Polynomial Hulls of Sets Fibered Over the Circle

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0. Introduction. Let $D = \{z \in \mathbf{C} : |z| < 1\}$ and $T = bD = \{z \in \mathbf{C} : |z| = 1\}$. We denote by $A(D)$ the algebra of all continuous functions on $\bar{D}$ that are holomorphic in D.

Let M be a two–dimensional, connected, compact, totally real C^2 submanifold of $T \times \mathbf{C}$. Then for each $\vartheta \in \mathbf{R}$ the fiber

$$M_\vartheta = \{z \in \mathbf{C} : (e^{i\vartheta}, z) \in M\}$$

is a simple closed curve in $\mathbf{C}$ bounding a simply connected region $Y_\vartheta \subset \mathbf{C}$. We assume, in addition, that each Y_ϑ contains the point 0, and we set $Y = \bigcup_\vartheta \{e^{i\vartheta}\} \times Y_\vartheta$.

Our main result is a precise description of the polynomially convex hull $\widehat{M}$ of M. Recall that

$$\widehat{M} = \{z \in \mathbf{C}^2 : |f(z)| \leq \sup_M |f| \text{ for all } f \in O(\mathbf{C}^2)\}.$$

Clearly $\widehat{M}$ projects onto the closed disc $\bar{D} \subset \mathbf{C}$. If the boundary values of a function $f \in A(D)$ satisfy the condition

$$f(e^{i\vartheta}) \in M_\vartheta \qquad \text{for all } \vartheta \in \mathbf{R}, \tag{0.1}$$

then its graph

$$G_f = \{(z, f(z)) : z \in D\}$$

is an analytic variety with boundary in M, so G_f is contained in $\widehat{M}$ by the maximum principle. We shall prove the converse to this: for each point $(a,b) \in \widehat{M}$ with $a \in D$ there exists an $f \in A(D)$ satisfying (0.1) and $f(a) = b$. In other words, the graphs of solutions of (0.1) fill the entire polynomial hull of M except the set $Y \subset \partial\widehat{M}$. We shall prove, moreover, that the topological boundary of $\widehat{M}$ is piecewise smooth. The part of $\partial\widehat{M}$ over D, denote it by Σ, is a smooth

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Levi flat hypersurface foliated by the graphs G_f of those solutions of (0.1) that are nonvanishing on $\bar{D}$. Thus, $\partial\widehat{M} = Y \cup M \cup \Sigma$, where Y and Σ are smooth hypersurfaces with common boundary M.

These results generalize our previous work [10] where each fiber M_ϑ was assumed to be the boundary of a convex set Y_ϑ. The proof in [10] was based on the work of Alexander and Wermer [2] and Slodkowski [22] who gave a similar description of the hull $\widehat{M}$ when $M \subset bD \times \mathbf{C}$ is compact and each fiber M_ϑ is convex (no smoothness of M was required). The hull $\widehat{M}$ is then filled by the graphs of bounded holomorphic functions $f \in H^\infty(D)$ with boundary values $f(e^{i\vartheta}) \in MY_\vartheta$ almost everywhere on T.

In this paper we find solutions of (0.1) by the *continuity method.* To solve the problem for small perturbations of the initial manifold M we consider an analogue of the Bishop equation [6] for finding analytic discs with boundaries in M. We combine this with the a priori Hölder estimates to obtain a homotopy lifting theorem for solutions of (0.1). (See Theorem 1.)

The problem of finding functions $f \in A(D)$ with boundary values $f(e^{i\vartheta})$ in prescribed curves $M_\vartheta \subset \mathbf{C}$ is known as the *Hilbert boundary value problem.* The special case where each $M_\vartheta \subset \mathbf{C}$ is an affine real line in $\mathbf{C}$ was mentioned by Riemann in 1951 and was solved by Hilbert in 1905 [16] by an explicit integral formula. For the history and results concerning this linear Hilbert problem, consider Chapter 4 of [12], containing historical remarks, as well as the monographs [14, 15, 19, 20, 21]. A survey of results on the general nonlinear Hilbert problem can be found in the introduction to [15] and in [24]. In our case, when each fiber M_ϑ is a smooth closed Jordan curve, the solutions of (0.1) were found before by Shnirelman [23] by a method that is substantially different from our method. He used the so-called quasilinear Fredholm operators. Moreover, Shnirelman did not consider the polynomial hull of M and the regularity of its boundary.

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1. Results. Let D be the unit disc and $T = bD$ the unit circle in $\mathbf{C}$. If K is either $\bar{D}$ or T and $0 \le \alpha \le 1$, we denote by $C^\alpha(K) = C^{0,\alpha}(K)$ the space of all continuous functions on K with finite norm

$$\|u\|_\alpha = \sup_{x\in K} |u(x)| + \sup_{\substack{x,y\in K\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty.$$

For every $k \in \mathbf{Z}_+$ we define the space

$$C^{k,\alpha}(K) = \left\{u \in C^k(K) : \|u\|_{k,\alpha} = \sum_{|\beta|\le k} \|D^\beta u\|_\alpha < \infty\right\}.$$

The space $C^{k,\alpha}(K)$ is a Banach algebra. Our functions can be either real or complex valued; it will be always clear from the context which one we use.

Let $A(D)$ be the algebra of all continuous function on $\bar{D}$ that are holomorphic in D (the disk algebra) and $A(T) = \{f|_T \colon f \in A(D)\}$. We also set

$$A^{k,\alpha}(D) = C^{k,\alpha}(\bar{D}) \cap A(D)$$

and

$$A^{k,\alpha}(T) = C^{k,\alpha}(T) \cap A(T).$$

If $f \in A(D)$, then $f \in A^{k,\alpha}(D)$ if and only if $f|_T \in A^{k,\alpha}(T)$ [14, pp. 363–364]. $A^{k-0}(D)$ denotes the intersection $\bigcap_{0<\alpha<1} A^{k-1,\alpha}(D)$, and similarly $A^{k-0}(T)$.

Denote by $\pi : \mathbf{C}^2 \to \mathbf{C}$ the projection $\pi(\zeta,z) = \zeta$. Throughout this paper M will denote a subset of $\mathbf{C}^2$ satisfying

(1.1)
- (i) M is a compact connected submanifold of $\mathbf{C}^2$ of class C^k $(k \geq 2)$, $\dim_{\mathbf{R}} M = 2$.
- (ii) $\pi(M) = T$, and $\pi : M \to T$ is a submersion.
- (iii) For each $\vartheta \in \mathbf{R}$, the fiber

$$M_\vartheta = \{z \in \mathbf{C} : (e^{i\vartheta},z) \in M\}$$

is a simple closed curve in $\mathbf{C}$ with 0 in the bounded part Y_ϑ of $\mathbf{C} \setminus M_\vartheta$.

Note that such an M is a torus embedded as a totally real submanifold of $\mathbf{C}^2$. For such M we consider the problem of finding functions

$$f \in A(D),\ f(e^{i\vartheta}) \in M_\vartheta \qquad \text{for all } \vartheta. \tag{1.2}$$

Each solution of (1.2) is of class $A^{k-0}(D)$ according to Čirka [7], so we shall fix an $\alpha \in (0,1)$ and look for $f \in A^\alpha(S)$ solving (1.2).

