}^n/\Gamma \to f(\mathsf{B}^n)$ to be the normalization of $f(\mathsf{B}^n)$.

Thus far we proved that $f = \eta \circ Q$, where Q is the quotient projection $\mathsf{B}^n \to \mathsf{B}^n/\Gamma$ for a finite subgroup $\Gamma \subset \operatorname{Aut}(\mathsf{B}^n)$. There is a common fixed point

$a \in \mathbf{B}^n$ of all $\gamma \in \Gamma$ [22]; interchanging 0 and a by an automorphism ϕ of $\mathbf{B}^n$ maps Γ into a finite unitary group, and we have $f = \eta \circ Q \circ \phi$.

It remains to show that the group Γ is fixed point free. Since the branching locus of f is finite, the branching locus L_Q of the quotient projection $Q : \mathbf{B}^n \to \mathbf{B}^n/\Gamma$ is also finite. On the other hand, Q is branched on the set $H_\gamma = \{z \in \mathbf{C}^n \mid \gamma(z) = z\}$ for every $\gamma \in \Gamma \setminus \{1\}$ because Q is not locally one–one near H_γ. Since H_γ is a linear subspace of $\mathbf{C}^n$, finiteness implies $H_\gamma = \{0\}$ for every $\gamma \in \Gamma \setminus \{1\}$. This means that 0 is the only fixed point of any $\gamma \in \Gamma \setminus \{1\}$. Theorem 1.1 is proved.

3. Proper maps of balls and fixed point free groups. The main step in the proof of Theorem 1.2 is the following

3.1. PROPOSITION. *If Γ_k is the cyclic subgroup of $U(2)$ generated by the matrix $A_k = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix}$, where $\epsilon = e^{\pi i/2k}$, then there exist no proper Γ_k-invariant rational maps from $\mathbf{B}^2$ to any finite dimensional ball.*

Assuming this for a moment we shall prove Theorem 1.2. Since every fixed point free representation $\pi : \Gamma \to \mathrm{U}(n)$ is a direct sum of irreducible representations of the same degree m, it follows that m divides n. Thus, if $n = 2^k$ and m is odd, we have $m = 1$, and hence Γ is cyclic. This proves the last assertion of Theorem 1.2 provided that the rest of the theorem holds.

To prove the first part of the theorem we may assume that the given representation $\pi : \Gamma \to \mathrm{U}(n)$ is irreducible. We shall identify Γ with its image $\pi(\Gamma) \subset \mathrm{U}(n)$. If Γ' is a subgroup of Γ, then every Γ-invariant map is also Γ'-invariant. Moreover, if Σ is a complex linear subspace of $\mathbf{C}^n$ that is Γ'-invariant, then the restriction of f to Σ is also Γ'-invariant. To prove that there is no rational, proper, Γ-invariant map $f : \mathbf{B}^n \to \mathbf{B}^N$ it therefore suffices, in view of Proposition 3.1, to find a subgroup Γ' of Γ and a two-dimensional complex subspace $\Sigma \subset \mathbf{C}^n$ such that the action of Γ' on Σ is equivalent to the action of the group Γ_k on $\mathbf{C}^2$.

Suppose first that Γ is of type A, i.e., all of its Sylow subgroups are cyclic. The general form of such groups is given in [26, p. 168]. If Γ is not cyclic, it contains a cyclic subgroup generated by a matrix of the form

$$U = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ \delta & & & & 0 \end{bmatrix},$$

where δ is a root of 1. The eigenvalues of U are $\delta^{1/n}, \delta^{1/n}\zeta, \ldots, \delta^{1/n}\zeta^{n-1}$, where ζ is a primitive nth root of 1. If n is even, $n = 2r$, then U^r has eigenvalues $\delta^{1/2}, \delta^{1/2}\zeta^r = -\delta^{1/2}, \ldots$. Hence there is a 2-dimensional complex subspace Σ

of $\mathbf{C}^n$ in which U^r acts as the matrix $T = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix}$, with $\epsilon = \delta^{1/2}$. The group generated by T is fixed point free if and only if ϵ is a primitive $4k$th root of 1 for some integer k. Hence we obtained the group Γ_k. Proposition 3.1 implies that there is no Γ-invariant map $\mathbf{B}^n \to \mathbf{B}^N$. Thus the degree of irreducible representations of Γ is odd in this case.

