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Extending proper holomorphic mappings of positive codimension

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Introduction

In this paper we obtain results on holomorphic continuation of proper holomorphic mappings between pseudoconvex domains with real-analytic boundaries in complex spaces of different dimensions. Equivalently, we obtain results concerning the analyticity of Cauchy-Riemann mappings between real-analytic pseudoconvex hypersurfaces in complex spaces of different dimensions.

To begin with, we recall the corresponding results for mappings of equidimensional domains. Let D and D' be bounded pseudoconvex domains with smooth boundaries in C^n. If the boundaries of D and D' are strictly pseudoconvex or, more generally, of finite type in the sense of D'Angelo [15], then every proper holomorphic map of D onto D' extends smoothly to $\bar{D}$ according to the results of Bell and Catlin [7, 8] and Diederich and Fornæss [19]. For mappings between strictly pseudoconvex domains this was proved by Fefferman [26] and Nirenberg, Webster, and Yang [37].

If the boundaries of the pseudoconvex domains D, $D' \subset \mathbf{C}^n$ are real-analytic, then every proper holomorphic mapping of D onto D' extends holomorphically to a neighborhood of $\bar{D}$ according to Baouendi and Rothschild [2] and Diederich and Fornæss [21]. This 'reflection principle' was first discovered by Lewy [34] and Pinčuk [38] for mappings between strictly pseudoconvex domains. In the case of biholomorphic mappings between weakly pseudoconvex domains with real-analytic boundaries the result follows from the work of Baouendi, Jacobowitz, and Treves [5]. Results in this direction were obtained in recent years by several authors; see the papers [4, 6, 18, 22, 33, 47, 49].

In this paper we are treating the case when the domains D and D' have different dimensions. To be specific, we assume that $D \subset \mathbf{C}^n$ and $D' \subset \mathbf{C}^N$ are bounded pseudoconvex domains with real-analytic boundaries and $N > n > 1$. In this situation a proper holomorphic map $f\colon D \to D'$ need not be regular at the boundary. For instance, there exist proper holomorphic maps of balls of different dimensions

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that do not extend continuously to the boundary [27, 35]. Moreover, for each n there exist proper holomorphic maps from the unit ball $\mathbf{B}^n \subset \mathbf{C}^n$ to $\mathbf{B}^{n+1} \subset \mathbf{C}^{n+1}$ that extend continuously to the closed ball $\bar{\mathbf{B}}^n$, but the extension is not of class $\mathscr{C}^2$ on any open subset of the sphere $b\mathbf{B}^n$ [23, 32]. Furthermore, if $N = N(n)$ is sufficiently large, there exist proper holomorphic maps $f\colon \mathbf{B}^n \to \mathbf{B}^N$ that extend continuously to $\bar{\mathbf{B}}^n$ and map the sphere $b\mathbf{B}^n$ onto $b\mathbf{B}^N$ [30].

On the other hand, if we assume that f extends to a map which is smooth of class $\mathscr{C}^s$ on $\bar{D}$ for some sufficiently large s, then f will continue holomorphically across large subsets of bD. A result of this type has been proved by Webster [48]: If $n \geqq 3$, $N = n+1$, $D \subset \mathbf{C}^n$ is a strictly pseudoconvex domain with real-analytic boundary, and $f\colon D \to \mathbf{B}^{n+1}$ is a holomorphic map which is of class $\mathscr{C}^3$ on $\bar{D}$, then f extends holomorphically across every point in an open dense subset of the boundary bD. Related results for proper holomorphic mappings of balls $f\colon \mathbf{B}^n \to \mathbf{B}^N$ were proved by Faran [24], Cima and Suffridge [11, 12], Cima, Krantz, and Suffridge [13], and the author [28]. These results were obtained by a generalization of the method developed in the equi-dimensional case by Lewy [34] and Pinčuk [38]. The codimension $N - n$ of the mapping was assumed to be low in these results.

1. Results

Our first main result is the following.

1.1. **Theorem.** *Let $M \subset \mathbf{C}^n$ and $M' \subset \mathbf{C}^N$ $(N > n > 1)$ be smooth real-analytic pseudoconvex hypersurfaces, M of finite type, and M' strictly pseudoconvex. Let $D \subset \mathbf{C}^n$ be a domain which contains M in its boundary and is pseudoconvex along M. If $f\colon D \cup M \to \mathbf{C}^N$ is a mapping of class $\mathscr{C}^\infty$ that is holomorphic on D and maps M to M', then f extends holomorphically to a neighborhood of an open, everywhere dense subset M_0 of M.*

Recall that a pseudoconvex real-analytic hypersurface is of finite type in the sense of D'Angelo [15] if and only if it contains no germs of positive dimensional complex varieties.

The domain D plays no role in Theorem 1.1. We can formulate the result in terms of C-R mappings between real-analytic hypersurfaces. Recall that a mapping $f\colon M \to \mathbf{C}^N$ is C-R if it satisfies the tangential Cauchy-Riemann equations on M.

1.2. **Theorem.** *Let $M \subset \mathbf{C}^n$ and $M' \subset \mathbf{C}^N$ $(N > n > 1)$ be smooth real-analytic pseudoconvex hypersurfaces, where M is of finite type and M' is strictly pseudoconvex. If $f\colon M \to M'$ is a smooth C-R map, then f is real-analytic at every point of an open, everywhere dense subset M_0 of M.*

Under the conditions of Theorem 1.2 the map f extends holomorphically to a domain $D \subset \mathbf{C}^n$ in the pseudoconvex side of M [3, 45]. The extended map $f\colon D \cup M \to \mathbf{C}^N$ continues holomorphically to a neighborhood of a point $p \in M$ if and only if $f|_M$ is real-analytic at p. This shows that the above two theorems are equivalent.

We can replace the condition that M' is strictly pseudoconvex by a somewhat weaker condition (A) introduced by (2.4) in Sect. 2 below. In this case we have to require in addition that f maps D to a domain $D' \subset \mathbf{C}^N$ in the pseudoconvex side of M' and bounded in part by M'.

There is no restriction on the codimension $N-n$ of the mapping. The above theorems are clearly false if $n=1$ and $N>1$.

We obtain a somewhat more precise result, Theorem 6.1, concerning the extension of f at a given point of M. In Sect. 4 we construct an upper semicontinuous, integer valued function $\nu: M \to \mathbf{Z}_+$, called the *deficiency* of f, which is invariantly associated to f and which measures a type of degeneracy of the mapping f. If ν is constant on a neighborhood of a point $z \in M$ in M, then f extends holomorphically to a neighborhood of z in $\mathbf{C}^n$ (Theorem 6.1). In particular, f extends holomorphically to a neighborhood of each $z \in M$ at which $\nu(z)=0$. The condition $\nu(z)=0$ is satisfied if the restriction of f to a certain complex hypersurface $Q_z \subset \mathbf{C}^n$ associated to M has a maximal number of linearly independent derivatives (of higher order) at z.

It remains an open problem whether f extends holomorphically across each point of M. Examples in Sect. 8 show that the function ν need not be constant on M even when f is a proper polynomial map between balls. However, it seems that for 'most' hypersurfaces M' we have $\nu \equiv 0$ for each mapping f, so f extends holomorphically across M. We shall not pursue this question in the present paper.

We denote by $\mathbf{B}^n$ the open unit ball in $\mathbf{C}^n$,

$$\mathbf{B}^n = \left\{ z \in \mathbf{C}^n : \sum_{j=1}^{n} z_j \bar{z}_j < 1 \right\},$$

and by $b\mathbf{B}^n$ its boundary, the unit sphere in $\mathbf{C}^n$. In the special case when the hypersurface M' is the unit sphere $b\mathbf{B}^N$ we can prove a similar extension result under a weaker smoothness assumption on the mapping f.

