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Polynomially Convex Hulls with Piecewise Smooth Boundaries*

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0. Introduction

We denote by D the unit disk $\{z \in \mathbb{C}: |z| < 1\}$, by bD the unit circle and by $\pi: \mathbb{C}^{m+1} \to \mathbb{C}$ the projection $(z, z_1, \ldots, z_m) \to z$. Let M be a compact subset of $\mathbb{C}^{m+1}$ such that $\pi(M) = bD$, and for each $z \in bD$ the set

$$M(z) = \{w \in \mathbb{C}^m: (z, w) \in M\} \tag{0.1}$$

is the boundary of a compact convex set $Y(z) \subset \mathbb{C}^m$ with nonempty interior. Set $Y = \bigcup_{z \in bD} \{z\} \times Y(z)$. Alexander and Wermer [1] and Slodkowski [8] proved independently that each point p in the polynomial convex hull $\hat{M}$ of M, $\pi(p) \in D$, lies in the graph of a bounded analytic disk $F(z) = (z, f_1(z), \ldots, f_m(z))$ ($f_j \in H^\infty(D)$ for $1 \leqq j \leqq m$) with image in $\hat{M}$. Alexander and Wermer obtained a more precise information in the case when $m = 1$ and each fiber $M(z)$ is a circle [1, Theorems 2 and 3].

In this paper we shall assume the existence of an analytic disk $F_0(z) = (z, f_0(z))$ on D, continuous on $\bar{D}$, such that $f_0(z)$ is an interior point of $Y(z)$ for each $z \in bD$. Then each bounded analytic disk $F(z) = (z, f(z))$ in $\hat{M}$ which passes through a boundary point of $\hat{M}$ lying over D is entirely contained in the boundary of $\hat{M}$ and its cluster set $\overline{F(D)} \setminus F(D)$ is contained in M (Theorem 1.1). If $m = 1$ and M is a smooth submanifold of $\mathbb{C}^2$ as above, then the hull $\hat{M}$ has piecewise smooth boundary and is filled by the images of analytic disks with smooth boundary values contained in M (Theorem 1.2). We also obtain a uniqueness result for the representing measures (Corollary 1.3).

1. Results

We denote by $A(D)$ the algebra of all continuous functions on $\bar{D}$ that are holomorphic on D. If $k \in \mathbb{Z}_+$ and $0 < c < 1$, we denote by $A^{k,c}(D)$ the set of all $f \in A(D)$ which are k times continuously differrentiable on $\bar{D}$ and whose derivative

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of order k satisfies the Hölder condition of order c on D. $A^{k-0}(D)$ is the intersection of all spaces $A^{k-1,c}(D)$ for $0<c<1$. Our main results are the following.

1.1. Theorem. *Let M be a compact subset of $\mathbb{C}^{m+1}$ such that $\pi(M)=bD$, and for each $z\in bD$ the set $M(z)$ defined by (0.1) is the boundary of a compact convex set $Y(z)$ with nonempty interior. Assume that there exists an analytic disk $F_0(z)=(z,f_0(z))$, $f_0\in A(D)^m$, such that $f_0(z)$ is an interior point of $Y(z)$ for each $z\in bD$. If $F(z)=(z,f(z))$ is a bounded analytic disk in $\hat{M}$ such that $F(z_0)$ is a boundary point of $\hat{M}$ for some $z_0\in D$, then $F(D)$ is contained in the boundary of $\hat{M}$ and $\overline{F(D)}\setminus F(D)$ is contained in M.*

1.2. Theorem. *Let M be a compact submanifold of $\mathbb{C}^2$ of class C^k, $k\geqq 2$, satisfying the hypotheses of Theorem 1.1. For each point (a,b) in $\hat{M}$, $a\in D$, there exists an analytic disk $F(z)=(z,f(z))$ in $\hat{M}$, $f\in A^{k-0}(D)$, satisfying $f(a)=b$ and $f(z)\in M(z)$ for each $z\in bD$. If $k\geqq 3$, then the set $S=b\hat{M}\cap\pi^{-1}(D)$ is a real hypersurface of class C^{k-1} in $\mathbb{C}^2$ foliated by analytic disks, and $S\cup M$ is a manifold with boundary of class C^s, where s is the integer part of $(k-1)/2$.*

We do not know whether the loss of smoothness in S is indeed the case or it is a consequence of our techniques. Each individual leaf of S is of class $A^{k-0}(D)$.