We shall first prove that certain families of solutions of (1.2) satisfy the homotopy lifting property. Suppose that $\{M^t : 0 \leq t \leq 1\}$ is a homotopy of class C^k such that each M^t satisfies the conditions (1.1). More precisely, there is a C^k function

$$r : T \times \mathbf{C} \times [0,1] \to \mathbf{R}$$

satisfying

(1.3)

(i) $M^t = \{(e^{i\vartheta},z) \in \mathbf{C}^2 : r(\vartheta,z,t) = 0\}$ satisfies (1.1) for each $t \in [0,1]$.

(ii) The gradient

$$\nu(\vartheta,x+iy,t) = (r_x + ir_y)(\vartheta,x+iy,t)$$

of r with respect to $x+iy$ is nonzero on M^t, $t \in [0,1]$.

Let $S = \mathbf{R}/\mathbf{Z}$. We also choose a C^k map $\varphi : S \times [0,1] \to \mathbf{C}$ such that for each $t \in [0,1]$,

$$\varphi(\cdot,t) : S \to M_0^t \text{ is a } C^k \text{ diffeomorphism.} \tag{1.4}$$

Under these hypotheses we have:

Theorem 1. *Suppose that $f^0 : S \to A(D)$ $(0 < \alpha < 1)$ is a continuous map satisfying*

(i) $f^0(s)(e^{i\vartheta}) \equiv f^0(s,\vartheta) \in M_\vartheta^0$.

(ii) $f^0(s,0) = \varphi(s,0) \in M_0^0$.

(iii) $f^0(s)$ *has no zeros on* $\bar{D}$ *for some (hence for all)* $s \in S$.

Then there exists a map $f : S \times [0,1] \to A^\alpha(D)$ of class C^{k-1} such that

(a) $f(s,t)(e^{i\vartheta}) \equiv f(s,t,\vartheta) \in M_\vartheta^t$.

(b) $f(s,t,0) = \varphi(s,t) \in M_0^t$.

(c) $f(s,0,\cdot) = f^0(s,\cdot)$.

Given any M satisfying (1.1), there exists a C^k homotopy $\{M^t : 0 \le t \le 1\}$ satisfying (1.3) such that $M^0 = T \times T$ is the distinguished boundary of the bidisc $D \times D$ and $M^1 = M$. Applying Theorem 1 with the initial family of solutions of (1.2) for M^0

$$f^0(s)(z) = e^{2\pi is}, \qquad s \in S,\ z \in \bar{D},$$

we obtain:

Corollary 2. *For each M satisfying* (1.1) *and each C^k diffeomorphism $\varphi : S \to M_0$ there exists a C^{k-1} map $f : S \to A^\alpha(D)$ such that*

(a) $f(s)(e^{i\vartheta}) \equiv f(s,\vartheta) \in M_\vartheta$.

(b) $f(s,0) = \varphi(s) \in M_0$.

(c) $f(s) \in A^\alpha(D)$ *is zero-free on* $\bar{D}$ *for all* $s \in S$.

This family of solutions of (1.2) has a number of other properties which we formulate in the next theorem.

Theorem 3. *Let M and f be as in Corollary 2. Denote by Σ the set $\{(z,f(s)(z)) : z \in D,\ s \in S\}$. Then the following hold:*

(i) *Σ is an embedded hypersurface of class C^{k-1}.*
(ii) *$\bar{\Sigma} = \Sigma \cup M$, and the pair (Σ,M) is a C^{k-1} hypersurface with boundary.*
(iii) *Let $\widehat{M}$ denote the polynomially convex hull of M. Then $\Sigma = \partial\widehat{M} \cap \pi^{-1}(D)$.*
(iv) *For each $(a,b) \in \widehat{M}$, $a \in D$, there exists a solution g of* (1.2) *such that $g(a) = b$.*
(v) *If $g \in A(D)$ is a solution of* (1.2) *that is zero-free on $\bar{D}$, then $g = f(s)$ for some $s \in S$.*

Thus, the hull $\widehat{M}$ is the compact set over $\bar{D}$ bounded by the hypersurfaces Σ and $Y = \bigcup_\vartheta \{e^{i\vartheta}\} \times Y_\vartheta$, where Y_ϑ is the region in $\mathbf{C}$ bounded by M_ϑ. The hypersurfaces Σ and Y have the common edge M, and $\partial\widehat{M}$ is smooth away from M. For related results on $\widehat{M}$ see [1, 3, 10].

Our solutions of (1.2) are obtained by the continuity method and are not explicit. In certain cases we have explicit formulas for solution. E.g., if each fiber M_ϑ is a circle of radius $r(\vartheta) > 0$ centered at 0, the outer functions

$$f(z) = e^{is} \exp\left(\int_{-\pi}^{\pi} \frac{e^{i\vartheta}+z}{e^{i\vartheta}-z} \log r(\vartheta)\, \frac{d\vartheta}{2\pi}\right), \qquad s \in \mathbf{R} \tag{1.5}$$

solve (1.2). In fact, these are the only solutions that are nonvanishing on $\bar{D}$. It would be of interest to have such explicit expression for solutions in general. See also [2].

Some smoothness of M is needed for our results to hold. Indeed, if the function r in our example above is merely continuous, the harmonic conjugate of $\log r$ may be discontinuous everywhere on T, so the solutions (1.5) have discontinuous argument. There are even no local solutions of (1.2) in this case. It is likely, however, that the smoothness requirement $M \in C^2$ could be weakened, but this would only result in additional technical complications which are not central to our problem.

We can justify the hypothesis that 0 lies in the region Y_ϑ for each ϑ. If

$$M = \{(e^{i\vartheta},z) \in \mathbf{C}^2 : |z - e^{-i\vartheta}| = c\}$$

and $c < 1$, every $f \in C(T)$ satisfying $f(e^{i\vartheta}) \in M_\vartheta$ has winding number -1, so it is not in $A(T)$. In the limit case $c = 1$ the only solution of (1.2) is $f \equiv 0$ [11, Proposition 14]. There are plenty of solutions for $c > 1$.

A more general condition which yields the same results is: There exists a $g \in A(D)$ with $g(e^{i\vartheta}) \in Y_\vartheta$ for all ϑ. This later condition can be reduced to the former by a simple change of coordinates $(\zeta,z) \to (\zeta, z - g(\zeta))$.

We shall outline the construction of the map f in Theorem 1. First, we extend the initial map $f^0 : S \to A^\alpha(D)$ given at $t = 0$ to an interval $0 \le t < t_1$ using the implicit function theorem in an appropriate Banach space (Section 2). In Section 3 we show that, with a suitable choice of parametrization, the resulting map is of class C^{k-1} and satisfies Theorem 1 on $0 \le t < t_1$.

This well-known method of constructing so-called analytic discs with boundaries in a given submanifold of $\mathbf{C}^n$ was initiated by Bishop [6]. Subsequently, it has been used by several authors ([17, 3, 4, 5, 11, 18], to mention just a few). We include a brief exposition to make the paper self contained and to obtain the specific result we need.

