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Localization of the Kobayashi Metric and the Boundary Continuity of Proper Holomorphic Mappings

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0. Introduction

A classical theorem of Carathéodory [8] states that every biholomorphic map $f: D_1 \to D_2$ between domains in the complex plane $\mathbb{C}$ bounded by simple closed Jordan curves extends to a homeomorphism of $\bar{D}_1$ onto $\bar{D}_2$.

There are some well-known generalizations of this result to domains in $\mathbb{C}^n$. If D_1 and D_2 are bounded pseudoconvex domains in $\mathbb{C}^n$ with $\mathscr{C}^2$ boundary and if the infinitezimal Kobayashi metric on D_2 grows sufficiently fast near the boundary of $D_2 (K_{D_2}(z; X) \geqq |X|/d(z, bD_2)^\varepsilon$ for some $\varepsilon \in (0,1)$), then every proper holomorphic map $f: D_1 \to D_2$ extends to a Hölder-continuous map of $\bar{D}_1$ onto $\bar{D}_2$ [2, 3, 10, 25, 29–32]. This holds in particular if D_2 is strictly pseudoconvex or if it is pseudoconvex with real-analytic boundary. The same result holds if D_2 is only piecewise smooth strongly pseudoconvex [31].

Further results treat exceptional cases such as the balls [1], Reinhardt domains [5, 24], domains with many symmetries [4], and analytically bounded Hartogs domains in $\mathbb{C}^2$ [11]. Biholomorphic maps between certain types of non-pseudoconvex domains with real-analytic boundaries were treated in [13] and [27]. Besides the papers mentioned above there is a vast literature concerning smooth extension of proper holomorphic maps of smoothly bounded pseudoconvex domains ([6, 7, 12, 14], to mention just a few).

In the present paper we obtain some results on the continuous extension of proper holomorphic maps $f: D_1 \to D_2$ under local assumptions on the boundaries bD_1 and bD_2. One of the main results is

Theorem. *Let $f: D_1 \to D_2$ be a proper holomorphic map of a domain $D_1 \subset \mathbb{C}^n$ onto a bounded domain $D_2 \subset \mathbb{C}^n$ and let bD_1 be of class $\mathscr{C}^2$ and strictly pseudoconvex near a point $z^\circ \in bD_1$. If there exists a sequence $\{z^j\} \subset D_1$ such that $\lim_{j\to\infty} z^j = z^\circ$, the limit $\lim_{j\to\infty} f(z^j) = w^\circ \in bD_2$ exists, and bD_2 is of class $\mathscr{C}^2$ and strictly pseudoconvex near w°, then f extends to a Hölder continuous map with the exponent $1/2$ on a neighborhood of z° in $\bar{D}_1$.*

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Note that there are no assumptions on D_1 and D_2 away from the points z° resp. w°; in particular, the domains are not assumed to be pseudoconvex away from these points. The theorem holds under weaker assumptions on z° resp. w° (see Theorem 1.1 in Sect. 1).

As an application we prove the following result (Corollary 1.3). Let $D_1 \subset \mathbb{C}^n$ be a domain with a $\mathscr{C}^2$ plurisubharmonic defining function, and let $D_2 \subset\subset \mathbb{C}^n$ be a bounded domain whose boundary is $\mathscr{C}^2$ strictly pseudoconvex outside a closed totally disconnected subset $E \subset bD_2$. Then every proper holomorphic map of D_1 onto D_2 extends continuously to $\bar{D}_1$. An example shows that this does not hold if the "singular set" E has a nontrivial connected component (Sect. 4).

Our results are obtained from a precise quantitative estimate concerning the localization of the infinitesimal Kobayashi metric near the boundary (Sect. 2). We hope that this localization estimate will be useful in other problems. The results generalize to domains in Stein manifolds via the embedding theorem [23, p. 124]; we omit the details.

We wish to point out that we are using techniques quite similar to those used by Vormoor [37] in a paper which, however, does not seem to be entirely correct.

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1. Results

If $f: D_1 \to \mathbb{C}^N$ is a continuous map on a domain $D_1 \subset \mathbb{C}^n$ and $z^\circ \in bD_1$ is a boundary point of D_1, we denote by $C(f, z^\circ)$ the cluster set of f at z°, i.e., the set of all points $\lim_{j\to\infty} f(z_j)$, where $\{z_j\} \subset D_1$ is a sequence converging to z°. If f is bounded on D_1, then $C(f, z^\circ)$ is compact, and f extends continuously to z° if and only if $C(f, z^\circ)$ consists of a single point. If there is a basis of neighborhoods $\{U_j\}$ of z° such that $U_j \cap D_1$ is connected for all j, then $C(f, z^\circ)$ is also connected. If $f : D_1 \to D_2$ is a proper mapping, the cluster set $C(f, z^\circ)$ is contained in bD_2 for each $z^\circ \in bD_1$. We denote by $C(f)$ the global cluster set of f:

$$C(f) = \overline{f(D_1)} \setminus f(D_1) \ .$$

Next we introduce *Condition* (P). *A point* $z^\circ \in bD_1$ *satisfies Condition* (P) *if the boundary* bD_1 *is of class* $\mathscr{C}^{1+\varepsilon}$ *near* z° *for some* $\varepsilon > 0$ *and if there exist a continuous negative plurisubharmonic function* ϱ *on* D_1 *and a neighborhood* U *of* z° *in* $\mathbb{C}^n$ *such that*

$$\varrho(z) \geqq -c_1 d(z, bD_1) \ , \qquad z \in D_1 \cap U$$

for some constant $c_1 > 0$.

