EXPOSITIONES
MATHEMATICAE

Expo. Math. 4 (1986), 243–255

On totally real embeddings into $\mathbb{C}^n$

Franc Forstnerič*

Summary. An immersion or embedding $f: M \to \mathbb{C}^n$ of a smooth real manifold into a complex Euclidean space $\mathbb{C}^n$ is called totally real if for each point $p \in M$ the tangent space to $f(M)$ at $f(p)$ contains no nontrivial complex linear subspace. In this paper we will apply a theorem of Gromov on convex integration of differential relations [5, p. 331, 1.3.1.] to the problem of constructing totally real embeddings of smooth n-dimensional manifolds into $\mathbb{C}^n$ for $n \geq 3$. Our main result (Theorem 1.2 and Corollary 1.3) is that a totally real immersion which is regularly homotopic to an embedding is also regularly homotopic to a totally real embedding. This result was stated by Gromov [5, p. 332], but to our knowledge no proof was ever published. Corollary 1.3 turns the problem of constructing totally real embeddings into a problem in homotopy theory. Since totally real embeddings are of considerable interest to complex analysis, we shall prove the result in this paper and obtain some applications.

To prove Theorem 1.2 we combine Gromov's main theorem in [5] with Whitney's method [16] to show that one can remove certain pairs of double points of a totally real immersion without losing total reality. This method was suggested by Gromov in [5, p. 332]. Thus each totally real immersion $f: M \to \mathbb{C}^n$ can be changed by a regular homotopy through totally real immersions into a totally real immersion f_1 with the minimal number of double points. In particular, if the self intersection number of f equals zero, then f_1 is a totally real embedding. As an application we prove that every orientable compact three dimensional manifold admits a totally real embedding into $\mathbb{C}^3$.

1. Results

We first recall some terminology. Let M be a smooth n-dimensional manifold. A *regular homotopy of immersions* of M into $\mathbb{R}^m$ is a parametrized family $\{f_t\}$, $t \in [0,1]$, of C^1 immersions $f_t: M \to \mathbb{R}^m$ whose derivatives $df_t: TM \to f_t^*(T\mathbb{R}^m) = M \times \mathbb{R}^m$ depend continuously on t.

An immersion $f: M \to \mathbb{C}^m$ is called *totally real* if for each point p in M the n-dimensional real-linear subspace $df(T_pM)$ of $\mathbb{C}^m$ contains no nontrivial complex subspace; this requires $m \geq n$. Equivalently, f is totally real if the complexified derivative $df^{\mathbb{C}}: TM \otimes \mathbb{C} = T^{\mathbb{C}}M \to M \times \mathbb{C}^m$ defined by

*) Research supported by a fellowship from the Alfred P. Sloan foundation and by a grant from the National Science Foundation.

0723-0869/86/030243-13 $4.10

$$df_p^{\mathbb{C}}(v+iw)=df_p(v)+i\,df_p(w), \qquad p\in M, \qquad v,w\in T_pM$$

is one-one on each fiber, i.e., a complex vector bundle map. If $m=n$, the above implies that $df^{\mathbb{C}}$: $T^{\mathbb{C}}M\to M\times\mathbb{C}^n$ is an isomorphism of complex vector bundles over M. A regular homotopy f_t: $M\to\mathbb{C}^n$ which is totally real at each stage induces a homotopy of complex bundle isomorphisms $df_t^{\mathbb{C}}$: $T^{\mathbb{C}}M\to M\times\mathbb{C}^n$.

We denote by $G_{2n,n}$ the Grassman manifold of n-dimensional real subspaces of $\mathbb{C}^n$ and by $G_{2n,n}^{tr}$ the open subset of $G_{2n,n}$ consisting of totally real n-dimensional subspaces, i.e., subspaces that contain no complex line. To every immersion f: $M\to\mathbb{C}^n$ we associate its Gauss map τ_f: $M\to G_{2n,n}$ defined by $\tau_f(p)=df(T_pM)$. The work of Gromov [5, p. 332] and Lees [9] yields the following results. (See also Section 2 below.)

1.1 Theorem. *Let M be a smooth n-dimensional manifold.*

(a) The regular homotopy classes of totally real immersions of M into $\mathbb{C}^n$ are in one-one correspondence with the homotopy classes of complex bundle isomorphisms $T^{\mathbb{C}}M\mapsto M\times\mathbb{C}^n$.

(b) Let f: $M\to\mathbb{C}^n$ be an immersion. If there is a homotopy τ_t: $M\to G_{2n,n}$ starting at $\tau_0=\tau_f$ such that the image of τ_1 is contained in $G_{2n,n}^{tr}$, then there is a regular homotopy of f into a totally real immersion of M into $\mathbb{C}^n$.

When $T^{\mathbb{C}}M$ is trivial the set of homotopy classes of complex bundle isomorphisms $T^{\mathbb{C}}M\mapsto M\times\mathbb{C}^n$ is in one-one correspondence with the elements of the complex K-group $K^1(M)$ [3]. Thus the totally real immersions of M into $\mathbb{C}^n$ are classified by $K^1(M)$.

Suppose now that M is compact and without boundary. We can approximate every immersion f: $M\to\mathbb{C}^n$ by an immersion with only finitely many regular double points. (Whitney called such an immersion completely regular [16, p. 221].) Whitney introduced the self-intersection number I_f of such immersion [16, p. 233]; I_f counts algebraically the number of double points of f and is invariant under regular homotopy. I_f is an integer if M is orientable and n is even and is an integer modulo 2 otherwise. If $n\geq 3$ and f: $M\to\mathbb{C}^n$ is an immersion whose number of double points exceeds $|I_f|$, Whitney proved that we can remove a pair of double points of f by a regular homotopy, thus reducing their number by two. Hence every immersion is regularly homotopic to an immersion with the minimal number of double points. We will show that the same holds for totally real immersions. More precisely, we have