The main part of the paper is Section 4, where we obtain a priori estimates on solutions of (1.2) in Hölder norms. The estimates enable us to extend the map f continuously to the closed interval $0 \le t \le t_1$. If $t_1 < 1$, we can again extend f past $t = t_1$ using Sections 2 and 3. Continuing this process, we extend f to the whole interval $0 \le t \le 1$.

Similar a priori estimates in certain related problems were obtained in [4, 5, 18].

2. A local perturbation theorem. Let $S = \frac{\mathbf{R}}{\mathbf{Z}}$ be the circle. For each Banach space E we denote by ΩE the Banach space of all continuous maps $g : S \to E$ equipped with the norm $\|g\|_{\Omega E} = \sup_{s\in S} \|g(s)\|_E$. If U is an open subset of E, ΩU is the open subset of ΩE consisting of all $g \in \Omega E$ with image in U.

Given a continuously differentiable map $F : U \subset E \to E'$ into a Banach space E', the induced map

$$\widetilde{F} : \Omega E \to \Omega E', \qquad (\widetilde{F}g)(s) = F\big(g(s)\big) \text{ for } s \in S$$

is also continuously differentiable, with the derivative

$$\big(D\widetilde{F}(g)h\big)(s) = DF\big(g(s)\big)h(s), \qquad s \in S.$$

This follows easily from the fact that each continuously differentiable map is also strictly differentiable [8, pp. 53–54]. If F is a bounded linear map, so is $\widetilde{F}$.

In the sequel we shall use spaces of functions on $T = bD$ instead of on $\bar{D}$. Let $\{M^t : 0 \le t \le 1\}$ be a C^k homotopy ($k \ge 2$) satisfying (1.3). Given a $g^\circ \in \Omega A^\alpha(T)$ satisfying

$$g^\circ(s)(e^{i\vartheta}) \equiv g^\circ(s,\vartheta) \in M_\vartheta^{t_\circ} \tag{2.1}$$

for all values of s and ϑ and for a fixed $t_0 \in [0,1]$, our aim is to find a C^1 map $t \to g(s,t,\vartheta)$ from a neighborhood J of t_0 in $[0,1]$ into the space $\Omega A^\alpha(T)$ satisfying

$$(2.2)\qquad \left|\begin{array}{l} \text{(i) } g(s,t,\vartheta) \in M^t_\vartheta \text{ for } t \in J\text{, and} \\ \text{(ii) } g(s,t_0,\vartheta) = g^\circ(s,\vartheta). \end{array}\right.$$

We shall assume in addition that $g^\circ(s)$ has winding number 0 for some (hence for all) $s \in S$. Equivalently, its holomorphic extension to $\bar{D}$ is nonvanishing.

We define

$$Y(s,\vartheta) = \nu\big(\vartheta, g^\circ(s,\vartheta), t_0\big),$$

where ν is as in (1.3). Since ν is of class C^{k-1}, $Y(s,\cdot)$ is in $C^\alpha(T)$ for each s, and $s \to Y(s,\cdot) \in C^\alpha(T)$ is continuous [17, p. 340]. Thus, $Y \in \Omega C^\alpha(T)$.

The geometrical assumptions (1.1) on M^{t_0} imply that the winding number of $Y(s,\cdot)$ equals that of $g^\circ(s)$ which is zero. So

$$Y(s,\vartheta) = e^{a(s,\vartheta)+ib(s,\vartheta)}$$

for some continuous maps

$$a : S \to C^\alpha(T), \qquad b : \mathbf{R} \to C^\alpha(T).$$

Recall that the harmonic conjugate $u \to \tilde{u}$ is a bounded linear map of $C^\alpha(T)$ into itself [13, p. 106] which annihilates the constants. Since b satisfies the periodicity condition

$$b(s+1,\vartheta) = b(s,\vartheta) + 2\pi\ell, \qquad \ell \in \mathbf{Z},$$

its harmonic conjugate $\tilde{b}(s,\cdot)$ with respect to ϑ is a well-defined continuous map $S \to C^\alpha(T)$, i.e., $\tilde{b} \in \Omega C^\alpha(T)$. Multiplying Y by the positive function $e^{-a-\tilde{b}}$, we obtain the function

$$X(s,\vartheta) = e^{-\tilde{b}(s,\vartheta)+ib(s,\vartheta)}$$

in $\Omega A^\alpha(T)$ such that $X(s,\vartheta)$ is real orthogonal to the curve M^t_ϑ at the point $g^\circ(s,\vartheta)$.

Let $G : \Omega C^\alpha(T) \to \Omega A^\alpha(T)$ be the linear map

$$G(u)(s,\vartheta) = g^\circ(s,\vartheta) + \big(u(s,\vartheta) + i\tilde{u}(s,\vartheta)\big)X(s,\vartheta), \qquad u \in \Omega C^\alpha(T).$$

Here $\tilde{u}(s,\vartheta)$ is the harmonic conjugate of $u(s,\vartheta)$ with respect to ϑ.

Consider the composition

$$\Phi : \Omega C^\alpha(T) \times [0,1] \to \Omega C^\alpha(T),$$

$$\Phi(u,t)(s,\vartheta) = r\big(\vartheta, G(u)(s,\vartheta), t\big)$$

where r is as in (1.3). Since r is of class C^2, the map Φ is of class C^1 in (u,t)

according to [17, p. 340]. Condition (2.1) is equivalent to

$$\Phi(0,t_0)(s,\vartheta) = r\big(\vartheta,g^\circ(s,\vartheta),t_0\big) = 0.$$

The partial derivative of Φ with respect to u at the point $u = 0$, $t = t_0$, applied to $v \in \Omega C^\alpha(T()$, equals

$$\begin{aligned}(2.3)\qquad D_u\Phi(0,t_0)v(s,\vartheta) &\\ &= \mathrm{Re}\big(\big(v(s,\vartheta)+i\tilde{v}(s,\vartheta)\big)X(s,\vartheta)\,\overline{\nu(s,g^\circ(s,\vartheta),t_0)}\big)\\ &= h(s,\vartheta)v(s,\vartheta)\end{aligned}$$

where

$$\begin{aligned}h(s,\vartheta) &= X(s,\vartheta)\,\overline{\nu\big(s,g^\circ(s,\vartheta),t_0\big)}\\ &= e^{a(s,\vartheta)+\tilde{b}(s,\vartheta))}|X(s,\vartheta)|^2\end{aligned}$$

is a positive function in $\Omega C^\alpha(T)$. Here we used the fact that $iX(s,\vartheta)$ is real orthogonal to $\nu\big(s,g^\circ(s,\vartheta),t_0\big)$, so $\tilde{v}$ does not enter into (2.3).

It follows that the derivative (2.3) is a linear isomorphism of $\Omega C^\alpha(T)$ onto itself. By the implicit function theorem in Banach spaces [8, p. 61] there is a C^1 map $u : J \subset [0,1] \to C^\alpha(T)$, where J is a neighborhood of t_0 in $[0,1]$, such that

$$\Phi\big(u(t),t\big) \equiv 0 \qquad \text{and} \qquad u(t_0) = 0.$$

Then $G\big(u(t)\big) : J \to \Omega A^\alpha(T)$ is a C^1 map, and the function $g(s,t,\vartheta) \equiv G\big(u(t)\big)(s,\vartheta)$ satisfies (2.2).