Here, $d(z, bD_1) = \inf\{|z-w| : w \in bD_1\}$.

All strictly pseudoconvex boundary points of D_1 satisfy Condition (P). Also, if ϱ is a $\mathscr{C}^2$ plurisubharmonic function on a domain $\Omega \subset \mathbb{C}^n$ and $D_1 = \{z \in \Omega : \varrho(z) < 0\} \subset\subset \Omega$, then each point $z^\circ \in bD_1$ at which $d\varrho(z^\circ) \neq 0$ satisfies Condition (P).

Our main result is

1.1. Theorem. *Let $f: D_1 \to D_2$ be a proper holomorphic map of a domain $D_1 \subset \mathbb{C}^n$ onto a bounded domain $D_2 \subset \mathbb{C}^n (n \geqq 1)$. If a point $z^\circ \in bD_1$ satisfies condition (P) and if the cluster set $C(f, z^\circ)$ contains a point $w^\circ \in bD_2$ at which bD_2 is $\mathscr{C}^2$ strictly pseudoconvex, then f extends continuously to z°.*

Note that D_1 and D_2 need *not* be smoothly bounded or pseudoconvex away from z° resp. w°.

If f extends continuously to a point $z^\circ \in bD_1$ satisfying Condition (P) and if bD_2 is strictly pseudoconvex near $w^\circ = f(z^\circ)$, then the hypothesis of Theorem 1.1 is also fulfilled for all points $z \in bD_1$ sufficiently close to z°. Thus f extends continuously to a neighborhood U of z° in $\bar{D}_1$. Applying the usual Khenkin's technique on U [10, 25, 30] we conclude that under the hypothesis of Theorem 1.1 the map f is Hölder continuous with the exponent $1/2$ near z° on $\bar{D}_1$.

1.2. Corollary. *Let D_2 be a bounded domain in $\mathbb{C}^n$ $(n \geqq 1)$ that is $\mathscr{C}^2$ strictly pseudoconvex at each point $w \in bD_2$ outside a closed, totally disconnected subset E of bD_2. Then every proper holomorphic map $f: D_1 \to D_2 (D_1 \subset \mathbb{C}^n)$ extends continuously to each point $z \in bD_1$ satisfying the Condition (P).*

Proof. Recall that $C(f, z^\circ)$ is contained in bD_2. If $C(f, z^\circ)$ is contained in the totally disconnected set E, it contains only one point (since $C(f, z^\circ)$ is connected) whence f extends continuously to z°. If on the other hand $C(f, z^\circ)$ intersects $bD_2 \setminus E$ then f extends continuously to z° according to Theorem 1.1.

1.3. Corollary. *Let D_2 be as in Corollary 1.2, and let $D_1 \subset \mathbb{C}^n$ be a domain with a $\mathscr{C}^2$ plurisubharmonic defining function. Then every proper holomorphic map $f: D_1 \to D_2$ extends continuously to $\bar{D}_1$.*

Proof. If $D_1 = \{\varrho < 0\}$, where ϱ is $\mathscr{C}^2$ plurisubharmonic on a neighborhood of $\bar{D}_1$ and $d\varrho \neq 0$ on bD_1, then every point $z \in bD_1$ satisfies the Condition (P). Thus Corollary 1.2 applies.

If the set $E \subset bD_2$ where bD_2 is not smooth strictly pseudoconvex has a connected component containing more than one point, f may not extend continuously to $\bar{D}_1$ even when D_1 is real-analytic and strictly pseudoconvex (see Example 1 in Sect. 4). Also, there is no reason why f should extend continuously to non-smooth points of bD_1 (Sect. 4, Example 2). However, if f does not extend continuously to $\bar{D}_1$ and bD_1 is $\mathscr{C}^2$ pseudoconvex, then bD_2 cannot be as nice as piecewise smooth strictly pseudoconvex according to Range [31]. The question remains what minimal regularity of bD_2 is required in order to have a continuous extension of f to $\bar{D}_1$.

We shall outline the proof of Theorem 1.1. Suppose that $z^\circ \in bD_1$ satisfies Condition (P), the boundary of D_2 is $\mathscr{C}^2$ strictly pseudoconvex near $w^\circ \in C(f, z^\circ)$, but f does not extend continuously to z°. Then, since $C(f, z^\circ)$ is connected, there is another point $w^1 \in bD_2 \cap C(f, z^\circ)$ dinstinct from w° at which bD_2 is $\mathscr{C}^2$ strictly pseudoconvex. Choose sequences $\{z_j^k : j \in Z_+\}$ in $D_1 (k = 0, 1)$ such that

$$\lim_{j \to \infty} z_j^k = z^\circ , \quad \lim_{j \to \infty} f(z_j^k) = w^k , \quad k = 0, 1 .$$

Let $w_j^k = f(z_j^k)$. We denote by $\operatorname{Kob\,dist}_{D_1}(z, z')$ the Kobayashi distance in D_1 between the points $z, z' \in D_1$, and similarly for D_2. For the definition of the Kobayashi metric and distance see Sect. 2. The following three steps provide a contradiction, thus proving Theorem 1.1.

Step 1. There is a constant $K \in \mathbb{R}$ such that

$$\operatorname{Kob\,dist}_{D_2}(w_j^0, w_j^1) \geqq -1/2 \log d(w_j^0, bD_2) \\ -1/2 \log d(w_j^1, bD_2) - K\ , \quad j \in \mathbb{Z}_+$$

Step 2.

$$\operatorname{Kob\,dist}_{D_1}(z_j^0, z_j^1) \leqq -1/2 \log d(z_j^0, bD_1) \\ -1/2 \log d(z_j^1, bD_1) - l(z_j^0, z_j^1)\ ,$$

where $\lim_{j \to \infty} l(z_j^0, z_j^1) = \infty$.