1.2 Theorem. *Let f: $M\to\mathbb{C}^n$ be a totally real immersion of a smooth compact manifold M of dimension $n\geq 3$ into $\mathbb{C}^n$. There exists a regular homotopy f_t: $M\to\mathbb{C}^n$ of totally real immersions with $f_0=f$ such that the number of double points of f_1 equals the self-intersection number $|I_f|$. If f is regularly homotopic to an embedding, then f is also regularly homotopic to a totally real embedding by a homotopy through totally real immersions.*

Part (b) of Theorem 1.1 and Theorem 1.2 imply

1.3 Corollary. *Let M be a smooth compact manifold of dimension $n \geq 3$. If the Gauss map τ_f: $M \to G_{2n,n}$ of an embedding f: $M \to \mathbb{C}^n$ is homotopic to a map with image in $G^{tr}_{2n,n}$, then f is regularly homotopic to totally real embedding.*

In a footnote on page 332 in [5] Gromov claims that the same result holds also for $n=2$. We do not know how to prove this since Whitney's method only works for $n \geq 3$.

As an application we shall prove that all compact orientable three-manifolds admit totally real embeddings into $\mathbb{C}^3$.

1.4. Theorem. *Every immersion of a compact orientable three dimensional manifold M into $\mathbb{C}^3$ is regularly homotopic to a totally real immersion. There exists a totally real embedding of M into $\mathbb{C}^3$.*

It is known that for n different from one and three the n-sphere S^n does not admit totally real embeddings into $\mathbb{C}^n$ [8, 14]; hence Theorem 1.4 only holds in dimension three (and of course in dimension one). Explicit totally real embeddings of S^3 and certain quotients of S^3, including the real projective space $\mathbb{RP}^3$, can be found in [1] and [4]. For n different from 1, 3 or 7 the projective space $\mathbb{RP}^n$ does not admit a totally real embedding into $\mathbb{C}^n$. Since $\mathbb{RP}^7$ is parallelizable, it admits a totally real immersion into $\mathbb{C}^7$, but we do not know whether this immersion is regularly homotopic to an embedding. We shall prove that

1.5 Proposition. *All totally real immersions of $\mathbb{RP}^7$ into $\mathbb{C}^7$ are regularly homotopic.*

There are two distinct regular homotopy classes of immersions $\mathbb{RP}^7 \to \mathbb{C}^7$; one of them can be represented by an embedding, the other one by an immersion with exactly one double point. (See Section 4.) The problem is to decide which class contains a totally real immersion. For a survey of known totally real embeddings of other manifolds see [15].

The paper is organized as follows. In Section 2 we recall the theorem of Gromov, and in Section 3 we outline Whitney's method of removing pairs of double points of an immersion. In Section 4 we prove Theorem 1.2. In Section 5 we discuss classification of immersions of n dimensional manifolds into $\mathbb{C}^n$, we prove Theorem 1.1 (b), Theorem 1.4, and Proposition 1.5.

I wish to thank Edgar Lee Stout for several helpful conversations about this problem and Rob Thompson for his help in homotopy theory.

2. The theorem of Gromov.

We recall Gromov's theorem [5]. Let π: $X \to M$ be a smooth fibre bundle with n-dimensional base manifold M and q-dimensional fiber. Denote by X^1 the manifold of 1-jets of sections of $\pi\colon X \to M$, and let $\pi_0\colon X^1 \to X$ and $\pi_1\colon X^1 \to M$ be the natural bundles, the first of which is an affine bundle. An open subset Ω of X^1 is called an *open differential relation of order one.* A *solution* of the relation Ω is a C^1 section $\sigma\colon M \to X$ whose one-jet $j^1(\sigma)\colon M \to X^1$ has values in Ω. Clearly a necessary condition for the existence of solutions of Ω is the existence of sections of Ω over M. Gromov proved [5, p. 231, 1.3.] that under certain condition on Ω which we shall describe below the natural map from the space $\Gamma_\Omega(M,X)$ of solutions of Ω into the space $\Gamma(M,\Omega)$ of sections of Ω is a *weak homotopy equivalence*, i.e., the path connected components of both spaces are in one-one correspondence and the map induces isomorphism of all homotopy groups of the two spaces.

Since Gromov's condition on Ω is local on X, it suffices to describe it in the case when $\pi\colon X = M \times \mathbb{R}^q \to M$ is a trivial bundle. There is a natural decomposition of each tangent space $T_x X, x \in X$, into a direct sum $T_{\pi(x)} M \oplus T_x \mathbb{R}^q = T_{\pi(x)} M \oplus \mathbb{R}^q$ of horizontal and vertical subspaces. Every section σ of X is of the form $\sigma(p) = (p, f(p))$ for some map $f\colon M \to \mathbb{R}^q$. If we choose local coordinates $u_1, \ldots, u_n$ in an open subset U of M, then the one-jet $j^1(\sigma)$ of a section $\sigma(p) = (p, f(p))$ is determined by $f\colon U \to \mathbb{R}^q$ and by the partial derivatives $\dfrac{\partial f}{\partial u_j}\colon U \to \mathbb{R}^q$, $1 \le j \le n$. Thus each one-jet $x^1 \in X^1$ over U is given by $x^1 = (x; v_1, \ldots, v_n)$, where x is a point in X and $\{v_j\}$ are vectors in $\mathbb{R}^q$. The restriction of a one-jet to the hyperplane $u_k = u_k^0$ through a point $u^0 \in U$ is determined by

$$x_k^1 = (x; v_1, \ldots, \hat{v}_k, \ldots, v_n), \qquad \pi(x) = u_0, \tag{2.1}$$

where the hat indicates that the k-th component is omitted. The relation $\Omega \subset X^1$ is said to be *ample in the coordinate direction* u_k if for each restricted one-jet x_k^1 of the form (2.1) the set of all vectors $v \in \mathbb{R}^q$ for which the one-jet $x^1 = (x; v_1, \ldots, v, \ldots, v_n)$ (v on the k-th spot) lies in Ω is either empty or else the convex hull of each of its path connected components equals $\mathbb{R}^q$. The relation Ω is called *ample in the coordinate directions* if each point $x \in X$ has an open neighborhood Y and local coordinates $u_1, \ldots, u_n$ in the projection $U = \pi(Y) \subset M$ such that $Y \mapsto U$ is a trivial bundle over U and the relation $\Omega \cap \pi_0^{-1}(Y)$ is ample in all coordinate directions u_j at all points of Y. Gromov's main result in [5, p. 231] is that for every open differential relation $\Omega \subset X^1$ which is ample in the coordinate directions the map $\Gamma_\Omega(M,X) \to \Gamma(M,\Omega)$ is a weak homotopy equivalence.