1.3. **Theorem.** *Let D and M be as in Theorem 1.1. If $f: D \cup M \to \mathbf{C}^N$ is a mapping of class $\mathscr{C}^{N-n+1}$ that is finite holomorphic on D and satisfies $|f(z)|=1$ for each $z \in M$, then f extends holomorphically to a neighborhood of an open, everywhere dense subset M_0 of M.*

In the special case when both domains are unit balls we have the following stronger result.

1.4. **Theorem.** *Let U be an open ball centered at a point $p \in b\mathbf{B}^n$, and let $M = U \cap b\mathbf{B}^n$. If $N > n > 1$ and $f: \bar{\mathbf{B}}^n \cap U \to \mathbf{C}^N$ is a mapping of class $\mathscr{C}^{N-n+1}$ that is holomorphic on $\mathbf{B}^n \cap U$ and takes M to the unit sphere $b\mathbf{B}^N$, then f is rational, $f=(p_1, \ldots, p_N)/q$, where the p_j and q are holomorphic polynomials of degree at most $N^2(N-n+1)$. The extended map is holomorphic on $\mathbf{B}^n$, it maps $\mathbf{B}^n$ to $\mathbf{B}^N$, and it has no poles on $b\mathbf{B}^n$.*

In particular, if $f: \mathbf{B}^n \to \mathbf{B}^N$ is a proper holomorphic map $(N > n > 1)$ that extends to a $\mathscr{C}^{N-n+1}$ map on $\bar{\mathbf{B}}^n$, then f is rational. This theorem shows that the space of all rational proper mappings $f: \mathbf{B}^n \to \mathbf{B}^N$ is finite dimensional. The bound $N^2(N-n+1)$ on the degree of f is not sharp.

An unpublished result due to Pinčuk implies that the extended rational map has no singularities on $\bar{\mathbf{B}}^n$.

The equidimensional case $N=n$ of Theorem 1.4 is due to Poincaré [41], Tanaka [44], Alexander [1], and Pinčuk [38]. The map f extends to an automorphism of $\mathbf{B}^n$. Poincaré and Tanaka considered the case when f is biholomorphic on a neighborhood of a point $p\in b\mathbf{B}^n$. Alexander assumed that f is $\mathscr{C}^\infty$ on $\bar{\mathbf{B}}^n\cap U$. Pinčuk reduced the smoothness requirement to $\mathscr{C}^1$. Alexander proved as well that every proper holomorphic map $f\colon \mathbf{B}^n\to\mathbf{B}^n$ is an automorphism of $\mathbf{B}^n$. See also [42, p. 313]. More generally, every proper holomorphic map between strictly pseudoconvex domains with smooth algebraic boundaries in $\mathbf{C}^n$ is a branch of an algebraic map according to Webster [47]. He also found sufficient conditions on the two boundaries that force the map to be rational.

If $N>n$, there exist non-rational proper holomorphic maps of $\mathbf{B}^n$ into $\mathbf{B}^N$ [23, 27, 30, 32, 35]. Such a map need not be continuous at the boundary [27], and even when it is continuous, it may take the boundary $b\mathbf{B}^n$ onto $b\mathbf{B}^N$ [30]. Hence some regularity of f at the boundary is necessary for f to be rational.

Proper mappings of balls of small codimension $N-n$ were studied by Webster [48], Faran [24, 25], Cima and Suffridge [11, 12], and the author [28]. In the case when $N\leqq 2n-1$, Theorem 1.4 was proved by the author in [28, Theorem 1.3], using inductively the method of Cima and Suffridge [11]. Their method, which is a generalization of the reflection principle of Lewy [34] and Pinčuk [38], shows that f is rational provided that it satisfies certain nondegeneracy condition at a point of $b\mathbf{B}^n$. The examples in Sect. 8 show that the required condition fails in general.

For the results on classification of proper mappings between balls see the papers [1, 11, 12, 16, 17, 24, 25, 28, 48].

Although Theorem 1.1 is of local nature, we can combine it with the main result of [29] to obtain the following result under global hypotheses on the mapping.

1.5. **Corollary.** *Let $D\subset\mathbf{C}^n$ and $D'\subset\mathbf{C}^N$, $N>n>1$, be bounded strictly pseudoconvex domains with smooth real-analytic boundaries, and let $\Gamma\subset bD'$ be a closed $\mathscr{C}^\infty$ submanifold of real dimension $2n-1$. If $f\colon D\to D'$ is a proper holomorphic map with nontangential boundary values $f^*(z)\in\Gamma$ for almost every $z\in bD$, then f extends holomorphically across every point in an open, everywhere dense subset of bD.*

We proved in [29] that the map f extends smoothly to the closure of D. Therefore Corollary 1.5 follows immediatelly from Theorem 1.1.

We mention another consequence of Theorem 1.1. In [27] we proved that for each $n>1$ there exist real-analytic strictly pseudoconvex hypersurfaces $M\subset\mathbf{C}^n$ that do not admit analytic C-R embeddings to any finite dimensional sphere. A similar result was established independently by Faran (unpublished). Theorem 1.3 implies

1.6. **Corollary.** *For each $N>n>1$ there exist bounded strictly pseudoconvex domains $D\subset\mathbf{C}^n$ with real-analytic boundary such that no proper holomorphic map $f\colon D\to\mathbf{B}^N$ extends to a map of class $\mathscr{C}^{N-n+1}$ on $\bar{D}$.*

The present work leaves several open questions. In the context of Theorem 1.1 one would like to know whether f extends holomorphically across each point of M. Second, our method does not cover the case when M' is an arbitrary real-analytic pseudoconvex hypersurface of finite type in $\mathbf{C}^N$. Third, it is not known how much smoothness of the map f on $D\cup M$ one needs in order to obtain a holomorphic extension. We have mentioned above that the continuity of f on $D\cup M$ is not sufficient. It seems that Theorem 1.1 should hold under the assumption $f\in\mathscr{C}^{N-n+1}(D\cup M)$, but at the moment we do not know how to prove this. On the other hand, it is an open problem to construct a proper holomorphic map $f\colon \mathbf{B}^n\to\mathbf{B}^N$ $(N>n>1)$ that is of class $\mathscr{C}^s(\bar{\mathbf{B}}^n)$ for some s with $0<s<N-n+1$, but which is not rational.

Although most strictly pseudoconvex real-analytic hypersurfaces do not admit smooth C-R embeddings into any finite dimensional sphere according to [27], it would be of interest to construct such embeddings with a finite degree of smoothness, say $\mathscr{C}^2$.

We shall now explain the main idea in the proof of Theorems 1.1, 1.3, and 1.4. The proof of Theorem 1.1 relies on methods developed by Webster [47, 49], Diederich and Webster [22], and Diederich and Fornæss [21]. Our proof of Theorems 1.3 and 1.4 uses similar ideas, but it is much simpler.

Let M be an analytic real hypersurface in $\mathbf{C}^n$. For each point $z^0\in M$ there is a neighborhood U of z^0 in $\mathbf{C}^n$ and an analytic real function $r(z,\bar z)$ on U with nonvanishing gradient such that

$$M\cap U=\{z\in U: r(z,\bar z)=0\}.$$

If $M'\subset\mathbf{C}^N$ is another analytic real hypersurface defined locally by

$$M'\cap U'=\{z'\in U': r'(z',\bar z')=0\},$$

and if $f\colon U\to U'$ is a holomorphic mapping taking $M\cap U$ into $M'\cap U'$, then we have a relation

$$r'(f(z),\overline{f(z)})=p(z,\bar z)r(z,\bar z),\quad z\in U, \tag{1.1}$$

where p is a real-valued analytic function on U.

Let $z=x+iy$, where x and y are the real coordinates on $\mathbf{C}^n=\mathbf{R}^{2n}$. Since the functions in (1.1) are represented by convergent power series in x and y in a neighborhood of z^0, they are still defined for complex valued x and y in a suitable domain, and the relation (1.1) persists. Equivalently, we may set $\bar z=\bar w$ and vary z and w independently to obtain the identity

$$r'(f(z),\overline{f(w)})=p(z,\bar w)r(z,\bar w) \tag{1.2}$$

for z, w in a neighborhood U_0 of z^0 in $\mathbf{C}^n$.