Step 3. There is a constant $c > 0$ such that

$$d(w_j^k, bD_2) \leqq c\, d(z_j^k, bD_1)\ , \quad j \in \mathbb{Z}_+\ , \quad k = 0, 1\ .$$

Suppose that Steps 1–3 hold. The Kobayashi distance is decreasing under holomorphic mappings so

$$\operatorname{Kob\,dist}_{D_1}(z_j^0, z_j^1) \geqq \operatorname{Kob\,dist}_{D_2}(w_j^0, w_j^1)\ .$$

Steps 1–3 now imply

$$l(z_j^0, z_j^1) \leqq K + \log c\ , \quad j \in \mathbb{Z}_+\ ,$$

which is a contradition to the fact that

$$l(z_j^0, z_j^1) \to \infty \quad \text{as} \quad j \to \infty\ .$$

This concludes the proof of Theorem 1.1 provided that Steps 1–3 hold.

Step 3 follows from the Hopf lemma applied to the negative continuous plurisubharmonic function

$$\tau(w) = \max\{\varrho(z)\,;\ z \in D_1, f(z) = w\}\ , \quad w \in D_2\ ,$$

where $\varrho : D_1 \to (-\infty, 0)$ is as in the Condition (P). See [2, 10, 25] for the details. Step 2 is proved in Sect. 2, Proposition 2.5.

The main problem is to obtain the estimate from below in Step 1. Assuming the existence of good local holomorphic peaking functions for points $A \in bD_2$ near w° we first prove an estimate concerning the localization of the infinitesimal Kobayashi metric near w° (Theorem 2.1 and Lemma 2.2). The estimate of Step 1 follows by integration, using a stronger hypothesis on the local holomorphic peak functions (Theorem 2.3 and Corollary 2.4).

Using the techniques of this paper we also obtain a result on continuous extension of proper holomorphic maps $f: D_1 \to D_2$ of a domain $D_1 \subset \mathbb{C}^n$ into a bounded strictly pseudoconvex domain $D_2 \subset \mathbb{C}^N$, $N > n$, provided that the global cluster set $C(f) = \overline{f(D_1)} \setminus f(D_1) \subset bD_2$ is not too large. The following theorem generalizes the main result of [15] (see also [19]).

1.4. Theorem. *Let $D_1 \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with $\mathscr{C}^2$ boundary and let $f: D_1 \to D_2$ be a proper holomorphic map into a bounded, strictly pseudoconvex domain $D_2 \subset \mathbb{C}^N$ with $\mathscr{C}^2$ boundary ($N > n$). Assume that the global cluster set $C(f)$ is contained in the union of a closed, totally disconnected subset $E \subset bD_2$ and a closed $\mathscr{C}^2$ submanifold M of $bD_2 \setminus E$ of real dimension $2n-1$. Then f extends continuously to each point $z^\circ \in bD_1$ satisfying Condition (P).*

Thus, if D_1 admits a $\mathscr{C}^2$ plurisubharmonic defining function, every such map f extends continuously to $\bar{D}_1$. If we do not assume anything on the cluster set $C(f)$, f need not extend continuously even if both D_1 and D_2 are smooth strictly pseudoconvex [16, 28]. Indeed, there exist proper holomorphic maps from the unit disk $\Delta \subset \mathbb{C}$ into the unit ball B^N of $\mathbb{C}^N$ ($N > 1$) that extend continuously (even smoothly) to $\bar{\Delta} \setminus \{1\}$, but whose cluster set at the point 1 is the entire sphere bB^N [20]. On the other hand, a proper holomorphic map may have a large global cluster set and yet extend continuously to the closure of the domain [18].

It would be of interest to know whether Theorem 1.4 holds if we only assume that the manifold $M \subset bD_2 \setminus E$ has Cauchy-Riemann dimension $n-1$ at each point (= the CR dimension of bD_1). If $n=1$ and the set E is empty, this holds by a theorem of Čirka [9].

2. Localization of the Kobayashi Metric

Let $\Delta(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$, $\Delta(r) = \Delta(0,r)$ and $\Delta = \Delta(1)$. If D is a domain in $\mathbb{C}^n$, we denote by $K_D(z,X)$ the infinitesimal Kobayashi length of the tangent vector $X \in T_z\mathbb{C}^n$ at the point $z \in D$: $K_D(z,X) = \inf\{1/r > 0 : \exists\, h: \Delta \to D$ holomorphic, $h(0) = z,\ h'(0) = r \cdot X\}$. If $\gamma: [0,1] \to D$ is a $\mathscr{C}^1$ curve in D, its Kobayashi length is

$$l_D(\gamma) = \int_0^1 K_D(\gamma(t);\, \gamma'(t))\,dt \ .$$

The Kobayashi distance between the points $a, b \in D$ is

$$\text{Kob dist}_D\,(a,b) = \inf\{l_D(\gamma) : \gamma : [0,1] \to D \text{ of class } \mathscr{C}^1 \ , \quad \gamma(0) = a, \gamma(1) = b\} \ .$$

If $f: D_1 \to D_2$ is a holomorphic map of domains, then the Schwarz-Pick lemma holds:

$$\text{Kob dist}_{D_1}(a,b) \geqq \text{Kob dist}_{D_2}\,(f(a), f(b)) \ , \qquad a, b \in D_1 \ ,$$

and

$$K_{D_1}(z; X) \geqq K_{D_2}(f(z); f'(z)X) \ , \qquad z \in D_1 \ , \qquad X \in T_z\mathbb{C}^n \ .$$

For further properties of the Kobayashi metric see [26] and [33].