If Γ is of type B, its Sylow 2-subgroups are isomorphic to a generalized quaternionic group $Q2^a$ given by [26, p. 171, Lemma 5.6.2]. Each irreducible representation of $Q2^a$ contains a cyclic subgroup generated by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Its eigenvalues are $\pm i$, and hence this subgroup is conjugate to the group Γ_1. Proposition 3.1 implies that there is no Γ-invariant map $\mathbf{B}^n \to \mathbf{B}^N$.

This shows that Theorem 1.2 will be proved when we prove Proposition 3.1.

Proof of Proposition 3.1. Recall that $\Gamma_k \subset \mathrm{U}(2)$ is the group of order $4k$ generated by the matrix $\begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix}$, $\epsilon = e^{i\pi/2k}$. The proof of the Hilbert basis theorem due to Noether [17] shows that the polynomials $M_{\alpha,\beta}(z) = \sum_{\gamma \in \Gamma} m_{\alpha,\beta} \circ \gamma(z)$, where $m_{\alpha,\beta}(z)$ is the monomial $z_1^\alpha z_2^\beta$ of order $\alpha + \beta \leqslant |\Gamma_k| = 4k$, generate the algebra of holomorphic Γ_k-invariant polynomials. We have

$$M_{\alpha,\beta}(z) = \sum_{j=1}^{4k} \epsilon^{\alpha j}(-\epsilon)^{\beta j} z_1^\alpha z_2^\beta = \left(\sum_{j=1}^{4k} \delta^j \right) z_1^\alpha z_2^\beta,$$

where $\delta = \epsilon^\alpha(-\epsilon)^\beta$ is a $4k$th root of one. Clearly $\sum_{j=1}^{4k} \delta^j = 0$ unless $\delta = 1$ which happens if and only if (α, β) equals: $(4k, 0), (2k-1, 1), \ldots, (1, 2k-1), (0, 4k)$; $(4k-2, 2), \ldots, (2, 4k-2)$. It follows that the monomials

$$q_1 = z_1^{4k}, \qquad q_2 = z_1^{2k-1} z_2, \qquad q_2 = z_1^{2k-3} z_2^3, \ldots, q_k = z_1 z_2^{2k-1}, \qquad q_{k+1} = z_2^{4k} \tag{3.1}$$

generate the algebra of Γ_k-invariant polynomials.

Suppose that $f = (P_1/_q, \ldots, P_s/_q)$ is a proper rational map from $\mathbf{B}^2$ into $\mathbf{B}^s$. Composing f with an automorphism of $\mathbf{B}^s$ we may assume that $f(0) = 0$. We may also assume that the constant term of q equals 1. Thus

$$P_i(z) = \sum_{j=1}^{d} p_{i,j}(z) \quad \text{and} \quad q(z) = 1 + \sum_{j=1}^{d'} q_j(z),$$

where $p_{i,j}$ and q_j are homogeneous polynomials of degree j. We choose d and d' such that $q_{d'}$ and at least one of $p_{i,d}$'s are nonzero. Since $f\colon \mathbf{B}^2 \to \mathbf{B}^s$ is proper, we have

$$\sum_{i=1}^{s} P_i(z)\overline{P_i(z)} = q(z)\overline{q(z)} \qquad \text{if} \quad z_1\bar{z}_1 + z_2\bar{z}_2 = 1. \tag{3.2}$$

We claim that (3.2) implies $d' < d$. Fix a point $z \in b\mathbf{B}^2$ for which $q_{d'}(z) \neq 0$ and consider the equation $\sum_{i=1}^{s} |P_i(\lambda z)/q(\lambda z)|^2 = 1$ which holds on the circle $|\lambda| = 1$.

Rewrite this as

$$\sum_{i=1}^{s} P_i(\lambda z)\overline{P_i(\lambda z)} = q(\lambda q)\overline{q(\lambda z)}. \tag{3.3}$$

The right hand side of (3.3) is a polynomial in λ and $\bar{\lambda}$ of degree d'. On the other hand, since P_i contains no constant term and $\lambda\bar{\lambda} = 1$, the left hand side is a polynomial of degree at most $d-1$ in λ and $\bar{\lambda}$. Since the higher powers of λ and $\bar{\lambda}$ are independent of the lower powers on the circle $|\lambda| = 1$, the degrees on both sides must agree, and hence $d' \leqslant d-1$ as claimed.

