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# STABILITY OF POLYNOMIAL CONVEXITY OF TOTALLY REAL SETS

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ABSTRACT. We show that certain compact polynomially convex subsets of $\mathbf{C}^n$ remain polynomially convex under sufficiently small $\mathbf{C}^2$ perturbations.

**1. Statement of the results.** Let $M$ be a Stein manifold. Denote by $\mathcal{O}(M)$ the algebra of all holomorphic functions on $M$ with the standard topology of uniform convergence on compact subsets. A compact subset $K$ of $M$ is said to be $\mathcal{O}(M)$-convex if for every point $x \in M \setminus K$ there is a holomorphic function $f \in \mathcal{O}(M)$ such that

$$|f(x)| > \sup_{y \in K} |f(y)|.$$

Since the holomorphic polynomials are dense in the algebra $\mathcal{O}(\mathbf{C}^n)$ of holomorphic functions on $\mathbf{C}^n$, an $\mathcal{O}(\mathbf{C}^n)$-convex subset of $\mathbf{C}^n$ is just a polynomially convex subset.

Given a compact $\mathcal{O}(M)$-convex subset $K$ of $M$, an open neighborhood $U$ of $K$ and a $\mathbf{C}^k$ diffeomorphism $\Psi$ of $U$ onto an open subset $\Psi(U)$ in $M$, we ask whether the set $\Psi(K)$ is also $\mathcal{O}(M)$-convex provided that $\Psi$ is sufficiently close to the identity on $U$ in the $\mathbf{C}^k$ sense. In other words, is $\mathcal{O}(M)$-convexity a stable property under smooth perturbations? In general this is not so as the following example shows.

EXAMPLE 1. *Let* $M = \mathbf{C}^2$ and $K = \{(z,0) \in \mathbf{C}^2\colon |z| \leqslant 1\}$. Clearly $K$ is convex and hence polynomially convex. The diffeomorphisms $\Psi_\varepsilon\colon \mathbf{C}^2 \to \mathbf{C}^2$ given by

$$\Psi_\varepsilon(z,w) = \left(z, w + \varepsilon|z|^2\right), \qquad \varepsilon \geqslant 0,$$

are close to the identity in the $\mathbf{C}^\infty$ sense for small $\varepsilon$, but the set

$$\Psi_\varepsilon(K) = \left\{\left(z, \varepsilon|z|^2\right)\colon |z| \leqslant 1\right\}$$

is not polynomially convex for any $\varepsilon > 0$ since it contains the boundary of the analytic disk $\Delta_{\delta,\theta} = \{(z, \varepsilon\delta^2 e^{i\theta})\colon |z| \leqslant \delta\}$ for each $\delta \in [0,1]$ and $\theta \in \mathbf{R}$. These disks fill an open subset of $\mathbf{C}^2$ that is contained in the polynomial hull of $\Psi_\varepsilon(K)$ according to the maximum principle.

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Recall that a $\mathbf{C}^1$ submanifold $\Sigma$ of a complex manifold $M$ is called totally real if for each point $x \in \Sigma$ the tangent space $T_x\Sigma$ contains no nontrivial complex subspace. If $K$ is a compact subset of a totally real submanifold $\Sigma$, then by [**2**, p. 300] there is an open neighborhood $U$ of $K$ in $M$ and a $\mathbf{C}^2$ strictly plurisubharmonic function $\rho\colon U \to \mathbb{R}_+$ such that

$$(1.1) \quad K = \{x \in U \mid \rho(x) = 0\}, \qquad \rho \geqslant 0 \text{ strictly plurisubharmonic on } U.$$

Conversely, every compact subset $K$ of $M$ of the form (1.1) is locally contained in a $\mathbf{C}^1$ totally real submanifold of $M$ [**3**]. Therefore we shall say that a compact subset $K$ of $M$ is *totally real* if it is of the form (1.1).