We consider D a bounded domain in $\mathbb{C}^n$ and $D_0 \subset D_1$ open subsets of D such that each point $z \in bD_0$ has an open neighborhood U_z in $\mathbb{C}^n$ for which $U_z \cap D_1 = U_z \cap D$ or, more generally, $U_z \cap D_1$ is a reunion of connected components of $U_z \cap D$. In particular D_0 has a positive distance from the complement of D_1 in D. We denote by $d(z)$ the Euclidean distance from $z \in D$ to bD.

Clearly $K_{D_1}(z,X) \geqq K_D(z;X)$, $z \in D_1$, $X \in T_z\mathbb{C}^n$. Conversely, if we assume that every holomorphic map $h: \Delta \to \bar{D}$ such that $h(0) \in bD \cap bD_0$ is constant, then Montel's theorem implies:

$$\lim_{\substack{z \in D_0 \\ z \to bD}} K_D(z;X)/K_{D_1}(z;X) = 1 \ , \quad X \in \mathbb{C}^n \setminus \{0\} \ . \tag{2.1}$$

We will show that the seemingly rather weak quantitative version of the non-existence of these mappings h implies an estimate (2.2) precising (2.1) which is sharp already in the case of strictly pseudoconvex domains (and is valid, for example, in the case of strict geometric convexity).

We introduce the *Property* (*):
(*) *Every point $z \in bD_0 \cap bD$ has an open neighborhood V_z in $\mathbb{C}^n$ which verifies the following:*
For every $\eta > 0$ there exists $C > 0$ depending on η such that for each holomorphic map $h: \Delta \to V_z \cap D_1$ we have

$$(|\zeta| \leqq 1 - Cd(h(0))) \Rightarrow |h(\zeta) - h(0)| \leqq \eta \ .$$

2.1. Theorem. *If (*) holds, there exists a constant $c > 0$ such that for each $z \in D_0$ and $X \in T_z\mathbb{C}^n$*

$$K_D(z;X) \geqq (1 - cd(z)) \cdot K_{D_1}(z;X) \ . \tag{2.2}$$

The estimate is the best possible, except of course for the size of the constant c which depends on D_0, D_1 and D.

An example of the situation where (*) holds is obtained in the following way. Start from a bounded domain D (not supposed to be pseudoconvex), and Σ a relatively open subset of bD. Assume that Σ is a $\mathscr{C}^2$ strictly pseudoconvex hypersurface. If $D_1 \subset D$ is a domain in the pseudoconvex side of Σ and $bD_0 \cap bD$ is a compact subset of Σ, then (*) holds. This follows from the following lemma which applies to much more general situations, e.g., in the case of strict geometric convexity.

2.2. Lemma. *Let $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous increasing function, $\varphi(0) = 0$. Assume that for every point $A \in bD_1 \cap bD$ there exists a function $P_A \in C(\bar{D}_1) \cap \mathscr{O}(D_1)$, $|P_A| < 1$ on $\bar{D}_1 \setminus \{A\}$, which peaks at A and satisfies*

$$c_1|1 - P_A(z)| \leqq |z - A| \leqq \varphi(|1 - P_A(z)|) \ , \quad z \in D_1 \ . \tag{2.3}$$

Then (*) holds. ($c_1 > 0$ is constant independent of A.)

Remark. It is important to notice that P_A is a peaking function for D_1, not for D. Near every strictly pseudoconvex boundary point of D the condition (2.3) holds with $\varphi(x) = \sqrt{x}$ which we can see be convexifying bD locally near the given point.

Proof. The following is a simple consequence of the Schwarz lemma on Δ and we omit the proof:

For each $\lambda > 0$ there exists a constant $C_\lambda > 0$ such that for each $\varepsilon > 0$ and each holomorphic map $g: \Delta \to \Delta$ satisfying $|1 - g(0)| \leqq \varepsilon$ we have

$$(|\zeta| \leqq 1 - C_\lambda \cdot \varepsilon) \Rightarrow |1 - g(\zeta)| \leqq \lambda \ . \tag{2.4}$$

Given $\eta > 0$ as in (*) we choose $\lambda > 0$ satisfying $2\varphi(\lambda) = \eta$. Let C_λ be the constant for which (2.4) holds.

Assume that $h: \Delta \to D_1$ is holomorphic, $h(0) \in D_0$ and $\varepsilon = d(h(0))$. Choose a point $A \in bD_1 \cap bD$ such that $|h(0) - A| = \varepsilon$. We only need to consider small values of ε; let $\varepsilon < \eta/2$. By (2.3) we have

$$|1 - P_A(h(0))| \leqq \varepsilon/c_1$$

whence (2.4) applied to the function $g = P_A \circ h: \Delta \to \Delta$ implies

$$(|\zeta| \leqq 1 - C_\lambda \varepsilon/c_1) \Rightarrow |1 - P_A(h(\zeta))| \leqq \lambda \ .$$

Thus for $|\zeta| \leqq 1 - C_\lambda \varepsilon/c_1$ we have

$$\begin{aligned} |h(\zeta) - h(0)| &\leqq |h(\zeta) - A| + |h(0) - A| \\ &\leqq \varphi(|1 - P_A(h(\zeta)|) + \varepsilon \\ &\leqq \varphi(\lambda) + \eta/2 \\ &\leqq \eta \ . \end{aligned}$$

Lemma 2.2 is proved.