3. Uniqueness and regularity of solutions. Let $\{M^t : 0 \le t \le 1\}$ and $\varphi : S \times I \to \mathbf{C}$ be as in (1.3) (resp. (1.4)). In this section we prove that the family of solutions $\{g(s,t),\cdot)\} \subset A^\alpha(T)$, constructed in Section 2 (2.2), can be reparametrized such that the resulting family $\{f(s,t,\cdot)\} \subset A^\alpha(T)$ satisfies the conclusions of Theorem 1 on the interval J containing t_0. A necessary additional assumption is that the map $s \in S \to g(s,t_0,0) \in M_0^{t_0}$ is homotopic to $s \to \varphi(s,t_0) \in M_0^{t_0}$ in $M_0^{t_0}$.

Denote by $B^t \subset A^\alpha(T)$ the set of all solutions of (1.2) for M^t that extend to zero-free functions on $\bar{D}$. (Equivalently, they have winding number zero.) We denote by $\psi : A^\alpha(T) \to \mathbf{C}$ the linear evaluation map $\psi(f) = f(0)$. We also set $B = \bigcup\{t\} \times B^t \subset J \times A^\alpha(T)$, the union over $t \in J$. The main results of this section are:

(a) B is a one-one immersed C^{k-1} submanifold of $J \times A^\alpha(T)$ with one-dimensional fibers B^t.

(b) If C^t $(t \in J)$ is the connected component of B^t containing $\{g(s,t,\cdot) : s \in S\}$, then the restriction

$$\psi\Big|_{C^t} : C^t \to M_0^t \tag{3.1}$$

is a C^{k-1} diffeomorphism.

We shall see in Section 5 that in fact B is connected.

Once we know that (a) and (b) hold, we set

$$f(s,t) = \big(\psi|_{C^t}\big)^{-1} \circ \varphi(s,t), \qquad s \in S,\, t \in J \tag{3.2}$$

to obtain a family of solutions satisfying Theorem 1 on $t \in J$.

Fix a $t_1 \in J$ and an $h_0 \in B^{t_1}$. As in Section 2, we can find an invertible function $X \in A^\alpha(T)$ such that $X(\vartheta)$ is real orthogonal to $M_\vartheta^{t_1}$ at the point $h_0(\vartheta)$. We can write every function in $C^\alpha(T)$ in the form

$$H(u,v) = h_0 + (u + i\tilde{u})X + viX$$

for some uniquely determined real–valued functions $u,\, v \in C^\alpha(T)$.

Since X is invertible and $(u + i\tilde{u}) \in A^\alpha(T)$, the function $H(u,v)$ belongs to $A^\alpha(T)$ if and only if $v = s \in \mathbf{R}$ is a constant. So we shall only consider H on $C^\alpha \times \mathbf{R}$.

The image of $H(u,s)$ lies in M^t if and only if

$$\Psi(u,s,t)(\vartheta) \equiv r\big(\vartheta, H(u,s)(\vartheta), t\big) = 0, \qquad \vartheta \in \mathbf{R}. \tag{3.3}$$

By the hypothesis, this holds at $u = 0$, $s = 0$, $t = t_1$. The map $\Psi : C^\alpha(T) \times \mathbf{R}^2 \to C^\alpha(T)$ is of class C^{k-1} [17, p. 340]. Its partial derivative $D_u\Psi$ at the point $u = 0$, $s = 0$, $t = t_1$ is a linear isomorphism of $C^\alpha(T)$ onto itself; we shall omit the computation since it is similar to the one for Φ in Section 2.

By the implicit function theorem [8, p. 61] there is a C^{k-1} map $u = u(s,t)$ into $C^\alpha(T)$ defined near $s = 0$, $t = t_1$, such that the corresponding C^{k-1} map

$$h(s,t) = H\big(u(s,t),s\big)$$

into $A^\alpha(T)$ satisfies (3.3), and every solution $f \in B^t$ for t close to t_1 and f close to h_0 equals $f = h(s,t)$ for some $s \in \mathbf{R}$. In other words, the C^{k-1} map

$$(t,s) \to \big(t, h(s,t)\big) \in B \subset J \times A^\alpha(T) \tag{3.4}$$

locally parametrizes the set B in a neighborhood of $h_0 \in B^{t_1}$.

Differentiating (3.3) with respect to s at $s = 0$, $t = t_1$, we conclude that $\frac{\partial u}{\partial s}(0,t_1) = 0$ and, therefore,

$$\frac{\partial h}{\partial s}(0,t_1)(\vartheta) = iX(\vartheta) \neq 0 \qquad \text{for all } \vartheta.$$

Hence the map (3.4) and also the composition

$$s \to \psi\big(h(s,t)\big) = h(s,t)(0) \in M_0^t$$

are C^{k-1} immersions near $s = 0$, $t = t_1$. Consequently, $\psi : B^t \to M_0^t$ is a local C^{k-1} diffeomorphism. This proves (a).

The connected component C^t of B^t is a connected one–dimensional manifold without boundary, so it is homeomorphic to either $\mathbf{R}$ or S. The composition $s \to \psi \circ g(s,t,\cdot) = g(s,t,0) \in M_0^t$ is homotopic to the homeomorphism $s \to \varphi(s,t)$ of S onto M_0^t for $t = t_0$ and thus for all $t \in J$, so we conclude that C^t is homeomorphic to S and $\psi : C^t \to M_0^t$ is homotopic to a homeomorphism. Since this map is a local diffeomorphism, it follows that ψ is a global diffeomorphism for each $t \in J$. This proves (b). □

We summarize the results of Sections 2 and 3 in the following:

Theorem 4. *Let $\{M^t : 0 \le t \le 1\}$ be a C^k homotopy satisfying (1.3), and let $\varphi : S \times [0,1] \to \mathbf{C}$ be as in (1.4). Assume that for some $t_0 \in [0,1]$ there exists a continuous mapping $g^0 : S \to A^\alpha(T)$ such that*

$$g^0(s)(\vartheta) \in M_\vartheta^{t_0},$$

$$g^0(s)(0) = \varphi(s,t_0) \in M_0^{t_0}.$$

Then there exists a neighborhood J of t_0 in [0,1] and a mapping $f : S \times J \to A^\alpha(T)$ of class C^{k-1} satisfying

(a) $$f(s,t)(\vartheta) \in M_\vartheta^t,$$

(b) $$f(s,t)(0) = \varphi(s,t) \in M_0^t,$$

(c) $$f(s,t_0) = g^0(s).$$

The mapping f is unique in the following sense. If J_0 is a connected subset of J containing t_0 and if $h : J_0 \to A^\alpha(T)$ is a continuous map satisfying $h(t)(\vartheta) \in M_\vartheta^t$ and $h(t_0) = g^0(s)$ for some $s \in S$, then for each $t \in J_0$, $h(t) = f(s,t)$ for a unique $s = s(t)$.