We let $\bar{z}_1 = w_1$ and $\bar{z}_2 = w_2$ be independent variables. From (3.2) it follows that

$$\sum_{i=1}^{s} P_i(z)\bar{P}_i(w) = q(z)\bar{q}(w) \qquad \text{if} \quad z_1w_1 + z_2w_2 = 1, \tag{3.4}$$

where $\bar{f}(w) = \overline{f(\bar{w})}$. If we substitute $w_2 = (1 - z_1w_1)/z_2$ into (3.4), we obtain an identity among the independent variables z_1, z_2, and w_1. Notice that the terms of degree l in (3.4) give rise only to terms of degree not exceeding l after the substitution. Since q has no terms of maximal degree d, we have

$$\sum_{i=1}^{s} p_{i,d}(z)\bar{p}_{i,d}(w) = (\text{terms of degree} < 2d). \tag{3.5}$$

Consider a typical term $c_{\alpha,\beta}z_1^{\alpha}z_2^{\beta}\bar{c}_{\gamma,\delta}w_1^{\gamma}w_2^{\delta}$ of maximal degree in $p_{i,d}(z)\bar{p}_{i,d}(w)$; hence $\alpha + \beta = \gamma + \delta = d$. We substitute $w_2 = (1 - z_1w_1)/z_2$ and consider the terms of maximal degree $2d$ of the form const $\cdot\, z_1^dw_1^d$. We only obtain such terms when $\alpha = \gamma$ and $\beta = \delta$; the corresponding term is

$$|c_{\alpha,\beta}|^2(z_1w_1)^{\alpha}(-z_1w_1)^{\beta} = (-1)^{\beta}|c_{\alpha,\beta}|^2(z_1w_1)^d. \tag{3.6}$$

Suppose now that the map f is Γ_k-invariant. We may assume that q and the P_i's are Γ_k-invariant [24, p. 73]; hence each monomial $z_1^{\alpha}z_2^{\beta}$ that appears in one of the P_j's is a product of the monomials (3.1):

$$z_1^{\alpha}z_2^{\beta} = \left(z_1^{4k}\right)^{\alpha_1}\left(z_1^{2k-1}z_2\right)^{\alpha_2} \cdots \left(z_1z_2^{2k-1}\right)^{\alpha_k}\left(z_2^{4k}\right)^{\alpha_{k+1}}.$$

This implies $d = 4k\alpha_1 + 2k\alpha_2 + \cdots + 2k\alpha_k + 4k\alpha_{k+1}$ and $\beta = \alpha_2 + 3\alpha_3 + \cdots + (2k-1)\alpha_k + 4k\alpha_{k+1}$. It follows that

$$\beta = \frac{d}{2k} + 2(-\alpha_1 + 2\alpha_2 + \cdots + (2k-1)\alpha_{k+1}),$$

which shows that β is determined by d modulo 2. Thus $(-1)^{\beta}$ only depends on d; hence the coefficients of all terms (3.6) in (3.5) are either all positive or all negative, depending on d. This shows that the left side of (3.5) contains a nonzero term const$(z_1w_1)^d$ which is a contradiction. This proves Proposition 3.1.

We shall conclude this section by two examples of fixed point free groups $\Gamma \subset \mathrm{U}(n)$ for which there is a proper precisely Γ-invariant polynomial map from

B^n into a higher dimensional ball. It seems that these are the only presently known examples of such embeddings.

Example 1. Let C_k be a cyclic group of order $k \geqslant 2$. Choose a generator A of C_k, and let $\pi : C_k \to \mathrm{U}(n)$ be the representation given by

$$\pi(A) = \begin{bmatrix} \epsilon & & & \\ & \epsilon & & \\ & & \ddots & \\ & & & \epsilon \end{bmatrix}, \qquad \epsilon = e^{2\pi i/k}.$$

The monomials $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ of order $|\alpha| = \alpha_1 + \cdots + \alpha_n = k$ form a basis of invariants of $\pi(C_k)$. If we let

$$c_\alpha = \left(\frac{k!}{\alpha_1! \cdots \alpha_n!} \right)^{1/2} > 0,$$

then the map $f : \mathbf{C}^n \to \mathbf{C}^N$, whose components are the monomials $c_\alpha z^\alpha$ for all $|\alpha| = k$, induces a proper holomorphic embedding of $\mathsf{B}^n / \pi(C_k)$ to B^N. Here N is the number of all such monomials. In the case when $n = 2$ we obtain the following maps:

$$k = 2, \qquad f_2(x, y) = \left(x^2, \sqrt{2}\, xy, y^2\right) : \mathsf{B}^2 \to \mathsf{B}^3,$$

$$k = 3, \qquad f_3(x, y) = \left(x^3, \sqrt{3}\, x^2 y, \sqrt{3}\, xy^2, y^3\right) : \mathsf{B}^2 \to \mathsf{B}^4,$$

$$k = 4, \qquad f_4(x, y) = \left(x^4, 2x^3 y, \sqrt{6}\, x^2 y^2, 2xy^3, y^4\right) : \mathsf{B}^2 \to \mathsf{B}^5,$$

etc. The components of these maps are homogeneous polynomials of the same degree k. Rudin proved [23] that these are essentially the only proper maps of balls whose components are linearly independent homogeneous polynomials of the same degree.