Denote by $P(\hat{M})$ the closure of the set of restrictions of holomorphic polynomials to $\hat{M}$ in the algebra $C(\hat{M})$ of all continuous functions on $\hat{M}$. For every point $p\in\hat{M}$ there exists a positive Borel measure m_p of mass 1 supported on Y which represents p in the sense that $\int_Y g\cdot dm_p=g(p)$ for all $g\in P(\hat{M})$ (see [9, p. 106]). Such representing measure is not unique in general. Suppose now that each fiber $Y(z)$, $z\in bD$, is strictly linearly convex in the sense that for each $b\in bY(z)$ there is a real line l_b in $\mathbb{C}$ such that $l_b\cap Y(z)=\{b\}$. The construction of bounded analytic disks in $\hat{Y}$ given in [1], together with Theorem 1.2 above, imply

1.3. Corollary. *Let M be a C^3 submanifold of $\mathbb{C}^2$ satisfying the hypotheses of Theorem 1.2 and assume that each $Y(z)$ is strictly linearly convex. Then each point p in the boundary of $\hat{M}$, $\pi(p)\in D$, admits a unique positive representing measure for the algebra $P(\hat{M})$ supported on Y. This unique representing measure is indeed supported on M.*

One may ask whether we could remove the hypothesis that there be an analytic disk $F_0(z)=(z,f_0(z))$ in $\hat{M}$, continuous on $\bar{D}$, such that $f_0(z)$ is an interior point of $Y(z)$ for each $z\in bD$. The following examples show that at least some hypothesis of this kind is required. For each $c\geqq 0$ and $z\in bD$ let $M_c(z)$ be the circle of radius one centered at the point $c\bar{z}\in\mathbb{C}$, and let $M_c=\bigcup_{z\in bD}\{z\}\times M_c(z)$. If $c>1$, the corresponding set Y_c is polynomially convex. If $c=1$, the hull of M_1 equals $Y_1\cup(D\times\{0\})$ and so the fibers of $\hat{M}_1$ change discontinuously at points of bD. See Proposition 14 in [5] for further details.

A similar hypothesis was used by Globevnik in [6] to prove a result on the extension of analytic selections. On the other hand, Alexander and Wermer used the hypothesis that there exist two distinct bounded analytic disks $F_j(z)=(z,f_j(z))$ $(j=1,2)$ in $\hat{M}$ [1, Theorems 2 and 3] in order to obtain a description of the hull when each $M(z)$ is a circle.

We will show (Proposition 3.1) that at least in the case when $M \subset \mathbb{C}^2$ is of class C^2 the assumption on the existence of the disk F_0 is equivalent to the assumption that there exist two disks F_1, F_2 in $\hat{M}$ which are continuous on $\bar{D}$. However, we do not know whether it would suffice to assume only that there exist two distinct bounded analytic disks in $\hat{M}$.

It would be interesting to know whether Theorem 1.2 holds in the case when each fiber $Y(z) \subset \mathbb{C}$ is simply connected but not necessarily convex.

2. Proof of Theorem 1.1

If we approximate the map $f_0 \in A(D)^m$ close enough on $\bar{D}$ by a polynomial map $f = (f_1, \ldots, f_m)$ and apply the polynomial change of coordinates $(z, z_1, \ldots, z_m) \to (z, z_1 - f_1(z), \ldots, z_m - f_m(z))$ on $\mathbb{C}^{m+1}$, we may assume that $0 \in \mathbb{C}^m$ is an interior point of each fiber $Y(z)$, $z \in bD$. Since M is closed, the set $Y \setminus M$ is open in $bD \times \mathbb{C}^m$. Hence there is an $r > 0$ such that the ball $\mathbb{B}(0, r) \subset \mathbb{C}^m$ is contained in each fiber $Y(z)$, $z \in bD$.