We define for each $w\in U_0$ a complex hypersurface $Q_w\subset U_0$ by

$$Q_w=\{z\in U_0: r(z,\bar w)=0\}. \tag{1.3}$$

These hypersurfaces were first introduced by Segre [43] and were later used by several authors [21, 22, 47, 49]. Let $Q'_{w'}\subset U'_0\subset\mathbf{C}^N$ be the analogous hypersurface associated to M'. We assume also that $f(U_0)\subset U'_0$. The identity (1.2) implies $f(Q_w)\subset Q'_{f(w)}$ for each $w\in U_0$. If $N=n$ and $f\colon U_0\to U'_0$ is a biholomorphic map, we can apply the same argument to its inverse to conclude that f maps each Q_w biholomorphically onto $Q'_{f(w)}$. Hence the family of complex hypersurfaces $\{Q_w\}$ is invariantly attached to M.

Since $r(z,\bar{z})$ is real-valued, we have

$$r(z,\bar{w})=\bar{r}(\bar{w},z)=\overline{r(w,\bar{z})}\,,$$

so $z\in Q_w$ if and only if $w\in Q_z$. Moreover, $z\in Q_z$ if and only if $z\in M$. Hence the inclusion $f(Q_w)\subset Q'_{f(w)}$ is equivalent to

$$f(w)\in\bigcap\{Q'_{f(z)}: z\in Q_w\}. \tag{1.4}$$

Suppose now that M resp. M' is a part of the boundary of a pseudoconvex domain D resp. D', f is defined on $D\cup M$, it maps D holomorphically into D', and it takes M into M'. Let w be a point outside $D\cup M$ but close to M. If f extends holomorphically across M, its value $f(w)$ lies in the set

$$X_w=\bigcap\{Q'_{f(z)}: z\in Q_w\cap D\} \tag{1.5}$$

according to (1.4). This is the crucial observation on which the proof of Theorem 1.1 is based. It was used in a similar way by Webster [47, 49], Diederich and Webster [22], and Diederich and Fornæss [21].

We shall prove in Sect. 5 (Proposition 5.1) that the set $X\subset(\mathbf{C}^n\backslash\bar{D})\times\mathbf{C}^N$ with fibers X_w for w close to M is a complex variety of dimension at least n. In the equidimensional case $N=n$ treated by Diederich and Fornæss [21] the fibers X_w are finite, and the variety X extends the graph

$$\Gamma(f)=\{(z,f(z)): z\in D\}$$

of f across $M\times M'$. Therefore f extends holomorphically across M according to [6, Lemma 1].

In the case $N>n$ the fibers X_w may be positive dimensional. We will show that near almost every point $p^0=(z^0,f(z^0))$, $z^0\in M$, the top dimensional part Y of X is a branched analytic cover over a suitable domain in $(\mathbf{C}^n\backslash\bar{D})\times\mathbf{C}^m$, where m is the dimension of the fibers X_w for w close to z^0. Locally near p^0 the analytic extension $\tilde{Y}$ of Y across the pseudoconvex hypersurface $M\times\mathbf{C}^N$ intersects $M\times M'$ precisely in the graph $\Gamma(f|_M)=\{(z,f(z)): z\in M\}$ of the restriction $f|_M$. This implies that $\Gamma(f|_M)$ is real-analytic near p^0 and hence $f|_M$ is real-analytic near z^0. Consequently f extends holomorphically to a neighborhood of z^0. The details are given in Sect. 6.

In the special case when the hypersurface M' is the unit sphere $\{z'\in\mathbf{C}^N: |z'|=1\}$, the associated complex hypersurface $Q'_{w'}$ is the hyperplane

$$Q'_{w'}=\left\{z'\in\mathbf{C}^N: \sum_{j=1}^{N} z'_j\bar{w}'_j=1\right\}.$$

Hence each fiber X_w (1.5) is also an affine complex subspace of $\mathbf{C}^N$. In this case we have a simpler proof of the fact that X is a nonempty complex variety; the proof in Sect. 7 can be read independently of the Sect. 3–6.

A few words concerning the organization of the paper are appropriate. In Sects. 2–4 we develop the necessary tools: the Segre varieties in Sect. 2, the preimage map f^* in Sect. 3, and the deficiency function ν in Sect. 4. In Sect. 5 we construct an analytic variety X in $(\mathbf{C}^n\backslash\bar{D})\times\mathbf{C}^N$ associated to f, and we show that X behaves well near most points of the graph of $f|_M$. In Sect. 6 we extend a part of X across the hypersurface $bD\times\mathbf{C}^N$ as a branched covering, and we prove Theorem 1.1. In Sect. 7

we prove Theorems 1.3 and 1.4 with the ball as the target domain. Section 7 does not depend on the development in Sects. 3–6 and can be read independently. In Sect. 8 we calculate the variety X for several polynomial proper maps of balls.

Remark. After the completion of this work I was informed by Pinčuk that he had obtained results similar to Theorem 1.1 and 1.3 around the year 1978, but he had not published them. In the context of Theorem 1.3, with $M \subset \mathbf{C}^n$ strictly pseudo-convex, he proved that the mapping f extends holomorphically to each point of M. If M is compact and non-spherical, and if $f: M \to b\mathbf{B}^N$ is a smooth C-R map that is initially defined near a point $z \in M$, then f continues holomorphically along each path in M starting at z. Similar results for mappings of equi-dimensional hypersurfaces were obtained by Pinčuk [40] and Vitushkin [46].

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2. Invariant complex hypersurfaces

In this section we shall consider more carefully the complex hypersurfaces Q_w introduced by (1.3). Our exposition is similar to Sect. 1 of [21].

Fix a point $z^0 \in M$ and let r be a real-analytic local defining function of M near z^0. In suitable local holomorphic coordinates we have $z^0 = 0$, and the Taylor expansion of r at 0 is

$$r(z, \bar{z}) = \frac{i}{2}(z_1 - \bar{z}_1) + \sum_{j,k=1}^{n} c_{jk} z_j \bar{z}_k + h(z, \bar{z}), \tag{2.1}$$

where h contains no purely holomorphic terms and $|h| = O(|z|^3)$.

Choose a small polydisc U_0 of the form

$$U_0 = U_0(\varrho, \sigma) = \{z \in \mathbf{C}^n : |z_1| < \varrho,\ |z_j| < \sigma \text{ for } 2 \leqq j \leqq n\} \tag{2.2}$$

such that $\bar{D} \cap U_0 = (D \cup M) \cap U_0$, and the power series expansion of the complexification $r(z, \bar{w})$ at $(0,0)$ converges in $U_0 \times U_0$. Choosing U_0 sufficiently small we have $\partial r / \partial z_1 \neq 0$ on $U_0 \times U_0$, so the complexification

$$M_C = \{(z, w) \in U_0 \times U_0 : r(z, w) = 0\}$$

of M in $U_0 \times U_0$ is a smooth complex hypersurface of complex dimension $2n-1$. Clearly (z, w) is in M_C if and only if z is in $Q_{\bar{w}}$.

