manuscripta math. 55, 1 - 7 (1986)

manuscripta
mathematica

SOME TOTALLY REAL EMBEDDINGS OF THREE-MANIFOLDS

Franc Forstnerič *

Let V be a complex hypersurface in an open subset of $\mathbb{C}^3$, and let M be a smooth compact real hypersurface in V. Using a theorem of Gromov we prove that there exist small $\mathbf{C}^1$ perturbations $\tilde{M}$ of M in $\mathbb{C}^3$ such that $\tilde{M}$ is a totally real submanifold of $\mathbb{C}^3$. As a consequence we show that certain quotients of the three-sphere admit totally real embeddings into $\mathbb{C}^3$. In some special cases including the real projective three-space we find explicit totally real embeddings into $\mathbb{C}^3$. Our construction is similar to that of Ahern and Rudin who found a totally real embedding of the three-sphere into $\mathbb{C}^3$.

1. The main lemma

Let Ω be an open subset of $\mathbb{C}^3$, $h\colon\Omega\to\mathbb{C}$ a holomorphic function, and $\rho\colon\Omega\to\mathbf{R}$ a function of class $\mathbf{C}^k$, $k\geq 1$, such that the differential form $dh\wedge d\overline{h}\wedge d\rho$ does not vanish at any point of the set

$$M=\{x\in\Omega \mid h(x)=\rho(x)=0\}. \tag{1.1}$$

These conditions imply that M is a $\mathbf{C}^k$ real hypersurface contained in the complex hypersurface $V=\{h=0\}$ in Ω. Thus M is an orientable three-manifold whose tangent bundle $\mathrm{T}M$ contains a complex subbundle $\mathrm{T}^{\mathbb{C}}M=\mathrm{T}M\cap i\mathrm{T}M$ of complex dimension one.

Let $x=(x_1,x_2,x_3)$ be complex coordinates on $\mathbb{C}^3$. For $x\in M$ let $\mathrm{T}^{0,1}_xM$ be the space of all complex-valued tangent vectors $X=\sum_{j=1}^3 a_j\frac{\partial}{\partial\overline{x}_j}\in\mathrm{T}^{0,1}_x\mathbb{C}^3$ tangent to M. The space $\mathrm{T}^{0,1}_xM$ is the intersection of the zero spaces in $\mathrm{T}^{0,1}_x\mathbb{C}^3$ of the (0,1)-forms $\overline{\partial}\mathrm{Re}h$, $\overline{\partial}\mathrm{Im}h$ and $\overline{\partial}\rho$. Since h is holomorphic, $\overline{\partial}h=0$ whence $\mathrm{T}^{0,1}_xM$ equals the intersection of the zero space of $\overline{\partial}\,\overline{h}=\overline{\partial h}$ and $\overline{\partial}\rho$. Since the form

$$\begin{aligned} dh\wedge d\overline{h}\wedge d\rho &= \partial h\wedge\overline{\partial h}\wedge\left(\partial\rho+\overline{\partial}\rho\right)\\ &=2i\,\mathrm{Im}\left(\partial h\wedge\overline{\partial h}\wedge\overline{\partial}\rho\right)\end{aligned}$$

* *Research supported by a fellowship from the Alfred P. Sloan foundation*

is nonzero on M, it follows that $\overline{\partial h} \wedge \overline{\partial}\rho$ is nonzero on M, and thus the space $T_x^{0,1}M$ has complex dimension one for each $x \in M$. Let L be the differential operator on Ω such that

$$\overline{\partial} f \wedge \overline{\partial h} \wedge \overline{\partial}\rho = Lf \cdot d\overline{x}_1 \wedge d\overline{x}_2 \wedge d\overline{x}_3 \tag{1.2}$$

for every $\mathbf{C}^1$ function $f: \Omega \to \mathbb{C}$. Clearly L is an operator of type (0,1) on Ω, i.e., $L = \sum a_j(x) \frac{\partial}{\partial \overline{x}_j}$. The definition (2) implies that $Lh = L\overline{h} = L\rho = 0$, and therefore L is tangent to M. Hence L_x spans $T_x^{0,1}M$ for each $x \in M$.