Proof. The map $f(s,t)$ was defined by (3.2) above. It remains to prove the uniqueness part. Since J_0 is connected and h is continuous, the image of h is a connected subset of B. The assumption $h(t_0) = g^0(s) \in C^{t_0}$ implies that the image of h is contained in the connected component $C = \bigcup\{t\} \times C^t$ of B. Since the map (3.2) is a diffeomorphism, the result follows. This concludes the proof of Theorem 4. □

Remark. If we replace the space $A^\alpha(T)$ by $A^{m,\alpha}(T)$ for some

$$m \in \{0,1,\dots,k-2\},$$

the same methods yield a map $f : S \times J \to A^{m,\alpha}(T)$ of class C^n, $n = k - m - 1$, satisfying the conclusions of Theorem 1 on a neighborhood J of t_0 in [0,1]. The uniqueness part of Theorem 4 implies that this map coincides with the previously constructed map into $A^\alpha(T)$ if we consider $A^{m,\alpha}(T)$ as the natural subspace of $A^\alpha(T)$. We shall omit the details. (See also [11].)

4. The a priori estimates.

Theorem 5. *Let $M \subset \mathbf{C}^2$ be a submanifold of class C^k $(k \geq 2)$ satisfying* (1.1). *For each $\ell \in \{0,1,\ldots,k-1\}$ and $0 < \alpha < 1$ there is a constant $C_{\ell,\alpha}$ such that every solution f of* (1.2) *which is nonvanishing on $\bar{D}$ satisfies*

$$\|f\|_{\ell,\alpha} \leq C_{\ell,\alpha}. \tag{4.1}$$

Moreover, if $\{M^t : 0 \leq t \leq 1\}$ is a C^k homotopy satisfying (1.3), the constants $C_{\ell,\alpha}$ can be chosen to be independent of t.

Note. In the case when $M = T \times T$ is the distinguished boundary of the bidisc, the finite Blaschke products show that no estimate (4.1) is possible for solutions of (1.2) which have zeroes on D.

Proof. Let $r : T \times \mathbf{C} \to \mathbf{R}$ be a C^k defining function for M (see (1.3)) whose gradient

$$\nu(\vartheta,x+iy) = (r_x + ir_y)(\vartheta,x+iy)$$

is nonvanishing at every point $(e^{i\vartheta},x+iy) \in M$. The function $f(e^{i\vartheta}) \equiv f(\vartheta)$ is of class $A^{k-0}(T)$ [7] and satisfies

$$r\big(\vartheta,f(\vartheta)\big) = 0.$$

Differentiating with respect to ϑ, we obtain

$$r_\vartheta\big(\vartheta,f(\vartheta)\big) + \operatorname{Re}\left(\frac{\partial f}{\partial\vartheta}\,\overline{\nu(\vartheta,f(\vartheta))}\right) = 0.$$

We shall now use the hypothesis that f is nonvanishing on $\bar{D}$, so $f = e^g$ for some $g \in A^{k-0}(T)$. We also introduce the function

$$\eta(\vartheta,z) = z\,\overline{\nu(\vartheta,z)} \tag{4.2}$$

on $T \times \mathbf{C}$ whose restriction to M is nonvanishing. With this notation we have

$$r_\vartheta\big(\vartheta,f(\vartheta)\big) + \operatorname{Re}\left(\frac{\partial g}{\partial\vartheta}\,\eta\big(\vartheta,f(\vartheta)\big)\right) = 0. \tag{4.3}$$

The geometric hypotheses (1.1) on M imply that $\eta : M \to \mathbf{C}\setminus\{0\}$ is null homotopic in $\mathbf{C}\setminus\{0\}$, so

$$\eta(\vartheta,z) = e^{a(\vartheta,z)+ib(\vartheta,z)}.$$

We denote by $\tilde{b}(\vartheta,f(\vartheta))$ the harmonic conjugate of the function $b(\vartheta,f(\vartheta))$. Multiplying (4.3) by $e^{-a-\tilde{b}}$, we obtain our main identity

$$\text{(4.4)}\qquad \operatorname{Re}\left(\frac{\partial g}{\partial\vartheta}(\vartheta)e^{-\tilde{b}(\vartheta,f(\vartheta))+ib(\vartheta,f(\vartheta))}\right) = -e^{-\tilde{b}(\vartheta,f(\vartheta))}\,e^{-a(\vartheta,f(\vartheta))}\,r_\vartheta\big(\vartheta,f(\vartheta)\big),$$

which we shall exploit to prove the estimates (4.1).

Notice that the function $\frac{\partial g}{\partial\vartheta}e^{-\tilde{b}+ib}$ extends to a holomorphic function on D with value 0 at $0\in D$ (since $\frac{\partial g}{\partial\vartheta} = ie^{i\vartheta}g'(e^{i\vartheta})$ extends to $izg'(z)$), so its imaginary part is precisely the harmonic conjugate of the real part. If we can estimate the right–hand side of (4.4) in a L^p $(1<p<\infty)$ or a Hölder norm on T, then the boundedness of the harmonic conjugation in these norms will give a bound on $\frac{\partial g}{\partial\vartheta}e^{-\tilde{b}+ib}$ and on $\frac{\partial f}{\partial\vartheta}$ in the same norm. This will in turn yield a better estimate on f itself.

First, there is a constant C_1 such that

$$\text{(4.5)}\qquad \begin{cases} \big|f(\vartheta)\big| \le C_1, & \\ \big|b(\vartheta,z)\big| \le C_1, & \text{on } (e^{i\vartheta},z)\in M, \\ \big|e^{-a(\vartheta,f(\vartheta))}r_\vartheta(\vartheta,f(\vartheta))\big| \le C_1. & \end{cases}$$

Further, we will prove that for each $p\in(0,\infty)$, there is a constant $C_2 = C_2(p)$ such that

$$\text{(4.6)}\qquad \big\|e^{\pm\tilde{b}(\vartheta,f(\vartheta))}\big\|_{L^p(T)} \le C_2$$

for all f solving (1.2). The proof will be split into three lemmas.

Lemma 6. *Let $p\in(0,\infty)$. For each $u\in C(T)$ the function $e^{\tilde{u}}$ is in $L^p(T)$. If $p\|u\|_\infty < \frac{\pi}{2}$, then*

$$\|e^{\tilde{u}}\|_{L^p}^p \le \frac{1}{\cos(p\|u\|_\infty)}.$$

This lemma is well known; see, for instance, [14, p. 365].