1.1 THEOREM. *Let $M$ be a Stein manifold and $K$ a compact totally real subset of $M$ that is $\mathcal{O}(M)$-convex. Then every sufficiently small $\mathbf{C}^2$ perturbation of $K$ in $M$ is also $\mathcal{O}(M)$-convex.*

We need to specify what we mean by a small $\mathbf{C}^2$ perturbation of $K$. We embed the Stein manifold $M$ in a Euclidean space $\mathbb{C}^n$ [**4**, p. 125]. Let $U$ be an open neighborhood of $K$ in $\mathbb{C}^n$, and denote by $E$ the Banach space $\mathbf{C}^2(U)^n$ of all $n$-tuples of complex valued functions $\Psi = (\Psi_1, \dots, \Psi_n)$ of class $\mathbf{C}^2$ on $U$ which have finite norm

$$(1.2) \qquad \|\Psi\|_E = \sum_{j=1}^{n} \sup\left\{\left|D^\alpha \Psi_j(z)\right| : z \in U, |\alpha| \leqslant 2\right\}.$$

Theorem 1.1 asserts that the set $\Psi(K)$ is $\mathcal{O}(M)$-convex for each $\Psi$ in an open neighborhood of the identity map in $E$ such that $\Psi(K) \subset M$.

1.2 COROLLARY. *Let $M$ be a Stein manifold, let $N$ be a manifold of class $\mathbf{C}^2$ and let $B$ be an open neighborhood of $0$ in some $\mathbb{R}^m$. Suppose that $F\colon N \times B \to M$ is a $\mathbf{C}^2$ map such that $F_0 = F(\cdot, 0)$ is a totally real embedding of $N$ in $M$. If $K$ is a compact subset of $N$ such that $F_0(K)$ is $\mathcal{O}(M)$-convex, then $F_t(K)$ is $\mathcal{O}(M)$-convex for all $t$ in a neighborhood of $0$ in $\mathbb{R}^m$. (Here, $F_t = F(\cdot, t)$.)*

EXAMPLE 2. If $\Sigma$ is a totally real affine subspace of $\mathbb{C}^n$, then every compact subset $K$ of $\Sigma$ is polynomially convex. This follows from the Stone-Weierstrass approximation theorem and from the fact that the general linear group $GL(n, \mathbb{C})$ acts transitively on the set of totally real subspaces of $\mathbb{C}^n$ of dimension $k$ for each $1 \leqslant k \leqslant n$. Hence, by Theorem 1.1, every small $\mathbf{C}^2$ perturbation of a compact subset $K \subset \Sigma$ is polynomially convex.

We shall consider the same question in the case when $K$ is a subset with nonempty interior in a Stein manifold. Suppose that $D$ is an open relatively compact subset of $M$ whose topological boundary $\overline{D} \setminus D$ contains a strictly pseudoconvex hypersurface $\Gamma$ such that $D$ lies on the convex side of $\Gamma$. More precisely, we assume that there is an open subset $V$ of $M$ and a strictly plurisubharmonic function $\rho$: $V \to \mathbb{R}$ of class $\mathbf{C}^2$ such that

(i) $D \cap V = \{x \in V \mid \rho(x) < 0\}$,

(ii) $\Gamma \cap V = \{x \in V \mid \rho(x) = 0\} \subset\subset \Gamma$, and

(iii) $d\rho \neq 0$ on the set $\Gamma \cap V$.

We define the *support* of a diffeomorphism $\Psi$: $M \to M$ to be the closure of the set $\{x \in M \mid \Psi(x) \neq x\}$ where $\Psi$ differs from the identity map.

1.3 THEOREM. *Let $D$ be an open relatively compact subset of a Stein manifold $M$ that satisfies the properties* (i), (ii) *and* (iii) *above. If the set $K = \overline{D}$ is $\mathcal{O}(M)$-convex, then for every sufficiently small $\mathbf{C}^2$ perturbation $\Psi$: $M \to M$ supported in $V$ the set $\Psi(K)$ is also $\mathcal{O}(M)$-convex.*

1.4 COROLLARY. *If $D$ is a relatively compact strictly pseudoconvex domain in a Stein manifold $M$ such that $\overline{D}$ is $\mathcal{O}(M)$-convex, then every sufficiently small $\mathbf{C}^2$-perturbation of $\overline{D}$ in $M$ is also $\mathcal{O}(M)$-convex.*

In §2 we prove Theorem 1.1 and Corollary 1.2; in §3 we prove Theorem 1.3 and Corollary 1.4.