Proof of Theorem 2.1. Fix a point $p \in bD_0 \cap bD$ and V_p an open neighborhood of p such as in (*). Shrinking V_p if necessary we may assume that $V_p \cap D_1$ is a reunion of connected components of $V_p \cap D$. There is a smaller neighborhood V of p and a constant $d \in (0, 1)$ such that for each $z \in V \cap D$, $B(z, d) \cap D_1 \subset V_p \cap D_1$. (Here $B(z, d)$ is the ball of radius d centered at z.) The theorem will be proved if we find a constant $K > 0$ such that for every holomorphic mapping $h: \Delta \to D$ with $h(0) \in V$ we have $h(\Delta_{1 - Kd(h(0))}) \subset D_1$.

We may assume that D has diameter $\leqq 1$. Given $\varepsilon > 0$, let $\varrho = \varrho(\varepsilon)$ be the largest number in $[0, 1]$ such that $|h(\zeta) - h(0)| \leqq d$ whenever h is a holomorphic mapping $h: \Delta \to D$, $h(0) \in V$, $d(h(0)) \leqq \varepsilon$, and $|\zeta| \leqq \varrho$. So $h(\Delta_\varrho) \subset D_1$. Of course $\varrho \geqq d$ by the Schwarz lemma. According to the Property (*) there exists a $C > 0$ such that

$$|h(\zeta) - h(0)| \leqq d/2 \quad \text{whenever} \quad d(h(0)) \leqq \varepsilon \quad \text{and} \quad |\zeta| \leqq \varrho - C\varepsilon \ .$$

We have $\log |h(\zeta) - h(0)| < 0$ on the unit disk and, if $\varrho < 1$,

$$d = \sup_{|\zeta| = \varrho} |h(\zeta) - h(0)|$$

for some h. By Hadamard's three circles lemma the function $\log \sup_{|\zeta| = r} |h(\zeta) - h(0)|$ is a convex function of $\log r$. Therefore

$$\left| \frac{\log(\varrho - C\varepsilon)}{\log d/2} \right| \geqq \left| \frac{\log \varrho}{\log d} \right| \ .$$

If we restrict to the case where $\varepsilon \leqq d/2C$ (so $\varrho - C\varepsilon \geqq d/2$) we get:

$$\frac{|\log \varrho| + 2C\varepsilon/2}{|\log d/2|} \geqq \frac{|\log \varrho|}{|\log d|}$$

and therefore

$$1-\varrho \leqq |\log \varrho| \leqq \frac{2C|\log d|}{d\cdot \log 2}\cdot \varepsilon \ .$$

Hence $\varrho \geqq 1-K\cdot\varepsilon$ as desired. This proves Theorem 2.1.

Using the localization of the infinitesimal Kobayashi metric we shall prove an estimate concerning the rate of growth of the Kobayashi distance near the boundary which is sharp in the case of strictly pseudoconvex boundary points.

2.3. Theorem. *Let $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ be a continuous function satisfying the Dini condition $\int_0^1 \frac{\varphi(x)}{x}\,dx<\infty$. Assume that for every point $A\in bD_1\cap bD$ there exists a function $P_A\in C(\bar{D}_1)\cap\mathcal{O}(D_1)$, $|P_A|<1$ on $\bar{D}_1\setminus\{A\}$, which peaks at A and satisfies*

$$c_1|1-P_A(z)|\leqq|z-A|\leqq\varphi(1-|P_A(z)|)\ , \quad z\in D_1\ .$$

Then for every point p in the relative interior of $bD_1\cap bD$ there is a neighborhood U of p and a constant K such that

$$\text{Kob dist}_D(z;D\setminus D_1)\geqq 1/2\log 1/d(z)-K\ , \quad z\in U\cap D_1\ . \tag{2.5}$$

Remark. A comparison with the Kobayashi distance on a ball in D that is tangent to bD at one point of bD shows that Kob $\text{dist}_D(z;D\setminus D_1)$ cannot grow faster than $1/2\log 1/d(z)$ as z approaches bD. The optimal rate of growth (2.5) is attained in particular when bD is $\mathscr{C}^2$ strictly pseudoconvex near the given point p since the hypothesis of Theorem 2.3 is then satisfied with $\varphi(x)=\sqrt{x}$.

Proof of Theorem 2.3. By Lemma 2.2 and Theorem 2.1 we can choose $D_0\subset D_1$ such that the point p is in the relative interior of $bD_0\cap bD$ and the estimate (2.2) holds. Let U be a small neighborhood of p, $U\cap D_1$ contained in D_0, such that for each $z\in U\cap D_1$ there is a closest point $A_z\in bD_0\cap bD$ satisfying $\inf_{z'\in D\setminus D_0}|A_z-z'|\geqq\delta>0$ for some δ independent of z.

Let $\gamma:[0,1]\to\bar{D}_0$ be a $\mathscr{C}^1$ path, $\gamma(0)\in bD_0\cap D$, $\gamma(1)\in U\cap D_0$, $\gamma(t)\in D_0$ for $t>0$. Let $A\in bD_0$ be the closest point to $\gamma(1)$ on bD and $P_A\in C(\bar{D}_1)\cap\mathcal{O}(D_1)$ the peaking function for A. Denote by $\lambda(t)=P_A(\gamma(t))$ the corresponding path in Δ and set $\xi(t)=|\lambda(t)|$. We may assume that P_A has no zeroes on $\bar{D}_0$ whence ξ is of class $\mathscr{C}^1$. By Theorem 2.1 we have

$$\begin{aligned} l_D(\gamma)&=\int_0^1 K_D(\gamma(t);\gamma'(t))dt\\ &\geqq\int_0^1(1-cd(\gamma(t)))K_{D_1}(\gamma(t);\gamma'(t))dt\ . \end{aligned}$$