Lemma 7. *Let Q be the set of all functions on $T \times \mathbf{C}$ of the form*

$$q(\vartheta,z) = \sum_{\substack{-j_0 \le j \le j_0 \\ 0 \le \ell \le \ell_0}} c_{j\ell} e^{ij\vartheta} z^\ell, \qquad c_{j\ell} \in \mathbf{C}. \tag{4.7}$$

Then the set $\operatorname{Re} Q\big|_M = \{\operatorname{Re} q\big|_M : q \in Q\}$ *is dense in the space $C(M)$ of real-valued continuous functions on M.*

Proof. Note that $\operatorname{Re} Q\big|_M$ is a linear subspace of $C(M)$. Let μ be a finite real Borel measure on M that annihilates $\operatorname{Re} Q\big|_M$. Under the projection $\pi : M \to T$ the measure μ disintegrates in the sense that there exists a measure μ^* on T and, for almost every ϑ with respect to μ^*, there exists a measure σ_ϑ on M_ϑ such that, for all $f \in C(M)$,

$$\int_M f\, d\mu = \int_{-\pi}^{\pi} d\mu^*(\vartheta) \int_{M_\vartheta} f\, d\sigma_\vartheta .$$

Since Q contains functions $\{z^\ell \cos j\vartheta,\ z^\ell \sin j\vartheta : j\,,\ \ell \in \mathbf{Z}_+\}$, we have

$$0 = \int_{-\pi}^{\pi} \cos j\vartheta \int_{M_\vartheta} \operatorname{Re} z^\ell\, d\sigma_\vartheta\, d\mu^*(\vartheta),$$

$$0 = \int_{-\pi}^{\pi} \sin j\vartheta \int_{M_\vartheta} \operatorname{Re} z^\ell \cdot d\sigma_\vartheta\, d\mu^*(\vartheta), \qquad j \in \mathbf{Z}_+ .$$

It follows that

$$\left(\int_{M_\vartheta} \operatorname{Re} z^\ell\, d\sigma_\vartheta\right) d\mu^*(\vartheta)$$

is the zero measure on T for all $\ell \in \mathbf{Z}_+$, so the function

$$\vartheta \to \int_{M_\vartheta} \operatorname{Re} z^\ell\, d\sigma_\vartheta$$

is zero almost everywhere with respect to μ^* for all ℓ. The same is then true of

$$\int_{M_\vartheta} \operatorname{Re} h(z)\, d\sigma_\vartheta$$

for every holomorphic polynomial $h(z)$ on $\mathbf{C}$. Since $\{\operatorname{Re} h\big|_{M_\vartheta} : h \text{ polynomial}\}$ is dense in $C(M_\vartheta)$ by Mergelyan's theorem, we conclude that $\sigma_\vartheta = 0$ a.e. $[\mu^*]$. Thus $\mu = 0$, so $\operatorname{Re} Q\big|_M$ is dense in $C(M)$. Lemma 7 is proved. □

For each function q of the form (4.7), we denote by R_q the operator which assigns to each $f \in C(T)$ the harmonic conjugate of $\operatorname{Re} q(\vartheta, f(e^{i\vartheta}))$.

Lemma 8. *R_q is a bounded (nonlinear) operator from $A(T)$ into $C(T)$.*

Proof. Since R_q is linear in q, it suffices to prove the lemma for the operators $R_{j,\ell}$ associated to the functions $q_{j,\ell} = e^{ij\vartheta} z^\ell$. Moreover, since $f \to f^\ell$ is a bounded operator of $A(T)$ into itself, it suffices to consider the case $\ell = 1$.

An explicit computation with Fourier series shows

$$(R_{j,1} f)(e^{i\vartheta}) = \begin{cases} \operatorname{Im}\big(e^{ji\vartheta} f(e^{i\vartheta})\big), & j > 0, \\ \operatorname{Im} f(e^{i\vartheta}) - \operatorname{Im} f(0), & j = 0 \end{cases}$$

and

$$(R_{-j,1} f)(e^{i\vartheta})$$
$$= \operatorname{Im}\left[\left(f(e^{i\vartheta}) - \sum_{s=0}^{j} \frac{f^{(s)}(0)}{s!} e^{is\vartheta}\right) e^{-ji\vartheta} + \sum_{s=0}^{j-1} \overline{\frac{f^{(s)}(0)}{s!}}\, e^{(j-s)i\vartheta}\right], \quad j > 0.$$

This proves Lemma 8. □

The estimate (4.6) follows immediately from the preceding lemmas. By Lemma 7 we can write

$$b = \operatorname{Re} q + (b - \operatorname{Re} q) = \operatorname{Re} q + b',$$

where q is of the form (4.7) and $p\|b'\|_\infty < \frac{\pi}{2}$. Then,

$$e^{\tilde b(\vartheta, f(\vartheta))} = e^{\widetilde{\operatorname{Re} q}(\vartheta, f(\vartheta))}\, e^{\tilde b'(\vartheta, f(\vartheta))}$$
$$= e^{R_q f(\vartheta)}\, e^{\tilde b'(\vartheta, f(\vartheta))}.$$

The first term is uniformly bounded by Lemma 8, and the second is bounded in $L^p(T)$ according to Lemma 6. This proves (4.6).

Choose a $p \in (2,\infty)$. Recall that the harmonic conjugation $u \to \tilde u$ is a bounded linear map of $L^p(T)$ into itself [13, p. 113]. The identity (4.4) together with the estimates (4.5) and (4.6) implies

$$\left\| \frac{\partial g}{\partial \vartheta} e^{-\tilde b + ib} \right\|_{L^p} \le C_3, \qquad \text{all } f.$$

Since

$$\frac{\partial f}{\partial \vartheta} = f \frac{\partial g}{\partial \vartheta} = f e^{\tilde b - ib} \left(\frac{\partial g}{\partial \vartheta} e^{-\tilde b + ib} \right),$$

the Hölder inequality for the product of two L^p functions yields the estimate

$$\left\| \frac{\partial f}{\partial \vartheta} \right\|_{L^{p/2}} \le \|f\|_\infty \|e^{\tilde b}\|_{L^p} \left\| \frac{\partial g}{\partial \vartheta} e^{-\tilde b + ib} \right\|_{L^p} \le C_4.$$

Since $\frac{p}{2} > 1$ by the choice of p, a theorem of Hardy and Littlewood implies for $\alpha = 1 - \frac{2}{p} \in (0,1)$

$$\|f\|_\alpha \leq C_5 .$$

With this estimate we can go back to the identity (4.4) to obtain a Hölder estimate for $\frac{\partial f}{\partial \vartheta}$. Since r is of class C^k for $k \geq 2$, the functions η, a, b and r_ϑ are of class C^{k-1}, so the compositions $a(\vartheta,f(\vartheta))$, $b(\vartheta,f(\vartheta))$, $r_\vartheta(\vartheta,f(\vartheta))$ are uniformly bounded in the C^α norm. Also, the harmonic conjugation is a bounded operator on $C^\alpha(T)$ [13, p. 106]. Thus, (4.4) implies

$$\left\| \frac{\partial g}{\partial \vartheta} e^{-\tilde{b}+ib} \right\|_\alpha \leq C_6$$

and, therefore, $\left\|\frac{\partial f}{\partial \vartheta}\right\|_\alpha \leq C_7$. Equivalently,

$$\|f\|_{1,\alpha} \leq C_8 .$$

We can iterate this argument to prove all estimates (4.1).