If $X = M \times \mathbb{C}^n \mapsto M$ is a trivial bundle with fiber $\mathbb{C}^n$, then we can define an open differential relation $\Omega \subset X^1$ as follows. Choose local coordinates $u_1, \ldots, u_n$ on $U \subset M$. A one-jet $x^1 = (x; v_1, \ldots, v_n)$ is in Ω if and only if the vectors $v_1, \ldots, v_n \in \mathbb{C}^n$ are complex linearly independent, i.e., the real n dimensional space spanned by them contains no nontrivial complex subspaces. In this case a section $\sigma(p) = (p, f(p))$ of X is a solution of Ω if and only if the map $f\colon M \to \mathbb{C}^n$ is a totally real immersion of M into $\mathbb{C}^n$. It is easy to see that Ω is ample in the coordinate directions. Thus part (a) of Theorem 1.1 follows from Gromov's theorem.

The essential step in the proof of Gromov's theorem is the following lemma which we state here for the sake of completeness. (See [5, p. 339, Lemma 3.1.3].)

2.1 Lemma (Gromov). *Let V be the cube $[0,1]^n$ with coordinates $u_1, \dots, u_n$, $X = V \times \mathbb{R}^q \to V$ the trivial bundle and $\Omega \subset X^1$ an open subset, ample in the coordinate directions. Let f_0: $V \to X$ and ϕ_0: $V \to \Omega$ be smooth sections such that on the boundary of the cube $j^1(f_0) = \phi_0$. Then for any $\varepsilon > 0$ one can find a C^1 section f: $V \to X$ with the following properties:*

(a) On the boundary of the cube V the one-jet $j^1(f)$ coincides with ϕ_0.
(b) $|f - f_0| < \varepsilon$.
(c) The jet $j^1(f)$ carries V into Ω.
(d) There exists a deformation ψ_t, $t \in [0,1]$, consisting of sections $V \to \Omega$, coinciding on the boundary of the cube with ϕ_0, such that $\psi_0 = \phi_0$ and $\psi_1 = j^1(f)$.

We shall prove a lemma which we will need in the proof of Theorem 1.2. Let π: $X = M \times \mathbb{R}^n \to M$ be a trivial bundle over an n-dimensional manifold M. Suppose that we have an almost complex structure J on the total space X, i.e., an operator J: $TX \to TX$ such that $-J^2$ equals the identity operator on TX. We define a relation $\Omega_J \subset X^1$ as follows: A one-jet $x^1 \in X^1$ belongs to Ω if and only if the tangent plane $V_x \subset T_x X$ to the graph of any section σ: $M \to X$ representing x^1 contains no nontrivial J-linear subspaces. Notice that V_x is independent of the choice of representative of the jet x^1.

2.2 Lemma. *The relation Ω_J defined above is ample in the coordinate directions.*

Proof. Choose a point $x_0 \in X$ and a system of local coordinates $u_1, \dots, u_n$ near $\pi(x_0) = u^0 \in M$. Let $e_j = \dfrac{\partial}{\partial u_j}$ for $1 \le j \le n$. The tangent plane at the point $x \in X$ to a section of X whose one-jet is $x^1 = (x; v_1, \dots, v_n)$ is spanned by the vectors $e_j + v_j$, $1 \le j \le n$. Fix an index k and a restricted one-jet $x_k^1 = (x; v_1, \dots, \hat{v}_k, \dots, v_n)$. Denote by W' the real $(n-1)$-dimensional subspace of $T_x X$ spanned by the vectors $e_j + v_j$, $j \in \{1, \dots, \hat{k}, \dots, n\}$. For every $v_k \in \mathbb{R}^n$ let $W(v_k)$ be the real n-dimensional subspace of $T_x X$ spanned by $e_j + v_j$, $1 \le j \le n$; thus $W' \subset W(v_k)$ for each $v_k \in \mathbb{R}^n$.

Denote by Σ the subset of all $v_k \in \mathbb{R}^n$ for which $W(v_k)$ contains a nontrivial J-linear subspace. We have to show that the complement Ω' of Σ in $\mathbb{R}^n$ is either empty or else the convex hull of each of its connected components equals $\mathbb{R}^n$.

If W' contains a nontrivial J-linear subspace, then so does $W(v_k)$ for each $v_k \in \mathbb{R}^n$, and hence $\Omega' = \emptyset$. Suppose now that W' contains no nontrivial J-linear subspace. In this case a vector $v_k \in \mathbb{R}^n$ is in Σ if and only if the vector $e_k + v_k$ lies in the real $(2n-2)$-dimensional subspace $W' + JW'$ of $T_x X$. Here we used the relation $J^2 = -I$. For simplicity of notation we shall assume that $k = n$ and write $v_k = v$. Let $J(e_j + v_j) = b_j$ for $1 \le j \le n-1$. A vector $e_n + v$ lies in W if and only if there exist real numbers α_j, β_j, $1 \le j \le n-1$, satisfying

$$\sum_{j=1}^{n-1} \alpha_j (e_j + v_j) + \beta_j b_j = e_n + v. \tag{2.2}$$

Write b_j in the form $b_j=\sum_{i=1}^n \gamma_{j,i} e_i + b'_j$, where $b'_j \in \mathbb{R}^n$ is a vertical vector and $\gamma_{j,i} \in \mathbb{R}$. Inserting this into (2.2) and comparing the coefficients of vectors $e_1, \dots, e_n$ we obtain a system of n linear equations for the $2n-2$ unknowns α_j, β_j, $(1 \le j \le n-1)$:

$$\begin{aligned} &\alpha_i + \sum_{j=1}^{n-1} \beta_j \gamma_{j,i} = 0, \qquad 1 \le i \le n-1, \\ &\sum_{j=1}^{n-1} \beta_j \gamma_{j,n} = 1. \end{aligned} \tag{2.3}$$

From (2.2) we see that Σ consists of the vectors v of the form

$$v = \sum_{j=1}^{n-1} \alpha_j v_j + \beta_j b'_j,$$

where the numbers α_j, β_j solve (2.3).