Example 2. Let $k = 2r + 1$, $\epsilon = e^{2\pi i/k}$, and let $\tau : C_k \to \mathrm{U}(2)$ be the representation

$$\tau(A) = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^2 \end{pmatrix}.$$

A basis of invariants is

$$x^{2r+1}, x^{2r-1}y, x^{2r-3}y^2, \ldots, xy^r, y^{2r+1}.$$

We denote $|x|^2 = a$ and $|y|^2 = b$. Then $a + b = 1$ for $(x, y) \in b\mathsf{B}^2$. We can expand $(a + b)^{2r+1}$ in the form

$$(a + b)^{2r+1} = a^{2r+1} + b^{2r+1} + \sum_{j=1}^{r} \alpha_j a^{2(r-j)+1} b^j (a + b)^j \tag{3.7}$$

for some uniquely determined real constants α_k. If the α_j's are all nonnegative, the map

$$f_r(x, y) = \left(x^{2r+1}, \sqrt{\alpha_1}\, x^{2r-1}y, \ldots, \sqrt{\alpha_r}\, xy^r, y^{2r+1}\right)$$

is precisely $\tau(C_k)$-invariant and maps $\mathbf{B}^2$ properly to $\mathbf{B}^{r+2}$. To see this, observe that on the sphere $\{|x|^2 + |y|^2 = 1\}$ we have

$$|f_r(x, y)|^2 = a^{2r+1} + b^{2r+1} + \sum_{j=1}^{r} |\alpha_j| a^{2(r-j)+1} b^j$$

which equals 1 in view of (3.7). In the special cases $r = 1, 2, 3, 4$ we get the maps

$$r = 1, \qquad f_1(x, y) = \left(x^3, \sqrt{3}\, xy, y^3\right) : \mathbf{B}^2 \to \mathbf{B}^3,$$

$$r = 2, \qquad f_2(x, y) = \left(x^5, \sqrt{5}\, x^3y, \sqrt{5}\, xy^2, y^5\right) : \mathbf{B}^2 \to \mathbf{B}^4,$$

$$r = 3, \qquad f_3(x, y) = \left(x^7, \sqrt{7}\, x^5y, \sqrt{14}\, x^3y^2, \sqrt{7}\, xy^3, y^7\right) : \mathbf{B}^2 \to \mathbf{B}^5,$$

$$r = 4, \qquad f_4(x, y) = \left(x^9, 3x^7y, \sqrt{27}\, x^5y^2, \sqrt{30}\, x^3y^3, 3xy^4, y^9\right) : \mathbf{B}^2 \to \mathbf{B}^6.$$

We believe that such a map f_r exists for each integer r. The first three of these maps can be found in [6, p. 500], without reference to fixed point free groups.

4. Proper maps of balls in low codimensions are rational. In this section we shall prove Theorem 1.3. For each point $z \in \mathbf{C}^n$, $z \neq 0$, we denote by $z^{\perp}$ the complex hyperplane in $\mathbf{C}^n$ orthogonal to z:

$$z^{\perp} = \left\{ w \in \mathbf{C}^n \mid z \cdot \overline{w} = \sum_{j=1}^{n} z_j \overline{w}_j = 0 \right\}. \tag{4.1}$$

Let $f\colon U \to \mathbf{C}^N$ be as in Theorem 1.3. We denote by $D^j f(z)$ the Fréchet derivative of f of order j at the point $z \in U$. (See [5, Chapter 5].) Recall that $D^1 f(z)$ is just the complex Jacobian of f at z. We denote by $P_k(\cdot, z_0)$ the Taylor polynomial of order k of f at the point $z_0 \in U$. Since f is smooth on U and holomorphic on $U \cap \mathbf{B}^n$, the components of $P_k(\cdot, z)$ are holomorphic polynomials for each $z \in U \cap \overline{\mathbf{B}}^n$. A special case of the following lemma was proved in [6].