1.1 LEMMA. *If there exists a $\mathbf{C}^1$ function $f: M \to \mathbb{C}$ for which Lf is zero-free on M, then there is a small $\mathbf{C}^1$ perturbation $\tilde{M}$ of M in $\mathbb{C}^3$ such that $\tilde{M}$ is a totally real submanifold of $\mathbb{C}^3$.*

Note. Although the operator L depends on the choice of functions h and ρ, the condition in Lemma 1.1 is independent of these choices, since a change of h and ρ amounts to multiplying L by a nonvanishing function. Equivalently, $L_x f$ should be a nonzero number for each point $x \in M$ and each nonzero vector $L_x \in T_x^{0,1}M$.

Proof. We consider the manifolds of the form

$$\tilde{M} = \{x \in \Omega \mid h(x) + \epsilon f(x) = \rho(x) = 0\}, \qquad \epsilon > 0. \tag{1.3}$$

If ϵ is sufficiently small, then $\tilde{M}$ is a $\mathbf{C}^1$ compact submanifold of $\mathbb{C}^3$ diffeomorphic to M. The manifold $\tilde{M}$ is totally real if and only if the (0,3)-form

$$\alpha = \overline{\partial}(h + \epsilon f) \wedge \overline{\partial}(\overline{h} + \epsilon\overline{f}) \wedge \overline{\partial}\rho$$

does not vanish at any point of $\tilde{M}$. We have

$$\alpha = \epsilon\, \overline{\partial} f \wedge \overline{\partial h} \wedge \overline{\partial}\rho + \epsilon^2\, \overline{\partial} f \wedge \overline{\partial f} \wedge \overline{\partial}\rho.$$

The definition (1.2) of the operator L implies

$$\epsilon^{-1}\alpha = Lf \cdot d\overline{x}_1 \wedge d\overline{x}_2 \wedge d\overline{x}_3 + O(\epsilon). \tag{1.4}$$

If Lf has no zeroes on M, then (1.4) implies that α has no zeroes on M and hence on an open neighborhood U of M in $\mathbb{C}^3$, provided that ϵ is sufficiently small. Choosing ϵ even smaller if necessary we may assume that $\tilde{M}$ is contained in U, and hence $\tilde{M}$ is totally real. This proves Lemma 1.1.

At this point a natural quetion is which manifolds of type (1.1) satisfy the hypothesis of Lemma 1.1. Although in general we do not know of any simple construction of functions f for which Lf is zero-free, we shall show in Section 2 below that the existence of such functions follows from a theorem of Gromov [3, p.331, 1.3.3]; moreover, in Section 3 we shall construct such functions explicitly in the special case when M is the quotient of the unit sphere S in $\mathbb{C}^2$ modulo a cyclic subgroup of SU(2). In the case when M equals S such functions were found by Ahern and Rudin [1].

I wish to thank Professor Edgar Lee Stout for proposing this problem and for several helpful conversations.

2. Consequences of a theorem of Gromov

Since L is a nowhere vanishing vector field of type (0,1) on M, its real and imaginary components L_1 and L_2 are linearly independent at every point $p \in M$. To every function $f = f_1 + if_2 : M \to \mathbb{C}$ we associate the section $f = (f_1, f_2)$ of the trivial bundle $X = M \times \mathbb{R}^2 \mapsto M$. The condition that the function $Lf = (L_1 f_1 - L_2 f_2) + i(L_1 f_2 + L_2 f_1)$ be nonvanishing defines an open subset Ω in the manifold X^1 of 1-jets of sections of the bundle $X \mapsto M$. A jet $f^1 \in X^1$ is called *singular* if it lies in the complement $\Sigma = X^1 \backslash \Omega$. We have to see that Σ satisfies the condition of Gromov's theorem [3, p.331, 1.3.3]. For each point $p \in M$ we can find local coordinates u_1, u_2, u_3 on M centered at p such that in these coordinates we have

$$X_1(u) = \frac{\partial}{\partial u_1} \qquad \text{and} \qquad X_2(u) = \sum_{j=1}^{3} c_j(u) \frac{\partial}{\partial u_j},$$