Denote by $\hat{M}(z)$ the fiber of the hull $\hat{M}$ over $z \in \bar{D}$. If $z \in bD$, then $\hat{M}(z) = Y(z)$. The ball $\mathbb{B}(0, r)$ is also contained in $\hat{M}(z)$ for each $z \in \bar{D}$. Recall that for each $a \in D$ and $b \in \hat{M}(a)$ there exists an $f = (f_1, \ldots, f_m) \in H^\infty(D)^m$ such that $f(a) = b$, and for almost every $z \in bD$ the nontangential boundary value of f at z lies in the set $Y(z)$ [1,8]. Taking convex combinations of maps passing through different points of $\hat{M}(a)$ we conclude that each fiber $\hat{M}(a)$ is convex.

2.1. Lemma. *If $a \in \bar{D}$ and b is an interior point of $\hat{M}(a)$, then there are* $\varepsilon > 0$ and $\delta > 0$ such that the ball $\mathbb{B}(b, \varepsilon)$ *is contained in $\hat{M}(z)$ whenever $z \in \bar{D}$ and $|z - a| < \delta$.*

Proof. We shall distinguish two cases.

Case 1. $a \in D$. There is a $t > 1$ such that $tb \in \hat{M}(a)$. There exists a bounded analytic disk $F(z) = (z, f(z))$ in $\hat{M}$ passing through the point (a, tb). Then the disk $F_1(z) = (z, f(z)/t)$ passes through (a, b) and has boundary values in the set with fibers $1/t \cdot Y(z)$, $z \in bD$.

Choose $C > 0$ sufficiently large such that $Y(z) \subset \mathbb{B}(0, C)$ for each $z \in bD$. There is an $\varepsilon > 0$ with the property that for all $w \in \mathbb{B}(0, C)$ the ball $\mathbb{B}(w/t, 2\varepsilon)$ is contained in the convex hull of the set $\{w\} \cup \mathbb{B}(0, r)$. Since $Y(z)$ is convex and contains $\mathbb{B}(0, r)$, we have $\mathbb{B}(w/t, 2\varepsilon) \subset Y(z)$ when $z \in bD$ and $w \in Y(z)$. Consequently the nontangential boundary values of the disk $z \to (z, f(z)/t + b')$ at bD lie in Y for each $b' \in \mathbb{C}^m$, $|b'| < 2\varepsilon$, and so the disk lies in $\hat{M}$. Thus the ball $\mathbb{B}(f(z)/t, 2\varepsilon)$ is contained in $\hat{M}(z)$ for each $z \in D$. If we choose $\delta > 0$ such that $|f(z)/t - b| < \varepsilon$ when $|z - a| < \delta$, then $\mathbb{B}(b, \varepsilon) \subset \hat{M}(z)$ when $|z - a| < \delta$ and Lemma 2.1 is proved in this case.

Case 2. $a \in bD$. It suffices to find an analytic disk $F(z) = (z, f(z))$ in $\hat{M}$, continuous on $\bar{D}$, such that $f(z)$ is an interior point of $Y(z)$ for each $z \in bD$ and $f(a) = b$. The proof can then be concluded as above.

Choose an open neighborhood V of the segment $\{tb \in \mathbb{C}^m : 0 \leqq t \leqq 1\}$ in $\mathbb{C}^m$ and a $\delta > 0$ such that V is contained in $Y(z)$ for each $z \in bD$ satisfying $|z - a| < \delta$. Let Ω be a small open neighborhood of the interval $(0, 1) \subset \mathbb{R}$ in the disk D such that $\zeta b \in V$

for all $\zeta \in \Omega$. There is a conformal map g from D onto a subset Ω_1 of Ω, $g \in A(D)$, satisfying

(i) $g(a) = \|g\| = 1$, and

(ii) $|g(z)| < r_1$ if $|z-a| \geqq \delta$.

The analytic disk $F(z) = (z, g(z)\,b)$ clearly satisfies the above requirement provided that $r_1|b| \leqq r$. Lemma 2.1 is proved.