By the implicit function theorem we can find a polydisc $U \subset U_0$ of the form (2.2) such that

$$M_C \cap (U_0 \times U) = \{(z, w) \in U_0 \times U : z_1 = \Phi(\tilde{z}, w)\},$$

where Φ is a holomorphic function of $\tilde{z} = (z_2, \ldots, z_n)$ and w. The function Φ has a power series expansion

$$\Phi(\tilde{z}, w) = \sum_{\alpha \in \mathbf{Z}_+^{n-1}} \phi_\alpha(w) \tilde{z}^\alpha,$$

with the coefficients ϕ_α holomorphic on U. For each fixed $w \in U$ we have

$$Q_w = \left\{ \zeta \in U_0 : \zeta_1 = \sum_{\alpha \in \mathbf{Z}_+^{n-1}} \phi_\alpha(\bar{w}) \bar{\zeta}^\alpha \right\}. \tag{2.3}$$

Our choice of r implies $r(z, 0) = \frac{i}{2} z_1$, so

$$Q_0 = \{\zeta \in U_0 : \zeta_1 = 0\}.$$

Definition. We say that the hypersurface $M \subset \mathbf{C}^n$ satisfies condition (A) if for each $z^0 \in M$ there is a neighborhood V of z^0 and an $\varepsilon > 0$ such that

$$M \cap Q_z \cap B(z, \varepsilon) = \{z\}, \quad z \in V \cap M. \tag{2.4}$$

Here $B(z, \varepsilon)$ is the open ball of radius ε centered at z. In other words, we require that $\{Q_z : z \in M\}$ is a family of supporting complex hypersurfaces for M.

If M is strictly pseudoconvex, the local defining function r (2.1) may be chosen strictly plurisubharmonic, and the condition (A) is satisfied in this case. The hypersurface Q_z for $z \in M$ lies in the pseudoconcave side of M and has second order contact with M at z. We shall omit the details. Condition (A) holds on some but not all weakly pseudoconvex hypersurfaces of finite type. There is a pseudoconvex hypersurface in $\mathbf{C}^2$ with an isolated weakly pseudoconvex point of D'Angelo type four at which condition (A) fails. We shall not investigate this further.

For every $z \in \mathbf{C}^n$ and $\delta > 0$ we denote by $\mathscr{G}_z(\delta)$ the set of all formal complex hypersurfaces at z of the form

$$Q = \left\{ \zeta_1 - z_1 = \sum_{\alpha \in \mathbf{Z}_+^{n-1}} c_\alpha (\tilde{\zeta} - \tilde{z})^\alpha \right\}, \tag{2.5}$$

where the coefficients $c = (c_\alpha : \alpha = (\alpha_2, \dots, \alpha_n))$ satisfy the conditions

$$c_0 = 0, \quad |c_\alpha| < \delta \quad \text{when} \quad |\alpha| = \alpha_2 + \dots + \alpha_n = 1. \tag{2.6}$$

Here, $\tilde{z} = (z_2, \dots, z_n)$, and similarly for $\tilde{\zeta}$. The condition involving δ means that the tangent plane of Q at z is not tilted too far away from $\{\zeta_1 = 0\} = Q_0$.

Since the spaces $\mathscr{G}_z(\delta)$ for different z are naturally isomorphic, we shall drop z in our notation. It will always be clear from the context at which point a given hypersurface is based. We shall refer to the elements of $\mathscr{G}(\delta)$ as the jets (of infinite order) of complex hypersurfaces.

For each $k \in \mathbf{Z}_+$ we denote by $\mathscr{G}_k(\delta)$ the space of jets (2.5) for which $c_\alpha = 0$ whenever $|\alpha| > k$. The elements of $\mathscr{G}_k(\delta)$ are called jets of order k of complex hypersurfaces. Notice that $\mathscr{G}_k(\delta)$ is a complex manifold with complex coordinates c_α, $|\alpha| \leqq k$; in fact it is a product of the polydisc $\Delta^{n-1}(\delta)$ centered at 0 with $\mathbf{C}^l$ for certain $l = l(k)$. There is a natural projection

$$\tau_k : \mathscr{G}(\delta) \to \mathscr{G}_k(\delta),$$

$$\tau_k(c)_\alpha = \begin{cases} c_\alpha, & \text{for } |\alpha| \leqq k, \\ 0, & \text{for } |\alpha| > k. \end{cases}$$

We shall sometimes write $\mathscr{G}(\delta) = \mathscr{G}_\infty(\delta)$ to make the notations consistent.

For a given $\delta > 0$ and for U and U_0 sufficiently small we can define a mapping

$$\begin{aligned} &g: M_C \cap (U_0 \times U) \to U_0 \times \mathscr{G}(\delta), \\ &g(z, w) = (z, {}_zQ_{\bar{w}}). \end{aligned} \tag{2.7}$$

Here, ${}_zQ_{\bar{w}} \in \mathscr{G}(\delta)$ is the jet of the complex hypersurface $Q_{\bar{w}}$ at the point z.

To obtain an explicit expression for g we fix a point $z \in Q_{\bar{w}}$, we insert $\zeta = z + (\zeta - z)$ into the equation (2.3) and reorder the power series according to the powers of $(\zeta - z)$:

$${}_zQ_{\bar{w}} = \left\{ \zeta_1 - z_1 = \sum_{\alpha \in \mathbf{Z}_+^{n-1}} c_\alpha(z, w) (\tilde{\zeta} - \tilde{z})^\alpha \right\}. \tag{2.8}$$

Each coefficient $c_\alpha(z, w)$ is a holomorphic function of (z, w) since it is a holomorphic expression in z and $\{\phi_\beta(w)\}$. Thus the mapping g (2.7) is holomorphic.

Let $\pi_k : U_0 \times \mathscr{G}(\delta) \to U_0 \times \mathscr{G}_k(\delta)$ be the projection

$$\pi_k(z, c) = (z, \tau_k(c)), \quad k \in \mathbf{Z}_+ .$$

For each $k \in \mathbf{Z}_+$ we also consider the holomorphic mapping

$$g_k = \pi_k \circ g : M_C \cap (U_0 \times U) \to U_0 \times \mathscr{G}_k(\delta). \tag{2.9}$$

Notice that $\tau_1({}_zQ_{\bar{w}})$ is just the tangent space $T_zQ_{\bar{w}}$ of the $Q_{\bar{w}}$ at z. For z in M, $\tau_1({}_zQ_z) = T_zQ_z = T_z^C M$ is the maximal complex subspace of T_zM.

Assuming that the hypersurface M contains no germs of positive dimensional complex varities, Diederich and Fornæss proved [21, Lemma 3] that for k sufficiently large and $U \subset U_0$ sufficiently small the map g_k (2.9) is finite holomorphic, i.e., a branched holomorphic covering onto its image

$$\mathscr{B}_k = g_k(M_C \cap (U_0 \times U)) \subset U_0 \times \mathscr{G}_k(\delta).$$

If M is strictly pseudoconvex (or Levi nondegenerate), the above already holds for $k=1$, and g_1 is biholomorphic near $(0,0) \in M_C$. This follows from a result of Webster [49]. We shall not go into details since these properties of g_k will not be used in the sequel.

Remark. The constructions of this section can also be applied to the hypersurface $M' = bD' \subset \mathbf{C}^N$ in a neighborhood of $f(z^0)$. We will use the same notation for objects associated to M', only adding a prime.

3. The preimage map

Let $f = (f_1, \dots, f_N) : D \cup M \to \mathbf{C}^N$ be a $\mathscr{C}^\infty$ mapping which is holomorphic on D and maps M to a smooth real-analytic pseudoconvex hypersurfaces $M' \subset \mathbf{C}^N$. Assume that M' satisfies the condition (A) defined by (2.4) above. This holds in particular when M' is strictly pseudoconvex. Let D' be a pseudoconvex domain in $\mathbf{C}^N$ bounded in part by M'. If M' is strictly pseudoconvex, then every point of M' has a local holomorphic peak function for D', so the maximum principle implies $f(D) \subset D'$, provided that D is chosen sufficiently small. In general we shall assume this as a part of our hypothesis.

We fix a point $z^0 \in M$. Our goal is to prove that f extends holomorphically to points of M arbitrary close to z^0. By translation of coordinates we may assume that $z^0=0$ and $f(0)=0\in M'$. Choose local coordinates and local defining functions r resp. r' for M resp. M' of the form (2.1). Let $U\subset U_0\subset \mathbf{C}^n$ and $U'\subset U_0'\subset \mathbf{C}^N$ be polydisc neighborhoods of 0 as in Sect. 2 such that $\bar{D}\cap U_0=(D\cup M)\cap U_0$, $f(U_0\cap\bar{D})\subset U_0'$, and $f(U\cap\bar{D})\subset U'$.