From our proof it is clear that the constants $C_{\ell,\alpha}$ only depend on $\sup\{|z| : z \in M_\vartheta, \vartheta \in \mathbf{R}\}$ and on the C^{k-1} norms of the functions a, b, and r_ϑ restricted to M. Thus, if M^t is a C^k homotopy as in (1.3), the $C_{\ell,\alpha}$ are independent of t. Theorem 5 is proved. □

Let $f : S \times [0,t_1) \to A^\alpha(T)$ be the map constructed in Sections 2 and 3 which satisfies Theorem 1 on $0 \leq t < t_1$. We shall obtain a priori estimates on the derivatives $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

Recall that the extension of $f(s,t)$ to $\bar{D}$ has no zeros, so $f(s,t) = e^{g(s,t)}$ for a C^{k-1} map $g : \mathbf{R} \times [0,t_1) \to A^\alpha(T)$. We shall write $g(s,t)(e^{i\vartheta}) \equiv g(s,t,\vartheta)$.

Let r and ν be as in (1.3). The restriction of the function $\eta : T \times \mathbf{C} \times [0,1] \to \mathbf{C}$ given by

$$\eta(\vartheta,z,t) = z\,\overline{\nu(\vartheta,z,t)}$$

to M_t is null–homotopic as a map of M_t into $\mathbf{C} \setminus \{0\}$, so

$$\eta(\vartheta,z,t) = e^{a(\vartheta,z,t)+ib(\vartheta,z,t)}, \qquad z \in M_\vartheta^t .$$

We set

$$A(s,t,\vartheta) = a\big(\vartheta,f(s,t,\vartheta),t\big),$$

$$B(s,t,\vartheta) = b\big(\vartheta,f(s,t,\vartheta),t\big).$$

As a function of s, B is only well–defined on $s \in \mathbf{R}$, but its harmonic conjugate $\widetilde{B}(s,t,\vartheta)$ with respect to ϑ is well defined on $s \in S = \frac{\mathbf{R}}{\mathbf{Z}}$.

We differentiate the identity

$$r\big(\vartheta,f(s,t,\vartheta),t\big) = 0 \tag{4.7}$$

with respect to s (see also the proof of Theorem 5) to obtain

$$\operatorname{Re}\left(\frac{\partial g}{\partial s}(s,t,\vartheta)\eta(\vartheta,f(s,t,\vartheta),t)\right)=0.$$

Multiplying this by the real-valued function $e^{-A-\tilde{B}}$ we have

$$\operatorname{Re}\left(\frac{\partial g}{\partial s}e^{-\tilde{B}+iB}\right)=0.$$

Since the function under Re extends holomorphically to $\bar{D}$, it must be constant. Consequently,

$$\frac{\partial g}{\partial s}(s,t,\vartheta)e^{(-\tilde{B}+iB)(s,t,\vartheta)}\equiv\frac{\partial g}{\partial s}(s,t,0)\,e^{(-\tilde{B}+iB)(s,t,0)}.$$

Multiplying this by f and taking into account

$$\frac{\partial f}{\partial s}(s,t,0)=\frac{\partial\varphi}{\partial s}(s,t),$$

we obtain

$$\frac{\partial f}{\partial s}(s,t,\vartheta)=\frac{f(s,t,\vartheta)}{f(s,t,0)}\,\frac{\partial\varphi}{\partial s}(s,t)\,e^{(-\tilde{B}+iB)(s,t,0)-(-\tilde{B}+iB)(s,t,\vartheta)}. \tag{4.8}$$

Since $b(\vartheta,z,t)$ is of class C^1 and f satisfies the estimates of Theorem 5, the exp term in the right-hand side of (4.8) is uniformly bounded in $C^\alpha(T)$. Thus, (4.8) implies that there is a constant C_9 such that

$$\left\|\frac{\partial f}{\partial s}(s,t,\cdot)\right\|_\alpha\le C_9,\qquad s\in S,\ 0\le t<t_1. \tag{4.9}$$

Similarly, we can differentiate (4.7) with respect to t and use the relation $\frac{\partial f}{\partial t}(s,t,0)=\frac{\partial\varphi}{\partial t}(s,t)$ to prove an estimate

$$\left\|\frac{\partial f}{\partial t}(s,t,\cdot)\right\|_\alpha\le C_{10}. \tag{4.10}$$

We shall omit the details since they are similar to the proof of (4.9) given above. □

5. Proof of Theorems 1 and 3. In Sections 2 and 3 we have constructed a C^{k-1} map $f : S \times [0,t_1) \to A^\alpha(T)$ which satisfies the conclusions of Theorem 1 on $0 \le t < t_1$. (See Theorem 4.) The estimates (4.1), (4.9), and (4.10) imply that f extends to a continuous map $f : S \times [0,t_1] \to A^\alpha(T)$, and we have

$$f(s,t_1,0) = f(s,t_1)(e^{i0}) = \varphi(s,t_1), \qquad s \in S.$$

The uniqueness part of Theorem 4 implies that the extended map is of class C^{k-1} on $S \times [0,t_1]$. If $t_1 < 1$, then by the same theorem f can be extended to the interval $0 \le t < t_1 + \varepsilon$ for some $\varepsilon > 0$. Continuing this process, we extend f to the whole interval [0,1]. Theorem 1 is proved. □

We now turn to the proof of Theorem 3. We shall write $f(s)(z) \equiv f(s,z)$. The map

$$\Phi : \bar{D} \times S \to \mathbf{C}^2, \qquad \Phi(z,s) = \big(z, f(s,z)\big)$$

is the composition of the C^{k-1} map $(z,s) \to \big(z,f(s)\big)$ into $\bar{D} \times A^\alpha(D)$ and the map $\bar{D} \times A^\alpha(D) \to \mathbf{C}$, $(z,h) \to h(z)$. The last map is linear in $h \in A^\alpha(D)$, and for each fixed h it is holomorphic on $z \in D$. Thus, Φ is of class C^{k-1} on $D \times S$ and $\Phi(\cdot,s)$ is holomorphic on D.

From the identity (4.8) we conclude that $\frac{\partial f}{\partial s}(s,z)$ is nonvanishing on $\bar{D} \times S$, so Φ is an immersion. The map $S \to M_\vartheta$, $s \to f(s,e^{i\vartheta})$ is a homeomorphism at $\vartheta = 0$, so it is homotopic to a homeomorphism for each ϑ. Since this is an immersion according to (4.8), it is a diffeomorphism for all ϑ. Thus, $\Sigma = \Phi(D \times S)$ is an immersed hypersurface with $\bar{\Sigma} = \Sigma \cup M$.

To prove that Φ is one–to–one, we consider the difference $g(s,z) = f(s,z) - f(s_0,z)$. We know already that $g(s,e^{i\vartheta}) \ne 0$ for $s \in S \setminus \{s_0\}$ and $\vartheta \in \mathbf{R}$. Moreover, for s close to s_0, $g(s,z)$ is close to $(s-s_0)\frac{\partial f}{\partial s}(s_0,z)$ which is nonvanishing on $z \in \bar{D}$. By the argument principle, $g(s,z)$ is nonvanishing on $\bar{D}$ for all $s \in S \setminus \{s_0\}$. This proves that Φ is an embedding, and Part (i) of Theorem 3 holds.

Since the hypersurface $\Sigma = \Phi(D \times S)$ is foliated by analytic disks, it is Levi flat and pseudoconvex.