If all $\gamma_{j,n}$ equal zero, the system (2.3) has no solutions whence the set Σ is empty. If on the other hand at least one of the numbers $\gamma_{i,n}$ is nonzero, the system (2.3) has rank n and hence its general solution depends on $n-2$ parameters. Thus Σ is an affine subspace of $\mathbb{R}^n$ of real dimension $n-2$ and hence the complement Ω' of Σ is a connected set whose convex hull equals $\mathbb{R}^n$. This shows that Ω is ample in the coordinate directions and Lemma 2.2 is proved.

3. Whitney's method of removing pairs of double points.

Let $f\colon M \to \mathbb{R}^{2n}$ be an immersion of a smooth compact manifold M of dimension $n \ge 3$ into $\mathbb{R}^{2n}$. After a small C^1 perturbation we may assume that f has only a finite number of regular double points $f(p)=f(p')$, $p, p' \in M$, at which the tangent planes $df(T_p M)$ and $df(T_{p'} M)$ have trivial intersection. To each double point Whitney [16] associated the local self-intersection number which is $\pm 1 \in \mathbb{Z}$ in the case when M is orientable and n is even, and is $1 \in \mathbb{Z}/2$ in every other case. The number I_f equals the sum of local self intersection numbers at all double points.

Suppose that we have a pair of double points

$$f(p_1)=f(p_2)=q, \qquad f(p'_1)=f(p'_2)=q' \tag{3.1}$$

with the opposite types of self-intersections. If M is not orientable or if n is odd, then every two self-intersections are of the opposite types. Whitney showed how one can remove this pair of double points by a regular homotopy of f [16, Sections 8–12]. His method consists of a homotopic deformation of sections of certain trivial bundle $\pi\colon X = V \times \mathbb{R}^n \to V$ whose total space X is diffeomorphic to a subset of $\mathbb{R}^{2n}$ and whose base space V is a closed n-dimensional cube. By successive applications of this method we can reduce the number of double points of our immersion to the minimal number $|I_f|$. In the next section we will show that the same method can be used with totally real immersions $f\colon M \to \mathbb{C}^n$ in such a way that we do not loose total reality, thus proving Theorem 1.2.

We shall outline Whitney's method; see Sections 8–12 of [16] and Milnor [10, p. 71]. Assume that (3.1) is a pair of self-intersections of the opposite types. Let C_1 and C_2 be non-intersecting curves in M, C_i joining p_i and p_i', neither passing through any other point where f has self-intersections. Then $B_i = f(C_i)$ joins q to q', and $B = B_1 \cup B_2$ is a simple closed curve in $f(M)$. We extend each B_i smoothly past the endpoints q, q' to an open arc in $f(M)$.

Let A_1 and A_2 be open smooth arcs in $\mathbb{R}^2$ intersecting transversely in points a and b and enclosing a disk D with two corners. To be specific, let U_ε, $0<\varepsilon<0.1$, be the rectangle $\{(u_1,u_2)\in\mathbb{R}^2 \mid -\varepsilon \le u_1,\ u_2 \le 1+\varepsilon\}$, let $A_1 = (\mathbb{R}\times\{0\}) \cap U_\varepsilon$, and let A_2 be the intersection with U_ε of the circle of radius one with center $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Choose an embedding $\psi_1\colon A_1 \cup A_2 \to \mathbb{R}^{2n}$ so that $\psi_1(A_1)$ and $\psi_1(A_2)$ are the arcs B_1 and B_2 respectively, with the points a and b corresponding to q and q'. If $n \ge 3$ and the self-intersections (3.1) are of the opposite types, we can choose B_1 and B_2 such that the following holds [10, p. 73].

3.1 Lemma. *For some $\varepsilon>0$ we can extend the embedding $\psi_1\colon A_1\cup A_2 \to \mathbb{R}^{2n}$ to an embedding $\psi\colon U_\varepsilon \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \to \mathbb{R}^{2n}$ such that the preimage $\psi^{-1}(f(M))$ is the union of the submanifolds $N_1 = A_1 \times \mathbb{R}^{n-1} \times \{0\}$ and $N_2 = A_2 \times \{0\} \times \mathbb{R}^{n-1}$ of $U_\varepsilon \times \mathbb{R}^{2n-2}$.*

In order to remove the two double points (3.1) of f it suffices to remove the two intersections of N_1 and N_2 by moving N_2 within $U_\varepsilon \times \mathbb{R}^{2n-2}$ while keeping its boundary fixed.

Let $u=(u_1,u_2)$, $x=(x_1,\dots,x_{n-1})$, and $y=(y_1,\dots,y_{n-1})$. Let Λ be the n-plane with coordinates (u_1,y) and let $\pi\colon \mathbb{R}^{2n} \to \Lambda$ be the projection $\pi(u,x,y)=(u_1,y)$. Denote by V the closed n-dimensional cube $V = \left\{(u_1,y)\in\Lambda \,\middle|\, -\frac{\varepsilon}{2} \le u_1 \le 1+\frac{\varepsilon}{2},\ |y_j|\le 1,\ 1\le j\le n-1\right\}$ in Λ, and let $X=\pi^{-1}(V)$. Then $\pi\colon X\to V$ is a trivial bundle over the cube V, with u_2 and x the fiber coordinates. Lemma 3.1 implies that the submanifold $N_2 \cap X$ is the graph of a section $h_0\colon V\to X$ of the bundle $\pi\colon X\to V$.