4.1. LEMMA. *Let $f\colon U \to \mathbf{C}^n$ be as in Theorem* 1.3, *and let $z_0 \in b\mathbf{B}^n \cap U$. Put $s = N - n + 1$. Then the following hold*:

(i) *The Taylor polynomial $P_s(\cdot, z_0)$ maps the affine hyperplane $z_0 + z_0^{\perp} \subset \mathbf{C}^n$ into the affine hyperplane $f(z_0) + f(z_0)^{\perp} \subset \mathbf{C}^N$.*

(ii) *If $v_1, \ldots, v_j$ is any set of vectors in $z_0^{\perp}$, $j \leqslant s$, then*

$$D^j f(z_0)(v_1, \ldots, v_j) \in f(z_0)^{\perp}.$$

(iii) *If f is holomorphic in U, then*

$$f((z_0 + z_0^\perp) \cap U) \subset f(z_0) + f(z_0)^\perp.$$

Proof. We want to prove that $z \cdot \bar{z}_0 = 1$ implies $P(z) \cdot \overline{P(z_0)} = 1$, where $P(z) = P_s(z, z_0)$. The complex submanifold

$$\Sigma = \left\{(z,w) \in \mathbf{C}^n \times \mathbf{C}^n \mid z \cdot w = \sum z_j w_j = 1\right\}$$

is the complexification of the real-analytic submanifold

$$S = \left\{(z,\bar{z}) \in \mathbf{C}^n \times \mathbf{C}^n \mid \sum |z_j|^2 = 1\right\} = \left\{(z,\bar{z}) \mid z \in \mathbf{C}^n\right\} \cap \Sigma.$$

The function $G(z,w) = f(z) \cdot \overline{f(\overline{w})} - 1$, defined on a neighborhood of the point $(z_0, \bar{z}_0) \in \mathbf{C}^{2n}$, is of class $\mathbf{C}^s$ and vanishes on S. The function $g(z,w) = P(z) \cdot \overline{P(\overline{w})} - 1$ is a holomorphic polynomial of order $2s$ in the variables (z,w). Since the derivatives of f and P at z_0 agree up to order s, the functions G and g have the same Taylor polynomial $T(z,w)$ of order s centered at $(z_0, \bar{z}_0)$.

Choose a system of local real-analytic coordinates $t = (t_1, \ldots, t_{2n-1})$ on S centered at $(z_0, \bar{z}_0)$ and consider the functions G and g in these coordinates. There derivatives of order at most s in t agree at $t = 0$ since this property does not depend on the choice of coordinates. Since $G(t) \equiv 0$, the t-derivatives of T up to order s all vanish at $t = 0$. If we let t take complex values, we get a system of local holomorphic coordinates on Σ centered at the point $(z_0, \bar{z}_0)$. Since T is holomorphic on Σ, its complex t-derivatives up to order s vanish at $t = 0$. The same is then true for g since T is its Taylor polynomial. Therefore the function

$$z \to g(z, \bar{z}_0) = P(z) \cdot \overline{P(z_0)} - 1, \qquad z \in z_0 + z_0^\perp$$

vanishes up to order s at $z = z_0$. But this is a polynomial of degree s in z, and hence $g(z, z_0)$ is identically zero for $z \in z_0 + z_0^\perp$. This means that P maps $z_0 + z_0^\perp$ into $f(z_0) + f(z_0)^\perp$, and part (i) is proved.

Part (ii) follows from (i) since the derivatives $D^j f(z_0)(v_1, \ldots, v_j)$ are linear combinations of coefficients of the vector-valued polynomial $z \to P_s(z, z_0)$.

If f is holomorphic near z_0, the function $G(z,w) = f(z) \cdot \overline{f(\overline{w})} - 1$ is holomorphic on $\mathbf{C}^{2n}$ near $(z_0, \bar{z}_0)$ and vanishes on S. Since Σ is a complexification of S, G vanishes also on Σ. Substituting $w = \bar{z}_0$ we see that $f(z) \cdot \overline{f(z_0)} = 1$ for $z \in z_0 + z_0^\perp$ near z_0, and (iii) is proved. This proves Lemma 4.1.

We shall also need the following lemma.