By pulling back by f a bounded strictly plurisubharmonic exhaustion function ϱ on D' of the form $\varrho(z')=-(-r'(z'))^{1-\eta}$ with $\eta>0$ arbitrarilly small [20] and applying the Hopf lemma to $\varrho\circ f$ we conclude that the normal derivative of f is nonvanishing at each point of M. At $z=0$ this just means $\partial f_1/\partial z_1(0)\neq 0$. Thus the preimage by f of every smooth complex hypersurface at $f(z)\in U_0'$ with tangent space close to $\{z_1'=0\}$ is again a smooth complex hypersurface at $z\in D\cap U_0$.

On the basis of the above observation we shall associate to f a mapping of jets of complex hypersurfaces as in [21]. Fix a point $z\in D\cap U_0$ and set $z'=f(z)\in D'\cap U_0'$. Choose a jet $Q'\in\mathcal{G}'(\delta')$ at z' of the form (2.5):

$$Q'=\left\{\zeta_1'-z_1'=\sum_{\alpha\in\mathbf{Z}_+^{N-1}} c_\alpha'(\tilde{\zeta}'-\tilde{z}')^\alpha\right\}. \tag{3.1}$$

Assume for a moment that the power series is convergent, so Q' is a complex hypersurface at z'. Our aim is to find the jet of its preimage $f^{-1}(Q')$ of the form (2.5) at z. We set $\zeta'=f(\zeta)$ and substitute the power series expansion

$$f_j(\zeta)-f_j(z)=\sum_{k=1}^{n}\frac{\partial f_j}{\partial z_k}(z)(\zeta_k-z_k)$$
$$+\text{terms of order}\geqq 2 \text{ in } (\zeta-z)$$

into the equation of Q'. Let $e_j=(0,\dots,1,\dots,0)$, with 1 on j-th spot. The coefficient of (ζ_1-z_1) is

$$a(z)=\frac{\partial f_1}{\partial z_1}(z)-\sum_{j=2}^{N} c_{e_j}'\frac{\partial f_j}{\partial z_1}(z),$$

and the equation becomes

$$a(z)(\zeta_1-z_1)=\sum_{k=2}^{n}\left(\sum_{j=2}^{N} c_{e_j}'\frac{\partial f_j}{\partial z_k}(z)-\frac{\partial f_1}{\partial z_k}(z)\right)(\zeta_k-z_k)$$
$$+\text{terms of order }\geqq 2 \text{ in } (\zeta-z). \tag{3.2}$$

At $z=0$ we have $\partial f_1/\partial z_1(0)\neq 0$ and $\partial f_j/\partial z_1(0)=0$ for $2\leqq j\leqq N$, so $a(0)\neq 0$. Choosing U_0 sufficiently small we may assume that $a(z)\neq 0$ for all $z\in U_0\cap\bar{D}$. We divide the Eq. (3.2) by $a(z)$ and solve the resulting equation for (ζ_1-z_1) by iteration. The result is a power series

$$\zeta_1-z_1=\sum_{\beta\in\mathbf{Z}_+^{n-1}} c_\beta(\tilde{\zeta}-\tilde{z})^\beta. \tag{3.3}$$

Since the power series involved contain no constant term, the required operations increase the order of terms. Therefore each coefficient c_β is obtained by only $|\beta|$ iterations, so it is a universal polynomial in the following quantities:

$$\begin{array}{ll} c'_\alpha, & |\alpha|\leqq|\beta|\,; \\ \partial^{|\alpha|}f_j/\partial z^\alpha(z), & |\alpha|\leqq|\beta|,\ 1\leqq j\leqq n\,; \\ 1/a(z)\,. & \end{array}$$

A detailed explanation of this argument can be found in [27].

The procedure described above works also if Q' is a formal hypersurface, i.e., it is defined by a formal power series, and $f(\zeta)-f(z)$ is given by a formal power series in $(\zeta-z)$. Since f is smooth on $D\cup M$, it has a formal power series expansion of the required type at each point $z\in D\cup M$. Of course, if both Q' and $f(\zeta)-f(z)$ are given by convergent power series, the resulting series (3.3) is also convergent.

Let $z\in U_0\cap(D\cup M)$. For each jet $Q'\in\mathscr{G}'(\delta')$ of the form (2.5) based at the point $f(z)$ we denote by $f^*(z,Q')$ the jet of the formal hypersurface (3.3) at z obtained by the above procedure. Given positive numbers δ and δ', we can choose U_0 sufficiently small such that f^* gives a mapping

$$f^*:(U_0\cap\bar{D})\times\mathscr{G}'(\delta')\to\mathscr{G}(\delta)\,. \tag{3.4}$$

Similarly we define for each $k\in\mathbf{Z}_+$ the mapping

$$f_k^*=\tau_k\circ f^*:(U_0\cap\bar{D})\times\mathscr{G}'(\delta')\to\mathscr{G}_k(\delta)\,. \tag{3.5}$$

We have seen above that the k-th order jet of $f^*(z,Q')$ only depends on z and on the k-th order jet $\tau_k(Q')$ of Q' at $f(z)$. Thus we have

$$\tau_k\circ f^*=f_k^*=f^*\circ\pi'_k\,,$$

where π'_k is the natural projection of $(U_0\cap\bar{D})\times\mathscr{G}'(\delta')$ onto $(U_0\cap\bar{D})\times\mathscr{G}'_k(\delta')$. The map f_k^* is smooth on $(U_0\cap\bar{D})\times\mathscr{G}'(\delta')$ and holomorphic on $(U_0\cap D)\times\mathscr{G}'(\delta')$. Although the space on which it is defined is infinite dimensional, f_k^* only depends on a finite number of coordinates, so the notions of smoothness and holomorphy make sense.

If Q_w is a complex hypersurface (1.3) and $z\in Q_w$, we denote by ${}_zQ_w\in\mathscr{G}(\delta)$ the jet of Q_w at the point z.

3.1. **Proposition.** *For each point $z\in M\cap U$ we have*

$$f^*(z,{}_{z'}Q'_{z'})={}_zQ_z\,,$$

where $z'=f(z)\in M'\cap U'$, and Q_z resp. $Q'_{z'}$, are the hypersurfaces (1.3).

Proof. By translation of coordinates we may assume that $z=0$ and $f(0)=0$. We extend f smoothly from $\bar{D}$ to a neighborhood of 0. There is a smooth nonvanishing real function $p(\zeta,\bar{\zeta})$ near 0 such that we have the identity

$$r'(f(\zeta),\overline{f(\zeta)})=p(\zeta,\bar{\zeta})r(\zeta,\bar{\zeta})$$

for small ζ. When we develop all the functions involved into formal power series centered at $\zeta=0$, the above becomes a formal power series identity in ζ and $\bar{\zeta}$. The identity persists if we vary $\bar{\zeta}$ independently of ζ. Setting $\bar{\zeta}=0$ we obtain the following identity in ζ:

$$r'(f(\zeta),0)=p(\zeta,0)r(\zeta,0).$$

Both sides of the equation contain the term $a\zeta_1$ for some $a\neq 0$, so we can solve the equations $r'(f(\zeta),0)=0$ and $p(\zeta,0)r(\zeta,0)=0$ for ζ_1 by iteration. Since the two equations are identical, we obtain in both cases the equation of the same formal hypersurface

$$R=\left\{\zeta_1=\sum_{\alpha\in Z_+^{n-1}} c_\alpha\tilde{\zeta}^\alpha\right\}.$$

To prove the proposition we will show that R is the equation of both $f^*(0,Q_0')$ and ${}_0Q_0$.