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## 2. Polynomial convexity of totally real sets.

PROOF OF THEOREM 1.1. If we embed the Stein manifold $M$ in a complex Euclidean space $\mathbb{C}^n$ [**4**], then a compact subset $K$ of $M \subset \mathbb{C}^n$ is $\mathcal{O}(M)$-convex if and only if it is polynomially convex. Therefore it suffices to prove the theorem in the case when $M = \mathbb{C}^n$. Let $U$ be an open subset of $\mathbb{C}^n$, $\rho$ a nonnegative strictly plurisubharmonic function on $U$, and let $K = \{z \in U \mid \rho(z) = 0\}$ be a compact polynomially convex subset of $\mathbb{C}^n$. Choose a smooth function $\chi$ on $\mathbb{C}^n$, $0 \leqslant \chi \leqslant 1$, such that $\chi = 1$ on a neighborhood of $K$ and $\chi = 0$ outside a compact subset of $U$. Let $E = \mathbf{C}^2(U)^n$ be the Banach space with the norm (1.2). Given a $\psi \in E$ we consider the map $\Psi$: $\mathbb{C}^n \to \mathbb{C}^n$ given by

$$\Psi(z) = z + \chi(z)\psi(z). \tag{2.1}$$

Clearly $\Psi$ is proper. If the $E$-norm of $\psi$ is sufficiently small, then $\Psi$ is also regular and hence a covering projection. Since $\Psi$ is one-to-one outside a compact subset of $\mathbb{C}^n$, it has only one sheet and therefore it is a diffeomorphism of $\mathbb{C}^n$ onto $\mathbb{C}^n$. Every small perturbation of $K$ can be achieved within $U$ with a map of the form (2.1).

Choose a neighborhood $V$ of $K$, $\overline{V}$ contained in $U$, such that $\chi = 1$ on a neighborhood of $\overline{V}$. There exists a $\mathbf{C}^\infty$ strictly plurisubharmonic exhaustion function $\phi$ on $\mathbb{C}^n$ such that $\phi < 0$ on the set $K$ but $\phi > 0$ on $\mathbb{C}^n \setminus V$ [**4**, p. 110]. Choose a $\mathbf{C}^\infty$ function $h$: $\mathbb{R} \to [0, \infty)$ that is equal to 0 on $(-\infty, 0]$ and is strictly convex on $(0, \infty)$. Then the function

$$\rho' = \chi \circ \rho + ch \circ \phi \colon \mathbb{C}^n \to [0, \infty)$$

is a strictly plurisubharmonic exhaustion function of class $\mathbf{C}^2$ on $\mathbb{C}^n$ provided that the constant $c > 0$ is chosen sufficiently big, and $K = \{z \in \mathbb{C}^n \mid \rho'(z) = 0\}$.

If $\psi \in E$ is small, the function $\tau = \rho' \circ \Psi^{-1}$ is a small $\mathbf{C}^2$ perturbation of $\rho'$, and $\tau = \rho'$ outside a large compact subset $B$ of $\mathbb{C}^n$. Hence the Levi form $L_\tau = \Sigma(\partial^2\rho/\partial z_j \partial \bar{z}_k) dz_j \otimes d\bar{z}_k$ of $\rho$ is a small perturbation of the Levi form $L_{\rho'}$ of $\rho'$, and they agree outside $B$. Since the eigenvalues of $L_{\rho'}$ are positive on the compact set $B$, the same is true for $L_\tau$. This says that $\tau$ is a nonnegative strictly plurisubharmonic exhaustion function on $\mathbb{C}^n$. The approximation theorem [**4**, p. 119, Theorem 5.2.8] implies that the zero set of $\tau$ is polynomially convex. Since $\Psi(K) = \{z \in \mathbb{C}^n \mid \tau(z) = 0\}$, Theorem 1.1 is proved.