Since holomorphic mappings decrease the Kobayashi metric, we have

$$\begin{aligned} K_{D_1}(\gamma(t);\gamma'(t))&\geqq K_\Delta(\lambda(t);\lambda'(t))=|\lambda'(t)|/(1-|\lambda(t)|^2)\\ &\geqq(1/2)\xi'(t)/(1-\xi(t))\ . \end{aligned}$$

Also, by hypothesis,

$$1-cd(\gamma(t))\geqq 1-c|\gamma(t)-A|\geqq 1-c\varphi(1-|P_A(\gamma(t))|)$$
$$\geqq 1-c\varphi(1-\xi(t))$$

and we may assume that the last quantity is positive. Therefore

$$l_D(\gamma)\geqq 1/2\int_0^1 (1-c\varphi(1-\xi(t)))d\xi(t)/(1-\xi(t))$$
$$\geqq 1/2\int_0^1 d\xi(t)/(1-\xi(t))-c/2\int_0^1 \varphi(x)/x\cdot dx\ .$$

The first integral equals

$$1/2\log 1/(1-\xi(1))-1/2\log 1/(1-\xi(0))\ .$$

We have

$$1-\xi(1)=1-|P_A(\gamma(1))|\leqq|\gamma(1)-A|/c_1=d(\gamma(1))/c_1\ ,$$
$$1-\xi(0)=1-|P_A(\gamma(0))|\geqq\varphi^{-1}(|\gamma(0)-A|)$$
$$\geqq\varphi^{-1}(\delta)\ .$$

Thus

$$l_D(\gamma)\geqq 1/2\log 1/d(\gamma(1))-K\ ,$$

where

$$K=1/2(\log 1/c_1+\log 1/\varphi^{-1}(\delta))+c\int_0^1 \varphi(x)/x\cdot dx)\ .$$

This proves Theorem 2.3.

2.4. Corollary. *Let D be a bounded domain in $\mathbb{C}^n$ whose boundary bD is $\mathscr{C}^2$ strictly pseudoconvex in a neighborhood of two distinct points w^0, $w^1\in bD$. Then there is a constant K such that*

$$\text{Kob dist}_D\,(a,b)\geqq -1/2\log d(a,bD)-1/2\log d(b,bD)-K$$

for each point a sufficiently close to w^0 and b close to w^1.

Proof. Each path γ in D starting at a and ending at b must exit from neighborhoods of w^0 and w^1. Hence the corollary follows from Theorem 2.3.

In the proof of Theorem (1.1) we also require the following

2.5. Proposition. *If D is a domain whose boundary bD is of class $\mathscr{C}^{1+\varepsilon}(\varepsilon>0)$ near a point $A\in bD$, then there exist a neighborhood U of A and a constant $C\in\mathbb{R}$ such that for all $z_0,z_1\in D\cap U$:*

$$\text{Kob dist}_D\,(z_0,z_1)\leqq 1/2\sum_{j=0}^{1}\log 1/d(z_j)-1/2\sum_{j=0}^{1}\log 1/(d(z_j)+|z_0-z_1|)+C\ .$$

In the formula the second term is the one which shows a difference of behavior of the Kobayashi distance between z_0 and z_1 when at least one of the z_j approaches bD_1, according to whether $|z_0 - z_1|$ approaches 0 or does not.

Proof. We will take for U a ball of radius $\varrho > 0$ centered at A; $\tilde{U}$ will be the closed ball of radius 4ϱ centered at A. We choose ϱ small enough in order that $bD \cap \tilde{U}$ is a connected hypersurface of class $\mathscr{C}^{1+\varepsilon}$ and the following two properties hold:

i) if for every $p \in bD \cap U$ we denote by n_p the inner unit normal vector to bD at p, then $|n_p - n_A| < 1/8$.

ii) for every $\delta \in [0, 2\varrho]$ and $z \in D \cap U$, $p \in bD \cap \tilde{U}$:

$$z + \delta n_p \in D \quad \text{and} \quad d(z + \delta n_p) > 3\delta/4 \ .$$

Let z_0 and z_1 be points in $D \cap U$. For $j = 0, 1$, let α_j be any point in bD at the minimum distance from z_j, i.e. $|z_j - \alpha_j| = d(z_j)$. Set $z_j' = z_j + |z_0 - z_1| n_{\alpha_j}$. Then we have

$$\text{Kob dist}_D(z_0, z_1) \leqq \text{Kob dist}_D(z_0', z_1') + \sum_{j=0}^{1} \text{Kob dist}_D(z_j, z_j') \ .$$

In the right hand side the first term is easy to bound from above by a constant independent on z_0 and z_1. Notice that $|z_0 - z_1| < 2\varrho$ implies $d(z_j') \geqq 3/4|z_0 - z_1|$, according to ii), and $|z_0' - z_1'| \leqq 5/4|z_0 - z_1|$, according to i). Let ψ be the holomorphic mapping from $\mathbb{C}$ into $\mathbb{C}^n$ defined by $\psi(\zeta) = z_0' + \zeta(z_1' - z_0')$. Let $\mathcal{O}$ be the connected open set in $\mathbb{C}$, $\mathcal{O} = \{\zeta \in \mathbb{C}, \min(|\zeta|, |\zeta - 1|) < \frac{3}{5}\}$. Thus $\psi(\mathcal{O}) \subset D$, $\psi(0) = z_0'$, $\psi(1) = z_1'$. So we get:

$$\text{Kob dist}_D(z_0', z_1') \leqq \text{Kob dist}_{\mathcal{O}}(0, 1) \ .$$

To end the proof we have to bound the terms $\text{Kob dist}_D(z_j, z_j')$ $(j = 0, 1)$. Let ϕ_j be the holomorphic mapping from $\mathbb{C}$ to $\mathbb{C}^n$, $\phi_j : \zeta \to \alpha_j + \zeta n_{\alpha_j}$, where α_j is as above. Then $\phi_j(0) = \alpha_j$, $\phi_j(d(z_j)) = z_j$, $\phi_j(d(z_j) + |z_0 - z_1|) = z_j'$.