For each pair of integers $m \ge 1$, $n \ge 0$, $m+n \le k-1$, we can consider $s \to f(s)$ as a C^m map of S into the space $A^{n,\alpha}(D)$ $(0 < \alpha < 1)$. (See the remark at the end of Section 3.) The evaluation map $\bar{D} \times A^{n,\alpha}(D) \to \mathbf{C}$, $(z,h) \to h(z)$, is of class C^n in $z \in \bar{D}$ and is linear in h. It follows that the derivative

$$\frac{\partial^{m+n} f}{\partial^m s\, \partial^n z}(s,z)$$

exists and is continuous on $\bar{D} \times S$. In particular, f is of class C^{k-2} in both variables, so Φ defined by (5.1) is en embedding of class C^{k-2} on $\bar{D} \times S$. This proves (ii).

To prove (iii) we choose a C^k homotopy $\{M^t : 0 \le t \le 1\}$ satisfying (1.3) and

(i) $M^1 = M$.
(ii) $M^0 = \{(e^{i\vartheta}, Re^{i\varphi}) : \vartheta, \varphi \in \mathbf{R}\}$ for some large $R > 0$.
(iii) If $t_1 < t_2$, then $M_\vartheta^{t_2}$ is contained in the region bounded by $M_\vartheta^{t_1}$ for all ϑ.
(iv) $\frac{\partial r}{\partial t}$ is nonvanishing, where r is the defining function as in (1.3).

Such a homotopy exists if R is large enough.

Let $f : S \times [0,1] \to A^\alpha(D)$ be as in Theorem 1. Choose a smooth, strictly decreasing function $\psi : [0,1] \to \mathbf{R}$ such that $\psi(0) > 1$, $\psi(1) = 1$, and define

$$\Sigma_t = \{(\psi(t)z,\ f(s,t,z)) : z \in D,\ s \in S\}, \qquad 0 \le t \le 1.$$

This is a smooth pseudoconvex hypersurface with boundary

$$\{(\psi(t)e^{i\vartheta}, w) : w \in M_\vartheta^t,\ \vartheta \in \mathbf{R}\}.$$

Denote by Ω_t the pseudoconvex domain in $D(\psi(t)) \times \mathbf{C}$ bounded by Σ_t. Here $D(c) = \{z \in \mathbf{C} : |z| < c\}$.

Since the derivatives $\frac{\partial f}{\partial z}(s,t,z)$ are uniformly bounded, conditions (iii) and (iv) above on $\{M^t\}$ imply that, for each $t > t_0$, the boundary of Σ_t is contained in Ω_{t_0}, provided that the derivative $|\psi'(t)|$ is sufficiently small on $[0,1]$. We claim that, as a consequence, $\Sigma_t \cap \Sigma_{t_0} = \emptyset$, so $\bar{\Omega}_t \subset \Omega_{t_0}$. For a proof consider the difference

$$g(z) = f\left(s, t_0, \frac{z}{\psi(t_0)}\right) - f\left(s', t, \frac{z}{\psi(t)}\right)$$

on $|z| \le \psi(t)$ (s, $s' \in S$). There is a homotopy $H : T \times [0,1] \to \mathbf{C}$ such that for all $\vartheta \in \mathbf{R}$:

(i) $H(e^{i\vartheta}, 0) = f(s', t, e^{i\vartheta})$.
(ii) $H(e^{i\vartheta}, 1) = 0$.
(iii) $H(e^{i\vartheta}, \gamma)$ lies in the region bounded by M_ϑ^t for each $\gamma \in [0,1]$.

So the winding number of $g(z)$ on $|z| = \psi(t)$ equals that of $f\left(s, t_0, \frac{z}{\psi(t_0)}\right)$ which is zero. Consequently, $f(z)$ is nonvanishing on $\{|z| \le \psi(t)\}$, as claimed.

We recollect the relevant properties of the family of pseudoconvex domains $\{\Omega_t\}$:

(i) $\bar{\Omega}_t \subset \Omega_{t_0}$ if $t > t_0$.
(ii) $\bigcup_{t > t_0} \Omega_t = \Omega_{t_0}$.
(iii) $\bigcup_{0 \le t \le 1} \Omega_t = D(\psi(0)) \times D(R)$.
(iv) $\bigcap_{t > t_0} \Omega_t = \bar{\Omega}_{t_0}$.
(v) $\operatorname{Int} \bar{\Omega}_t = \Omega_t$.

Property (i) was proved above. The rest follows from the definition of Ω_t and from the smooth dependence of Σ_t on t.

A theorem of Docquier and Grauert [9] implies that each Ω_t ($t > 0$) is holomorphically convex in Ω_0, so $\bar{\Omega}_t$ is polynomially convex for each $t \in [0,1]$. This proves (iii) of Theorem 3.

Fix a point $(a,b) \in \widehat{M}$, $a \in D$. If (a,b) lines in $\partial\widehat{M}$, part (iii) implies that $f(s,a) = b$ for some $s \in S$, so (iv) holds. If, on the other hand, (a,b) is an interior point of $\widehat{M}$, we can use Theorem 1 to find an $h \in A(D)$ such that $h(a) = b$ and $h(e^{i\vartheta})$ is in the region bounded by M_ϑ for all ϑ. (This time we apply Theorem 1 to a homotopy $\{M^t\}$ with $M^0 = M$ and with "shrinking" fibers M^t_ϑ.) We seek g of the form

$$g(z) = h(z) + (z-a)k(z), \qquad k \in A(D).$$

If k is a solution of (1.2) for the manifold contained in $T \times \mathbf{C}$ with fibers

$$N_\vartheta = \left\{ \frac{z - h(e^{i\vartheta})}{e^{i\vartheta} - a} : z \in M_\vartheta \right\}, \qquad \vartheta \in \mathbf{R}$$

(such functions exist by Corollary 2 applied to N), then g satisfies Theorem 3(iv).

Finally, suppose that $g \in A(D)$ is a zero–free function on D that solves (1.2). Let $\{M^t\}$ be a homotopy as in the proof of (iii) above. The technique of this paper allows us to construct a map $G : [0,1] \to A^\alpha(D)$ which is continuous and satisfies

(i) $G(t)(e^{i\vartheta}) \in M^t_\vartheta$, and
(ii) $G(1)(z) = g(z)$.

Set

$$J = \{t \in [0,1] : G(t) = f(s,t) \text{ for some } s \in S\},$$

where $f(s,t)$ is as before. Clearly J is a closed subset of $[0,1]$ containing 0 (since the constants $e^{2\pi i s}$ are the only nonvanishing solutions of (1.2) for $M^0 = T \times T$). The uniqueness part of Theorem 4 in Section 3 implies that J is open in $[0,1]$. Consequently, $J = [0,1]$, which proves Theorem 3(v). We proved all results stated in Section 1. □

Note added in proof. The author is pleased to announce a new and very strong result by Z. Slodkowski (*Polynomial hulls in* $\mathbf{C}^2$ *and quasicircles*, to appear), which uses in an essential way the main result of the present paper, together with some ideas from quasiconformal geometry. Slodkowski describes the polynomial hull of each compact set $X \subset \mathbf{C}^2$ fibered over the circle, whose fibers are simply connected continua, as the union of graphs of bounded analytic functions in the disc $D \subset \mathbf{C}$. This is a far–reaching generalization of the results in [2, 22].

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