Denote by $S(z)$ the boundary of $\hat{M}(z)$ for each $z \in D$, and let $S = \bigcup_{z \in D} \{z\} \times S(z)$. Lemma 2.1 implies that S is the part of the boundary of $\hat{M}$ over D and $\bar{S} \subset S \cup M$. Moreover, the interior of $\hat{M}$ equals the union of the relative interiors of the fibers $\hat{M}(z)$:

$$\operatorname{Int} \hat{M} = \bigcup_{z \in D} \{z\} \times \operatorname{Int} \hat{M}(z). \tag{2.1}$$

2.2. Lemma. *If $F(z) = (z, f(z))$ is a bounded analytic disk in $\hat{M}$ such that $f(z_0) \in S(z_0)$ for some $z_0 \in D$, then $f(z) \in S(z)$ for all $z \in D$.*

Proof. From (2.1) it follows that the set

$$D' = \{z \in D \colon f(z) \in \operatorname{Int} \hat{M}(z)\}$$

is open. If Lemma 2.2 does not hold, then D' is nonempty. Note that $D' \neq D$ by hypothesis. Choose a boundary point $a \in (\bar{D}' \setminus D') \cap D$ and a $\varrho > 0$ such that the intersection of the circle $bD(a, \varrho)$ with D' contains a closed arc $L' \subset bD(a, \varrho)$ with nonempty interior. Let L be the closure of $bD(a, \varrho) \setminus L'$. Since L' is contained in D', there is an $\varepsilon > 0$ such that for all $z \in L'$ we have $w \cdot f(z) \in \hat{M}(z)$ for each $w \in \mathbb{C}$ satisfying $|w-1| < \varepsilon$.

Denote by K_α the cone

$$K_\alpha = \{w = x + iy \in D \colon |y| \leqq \alpha(1-x) \text{ and } x \geqq -\alpha\}.$$

If $\alpha > 0$ is sufficiently small, the set $K_\alpha \cdot b$ is contained in the convex hull of $\{b\} \cup \mathbb{B}(0, r)$ for each $b \in \hat{M}(z)$, $z \in \bar{D}$. Therefore $K_\alpha \cdot \hat{M}(z)$ is contained in $\hat{M}(z)$ for each $z \in \bar{D}$. Choose a holomorphic function g on $D(a, \varrho)$, continuous on $\bar{D}(a, \varrho)$, satisfying

(i) $g(a) = 1$,

(ii) $g(L) \subset K_\alpha$, and

(iii) $|g(z) - 1| < \varepsilon$ when $z \in L'$.

To construct g we choose a continuous function $h \colon L \to K_\alpha$, we approximate $(h(z)-1)/(z-a)$ on L by a holomorphic polynomial $h_1(z)$ and set $g(z) = 1 + \delta(z-a)\,h_1(z)$ for a sufficiently small $\delta > 0$. If $c > 1$ is close to 1, then $cg(z)$ also satisfies (ii) and (iii) above. Hence the analytic disk $G(z) = (z, cg(z)\,f(z))$, $z \in \bar{D}(a, \varrho)$, has boundary in $\hat{M}$, i.e., $G(z) \in \hat{M}$ when $|z-a| = \varrho$. Thus $G(a) = (a, cb)$ also lies in $\hat{M}$. This is a contradiction since $c > 1$ and $b \in S(a)$. Therefore D' is empty and Lemma 2.2 is proved.

Theorem 1.1 immediately follows from Lemma 2.2 and from the fact $\bar{S} \subset S \cup M$ that follows from Lemma 2.1.

3. Proof of Theorem 1.2 and Corollary 1.3

We shall first show that through each point (a, b) in $\hat{M}$, $a \in D$, passes an analytic disk $F(z) = (z, f(z))$ in $\hat{M}$ with boundary values $\overline{F(D)} \setminus F(D)$ contained in M. Since M is a totally real submanifold of $\mathbb{C}^2$ of class $\mathbf{C}^k$, each such disk is of class $A^{k-0}(D)$ according to [4, p. 293], [3, 7].