Let us now use the fact that bD is of class $C^{1+\varepsilon}$, that n_{α_j} is normal to bD at α_j, and that the condition i) holds. We can replace ε by $\min(1, \varepsilon)$ and therefore assume that $0 < \varepsilon \leqq 1$. Set $\omega_0 = \{\zeta = \xi + i\eta \in \mathbb{C}, |\zeta| < 4\varrho, \xi > C|\eta|^{1+\varepsilon}\}$. If C is a constant chosen large enough, then $\phi_j(\omega_0) \subset D \cap U$. Fix such a C, and for convenience fix a domain $\omega \subset \omega_0$, symmetric with respect to the real axis, obtained by smoothing $b\omega_0$ in a small neighborhood if its two angular points. We have

$$\text{Kob dist}_D(z_j, z_j') \leqq \text{Kob dist}_\omega(d(z_j), d(z_j) + |z_0 - z_1|) \ .$$

To conclude, we therefore only need to check that if a and b are real numbers which satisfy $0 < a < b < 3\varrho$, then

$$\text{Kob dist}_\omega(a, b) \leqq 1/2(\log 1/a - \log 1/b) + \mathcal{O}(1) \ .$$

Let τ be a conformal mapping from ω to the unit disk such that $\tau(0) = 1$ and τ is real on the real axis. Since $b\omega$ is of class $\mathscr{C}^{1+\varepsilon}$, τ extends to a diffeomorphism from the closure of ω to the closed unit disk (see [38, Theorem 6, p. 426]). So there exist $K > 1$ and $\theta \in (-1, 1)$ such that for every $c \in (0, 3\varrho)$

$$\max(\theta, 1 - Kc) \leqq \tau(c) \leqq 1 - c/K \ .$$

Recall that the Kobayashi distance between two points x and x', $-1<x'<x<1$, in the unit disk is equal to $1/2(\log(1+x)/(1-x)-\log(1+x')/(1-x'))$.

Since τ is an isometry for the Kobayashi distance, one gets

$$\text{Kob dist}_\omega(\mathrm{a},\mathrm{b})\leqq\frac{1}{2}\left[\log\left(\frac{2}{(1/K)a}\right)-\log\left(\frac{1+\theta}{Kb}\right)\right],$$

which yields the desired estimate. Proposition 2.5 is proved.

3. Proof of Theorem 1.4

We first consider the case $n\geqq 2$. For each $z^\circ\in bD_1$ the cluster set $C(f,z^\circ)$ is contained in $E\cup M\subset bD_2$ by hypothesis. Suppose that f does not extend continuously to z°. Since $C(f,z^\circ)$ is connected and E is totally disconnected, there is a point $w^\circ\in C(f,z^\circ)\cap M$. The proof of Theorem 1.1 in [15] can be applied to conclude that the pair $(f(D_1),M)$ is a $\mathscr{C}^2$ manifold with boundary in a neighborhood of w° and $f(D_1)$ intersects bD_2 transversely near w°. This also follows from the results of Harvey and Lawson [22, Theorem 4.7, p. 243]. Here we need to assume that $n\geqq 2$. Choose a point $w^1\in C(f,z^\circ)\cap M\subset bD_2$ sufficiently close to w° such that the pair $(f(D_1),M)$ is a manifold with boundary near w^1 also (such w^1 exists since $C(f,z^\circ)$ is connected), and choose sequences $\{z_j^k:j\in\mathbb{Z}_+\}\subset D(k=0,1)$ satisfying

$$\lim_{j\to\infty} z_j^k=z^0\ ,\qquad \lim_{j\to\infty} f(z_j^k)=w^k\ ,\qquad k=0,1\ .$$

Suppose now that $\varrho\in\mathscr{C}(D_1)$ is a negative plurisubharmonic function satisfying the condition (P) near $z^\circ\in bD_1$. Then the function

$$\tau(w)=\max\{\varrho(z):z\in D_1,f(z)=w\}\ ,\qquad w\in f(D_1)$$

is negative, continuous and plurisubharmonic on $f(D_1)$ near the points w° and w^1 since $f(D_1)$ is smooth there. (τ may not be continuous near singular points of the variety $f(D_1)$.) As in [15] we can apply the Hopf lemma to $\tau(w)$ near w° resp. w^1 to conclude

$$d(w_j^k,bD_2)\leqq c\,d(z_j^k,bD_1)\ ,\qquad j\in Z_+\ ,\qquad k=0,1\ .$$

This means that step 3 in the proof of Theorem 1.1 in Sect. 1 holds. Exactly as before we can reach a contradiction to the assumption that f does not extend continuously to z°.

The case $n=1$ requires a slightly different proof. The set $K=\{w\in M:T_wM\subset T_w^{\mathbb{C}}bD_2\}$ is a closed subset of the one-dimensional manifold M, hence it is the union of a closed, totally disconnected subset K_0 of M and at most a countable set of open arcs $M_j\subset M$. Then $E_0=E\cup K_0$ is a closed, totally disconnected subset of bD_2 and $M_0=M\setminus K$ is a closed $\mathscr{C}^2$ submanifold of $bD_2\setminus E_0$.