PROOF OF COROLLARY 1.2. We may take $M = \mathbb{C}^n$ as before. Choose an open relatively compact neighborhood $V$ of $K$ in $N$. For each $t \in \mathbb{R}^m$ close to 0 the set $V_t = F_t(V)$ is a totally real submanifold of $\mathbb{C}^n$, and the map $\Phi_t\colon V_0 \to V_t$, $\Phi_t = F_t \circ F_0^{-1}$, is close to the identity map on $V_0$ in the $\mathbf{C}^2$-sense. Since $\Phi_t(F_0(K)) = F_t(K)$, it suffices to show that there is an open neighborhood $U$ of $F_0(K)$ such that for each $t$ the map $\Phi_t$ can be extended to a map $\Psi^t$ on $U$ that is close to the identity in the $\mathbf{C}^2$-sense on $U$.

The map $\phi_t(z) = \Phi_t(z) - z$, $z \in V_0$, is small in the $\mathbf{C}^2$-sense. Using a smooth partition of unity we extend $\phi_t$ to a $\mathbf{C}^2$ map $\psi_t$ on $U$ such that

$$\|\psi_t\|_{\mathbf{C}^2(U)} \leqslant c\|\phi_t\|_{\mathbf{C}^2(V_0)},$$

where the constant $c$ is independent of $t$. The map $\Psi_t(z) = z + \psi_t$, $z \in U$, is the desired extension of $\Phi_t$. Corollary 1.2 now follows from Theorem 1.1.

**3. Perturbations on strictly pseudoconvex boundary points.** We shall first consider the perturbations of $D$ that are supported in small subsets of $V$. Fix a point $x_0 \in \Gamma \cap V$, an open neighborhood $V_0$ of $x_0$ such that $\overline{V}_0 \subset V$, and a strictly plurisubharmonic defining function $\rho$ for $D \cap V_0$. According to [**1**, p. 530, Proposition 1] there exist a bounded strictly convex open set $C \subset \mathbb{C}^n$ ($n = \dim M$) with $\mathbf{C}^2$ boundary, a holomorphic map $\Phi\colon M \to \mathbb{C}^n$ and an open set $U \subset M$, $x_0 \in U \subset\subset V_0$, such that the following hold:

(i) $\Phi(D) \subset C$,

(ii) $\Phi(\{z \in U \mid \rho(z) > 0\}) \subset \mathbb{C}^n \setminus \overline{C}$,

(iii) $\Phi^{-1}(\Phi(U)) = U$, and

(iv) the restriction $\Phi|_U$ is regular and one-to-one.

Let $W$ be a neighborhood of $x_0$ such that $\overline{W} \subset U$. If $\Psi$ is a small $\mathbf{C}^2$ perturbation of $D$ supported in $W$, then $\tilde{\Psi} = \Phi \circ \Psi \circ \Phi^{-1}|_{\Phi(U)}$ is a small $\mathbf{C}^2$ perturbation of $C$ supported in $\Phi(U)$. We choose $\Psi$ so close to the identity map that the set $\tilde{\Psi}(D)$ is still convex. For every point $x \in U \setminus \overline{\Psi(D)}$ we have $\Phi(x) \in \mathbb{C}^n \setminus \overline{\tilde{\Psi}(C)}$ and hence there is a holomorphic function $h$ on $\mathbb{C}^n$ such that $h(\Phi(x)) = 1$, but $|h| < \frac{1}{2}$ on $\tilde{\Psi}(C)$. Because of (i) and (iii) above it follows that the point $x$ does not lie in the $\mathcal{O}(M)$-hull of $\overline{\Psi(D)}$.