Now we can deform h_0 in the coordinate direction u_2 through a smooth homotopy $\{h_t\}$ of sections of X to a section $h_1\colon V\to X$ such that

(a) the homotopy is fixed near the boundary of V, and
(b) $h_1(V)$ does not intersect $N_1 \cap X$.

To obtain an explicit formula we first represent the arc A_2 as the graph $u_2 = \alpha(u_1)$ and then choose a function $\beta\colon \mathbb{R}\to\mathbb{R}$, vanishing outside the interval $\left(-\frac{\varepsilon}{3}, 1+\frac{\varepsilon}{3}\right)$ and satisfying

$$-\frac{\varepsilon}{2} < \alpha(u_1) - \beta(u_1) < 0 \qquad \text{for all } u_1 \in A_1.$$

We also choose a smooth function $\varrho: \mathbb{R} \to [0,1]$ such that $\varrho(s)=0$ if $s \geq \frac{2}{3}$ and $\varrho(s)=1$ if $s \leq \frac{1}{3}$. Define

$$h_t(u_1, y) = (u_1, \alpha(u_1) - t\varrho(|y|)\beta(u_1), 0, y), \qquad 0 \leq t \leq 1.$$

Clearly $\{h_t\}$ is a homotopy of sections of the bundle $X \mapsto V$ with the required properties.

Let $M_2 = f^{-1}(\psi(N_2))$. For $0 \leq t \leq 1$ we define

$$f_t(p) = \begin{cases} f(p), & \text{if } p \notin M_2; \\ \psi \circ h_t \circ \pi \circ \psi^{-1} \circ f(p), & \text{if } p \in M_2. \end{cases} \tag{3.2}$$

Then $\{f_t\}$ is a regular homotopy of immersions starting at $f_0 = f$ which removes the two double points (3.1) and does not introduce any new self-intersections.

4. Proof of Theorem 1.2.

Suppose that that $f: M \to \mathbb{C}^n$ is a totally real immersion. Since total reality is stable under small C^1 perturbations, we may assume that f has only regular double points.

We use the notation established in Section 3. Thus, $\pi: X = V \times \mathbb{R}^n \to V$ is a trivial bundle over a cube $V = [0,1]^n$ and $\psi: X \to \psi(X) \subset \mathbb{C}^n$ is a diffemorphism onto a subset of $\mathbb{C}^n$. Let $J: TX \to TX$ be the almost complex structure on X induced from the natural complex structure on $\mathbb{C}^n$ via the diffeomorphism ψ. Explicitly, we have for every $v \in T_x X$:

$$d\psi(Jv) = i \cdot d\psi(v).$$

Let $\Omega = \Omega_J \subset X^1$ be the corresponding differential relation on X defined in Section 2 above. If $h: V \to X$ is a section, the $\psi \circ h: V \to \mathbb{C}^n$ is an embedding. The definition of Ω implies that the embedding $\psi \circ h$ is totally real if and only if the section $h: V \to X$ is a solution of Ω, i.e., $j^1(h): V \to X^1$ has values in Ω. By Lemma 2.2 the relation Ω is ample in the coordinate directions and hence we can use Lemma 2.1.

Let N_1 and N_2 be as in Lemma 3.1, and let N_2 be the graph of $h_0: V \to X$. We will break the proof of Theorem 1.2 into three steps.

Step 1. We find a section $h': V \to X$ such that $h' = h_0$ near the boundary of V and $h'(V)$ does not intersect N_1. This was explained in Section 3 above.

Step 2. The section $h_0: V \to X$ is a solution of Ω since $\psi(h_0(V)) = f(M_2)$ is a totally real submanifold of $\mathbb{C}^n$. Denote by $\phi = j^1(h_0): V \to \Omega \subset X^1$ the induced section of Ω. We apply Lemma 2.1 to the pair h', ϕ to find a section $h_1: V \to X$ satisfying:

(a) $h_1 = h_0$ near the boundary of the cube V.
(b) The graph $h_1(V)$ does not intersect N_1.
(c) The one-jet $j^1(h_1)$ is a section of Ω.

(d) There exists a deformation θ_t, $t\in[0,1]$, consisting of sections $V\to\Omega$, coinciding on the boundary of V with $j^1(h_0)=\phi$, such that $\theta_0=j^1(h_0)$ and $\theta_1=j^1(h_1)$.

If h_t: $V\to X$, $0\le t\le 1$, is any homotopy of h_0 into h_1 which is fixed on the boundary of V, then the maps f_t: $M\to\mathbb{C}^n$ given by (3.2) form are regular homotopy of f_0 into a totally real immersion f_1 which has a smaller number of double points.

Step 3. It remains to show that a regular homotopy $\{f_t\}$ of f_0 into f_1 can be chosen such that every f_t is a totally real immersion. Equivalently, we have to find a homotopy h_t: $V\to X$ of h_0 into h_1 such that

(i) $h_t=h_0$ near the boundary of V for all t, and
(ii) $j^1(h_t)$ is a section of Ω for all t.

Let π': $X'=X\times[0,1]\to V'=V\times[0,1]$ be the bundle with the projection $\pi'(x,t)=(\pi(x),t)$, and let X'^1 be its first jet bundle. There is a natural projection τ: $X'^1\to X^1$ obtained as follows: If a one-jet $x^1\in X'^1$ is represented by a local section h of $X'\to V'$ near the point $(p,t)\in V'$, then $\tau(x^1)$ is represented by the restriction of h to the subset $V\times\{t\}$. The subset $\Omega'=\tau^{-1}(\Omega)$ of X'^1 is clearly ample in the coordinate directions whenever Ω is. A section H: $V'\to X'$ corresponds to the homotopy $h_t=H(\cdot,t)$: $V\to X$ of sections of the bundle $X\to V$. A section H: $V'\to X'$ is a solution of Ω' if and only if the induced homotopy $h_t=H(\cdot,t)$ is a solution of Ω for each $t\in[0,1]$.