where $c_2(u), c_3(u) \neq 0$ for all u near 0. The components of a 1-jet $f^1 \in X^1$ of a section f of $X \mapsto M$ at the point u are its value $f(u) = \big(f_1(u), f_2(u)\big)$ and its first-order partial derivatives $a^i_j = \frac{\partial f_i}{\partial u_j}(u)$, $1 \leq i \leq 2$, $1 \leq j \leq 3$. The jet f^1 is singular if and only its components satisfy the system of linear equations

$$\begin{aligned} a^1_1 - \sum_{j=1}^{3} c_j a^2_j &= a^1_1 - c_1 a^2_1 - c_2 a^2_2 - c_3 a^2_3 = 0 \\ a^2_1 + \sum_{j=1}^{3} c_j a^1_j &= c_1 a^1_1 + a^2_1 + c_2 a^1_2 + c_3 a^1_3 = 0. \end{aligned} \tag{2.1}$$

If we fix the components of f^1 corresponding to the restriction of f to a hyperplane $u_k = \textit{constant}$, then the set of all pairs $(a^1_k, a^2_k) \in \mathbb{R}^2$ which satisfy the equations (2.1) consists of one point in the plane $\mathbb{R}^2$. In Gromov's terminology this means that the set $\Omega \subset X^1$ is ample in the coordinate directions. Therefore Gromov's theorem [3, p.331, 1.3.3] implies

2.1 THEOREM. *For every manifold M of type (1.1) there is a small C^1 perturbation $\tilde{M}$ of M of the form (1.3) such that $\tilde{M}$ is a totally real submanifold of $\mathbb{C}^3$.*

2.2 COROLLARY. *For every compact C^1 hypersurface $M \subset \mathbb{C}^2$ there is a C^1 function $f: M \to \mathbb{C}$ such that $\tilde{M} = \{(z, f(z)) \in \mathbb{C}^3 \mid z \in M\}$ is a totally real submanifold of $\mathbb{C}^3$.*

Remark. Let $\tau: M \to G$ be the Gauss map into the Grassman manifold G of real 3-dimensional subspaces of $\mathbb{C}^3$ which sends every point $x \in M$ to the tangent space $T_x M$. If M is of the form (1.1), then the image of τ is contained in the complement Σ in G of the set of totally real 3-planes. In our case one can see that Σ is a smooth submanifold of real codimension two in G with trivial normal bundle. (In dimensions higher than three the set Σ has singularities.) Therefore the map τ can be homotopically deformed into a map τ' with image in the set of totally real 3-planes. Gromov asserts that, as a consequence, there exists a regular homotopy from the initial embedding $M \mapsto \mathbb{C}^3$ into a totally real embedding of M into $\mathbb{C}^3$ [3, p.332, Corollaries], but he provides no details. Our Lemma 1.1 turns this problem into a simpler one to which we can apply the main result of Gromov.

As an application we shall consider certain quotients of the three-sphere S thought of as the unit sphere in $\mathbb{C}^2$. If Γ is a finite unitary group which acts without fixed points on S, Γ is said to be *fixed point free.* Clearly the quotient S/Γ is a smooth compact three-manifold with fundamental group Γ. There exists a complete classification of fixed point free groups which was carried out in order to solve the Clifford–Klein *spherical space form problem.* A beautiful exposition of this can be found in Chapters 5–7 of Wolf's book [9]. It is known that every finite subgroup of SU(2) is fixed point free [9, p.87–88]. If Γ is the cyclic two-group generated by the map $z \mapsto -z$ on $\mathbb{C}^2$, then S/Γ is the projective space $\mathbb{RP}^3$.