If (a, b) is a boundary point of $\hat{M}$, such a disk exists by Theorem 1.1. Suppose now that b is an interior point of $\hat{M}(a)$. As in the proof of Theorem 1.1 we may assume that 0 is an interior point of each $Y(z)$, $z \in bD$, and so an interior point of each $\hat{M}(z)$, $z \in \bar{D}$. Choose a point $b' \in S(a)$ such that $b = tb'$ for some $0 \leqq t < 1$ and let $(z, g(z))$ be an analytic disk in $\hat{M}$ passing through (a, b'). Then the disk $(z, tg(z))$ passes through (a, b), and for each $z \in bD$ the point $t \cdot g(z)$ lies in the interior of $Y(z)$. If we approximate $t \cdot g(z)$ close enough on $\bar{D}$ by a holomorphic polynomial $h(z)$ and change the coordinates on $\mathbb{C}^2$ by $(z, w) \to (z, w - h(z))$, we may assume $b = 0$. Every disk passing through $(a, 0)$ is of the form $F(z) = (z, (z-a) f(z))$. The boundary of F lies in M if and only if $f(z) \in 1/(z-a) \cdot M(z) = N(z)$ for each $z \in bD$. The set $N = \bigcup_{z \in bD} \{z\} \times N(z)$ satisfies the conditions of Theorem 1.1, and therefore such functions $f(z)$ exist. This proves the existence of disks through $(a, 0)$ with boundary values in M.

In order to show the regularity of the hypersurface $S = b\hat{M} \cap \pi^{-1}(D)$ we will use the results of [5] to prove that the disks passing through nearby points of S are tied smoothly together. Let $k \geqq 3$. Fix a disk $F(z) = (z, f(z))$ with boundary in M and denote by $\mathrm{Ind}_F M$ the index of M along the closed path F: $bD \to M$ as defined in [5, Definition 1]. Notice that $f(z) \neq 0$ for each z in bD and hence the restriction of f to bD has a well-defined winding number $\mathrm{W}(f)$ around the origin.

3.1. Lemma. $\mathrm{Ind}_F M = 1 + \mathrm{W}(f)$.

Proof. Let α: $bD \to M$ be the unique path in M of the form $\alpha(z) = (z, a(z))$ for which $a > 0$, and let β: $bD \to M$ be a parametrization of the fiber $M(1)$ in the positive direction. Then the homology classes $[\alpha]$ and $[\beta]$ generate the first homology group $\mathbf{H}_1 M$ and we have

$$[F] = [\alpha] + \mathrm{W}(f) \cdot [\beta]. \tag{3.1}$$

Recall that the index $\mathrm{Ind}_\gamma M$ only depends on the homology class of γ in M and the induced map Ind: $\mathbf{H}_1 M \to \mathbb{Z}$ is a homomorphism of groups. (see [5, Sect. 2].) Thus (3.1) implies

$$\mathrm{Ind}_F M = \mathrm{Ind}_\alpha M + \mathrm{W}(f) \cdot \mathrm{Ind}_\beta M.$$

To prove Lemma 3.1 it suffices to show that $\mathrm{Ind}_\alpha M = \mathrm{Ind}_\beta M = 1$. This follows most easily from the following observation. There is a regular homotopy of M into the standard torus $T^2 = \{(z, w) \in \mathbb{C}^2 \colon |z| = |w| = 1\}$ obtained by deforming each fiber $M(z)$ radially into the circle $\{|w| = 1\}$. This regular homotopy carries α into the path $\gamma(e^{i\theta}) = (e^{i\theta}, 1)$ and β into a path homologous in T^2 to $\delta(e^{i\theta}) = (1, e^{i\theta})$. The index is invariant under a regular homotopy through totally real immersed submanifolds [5, Sect. 2], and a trivial computation shows $\mathrm{Ind}_\gamma T^2 = \mathrm{Ind}_\delta T^2 = 1$. (See also Example 1 in [5, Sect. 7].) This implies $\mathrm{Ind}_\alpha M = \mathrm{Ind}_\beta M = 1$ and Lemma 3.1 is proved.

Suppose now that $F(z) = (z, f(z))$ is an analytic disk in S. Then $f(z) \neq 0$ for each $z \in \bar{D}$ since 0 is an interior point of $\hat{M}(z)$, and therefore f has winding number zero. Lemma 3.1 implies $\mathrm{Ind}_F M = 1$. By [5, Theorem 3] there is an embedding σ: $\bar{D} \times (-1, 1) \to \mathbb{C}^2$ of class $\mathbf{C}^s$, where s is the integer part of $(k-1)/2$, such that

(i) $\sigma(z, 0) = F(z)$, and

(ii) for each fixed $t \in (-1, 1)$ the map $z \to \sigma(z, t)$ is an analytic disk with boundary in M.