4.2. Lemma. *Let $f: U \to \mathbf{C}^k$ be a holomorphic map defined on an open connected subset U of $\mathbf{C}^n$, and let z_0 be a point in U. Denote, for each $z \in U$ and for each positive integer j, by V_z^j the linear subspace of $\mathbf{C}^k$ spanned by all partial derivatives $\partial^\alpha f / \partial z^\alpha(z)$ of order $|\alpha|$, $1 \leqslant |\alpha| \leqslant j$. If there is an integer j such that*

$V_z^{j+1} = V_z^j$ *for all* $z \in U$, *then the image* $f(U)$ *lies in the affine subspace* $f(z_0) + V_{z_0}^j$ *of* $\mathbf{C}^k$.

Proof. In view of the Taylor formula for f at the point z_0 we need to prove that $V_{z_0}^{j+l} = V_{z_0}^j$ for all $l \in \mathbf{Z}_+$. By induction on j it suffices to show that the hypothesis $V_z^{j+1} = V_z^j$ for all $z \in U$ implies that $V_z^{j+2} = V_z^j$ for all z close to z_0. Let the vectors

$$X_r(z) = \frac{\partial^{\alpha_r} f}{\partial z^{\alpha_r}}(z), \qquad 1 \leqslant r \leqslant \dim V_z^j, \quad 1 \leqslant |\alpha_r| \leqslant j$$

be a basis of V_z^j for all points z close to z_0, and let $X(z) = \partial^\alpha f/\partial z^\alpha(z)$ for some multiindex α, $|\alpha| = j + 1$. The hypothesis $V_z^{j+1} = V_z^j$ implies that there are smooth functions $a_r(z)$, $1 \leqslant r \leqslant \dim V_z^j$, such that $X(z) = \sum a_r(z) X_r(z)$. Differentiating with respect to z_i we have

$$\frac{\partial X}{\partial z_i}(z) = \sum_r \frac{\partial a_r}{\partial z_i}(z) X_r(z) + \sum_r a_r(z) \frac{\partial X_r}{\partial z_i}(z).$$

Since $V_z^{j+1} = V_z^j$, each term on the right lies in V_z^j whence $\partial X/\partial z_i(z) \in V_z^j$ for all z near z_0. Since α and i were arbitrary, it follows that $V_z^{j+2} = V_z^j$ and Lemma 4.2 is proved.

Proof of Theorem 1.3. For every point $z \in b\mathbf{B}^n \cap U$ we denote by V_z the complex subspace of $\mathbf{C}^N$ spanned by all vectors of the form $D^j f(z)(v_1, \ldots, v_j)$, where $v_i \in z^\perp$ for $1 \leqslant i \leqslant j \leqslant s$. Lemma 4.1 implies that $V_z \subset f(z)^\perp$. There are two possibilities: either $V_{z_0} = f(z_0)^\perp$ for some point $z_0 \in b\mathbf{B}^n \cap U$ or else $V_z \neq f(z)^\perp$ for all $z \in b\mathbf{B}^n \cap U$.

Consider first the case when

$$V_{z_0} = f(z_0)^\perp \tag{4.2}$$

for some $z_0 \in b\mathbf{B}^n \cap U$. We may assume that $z_0 = (0, \ldots, 0, 1) \in b\mathbf{B}^n$. Let $u_i(z)$ be the vector

$$u_i(z) = (0, \ldots, 0, -\bar{z}_n, 0, \ldots, 0, \bar{z}_i), \qquad 1 \leqslant i \leqslant n-1,$$

where the entry $-\bar{z}_n$ is in the ith place. Clearly $u_i(z) \in z^\perp$. The assumption (4.2) implies that there are $N-1$ vectors $X_1(z), \ldots, X_{N-1}(z)$ in $\mathbf{C}^N$, where each $X_i(z)$ is of the form $D^j f(z)(v_1(z), \ldots, v_j(z))$ for some integer $j \leqslant s$ and for some vectors $v_k \in \{u_1, \ldots, u_{n-1}\}$, such that $X(z_0), \ldots, X_{N-1}(z_0)$ form a basis of the complex hyperplane $f(z_0)^\perp$. This implies that the system of linear equations

$$\begin{aligned} &F(z) \cdot \overline{f(z/|z|^2)} = 1, \\ &F(z) \cdot \overline{X_j(z/|z|^2)} = 0, \qquad 1 \leqslant j \leqslant N-1 \end{aligned} \tag{4.3}$$

for the unknown functions $F = (F_1, \ldots, F_N)$ has rank N at the point z_0, and hence it has a unique solution $F(z)$ in a neighborhood of z_0. By part (ii) of Lemma 4.1 the function $f(z)$ is a solution of (4.3) when $z \in b\mathbf{B}^n$ whence $F(z) = f(z)$ for $z \in b\mathbf{B}^n$ near z_0. If we redefine the function $f(z)$ outside $\overline{\mathbf{B}}^n$ by taking $f(z) = F(z)$, then f is holomorphic near z_0 [14, 25, 18]. The proof of Theorem 1 in [6] can be applied to conclude that f is rational, $f = (P_1/_2, \ldots, P_N/_q)$. The degrees of the polynomials P_j and q may differ from those given in [6].