2.3 COROLLARY. *For every finite subgroup Γ of SU(2) the quotient S/Γ admits a totally real embedding into $\mathbb{C}^3$.*

Proof. The algebra A_Γ of holomorphic Γ-invariant polynomials on $\mathbb{C}^2$ is finitely generated for every finite unitary group Γ [5]. If Γ is a finite subgroup of SU(2), then A_Γ is generated by three homogeneous polynomials p_1, p_2, p_3 satisfying one polynomial relation $h(p_1, p_2, p_3) = 0$ [4, p.50–63; 6, Chapter 4]. The associated polynomial map $P = (p_1, p_2, p_3): \mathbb{C}^2 \to \mathbb{C}^3$ induces a proper holomorphic embedding of the quotient space $\mathbb{C}^2/\Gamma$ onto the affine algebraic subvariety $V = P(\mathbb{C}^2) =$

$\{x \in \mathbb{C}^3 \mid h(x) = 0\}$ [2]. Since Γ acts without fixed points on S and hence on $\mathbb{C}^2\backslash\{0\}$, the restriction $P: \mathbb{C}^2\backslash\{0\} \to V\backslash\{0\}$ is an unbranched covering projection. Hence the image $M = P(S)$ of the sphere is a smooth hypersurface in V that is diffeomorphic to the quotient S/Γ. Thus M is of the form (1.1), and Corollary 2.3 follows from Theorem 2.1.

Note. Corollary 2.3 holds for every fixed point free subgroup of U(2) whose algebra of invariant polynomials is generated by three homogeneous polynomials.

3. Some explicit totally real embeddings

Let Γ be a finite subgroup of SU(2), $P = (p_1, p_2, p_3): \mathbb{C}^2 \to \mathbb{C}^3$ its associated polynomial map, and let $M = P(S)$ be the image of the unit sphere. (See the proof of Corollary 2.3 above.) In this case we can find a function $f: M \to \mathbb{C}$ satisfying the hypothesis of Lemma 1.1 in the following way. Let $z = (z_1, z_2)$ be complex coordinates in $\mathbb{C}^2$. We denote by D the differential operator $D = z_1 \frac{\partial}{\partial \overline{z}_2} - z_2 \frac{\partial}{\partial \overline{z}_1}$ of type (0,1) on $\mathbb{C}^2$. The identity $D(z_1\overline{z}_1 + z_2\overline{z}_2) = 0$ shows that D is tangent to the unit sphere S. Since the map P is holomorphic and nondegenerate on S, the image $P_* D_z$, $z \in S$, is a nonzero vector of type (0,1) tangent to $M = P(S)$ at the point $x = P(z) \in M$; hence $P_* D_z$ is a basis of $\mathrm{T}^{0,1}_x M$. (See Section 1.) Therefore it suffices to find a $\mathbf{C}^1$ function $f: M \to \mathbb{C}$ such that $P_* D_z f \neq 0$ for each $z \in S$. Using the identity $P_* D_z f = D_z(f \circ P)$ we see that the function $D(f \circ P)$ should be zero-free on S. Since we can write every Γ-invariant function $g: S \to \mathbb{C}$ in the form $g = f \circ P$ for some $f: M \to \mathbb{C}$, it suffices to find a Γ-invariant function $g(z)$ on S for which Dg is zero-free on S.

We shall find such function in the case when Γ is a finite cyclic subgroup of SU(2). If Γ has order n, we can choose coordinates z_1, z_2 in $\mathbb{C}^2$ such that in these coordinates Γ is generated by the matrix $\left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{smallmatrix}\right)$, $\epsilon = e^{2\pi i/n}$ [4, p.40]. The polynomials $p_1(z) = z_1^n$, $p_2(z) = z_2^n$, and $p_3(z) = z_1 z_2$ generate its algebra of invariants [4, p53; 6, p.95]. If we let $a(z) = \overline{z_1 z_2} = \overline{p}_3$ and $b(z) = \overline{z}_1 z_2^n \overline{z}_2^{n+1} = \overline{p}_3 p_2^n \overline{p}_2^n$, then the functions

$$\alpha(z) = Da(z) = |z_1|^2 - |z_2|^2$$

and

$$\beta(z) = Db(z) = |z_2|^{2n}\left((n+1)|z_1|^2 - |z_2|^2\right)$$

are real-valued and have no common zero on the sphere S. Therefore $g = a + ib$ is a Γ-invariant function on S for which Dg has no zeroes on S. Hence the function $f(x) = \overline{x}_3(1 + i|x_2|^{2n})$ on $\mathbb{C}^3$ satisfies the hypothesis of Lemma 1.1 on the submanifold $M = P(S)$ of $\mathbb{C}^3$. In the case when Γ is the trivial group such functions were found by Ahern and Rudin [1].