We claim that the global cluster set $C(f)=\overline{f(D_1)}\setminus f(D_1)$ is contained in $E_0\cup M_0$. Assuming this we can conclude the proof exactly as in the case $n\geqq 2$. Since the tangent space T_wM_0 is not contained in $T_w^{\mathbb{C}}bD_2$ for any $w\in M_0$, the proof of Theorem 1.1 in [15] shows that near each $w\in M_0$ the pair $(f(D_1),M_0)$ is a $\mathscr{C}^2$ manifold with boundary that intersects bD_2 transversely. The rest of the proof follows as before.

It remains to prove that $C(f)$ is contained in $E_0 \cup M_0$. Assume on the contrary that there is a point $w \in M_j \cap C(f)$ for some j. Since D_2 is strictly pseudoconvex, w is a peak point for the algebra $A(D_2) = \mathcal{O}(D_2) \cap C(\bar{D}_2)$. If p_w is a peak function for w, the maximum principle applied to the function $p_w \circ f \in H^\infty(D_1)$ implies that there is a subset $\Gamma \subset bD_1$ of positive surface measure such that for each $z \in \Gamma$ the non-tangential limit $f^*(z)$ exists and lies in M_j.

Recall that M_j is a complex tangential curve in bD_2. Thus by a theorem of Rudin [35] each compact subset L of M_j is a peak-interpolation set for the algebra $A(D_2)$. If $L \subset M_j$ is a sufficiently large compact subset and $\varphi \in A(D_2)$ is a peaking function for L, then $\varphi = f \in H^\infty(D_1)$ has nontangential boundary values equal to 1 for each z in a subset $\Gamma_0 \subset \Gamma$ of positive measure. The uniqueness principle [36] implies $\varphi \circ f = 1$, a contradiction. Thus $C(f)$ is contained in $E_0 \cup M_0$ and Theorem 1.4 is proved.

4. Examples

In the first example we construct a biholomorphic map $f: D_1 \to D_2$ of a bounded strictly pseudoconvex domain D_1 with real-analytic boundary onto a bounded domain D_2 whose boundary bD_2 is smooth strictly pseudoconvex outside a circle $E \subset bD_2$ such that f does not extend continuously to $\bar{D}_1$.

Example 1. Choose a holomorphic function $g = u + iv$ on the unit disk $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ whose real part u is continuous on $\bar{\Delta}$ and smooth on $\bar{\Delta} \setminus \{1\}$ and whose imaginary part v is unbounded at the point 1. Then the map

$$f: (\bar{\Delta} \setminus \{1\}) \times \mathbb{C}^{n-1} \to \mathbb{C}^n$$

$$f(z_1, \ldots, z_n) = (z_1, \ldots, z_{n-1}, e^{g(z_1)} z_n)$$

is smooth on $(\bar{\Delta} \setminus \{1\}) \times \mathbb{C}^{n-1}$ and biholomorphic on $\Delta \times \mathbb{C}^{n-1}$. Set

$$D_1 = \{(z', z_n) \in \mathbb{C}^n : |z'|^2 + |z_n|^2 + |z_n|^{-2} < 3\}$$

and

$$D_2 = f(D_1) = \{(w', w_n) \in \mathbb{C}^n : |w'|^2 + |e^{-u(w_1)} w_n|^2 + |e^{-u(w_1)} w_n|^{-2} < 3\} .$$

Then bD_2 is real-analytic and strictly pseudoconvex outside the circle

$$E = \{(1, 0, \ldots, w_n) : |w_n| = e^{u(1)}\} \subset bD_2 .$$

Clearly f does not extend continuously to any point of the circle

$$E_1 = \{(1, 0, \ldots, z_n) : |z_n| = 1\} \subset bD_1 .$$

If $z^\circ \in E_1$, the cluster set $C(f, z^\circ)$ equals E. The inverse $f^{-1}: D_2 \to D_1$ extends continuously to $\bar{D}_2 \setminus E$.

Example 2. We construct a bounded domain $D \subset \mathbb{C}^2$ whose boundary is real-analytic and strictly pseudoconvex except at one point $p \in bD$ such that D admits automorphisms that do not extend continuously to p. The idea is similar to Fridman's [17].

Let

$$\Omega=\{(z_1,z_2)\in\mathbb{C}^2:|z_1|^2+|z_2|^2+|z_2|^{-2}<3\}\ ,$$
$$F(z_1,z_2)=(z_1,(1-z_1)z_2)\ ,\quad \text{and}$$
$$D=F(\Omega)\ .$$

The boundary of D is real analytic and strictly pseudoconvex except at the point $p=(1,0)\in bD$.

For each $\theta\in\mathbb{R}$ the map $\Phi_\theta(z_1,z_2)=(e^{i\theta}z_1,z_2)$ is an automorphism of Ω and hence

$$f_\theta(w_1,w_2)=F\circ\Phi_\theta\circ F^{-1}(w_1,w_2)$$
$$=\left(e^{i\theta}w_1,\frac{1-e^{i\theta}w_1}{1-w_1}\,w_2\right)$$

is an automorphism of D. From the definition of Ω and F it follows that f_θ does not extend continuously to p if $\theta\neq k\cdot 2\pi$. On the other hand, the automorphisms $F\circ\psi_\theta\circ F^{-1}$ of D, where $\psi_\theta(z_1,z_2)=(z_1,e^{it}z_2)$, extend continuously to $\bar{D}$ and preserve the point p. According to Corollary 1.2 every automorphism of D extends continuously to $\bar{D}\setminus\{p\}$.

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