It remains to consider the second possibility when

$$V_z \neq f(z)^{\perp} \qquad \text{for all} \quad z \in b\mathbf{B}^n \cap U. \tag{4.4}$$

Let $\mathbf{P}^n$ be the projective space of complex affine hyperplanes in $\mathbf{C}^n$. We shall prove that there is an open set of hyperplanes $\Lambda \in \mathbf{P}^n$ such that $\Lambda \cap U \cap \mathbf{B}^n \neq \emptyset$ and the restriction of f to Λ is rational. Once this is proved, it follows from [3, p. 201] that f is rational.

By composing f with an automorphism of $\mathbf{B}^n$ we may assume that f is defined on an open neighborhood of the set

$$\overline{\mathbf{B}}^n \cap \{z = (z_1, \ldots, z_n) \in \mathbf{C}^n \mid z_1 = 0 \text{ or } z_2 = 0\}$$

and that f has maximal rank n at the point 0. Let $\mathbf{e}_j = (0, \ldots, 1, \ldots, 0)$ with 1 on the jth spot, and consider any set of $n-1$ vectors of the form

$$D^k f(\zeta \mathbf{e}_1)(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}), \qquad k \leqslant s, \quad 2 \leqslant i_l \leqslant n,$$

where $\zeta \in \overline{\Delta} = \{w \in \mathbf{C} : |w| \leqslant 1\}$. Since these vectors are holomorphic in $\zeta \in \Delta$, continuous on $\overline{\Delta}$, and linearly dependent for $\zeta \in b\Delta$ by (4.4), they are also linearly dependent at $\zeta = 0$. Hence there is a complex subspace $Z \subset \mathbf{C}^N$ of dimension $N-2$ such that

$$D^k f(0)(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}) \in Z \qquad \text{for all} \quad k \leqslant s, \quad 2 \leqslant i_l \leqslant n. \tag{4.5}$$

Let $z' = (z_2, \ldots, z_n)$. For each $2 \leqslant j \leqslant n$ let α_j be the number of indices i_l in (4.5) whose value equals j, and let $\alpha = (\alpha_2, \ldots, \alpha_n)$. Then the expression (4.5) is simply the partial derivative

$$\frac{\partial^{\alpha} f}{\partial z_2^{\alpha_2} \ldots \partial z_n^{\alpha_n}}(0, z'), \qquad |\alpha| = \alpha_2 + \cdots + \alpha_n = k \tag{4.6}$$

evaluated at the point $z' = 0$. For $j \in \{1, \ldots, s\}$ we denote by $V_{z'}^j$ the linear subspace of $\mathbf{C}^N$ spanned by all vectors of the form (4.6) for $1 \leqslant k \leqslant s$. (4.5) implies $\dim V_0^s \leqslant N-2$. We may apply the same argument to the map $f \circ \phi$, where ϕ is an automorphism of the ball $\mathbf{B}^n$ that preserves the hyperplane $\Lambda = \{z_1 = 0\}$ and is close to the identity map on $\mathbf{B}^n$, to conclude that $\dim V_{z'}^s \leqslant N-2$ for all points z' sufficiently close to 0. Such automorphisms exist, see [21, p. 30].

Consider the flag $V_{z'}^1 \subset V_{z'}^2 \subset \cdots \subset V_{z'}^s$ of $s = N - n + 1$ subspaces of $\mathbf{C}^N$. Since f has maximal rank n on $\mathbf{C}^n$ near 0, we have $\dim V_{z'}^1 = n - 1$ whence

$$\dim V_{z'}^s - \dim V_{z'}^1 \leqslant (N-2) - (n-1) = N - n - 1 = s - 2.$$

It follows that there is an integer $j = j(z') \in \{1, \ldots, s-1\}$ such that $V_{z'}^j = V_{z'}^{j+1}$. Choose $j(z')$ as small as possible. We can find a point z_0' near 0 such that $j(z')$ is constant in a neighborhood of z_0'. Let $z_0 = (0, z_0')$. Lemma 4.2 implies that

$$f(\Lambda \cap \mathbf{B}^n) \subset f(z_0) + V_{z_0}^s.$$

Hence f maps the $(n-1)$-ball $\Lambda \cap \mathbf{B}^n$ properly into the $(N-2)$-ball $(f(z_0) + V_{z_0'}^s) \cap \mathbf{B}^N$, and it is of class $\mathbf{C}^s$ to the boundary. By induction on the dimension n it follows that the restriction of f to Λ is rational.