Example. Let Γ be the cyclic two-group generated by the map $z \mapsto -z$; the quotient S/Γ is the real projective space $\mathbf{RP}^3$. The map $P(z) = \left(z_1^2, z_2^2, \sqrt{2}\, z_1 z_2\right)$ maps the sphere S onto the intersection M of the algebraic variety $P(\mathbb{C}^2) = \{x \in \mathbb{C}^3 \mid x_3^2 = 2x_1x_2\}$ with the unit sphere $\{|x| = 1\}$ in $\mathbb{C}^3$, and P induces a diffeomophism of $S/\Gamma \approx \mathbf{RP}^3$ onto M. If we let $f(x) = \overline{x}_3\left(1 + i|x_2|^4\right)$, then for small $\epsilon > 0$ the set

$$\tilde{M} = \left\{x \in \mathbb{C}^3 \mid |x| = 1,\ x_3^2 = 2x_1x_2 + \epsilon f(x)\right\}$$

is a totally real submanifold of $\mathbb{C}^3$ diffeomorphic to $\mathbf{RP}^3$.

The reason for considering cyclic subgroups of SU(2) is that they have very simple basis of invariant polynomials. We do not know of a systematic method to find a function $g \in A_\Gamma$ satisfying $Dg \neq 0$ on S for other subgroups of SU(2). It would be of interest to find a more direct construction of functions f satisfying the condition of Lemma 1.1, at least in the case when M is a compact hypersurface in $\mathbb{C}^2$.

The results of this paper only concern three-dimensional manifolds. In a different direction, Stout and Zame proved [7] that the seven sphere S^7 does not admit a totally real embedding into $\mathbb{C}^7$; this shows that Corollary 2.2 is false in dimension 7. It is unknown whether the real projective seven space $\mathbf{RP}^7$ embeds totally real into $\mathbb{C}^7$. If n is different from $1, 3$ or 7, the sphere S^n and the projective space $\mathbf{RP}^n$ do not admit totally real embeddings into $\mathbb{C}^n$. For these and related results see the survey in [8]. This shows that Corollary 2.2 only holds in dimensions one and three.

References

1. AHERN, P., RUDIN, W.: A totally real embedding of S^3 into $\mathbb{C}^3$. *Preprint*
2. CARTAN, H.: Quotient d'un espace analytique par un groupe d'automorphismes. *Algebraic Geometry and Topology*, a Symposium in honor of S. Lefschetz, 90–102 (1957); or Oeuvres **II, 51,** 789–811
3. GROMOV, M.L.: Convex integration of differential relations, I. *Math. USSR Izvestija* **7**, 329–343 (1973)
4. KLEIN, F.: *Lectures on the Icosahedron and the Solution of Equations of fifth Degree.* London: K. Paul 1913
5. NOETHER, E.: Der Endlichkeitssatz der Invarianten endlicher Gruppen. *Math. Ann.* **77**, 89–92 (1916)

6. SPRINGER, T.: Lectures on Invariant Theory. *Springer Lecture Notes in Mathematics* **585**, Berlin-Heidelberg-New York: Springer-Verlag 1977
7. STOUT, E.L., ZAME, W.R.: A Stein manifold topologically but not holomorphically equivalent to a domain in $\mathbb{C}^N$. To appear in *Adv. Math.*
8. STOUT, E.L., ZAME, W.R.: Totally real imbeddings and the universal covering spaces of domains of holomorphy: Some examples. *Manuscripta Math.* **50** (1985)
9. WOLF, J.A.: *Spaces of Constant Curvature*. New York: McGraw Hill 1967

Franc Forstnerič
Mathematics Department GN-50
University of Washington
Seattle, WA 98195, U.S.A.

(Received April 24, 1985)