Choose an initial homotopy of h_0 into h_1 which is fixed near the boundary of V and denote by H^0: $V'\to X'$ the corresponding section. Let θ_t: $V\to X^1$, $0\le t\le 1$, be as in Step 2(d) above, and denote by Θ: $V'\to X'^1$ the induced section. Clearly we have $j^1(H^0)=\Theta$ near the boundary of V'. We now apply Lemma 2.1 to the pair H^0, Θ to find a section H: $V'\to X'$ such that $j^1(H)$ is a section of Ω' and $H=H^0$ near the boundary of V'. This gives us the desired homotopy of h_0 into h_1 through solutions of Ω and hence a regular homotopy of f_1 into f_2 through totally real immersions of M into $\mathbb{C}^n$. Theorem 1.2 is proved.

5. On the classification of immersions of n-manifolds into $\mathbb{C}^n$

Let M be a compact n-dimensional manifold. If f: $M\to\mathbb{R}^k$ is an immersion, its derivative induces an injective map of real vector bundles df: $TM\to M\times\mathbb{R}^k$. By a theorem of Smale [11] and Hirsch [6] the map $f\to df$ induces a one-one correspondence between the regular homotopy classes of immersions $M\mapsto\mathbb{R}^k$ and homotopy classes of vector bundle injections $TM\mapsto M\times\mathbb{R}^k$, provided that $k>n$. This result also follows from Gromov's theorem [5, p. 332].

We recall briefly how the problem of classifiying vector bundle injections $TM\mapsto M\times\mathbb{R}^k$ can be turned into the problem of classifying sections of a bundle over M. (See [6] for the details.) Let $FM\mapsto M$ be the bundle over M whose fiber FM_p over $p\in M$ is the set of all order n-tuples $X=(X_1,\dots,X_n)$ of linearly independent vectors in T_pM. Every such n-tuple is called an n-frame based at p. The group $GL(n, \mathbb{R})$ acts on FM by multiplication on the right: To an n-frame $X = (X_1, \ldots, X_n)$ and a matrix $A = (a_{i,j})$ we associate the n-frame XA whose j-th component is $\sum_{i=1}^{n} a_{i,j} X_i$. With respect to this action FM is a principal $GL(n, \mathbb{R})$ bundle [12, p. 35].

Let $V_{k,n}$ be the Stiefel manifold consisting of all n-frames in $\mathbb{R}^k$ [12, p. 33]. The group $GL(n, \mathbb{R})$ acts on $V_{k,n}$ by right multiplication exactly as above. A vector bundle injection $TM \mapsto M \times \mathbb{R}^k$ determines in a natural way a $GL(n, \mathbb{R})$-equivariant map $FM \mapsto V_{k,n}$, and the problem is to classify such maps under equivariant homotopy.

Let $E \mapsto M$ be the bundle with fiber $V_{k,n}$ associated to the principal bundle $FM \mapsto M$ [12, p. 43]. Its total space E is the topological quotient of $FM \times V_{k,n}$ modulo the equivalence relation $(X, Y) \approx (XA, YA)$, where $X \in FM$, $Y \in V_{k,n}$, and $A \in GL(n, \mathbb{R})$. We denote the equivalence class of (X, Y) by $[X, Y]$. To every $GL(n, \mathbb{R})$-equivariant map $g\colon FM \to V_{k,n}$ we associate the section $f\colon M \to E$ by the formula $f(p) = [X_p, g(X_p)]$, where $X_p \in FM_p$ is any frame based at p. The equivariance of g implies that the definition of f does not depend on the choice of X_p. Conversely, each section of $E \mapsto M$ defines an equivariant map of FM into $V_{k,n}$. It follows that regular homotopy classes of immersions $M \mapsto \mathbb{R}^k$ are in one-one correspondence with the homotopy classes of sections of the bundle $E \mapsto M$.

Since we are interested in totally real immersions $M \mapsto \mathbb{C}^n$, we take $k = 2n$ from now on and identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ in the usual way. Assume also that $n \geq 2$. Every n-frame $X \in V_{2n,n}$ is represented by a complex $n \times n$ matrix $(x_{i,j})$ whose columns are linearly independent over $\mathbb{R}$. An n-frame $X = (x_{i,j})$ is called totally real if the real n-plane spanned by X is totally real. Equivalently, $\det(x_{i,j}) \neq 0$. Thus the set $V_{2n,n}^{tr}$ of totally real n-frames equals the group $GL(n, \mathbb{C})$. The group $GL(n, \mathbb{C})$ also acts on the bundle E according to the formula $B[X, Y] = [X, BY]$, where $B \in GL(n, \mathbb{C})$, $[X, Y] \in E$, and BY is the matrix product.

For each $p \in M$ we denote by E_p^{tr} the subset of E_p consisting of all $[X, Y] \in E_p$ with Y a totally real n-frame in $\mathbb{C}^n$. Clearly E_p^{tr} is isomorphic to $GL(n, \mathbb{C})$, and $E^{tr} = \bigcup_{p \in M} E_p^{tr}$ is an open subbundle of E. A section of the subbundle $E^{tr} \to M$ corresponds to a totally real immersion of M into $\mathbb{C}^n$. The theorem of Gromov (Theorem 1.1(a)) implies that the regular homotopy classes of totally real immersions are in one-one correspondence with the homotopy classes of sections of the bundle $E^{tr} \mapsto M$. Moreover, an immersion $f\colon M \to \mathbb{C}^n$ is regularly homotopic to a totally real immersion if and only if the associated section $\tilde{f}\colon M \to E$ is homotopic to a section of the subbundle E^{tr}.

If $f_0, f_1\colon M \to E^{tr}$ are two sections of E^{tr}, there is a unique map $h\colon M \to GL(n, \mathbb{C})$ such that $f_1 = h \cdot f_0$. Thus, if M admits a totally real immersion into $\mathbb{C}^n$, the regular homotopy classes of totally real immersions $M \mapsto \mathbb{C}^n$ are in one-one correspondence with the homotopy classes of maps of M into $GL(n, \mathbb{C})$, i.e., with elements of $[M, \mathrm{GL}(n, \mathbb{C})] = [M, \mathrm{U}(n)] = K^1(M)$. This argument is due to Duchamp [3].