The hypersurface equation of Q_0' of the form (2.5) is obtained by solving $r'(z',0)=0$ for z_1 by iteration:

$$Q_0'=\left\{z_1'=\sum_{\alpha\in Z_+^{N-1}} a_\alpha\tilde{z}'^\alpha\right\}.$$

Into this equation we substitute the formal power series expansion $z'=f(\zeta)$ and solve for ζ_1 by iteration to obtain

$$f^*(0,Q_0')=\left\{\zeta_1=\sum_{\alpha\in Z_+^{n-1}} b_\alpha\tilde{\zeta}^\alpha\right\}.$$

On the other hand, we obtain the equation of R by first substituting $z'=f(\zeta)$ into $r'(z',0)=0$ and then solving for ζ_1.

We claim that the order of operations does not matter and we get the same result in both cases. Clearly this is so if $f(\zeta)$ is a convergent power series in ζ. The key point now is that the power series involved contain no constant terms, hence coefficients b_α and c_α do not depend on the terms of order more than $|\alpha|$ of the power series $f(\zeta)$. Thus, if we replace the full series for $f(\zeta)$ by its convergent Taylor polynomial $P_l(\zeta)$ in the above procedures, the conclusion is that $b_\alpha=c_\alpha$ for $|\alpha|\leq l$. This proves that $R=f^*(0,Q_0')$.

The proof of the second case is completely analogous: ${}_0Q_0$ is obtained by solving $r(\zeta,0)=0$ for ζ_1 by iteration, and R is obtained by solving $p(\zeta,0)r(\zeta,0)=0$. Recall that the constant term of p is nonzero. If $p(\zeta,0)$ is a convergent power series, the result is the same in both cases. However, the terms of order $\leq l$ of R do not depend on the terms of order more than l of $p(\zeta,0)$, so $R={}_0Q_0$.

Consequently ${}_0Q_0=f^*(0,Q_0')$ and Proposition 3.1 is proved.

4. Deficiency of the mapping

In this section we shall introduce the deficiency function associated to the mapping f and prove some technical results.

Since we have $\partial r'/\partial w_1'\neq 0$ on $U_0'\times U'$, the set

$$\Sigma=\{(z,w'):z\in U_0\cap\bar{D},\ w'\in U',\ r'(f(z),w')=0\}$$

is a complex manifold with smooth boundary. Under the natural projection $\Sigma \to U_0 \cap \bar{D}$, Σ is a smooth fiber bundle that is holomorphic over $U_0 \cap D$. We associate to Σ the mapping

$$g^\Sigma : \Sigma \to (U_0 \cap \bar{D}) \times \mathscr{G}'(\delta'),$$
$$g^\Sigma(z, w') = (z, {}_{f(z)}Q^{\prime}_{\bar{w}'}).$$

From (2.8) in Sect. 2 we know that the mapping

$$g' : M'_C \to U'_0 \times \mathscr{G}'(\delta'), \quad g'(z', w') = (z', {}_{z'}Q^{\prime}_{\bar{w}'})$$

is holomorphic. Therefore g^Σ is smooth on Σ and holomorphic on its interior Int Σ.

For each $k \in \mathbf{Z}_+$ we define the mappings

$$h_k(z, w') = f_k^* \circ g^\Sigma(z, w') = f_k^*(z, {}_{f(z)}Q^{\prime}_{\bar{w}'})$$

and

$$F_k : \Sigma \to (U_0 \cap \bar{D}) \times \mathscr{G}_k(\delta),$$
$$F_k(z, w') = (z, h_k(z, w')).$$

Every F_k is a smooth map on Σ that is holomorphic in the interior of Σ and on each fiber

$$\Sigma_z = \{w' \in U' : (z, w') \in \Sigma\}, \quad z \in U_0 \cap \bar{D}.$$

For each point $z \in M \cap U$ we have $r'(f(z), \overline{f(z)}) = 0$, so $(z, \overline{f(z)}) \in \Sigma$. The fibers

$$Z_{z,k} = \{w' \in \Sigma_z : F_k(z, w') = F_k(z, \overline{f(z)})\}, \quad k \in \mathbf{Z}_+$$

form a decreasing sequence of complex subvarieties of Σ_z containing the point $\overline{f(z)}$. Let

$$\nu_k(z) = \dim_{\overline{f(z)}} Z_{z,k}, \quad z \in M \cap U.$$

Since F_k is holomorphic, the function ν_k is upper semicontinuous on M, and we have $\nu_1 \geqq \nu_2 \geqq \nu_3 \geqq \dots$.

Definition. The number $\nu(z) = \inf_k \nu_k(z)$ is called the deficiency of f at the point $z \in M$.

This function plays a very important role in our proof. Since $\nu : M \to \mathbf{Z}_+$ is the infimum of a decreasing sequence of upper semicontinuous functions on $M \cap U$, it is upper semicontinuous. We set

$$M_0 = \{z \in M \cap U : \nu \text{ is constant in a neighborhood of } z \text{ in } M\}. \tag{4.1}$$

We shall prove in Sect. 6 that f extends holomorphically across M_0. This will imply Theorem 1.1.

Fix a point $z^0 \in M \cap U$ and let $m = \nu(z^0)$. If $k_0 \in \mathbf{Z}_+$ is sufficiently large, then $m = \nu_k(z)$ for $k \geqq k_0$. Let $\zeta = (\zeta_1, \dots, \zeta_{N-1})$ be local holomorphic fiber coordinates on Σ near the point $(z^0, \overline{f(z^0)})$. (We may take $\zeta_j = w'_{j+1}$.) Then (z, ζ) are local $\mathscr{C}^\infty$ coordinates on Σ that are holomorphic on Int Σ. After an affine change of coordinates we may assume that $\overline{f(z^0)}$ corresponds to $\zeta = 0$, and that 0 is an isolated point of the intersection

$$Z_{z^0,k_0} \cap \{\zeta_1=0,\ldots,\zeta_m=0\}.$$

Consequently there are arbitrary small polydiscs $P=P'\times P''$ in the ζ-space, with $P'\subset\mathbf{C}^m$ and $P''\subset\mathbf{C}^{N-m-1}$, satisfying $Z_{z^0,k_0}\cap(\bar{P}'\times\partial P'')=\emptyset$. It follows that for a sufficiently small neighborhood V of z^0 we have

$$Z_{z,k}\cap(\bar{P}'\times\partial P'')=\emptyset, \quad z\in M\cap V,\ k\geqq k_0. \tag{4.2}$$

This shows that $v_k(z)\leqq m$ for all $z\in M\cap V$, so the function v_k is upper semicontinuous.

Suppose now that $z^0\in M_0$. Shrinking V if necessary we may assume that $v(z)=m$ for all $z\in M\cap V$, so the variety $Z_{z,k}\cap(\bar{P}'\times\partial P'')$ is m-dimensional for each $z\in M\cap V$ and for all large k. From (4.2) it follows that $Z_{z,k}\cap P$ is a branched analytic covering onto P' with respect to the coordinate projection $P\to P'$ [31, p. 108].