Applying the same argument to $f \circ \phi$, where ϕ is an automorphism of $\mathbf{B}^n$ such that $\phi(\Lambda \cap \mathbf{B}^n)$ is close to $\Lambda \cap \mathbf{B}^n$, we conclude that f is rational on an open set of hyperplanes $\Lambda \in \mathbf{P}^n$. By [3, p. 201] this proves that the map f is rational.

It remains to show that the map f is holomorphic on $\mathbf{B}^n$ and maps $\mathbf{B}^n$ into $\mathbf{B}^N$. Let $f = (p_1, \ldots, p_N)/q$, where $p_1, \ldots, p_N$ and q are polynomials in $z = (z_1, \ldots, z_n)$. Since the ring $\mathbf{C}[z]$ is a unique factorization domain, we may assume that these polynomials have no common factor. Assume that there is a point $a \in \mathbf{B}^n$ such that $q(a) = 0$, and let q_0 be an irreducible factor of q such that $q_0(a) = 0$. The condition $|f| = 1$ on $b\mathbf{B}^n \cap U$ implies

$$|p_1|^2 + \cdots + |p_N|^2 = |q|^2, \tag{4.7}$$

first on $b\mathbf{B}^n \cap U$ and then by the identity principle on all of $b\mathbf{B}^n$. From (4.7) we see that

$$\{q_0 = 0\} \cap b\mathbf{B}^n \subset \{p_j = 0\} \cap b\mathbf{B}^n$$

for all $1 \leqslant j \leqslant N$. There is a point $z_0 \in b\mathbf{B}^n$ such that $q_0(z_0) = 0$, and near z_0 the set $\{q_0 = 0\}$ is a complex $(n-1)$-manifold that intersects $b\mathbf{B}^n$ transversally. (4.7) shows that $p_j = 0$ on $\{q_0 = 0\}$ near z_0 for $1 \leqslant j \leqslant N$. Since the set $\{q_0 = 0\}$ is an irreducible algebraic variety, this implies that $p_j = 0$ on $\{q_0 = 0\}$ for all j. By the Nullstellensatz each p_j is divisible by q_0 in the ring $\mathbf{C}[z]$. This is a contradiction to our initial assumption that $p_1, \ldots, p_N, q$ have no common factor. Thus f is holomorphic on $\mathbf{B}^n$. Moreover, since $|f| = 1$ on the set of point of $b\mathbf{B}^n$ where f is holomorphic, it follows that f has no poles on $b\mathbf{B}^n$.

We claim that the restriction of f to every complex line l in $\mathbf{C}^n$ intersecting $\mathbf{B}^n$ is holomorphic on a neighborhood of $l \cap \overline{\mathbf{B}}^n$. Since $q \neq 0$ on $\mathbf{B}^n$, the restriction of q to l can only have a finite number of zeroes $a_1, \ldots, a_r$ on the circle $l \cap b\mathbf{B}^n$. Since each f_j is bounded in absolute value by 1 on $l \cap b\mathbf{B}^n \backslash \{a_1, \ldots, a_r\}$, the restriction $f_j|_l$ has a removable singularity at each a_k and so f_j is holomorphic on $l \cap \overline{\mathbf{B}}^n$ as claimed. The maximum modulus principle implies that f maps $\mathbf{B}^n$ into $\mathbf{B}^N$. This concludes the proof of Theorem 1.3.

Note. In the case when $N \leqslant 2n-2$ Corollary 1.4 implies that f has no singularities on the sphere $b\mathbf{B}^n$. We do not know whether the same is true when $N = 2n-1$. Note that there exist rational functions with singularities on $b\mathbf{B}^n$ that are bounded on $\overline{\mathbf{B}}^n$. For example, $f(z_1, z_2) = z_2^2/(1-z_1)$ is bounded in absolute value by $|z_2|^2$ on $\overline{\mathbf{B}}^2$ and has a point of indeterminacy at $(1,0) \in b\mathbf{B}^2$.

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