Consider now the problem of homotopy classification of sections of the bundle $E \mapsto M$. (See Part III of [12].) Recall that $\pi_i(V_{2n,n}) = 0$ if $1 \le i \le n-1$, and $\pi_n(V_{2n,n})$ equals $\mathbb{Z}$ if n is even and $\mathbb{Z}/2$ if $n \ge 3$ is odd [12, p. 132]. Denote by $\{\pi_n(E_p)\} \mapsto M$ the bundle whose fiber over $p \in M$ is the n-th homotopy group $\pi_n(E_p)$. Since the fibers E_p are simply connected, we do not need to specify a base point in $\pi_n(E_p)$. We represent the compact n-manifold M as a finite cell complex [12, p. 100]. Since the first nontrivial homotopy group of the fiber $V_{2n,n}$ occurs in dimension n, every section of E over a lower dimensional skeleton $M^{(i)}$ of M can be extended to a section over M [12, p. 178], and every two sections f_0, f_1: $M \to E$ are homotopic over the $(n-1)$-skeleton $M^{(n-1)}$ of M [12, p. 181]. The obstruction to extending this homotopy to the n-th skeleton $M^{(n)} = M$ is an element $\bar{d}(f_0, f_1)$ of the n-th cohomology group $\mathrm{H}^n(M, \{\pi_n(E_p)\})$ with coefficients in $\{\pi_n(E_p)\}$, called the primary difference of f_0 and f_1.

Choose a closed oriented n-cell σ_0 in the given cellular decomposition of M. Since M is an n-dimensional manifold, every $(n-1)$-cell lies in the faces of exactly two n-cells. This implies that every element of $\mathrm{H}^n(M, \{\pi_n(E_p)\})$ can be represented by a cocycle α which is nontrivial only on the reference cell σ_0; its value on σ_0 is an element of $\pi_n(E_{p_0})$ for a fixed reference point $p_0 \in \sigma_0$. (See [12, p. 201].) The construction of E implies that the bundle of coefficients $\{\pi_n(E_p)\}$ is twisted along a closed path in M precisely when the path reverses the orientation of M; in other words, $\{\pi_n(E_p)\}$ is isomorphic to the orientation bundle of M. This implies that for every compact n manifold

$$\mathrm{H}^n(M, \{\pi_n(E_p)\}) = \pi_n(E_p) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even;} \\ \mathbb{Z}/2, & \text{if } n \text{ is odd and } n \ge 3. \end{cases}$$

Thus there are $\mathbb{Z}$ distinct regular homotopy classes of immersions of M into $\mathbb{R}^{2n}$ if n is even and two distinct classes if $n \ge 3$ is odd.

Proof of Theorem 1.1 (b). Recall that $G_{2n,n}$ is the Grassman manifold of real n-dimensional subspaces of $\mathbb{C}^n$, and $G^{tr}_{2n,n}$ is the subset consisting of all totally real subspaces. The map $V_{2n,n} \mapsto G_{2n,n}$ which sends each n-frame to its linear span is a fibration. Since this map is invariant under the action of $\mathrm{GL}(n, \mathbb{R})$ on $V_{2n,n}$, it induces a map Φ: $E \mapsto M \times G_{2n,n}$ which is also a fibration and maps E^{tr} onto $M \times G^{tr}_{2n,n}$. If $\tilde{f}$: $M \to E$ is a section corresponding to an immersion f: $M \to \mathbb{C}^n$, then $\Phi \circ \tilde{f}$: $M \to G_{2n,n}$ is the Gauss map of f, i.e., $\Phi(\tilde{f}(p)) = (p, df(\mathrm{T}_p M))$. If there is a homotopy of $\Phi \circ \tilde{f}$ to a section of $M \times G^{tr}_{2n,n}$, then the homotopy lifting property implies that $\tilde{f}$ is homotopic to a section of E^{tr}. The converse is obvious, and part (b) of Theorem 1.1 is proved.

Proof of Theorem 1.4. Since the tangent bundle of an orientable three-dimensional manifold M is trivial [13], there exists a totally real immersion of M into $\mathbb{C}^3$.

Denote by f_0: $M \to E^{tr}$ the corresponding section. We will show that any other section f_1: $M \to E$ is homotopic to a section of E^{tr}.

Since every element of $H^3(M, \{\pi_3(E_p)\})$ can be represented by a cocycle supported on the reference cell σ_0, the definition of the primary difference $d(f_0, f_1)$ implies that f_1 can be changed homotopically into a section which agrees with f_0 outside σ_0. We denote the new section again by f_1. We can find a product representation ϕ: $\sigma_0 \times V_{6,3} \to E|_{\sigma_0}$ which carries $\sigma_0 \times V^{tr}_{6,3}$ onto $E^{tr}|_{\sigma_0}$ and satisfies $\phi^{-1} \circ f_0(p) = (p, X_0)$, $p \in \sigma_0$, for some constant frame $X_0 \in V^{tr}_{6,3}$. Denote by ψ: $\sigma_0 \times V_{6,3} \to V_{6,3}$ the projection onto the second factor. Then $\tilde{f}_1 = \psi \circ \phi^{-1} \circ f_1$: $\sigma_0 \to V_{6,3}$ maps the boundary $b\sigma_0$ into the point $X_0 \in V^{tr}_{6,3}$ and hence it defines an element of $\pi_3(V_{6,3}, X_0)$.