To simplify the notation we write $K = \overline{D}$ and $K' = \overline{\Psi(D)}$. The conclusion we just made is that

$$\hat{K}' \cap U = K' \cap U, \tag{3.1}$$

where $\hat{K}'$ is the $\mathcal{O}(M)$-convex hull of $K'$. Since the support of $\Psi$ is contained in $W$, we have $K' \setminus W = K \setminus W$, and hence (3.1) implies

$$\hat{K}' \cap (U \setminus W) = K \cap (U \setminus W). \tag{3.2}$$

We shall prove that $\hat{K}' = K'$. Assume that $\hat{K}' \neq K'$ in order to reach a contradiction. Because of (3.1) the two sets can differ only outside $U$. Since $K \setminus U = K' \setminus U$, the set $\hat{K}' \setminus U$ is strictly larger than $K \setminus U$. The polynomially convex set $K$ has a basis of open neighborhoods $\Omega$ that are smothly bounded strictly pseudoconvex domains with $\mathcal{O}(M)$-convex closure $\overline{\Omega}$. Thus we may choose $\Omega$ with these properties that does not contain the set $\hat{K}' \setminus U$. By an embedding theorem for strictly pseudoconvex domains due to Fornaess [**1**, p. 543] and Khenkin [**5**, p. 668] there exists a holomorphic embedding $F$: $M \to \mathbb{C}^N$ for some $N \in \mathbb{Z}_+$ and a bounded strictly convex domain $B \subset \mathbb{C}^N$ such that $F(\Omega) \subset B$ and $F(M \setminus \overline{\Omega}) \subset \mathbb{C}^N \setminus \overline{B}$. We may assume that $0 \in B$. Let

$$t_0 = \inf\{ t \in \mathbb{R}_+ \mid F(\hat{K}' \setminus U) \subset t\overline{B} \} \tag{3.3}$$

and replace $B$ by $t_0 B$. Then

$$F(\hat{K}' \setminus U) \subset \overline{B}, \tag{3.4}$$

and there is a point $p \in F(\hat{K}' \setminus U) \cap bB$. The set $A = \mathbb{C}^N \setminus \overline{F(W)}$ is open and contains the point $p$. Moreover, it follows from (3.3) and (3.4) that $F(\hat{K}') \cap A \subset \overline{B}$. This means that locally near $p$ the polynomially convex set $F(\hat{K}') = \widehat{F(K')}$ lies on the convex side of the smooth strictly convex hypersurface $bB$. According to [**1**, p. 530] there exists a holomorphic function $g$ defined on a neighborhood of $F(\hat{K}')$ in $\mathbb{C}^N$ such that

$$g(p) = 1 \quad \text{and} \quad |g(q)| < 1 \quad \text{for } q \in F(\hat{K}') \setminus \{p\}.$$

If $\varepsilon > 0$ is sufficiently small, the set

$$F(\hat{K}') \cap \{|g| \leqslant 1 - \varepsilon\} \subsetneqq F(\hat{K}')$$

is polynomially convex and contains $F(K')$. This is a contradiction since $F(\hat{K}') = \widehat{F(K')}$ is the polynomially convex hull of $F(K')$. This concludes the proof in the case when the support of the perturbation $\Psi$ is sufficiently small.

It remains to consider the general case. Let $\Gamma'$ be an open relatively compact subset of $\Gamma \cap V$. Using the methods introduced by Fornaess in [**1**] we can show that there exist an open set $U \subset\subset V$ such that $U \cap \Gamma = \Gamma'$, a holomorphic map $F$: $M \to \mathbb{C}^N$ and a bounded strictly convex domain $C \subset \mathbb{C}^N$ with $\mathbf{C}^2$ boundary such that the properties (i)–(iv) above hold. Moreover, the map $F$ is transversal to $bC$ at every point $x \in \Gamma'$. It follows that every small perturbation of $D$ supported in $U$ can be effected by a small perturbation of $C$ supported in a neighborhood of $F(\Gamma')$. The proof can be completed in the same way as above. We omit the details.

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