See also [2, p. 536] and [3, p. 993]. The image Σ of σ is an embedded, Levi flat hypersurface in $\mathbb{C}^2$ with boundary in M.

The map σ is of class $\mathbf{C}^{k-1}$ on $D\times(-1,1)$ [5] and hence Σ is of class $\mathbf{C}^{k-1}$ away from M. We can parametrize Σ by a map

$$\Phi(z,t)=(z,\ \varphi(z,t)),\ z\in\bar{D},\ t\in(-1,1) \tag{3.2}$$

which is holomorphic in $z\in D$ and is uniquely determined by Σ up to a diffeomorphism in the t variable.

The next step is to show that Σ is contained in the boundary of M. Let φ be as in (3.2). For each fixed $t=t_0$ the function $\psi(z)=\frac{\partial\varphi}{\partial t}(z,t_0)/\varphi(z,t_0)$ in $\mathrm{A}(D)$ is not real-valued at any point $z\in bD$ since $\varphi(z,t_0)$ lies in the convex curve $M(z)$ enveloping 0 and $\partial\varphi/\partial t$ is nonvanishing. Hence $\psi(z)$ is not real-valued at any point $z\in\bar{D}$ either. This implies that for each $(z,w)\in\Sigma$ the real half-line $r\to(z,rw)$, $r>0$, intersects Σ transversely at the point (z,w). The open regions in $\bar{D}\times\mathbb{C}$ defined by

$$\Sigma^-=\{(z,rw)\colon (z,w)\in\Sigma,\ 0<r<1\},$$
$$\Sigma^+=\{(z,rw)\colon (z,w)\in\Sigma,\ r>1\}$$

both contain Σ in the boundary and are pseudoconvex along Σ since Σ is Levi flat.

Suppose that there is a point $(z_0,\varphi(z_0,t_0))\in\Sigma$ contained in the interior of $\hat{M}$. We may assume that $w_0=\varphi(z_0,t_0)\neq 0$. If $c>1$ is close to 1, the point (z_0,cw_0) is also contained in $\hat{M}$ and hence there is an analytic disk $F(z)=(z,f(z))$ in $\hat{M}$, continuous on $\bar{D}$, passing through (z_0,cw_0). Replacing F by a convex combination $t\cdot F+(1-t)\cdot\Phi(.,t_0)$ for a sufficiently small $t>0$ we obtain a disk F satisfying:

(i) $F(\bar{D})\subset\Sigma^-\cup\Sigma\cup\Sigma^+$,

(ii) $F(bD)\subset\Sigma^-\cup\Sigma$, and

(iii) $F(z_0)\subset\Sigma^+$.

The family of disks $F_r(z)=(z,r\cdot f(z))$, $0<r<1$, provides a contradiction to the Kontinuitätssatz. Therefore Σ is contained in the boundary of $\hat{M}$ as claimed.

It follows that there is a neighborhood U of $\overline{F(D)}$ in $\bar{D}\times\mathbb{C}$ such that $(S\cup M)\cap U=\Sigma\cap U$, and so S is smooth near the disk $\overline{F(D)}$. Since S is the union of images of such disks according to Theorem 1.1, it is smooth of class $\mathbf{C}^{k-1}$ everywhere and $S\cup M$ is a manifold with boundary. This concludes the proof of Theorem 1.2.

Proof of Corollary 1.3. Choose a point p in $\hat{M}$ over D and a positive representing measure μ for p supported on Y. We recall briefly the construction due to Alexander and Wermer [1] of a bounded analytic disk in $\hat{M}$ passing through p. By a change of coordinates we may assume that $p=0$. For almost every $e^{i\theta}\in bD$ there is a probability measure μ_θ supported on $Y(e^{i\theta})$ such that

$$\int_0^{2\pi}\frac{d\theta}{2\pi}\cdot\int_{Y(e^{i\theta})}g(e^{i\theta},w)\cdot d\mu_\theta(w)=\int_Y g\cdot d\mu=g(0)$$

for each holomorphic polynomial $g(z,w)$. The function

$$f(e^{i\theta})=\int_{Y(e^{i\theta})}w\cdot d\mu_\theta(w), \tag{3.3}$$

which is defined a.e. on bD, extends to a bounded holomorphic function on D, and $f(0)=0$. Thus $F(z)=(z,f(z))$ is a bounded analytic disk passing through $p=0$. Since $Y(e^{i\theta})$ is convex and μ_θ is a probability measure, the point $f(e^{i\theta})$ lies in $Y(e^{i\theta})$.