From Proposition 3.1 it follows that

$$F_k(z,\overline{f(z)})=(z,\tau_k({}_zQ_z)), \quad z\in M\cap U,\ k\in\mathbf{Z}_+.$$

We have thus proved

4.1. **Lemma.** *Let $z^0\in M_0$, and let V and $P=P'\times P''$ be as above. Then for each $z\in M\cap V$ and $\zeta'\in P'$ there is a point $\zeta''\in P''$ satisfying*

$$F_k(z,(\zeta',\zeta''))=g_k(z,\bar{z}), \quad k\in\mathbf{Z}_+.$$

We shall consider the fibers of F_k over points of the set $\mathscr{B}_k=g_k(M_C\cap(U_0\times U))$. For each point $(z,w)\in M_C$ close to $(z^0,\bar{z}^0)$, with $z\in D\cup M$, we define a complex subvariety of Σ_z by

$$S_{(z,w)}=\{\zeta\in\Sigma_z : F_k(z,\zeta)=g_k(z,w), \quad k\in\mathbf{Z}_+\}. \tag{4.3}$$

For $z\in M\cap U$ we have $S_{(z,\bar{z})}=\bigcap Z_{z,k}$, so (4.2) implies $S_{(z,w)}\cap(\bar{P}'\times\partial P'')=\emptyset$ when (z,w) is close to $(z^0,\bar{z}^0)$. Thus $S_{(z,w)}\cap(P'\times P'')$ is at most m-dimensional complex subvariety of P. Apriori these varieties may be empty. The main result of this section is

4.2. **Proposition.** *If $z^0\in M_0$ and $m=v(z^0)$, there exist arbitrary small polydiscs $P\subset\mathbf{C}^{N-1}$ centered at 0 and neighborhoods W of $(z^0,\bar{z}^0)$ in $\mathbf{C}^{2n}$ such that*

$$\dim S_{(z,w)}\cap P=m, \quad (z,w)\in M_C\cap W,\ z\in\bar{D}.$$

Before we turn to the proof we introduce the notion of an A^∞ variety. Let $D\subset\mathbf{C}^n$ be a smoothly bounded domain and let $V\subset\mathbf{C}^n$ and $\Omega\subset\mathbf{C}^r$ be open subsets. We denote by $A^\infty((V\cap\bar{D})\times\Omega)$ the space of all functions that are smooth on $(V\cap\bar{D})\times\Omega$ and holomorphic on $(V\cap D)\times\Omega$.

Definition. A set $\mathscr{A}\subset(V\cap\bar{D})\times\Omega$ is said to be an A^∞ variety if for each $p\in\Omega$ there is a neighborhood $V'\times\Omega'$ of p in $\mathbf{C}^n\times\mathbf{C}^r$ and a finite collection of functions $a_j\in A^\infty((V'\cap\bar{D})\times\Omega')$ such that

$$\mathscr{A}\cap((V'\cap\bar{D})\times\Omega')=\{(z,\zeta) : a_j(z,\zeta)=0 \text{ for all } j\}.$$

We say that $\mathscr{A}$ is an A^∞ subvariety of $(V\cap\bar{D})\times\Omega$ if, in addition, $\mathscr{A}$ is a closed subset of $(V\cap\bar{D})\times\Omega$.

Clearly $\mathscr{A}\cap((V\cap D)\times\Omega)$ is a complex variety in the usual sense, and each fiber $\mathscr{A}_z$ of $\mathscr{A}$ is a complex variety in Ω, even when $z\in bD$.

Proof of Proposition 4.2. Choose a polydisc $P=P'\times P''$, a neighborhood V of z^0, and an integer k_0 such that (4.2) holds. For each $k\geqq k_0$ we consider the mappings

$$H_k:(V\cap\bar{D})\times P\to(V\cap\bar{D})\times\mathscr{G}_k(\delta)\times P',$$

$$H_k(z,\zeta)=(F_k(z,\zeta),\zeta')=(z,h_k(z,\zeta),\zeta')$$

and

$$\tilde{H}_k:(V\cap\bar{D})\times P\to(V\cap\bar{D})\times\mathscr{G}_k(\delta)\times P,$$

$$\tilde{H}_k(z,\zeta)=(z,h_k(z,\zeta),\zeta).$$

Let $\mathscr{H}_k$ resp. $\tilde{\mathscr{H}}_k$ be the images of H_k resp. $\tilde{H}_k$. The set $\tilde{\mathscr{H}}_k$ is an A^∞ subvariety since it is the graph of the A^∞ mapping h_k.

Denote by

$$\iota_k:(V\cap\bar{D})\times\mathscr{G}_k(\delta)\times P\to(V\cap\bar{D})\times\mathscr{G}_k(\delta)\times P'$$

the coordinate projection which deletes the last variable $\zeta''\in P''$. We have $H_k=\iota_k\circ\tilde{H}_k$, so $\mathscr{H}_k=\iota_k(\tilde{\mathscr{H}}_k)$.

If the point (z,w) satisfies $\iota_{k_0}(\tilde{H}_{k_0}(z,\zeta))=H_{k_0}(z^0,0)$, then $z=z^0$, $\zeta'=0'$, and $F_{k_0}(z^0,0',\zeta'')=F_{k_0}(z^0,0)$, i.e., $(0',\zeta'')\in Z_{z^0,k_0}$. Hence (4.2) implies that the fiber

$$\iota_{k_0}^{-1}(H_{k_0}(z^0,0))\cap\tilde{\mathscr{H}}_{k_0}$$

is finite.

Since $\tilde{\mathscr{H}}_{k_0}$ is closed in $(V\cap\bar{D})\times\mathscr{G}_{k_0}(\delta)\times P$, there are polydiscs $V_1\subset V\subset\mathbf{C}^n$ and $P_1=P_1'\times P_1''\subset P$ centered at z^0 resp. at 0, and there is a neighborhood G_{k_0} of $\tau_{k_0}({}_{z^0}Q_{z^0})$ in $\mathscr{G}_{k_0}(\delta)$ such that

$$((V_1\cap\bar{D})\times G_{k_0}\times P_1'\times\partial P_1'')\cap\tilde{\mathscr{H}}_{k_0}=\emptyset.$$

The same is then true for all $k\geqq k_0$ if we define

$$G_k=\tau_{k_0}^{-1}(G_{k_0})\cap\mathscr{G}_k(\delta).$$

If follows that for $k\geqq k_0$ the restriction of the projection ι_k to the A^∞ subvariety

$$\tilde{\mathscr{A}}_k=\tilde{\mathscr{H}}_k\cap((V_1\cap\bar{D})\times G_k\times P_1)$$

is a proper mapping of $\tilde{\mathscr{A}}_k$ into $(V_1\cap\bar{D})\times G_k\times P_1'$ with image

$$\mathscr{A}_k=\mathscr{H}_k\cap((V_1\cap\bar{D})\times G_k\times P_1').$$

We claim that, as a consequence, $\mathscr{A}_k$ is an A^∞ subvariety of $(V_1\cap\bar{D})\times G_k\times P_1'$. In the case of complex varieties this is a special case of the proper mapping theorem due to Remmert. The proof given by Čirka [14, p. 30] for this special case also applies to A^∞ varieties, with only trivial modifications in the statement and proof. The proof depends on the Weierstrass preparation theorem for functions of class

A^∞, but only with respect to the fiber variables. Again, the standard proof [31, p. 68] or [14, p. 11] applies with only trivial modifications. We shall not go into the details.

If W is a sufficiently small neighborhood of $(z^0, \bar{z}^0)$ in $\mathbf{C}^{2n}$, we have

$$g_k(z, w) \in (V_1 \cap \bar{D}) \times G_k$$

whenever $k \geqq k_0$, $z \in \bar{D}$, and $(z, w) \in M_C \cap W$. Let W be a product $W = W' \times W''$, with $W', W'' \subset \mathbf{C}^n$. Denote by R the smooth manifold with boundary

$$R = (M_C \cap ((W' \cap \bar{D}) \times W'')) \times P_1'.$$

Its interior is a complex manifold of dimension $(2n-1+m)$. We may assume that R is connected. The boundary of R contains the smooth generating submanifold

$$T = \{(z, \bar{z}, \zeta') : z \in M \cap W',\ \zeta' \in P_1'\}.$$

By a theorem of Pinčuk [39] T is a uniqueness set for continuous functions on R that are holomorphic in the interior of R.