We claim that the map $\pi_3(V^{tr}_{6,3}) \to \pi_3(V_{6,3})$ induced by the inclusion is onto. Recall that $V^{tr}_{6,3}$ is isomorphic to $GL(3, \mathbb{C})$ and $V_{6,3}$ is isomorphic to the quotient of $GL(6, \mathbb{R})$ modulo $GL(3, \mathbb{R})$. Since $U(3)$ is a retract of $GL(3, \mathbb{C})$ and the Stiefel manifold $V^0_{6,3} = O(6)/O(3)$ of orthonormal 3-frames is a retract of $V_{6,3}$, the inclusion $V^{tr}_{6,3} \to V_{6,3}$ is homotopy equivalent to the composition α: $U(3) \to V^0_{6,3}$ of the inclusion $U(3) \to O(6)$ and the quotient projection $O(6) \to V^0_{6,3}$. Kawashima proved [8, p. 292] that $\pi_3(\alpha)$: $\pi_3 U(3) \to \pi_3(V^0_{6,3})$ is onto. This establishes our claim.

It follows that the map $\tilde{f}_1$ is homotopic to a map $\tilde{f}$: $\sigma_0 \to V^{tr}_{6,3}$ with homotopy fixed on $b\sigma_0$. Define a section f: $\sigma_0 \to E|_{\sigma_0}$ by the formula $f(p) = \phi(p, \tilde{f}(p))$ and extend it to M by letting f equal f_0 outside σ_0. The definition implies that f is a section of E^{tr} satisfying $\bar{d}(f_0, f) = \bar{d}(f_0, f_1)$. Thus f is homotopic to f_1, and therefore the immersion corresponding to f_1 is regularly homotopic to a totally real immersion. Theorem 1.4 is proved.

Proof of Proposition 1.5. Since $\mathbb{RP}^7$ is parallelizable, the bundle $E \mapsto \mathbb{RP}^7$ constructed above is trivial and hence the immersions of $\mathbb{RP}^7$ into $\mathbb{C}^7$ are classified by the homotopy classes of maps $[\mathbb{RP}^7, V_{14,7}]$. Similarly, totally real immersions are classified by $[\mathbb{RP}^7, V^{tr}_{14,7}]$. Recall that the Stiefel variety $V = O(14)/O(7)$ of orthonormal 7-frames in $\mathbb{R}^{14}$ is a retract of $V_{14,7}$ and the unitary group $U(7)$ is a retract of $V^{tr}_{14,7} \equiv GL(7, \mathbb{C})$. Let α: $U(7) \to V$ be the composition of the inclusion $U(7) \to O(14)$ and the projection $O(14) \to O(14)/O(7) = V$. To prove Proposition 1.5 it suffices to show that for every map f: $M \to U(7)$ the composition $\alpha \circ f$: $M \to V$ is null homotopic.

The projective space $\mathbb{RP}^7$ has a cellular decomposition $C_0 \subset C_1 \subset \ldots \subset C_7 = \mathbb{RP}^7$ where C_i is obtained from C_{i-1} by attaching an i-cell with a map of degree two. The quotient C_i/C_{i-1} is an i-sphere whose double is contractible in C_7/C_{i-1} when $i < 7$. For $i \leq 7$ we have

$$\pi_i U(7) = \pi_i U(\infty) = \begin{cases} \mathbb{Z}, & \text{if } i \text{ is odd;} \\ 0, & \text{if } i \text{ is even} \end{cases}$$

since we are in the stable range [2, p. 314]. It follows that every map f: $\mathbb{R}\mathbb{P}^7 \to U(7)$ is null homotopic on the 6-skeleton C_6. Contracting $f|_{C_6}$ to a point in $U(7)$ we obtain a map $\tilde{f}$: $C_7/C_6 = S^7 \to U(7)$. Since $\pi_7(\alpha)$: $\pi_7(U(7)) \to \pi_7(V)$ is the zero map [8, p. 292; 14], the composition $\alpha \circ \tilde{f}$: $C_7/C_6 \to V$ is null homotopic and consequently $\alpha \circ f$: $\mathbb{R}\mathbb{P}^7 \to V$ is null homotopic. Proposition 1.5 is proved.

References

1. Ahern, P., Rudin, W.: A totally real embedding of S^3 into $\mathbb{C}^3$. *Preprint.*
2. Bott, R.: The stable homotopy of the classical groups. *Annals of Math.* **70** (1959), 313-337.
3. Duchamp, T.: A K-theoretic classification of totally real immersion into $\mathbb{C}^n$. *Preprint.*
4. Forstnerič, F.: Some totally real embeddings of three-manifolds. *Preprint.*
5. Gromov, M.L.: Convex integration of differential relations, I. *Math. USSR Izvestija* **7**, 329–343 (1973).
6. Hirsch, M.: Immersions of manifolds. *Trans. Amer. Soc.* **93** (1959), 242–276.
7. Husemoller, D.: *Fibre Bundles.* Mc-Graw Hill, New York 1966.
8. Kawashima, T.: Some remarks on lagrangian imbeddings. *J. Math. Soc. Japan* **33** (1981), 281–294.
9. Lees, J.A.: On the classification of lagrange immersions. *Duke math. J.* **43** (1976), 217–224.
10. Milnor, J.: *Lectures on the h-cobordism Theorem.* Notes by L. Siebenmann and J. Sondow. Princeton Mathematical Notes, Princeton 1965.
11. Smale, S.: Classification of immersions of spheres in Euclidean spaces. *Annals of Math.* **69** (1959), 327–344.
12. Steenrod, N.: *The Topology of Fibre Bundles.* Princeton University Press, Princeton 1951.
13. Stiefel, E.: Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten. *Comment. Math. Helvetici* **8** (1935-36), 305–353.
14. Stout. E.L., Zame, W.R.: A Stein manifold topologically but not holomorphically equivalent to a domain in $\mathbb{C}^N$. To appear in *Adv. Math.*
15. Stout, E.L., Zame, W.R.: Totally real imbeddings and the universal covering spaces of domains of holomorphy: Some examples. *Manuscripta Math.* **50** (1985).
16. Whitney, H.: The self-intersections of a smooth n-manifold in the $2n$-space. *Annals of Math.* **45** (1944), 220–246.

Received: 15.8.85

Department of Mathematics GN-50
University of Washington
Seattle, WA 98195, USA

Institute of Mathematics, Physics and Mechanics
University E.K. of Ljubljana
Jadranska 19
61000 Ljubljana
Yugoslavia