Suppose now that $p=0$ is a boundary point of $\hat{M}$ and M is of class $\mathbf{C}^3$. By Theorem 1.2 the disk F in $\hat{M}$ passing through p is unique and has boundary in M. Thus $f(e^{i\theta})$ is a boundary point of the strictly linearly convex set $Y(e^{i\theta})$. Since μ_θ is a probability measure, (3.3) implies that μ_θ is the point mass at $f(e^{i\theta})$. This proves that the representing measure μ for p is unique and is supported on M. Corollary 1.3 is proved.

The following Proposition shows that one of the hypotheses in Theorem 1.2 is equivalent to a seemingly weaker hypothesis.

3.1. Proposition. *Let M be as in Theorem 1.2. Then the following are equivalent.*

(i) *There is an analytic disk $F_0(z)=(z,f_0(z))$ in $\hat{M}$ which is continuous on $\bar{D}$ such that $f_0(z)$ is an interior point of $Y(z)$ for each $z\in bD$.*

(ii) *There are two distinct analytic disks $F_j(z)=(z,f_j(z))$ $(j=1,2)$, continuous on $\bar{D}$, with images in $\hat{M}$.*

Proof. Clearly (i) implies (ii) since the disks $z\to(z,f_0(z)+\varepsilon)$ lie in $\hat{M}$ for all small values of ε. To prove that (ii) implies (i) we shall need the following well known fact:

If $n\colon bD\to\mathbb{C}\setminus\{0\}$ is in the Lipschitz class $\mathbf{C}^\alpha$ for some $\alpha>0$, then the following are equivalent:

(a) The winding number of n around 0 is nonnegative.

(b) There exists a real-valued continuous function r on bD, r not identically zero, such that the product $r\cdot n$ is the boundary value of a function $f\in A(D)$.

If (a) holds, there is a positive function r satisfying (b).

Let f_1 and f_2 be as in (ii) above. Define $g(z)=(f_1(z)+f_2(z))/2$ $(z\in\bar{D})$ and $E=\{z\in bD\colon g(z)\in M(z)\}$. Note that $g(z)\in Y(z)$ for each $z\in bD$ since the fibers $Y(z)$ are convex. For $z\in E$ we denote by $n(z)$ the unit inward normal to the curve $M(z)$ at the point $g(z)$. We distinguish two cases.

Case 1: $E\neq bD$.

Since M is of class $\mathbf{C}^2$, there is an open neighborhood E_1 of E in bD such that $\bar{E}_1\neq bD$, and for each $z\in E_1$ the point $g(z)$ has a unique closest point $\tilde{g}(z)\in M(z)$. Extend n to E_1 by taking $n(z)$ to be the unit inward normal to $M(z)$ at the point $\tilde{g}(z)$. Finally we extend n from E_1 to a $\mathbf{C}^1$ function on bD with winding number around 0 equal zero. Choose a function $r\colon bD\to(0,\infty)$ such that $r\cdot n$ is the boundary value of a holomorphic function on D. The function $f_0=g+\varepsilon\cdot r\cdot n$ satisfies Proposition 3.1 (i) provided that $\varepsilon>0$ is sufficiently small.

Case 2: $E=bD$.

If the winding number of $n\colon bD\to bD$ is nonnegative, the proof can be completed as in Case 1. Suppose now that the winding number $W(n)$ of n is negative. The difference $f_1(z)-f_2(z)$ is orthogonal to $n(z)$ and thus equals $i\cdot r(z)\cdot n(z)$ for some continuous, real-valued function r on bD. Since f_1-f_2 is holomorphic on D and $W(i\cdot n)=W(n)<0$, the equivalence of (a) and (b) above implies $r\equiv 0$ and hence $f_1=f_2$. This is a contradiction to the assumption that the disks F_1 and F_2 are distinct. Proposition 3.1 is proved.

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