For each $k \geqq k_0$ we consider the associated A^∞ mapping

$$\tilde{g}_k : R \to (V_1 \cap \bar{D}) \times G_k \times P_1',$$

$$\tilde{g}_k((z, w), \zeta') = (g_k(z, w), \zeta').$$

For $z \in M \cap W'$ we have $(z, \bar{z}) \in M_C$ and

$$\tilde{g}_k((z, \bar{z}), \zeta') = (z, \tau_k({}_zQ_z), \zeta').$$

By choosing the polydisc $P_1 = P_1' \times P_1''$ correctly we may assume that Lemma 4.1 holds for P_1. Hence for each $\zeta' \in P_1'$ there is a point $\zeta'' \in P_1''$ such that

$$g_k(z, \bar{z}) = F_k(z(\zeta', \zeta'')), \quad k \geqq k_0.$$

The definition of H_k now implies

$$\tilde{g}_k((z, \bar{z}), \zeta') = H_k(z, (\zeta', \zeta'')) \in \mathscr{A}_k, \quad k \geqq k_0.$$

By pulling back the local A^∞ defining functions of $\mathscr{A}_k$ by the mapping $\tilde{g}_k$ we obtain local A^∞ functions on R that vanish on the submanifold T. The conclusion is that $\tilde{g}_k(R)$ is contained in $\mathscr{A}_k$ for all $k \geqq k_0$.

Since H_k is a finite mapping for $k \geqq k_0$ and the fibers of H_k are decreasing with k, we can find for each $(z, w, \zeta') \in R$ a finite set of points $\zeta'' \in P_1''$ satisfying

$$H_k(z, (\zeta', \zeta'')) = \tilde{g}_k((z, w), \zeta'), \quad k \in \mathbf{Z}_+.$$

This equation is equivalent to $(\zeta', \zeta'') \in S_{(z,w)} \cap P_1$. Hence the fiber $S_{(z,w)} \cap P_1$ is m-dimensional and Proposition 4.2 is proved.

5. Construction of an analytic variety

We now have all the necessary tools to construct the analytic variety $X \subset \mathbf{C}^{n+N}$ associated with f that will be used to extend f.

Let $U \subset U_0$ be polydisc neighborhoods of 0 as in Sects. 2 and 3. By the implicit function theorem we can solve the system of equations

$$r(z, \bar{w}) = 0,$$
$$z_j = w_j, \quad 2 \leqq j \leqq n,$$

for $z = z(w)$ when w is in a smaller neighborhood $U_1 \subset U$ of 0. The resulting mapping $w \to z(w)$ is real-analytic and has the properties

(i) $z(w) \in Q_w$.
(ii) $w \in U_1 \backslash \bar{D}$ implies $z(w) \in U \cap D$.
(iii) $w \in M \cap U_1$ implies $z(w) = w$.

The properties (i) and (iii) are obvious. Observe that $z(w)$ is antiholomorphic in the variable w_1, so the restriction of $z(w)$ to every complex line $L = \{w_2 = a_2, \ldots, w_n = a_n\}$ is the antiholomorphic reflection across the curve $M \cap L$. This implies (ii).

Let $z'(w) = f(z(w))$ for $w \in U_1 \backslash D$. Let $F_k : \Sigma \to (U_0 \cap \bar{D}) \times \mathscr{G}_k(\delta)$ be the mapping defined in Sect. 4, and let g_k be as in (2.9). We define

$$\begin{aligned} X = \{(w, w') \in (U_1 \backslash D) \times U' : r'(z'(w), \bar{w}') = 0, \\ F_k(z(w), \bar{w}') = g_k(z(w), \bar{w}) \text{ for all } k \in \mathbf{Z}_+\}. \end{aligned} \tag{5.1}$$

Notice that the first condition in the definition of X requires that the point $(z(w), \bar{w}')$ lies in the manifold Σ where the maps F_k are defined.

An equivalent definition of X is

$$X = \{(w, w') \in (U_1 \backslash D) \times U' : z'(w) \in Q'_{w'}, f^*(z(w), Q'_{w'}) =_{z(w)} Q_w\} \tag{5.2}$$

From the second definition of X and the construction of f^* in Sect. 3 it follows that for each $(w, w') \in X$, $w \in U_1 \backslash \bar{D}$, f maps the germ of the complex hypersurface Q_w at $z(w)$ into $Q'_{w'}$, so f maps the whole connected component $\tilde{Q}_w$ of $Q_w \cap D \cap U$ containing $z(w)$ into $Q'_{w'}$. Conversely, if $f(\tilde{Q}_w) \subset Q'_{w'}$, then $(w, w') \in X$. This shows that for each $w \in U_1 \backslash \bar{D}$ the fiber X_w of X equals

$$X_w = \{w' \in U' : f(\tilde{Q}_w) \subset Q'_{w'}\}. \tag{5.3}$$

This definition of X_w is almost the same as (1.5) in Sect. 1, except that Q_w is replaced by $\tilde{Q}_w$. The following proposition contains the relevant properties of X.

5.1. **Proposition.** (a) *The interior* $\operatorname{Int} X = X \cap ((U_1 \backslash \bar{D}) \times U')$ *is a closed complex subvariety of* $(U_1 \backslash \bar{D}) \times U'$.

(b) *The fiber* X_w *is a closed complex subvariety of* $Q'_{z'(w)} \cap U'$ *for each* $w \in U_1 \backslash D$.

(c) *If* $w \in M \cap U_1$, *then* $f(w) \in X_w \subset Q'_{f(w)}$.

(d) *If* z *belongs to the set* M_0 *defined by* (4.1) *and* $v(z) = m$, *there exist arbitrary small neighborhoods* $V = V' \times V''$ *of* $(z, f(z))$, *with* $V' \subset \mathbf{C}^n$ *and* $V'' \subset \mathbf{C}^N$, *such that*

$$\dim X_w \cap V'' = m, \quad w \in V' \backslash D.$$

Proof. Recall that g_k is holomorphic on the complexification M_C, and F_k is of class $A^\infty(\Sigma)$. Hence for each fixed $w \in U_1 \backslash D$ the defining equations (5.1) of X are antiholomorphic in $w' \in U'$. This proves (b).

Property (c) follows immediatelly from Proposition 3.1 and the definition of X.

We shall now prove (a). For each point $w^1 \in U_1 \backslash \bar{D}$ we can find a neighborhood $U_2 \subset U_1 \backslash \bar{D}$ of w^1 and an anti-holomorphic mapping $\xi: U_2 \to U \cap D$ satisfying

$$\xi(w) \in Q_w \quad \text{and} \quad \xi(w^1) = z(w^1).$$

If we write $a = z(w^1)$, we can obtain a ξ as above by solving the system of equations

$$\xi_j = a_j, \quad 2 \leqq j \leqq n,$$
$$r(\xi, \bar{w}) = 0$$

in a neighborhood of (a, w^1).

If the neighborhood U_2 is chosen sufficiently small, the points $\xi(w)$ and $z(w)$ lie in the same connected component $\tilde{Q}_w$ of $Q_w \cap D \cap U$ for each $w \in U_2$.

We now define a subset X_1 of $U_2 \times U'$ by the same equations as X (5.1), except that we replace $z(w)$ everywhere by $\xi(w)$ and $z'(w)$ by $f(\xi(w))$. Since ξ is anti-holomorphic in w, the new equations are anti-holomorphic in both variables (w, w'), hence X_1 is a complex subvariety of $U_2 \times U'$.

From the expression (5.3) for the fiber X_w it follows that X_1 coincides with $X \cap (U_2 \times U')$. Thus $X \cap (U_2 \times U')$ is a complex variety, and (a) is proved.

It remains to prove (d). Let

$$\eta: (U_1 \backslash D) \times U' \to (U \backslash D) \times U',$$
$$\eta(w, w') = (z(w), \bar{w}').$$

Recall that U' is symmetric with respect to 0, so $\bar{w}' \in U'$ when $w' \in U'$. For each fixed $w \in U_1 \backslash D$ the map $\eta(w, \cdot)$ provides an anti-holomorphic equivalence of the subvariety $Q'_{z'(w)} \cap U'$ onto the fiber $\Sigma_{z(w)}$ of Σ. The fiber X_w corresponds under $\eta(w, \cdot)$ to the variety $S_{(z(w), \bar{w})}$ defined by (4.3). Property (d) now follows from Proposition 4.2. This completes the proof of Proposition 5.1.

6. Proof of the main result

In this section we shall prove the following

6.1. **Theorem.** *Assume that the hypotheses of Theorem 1.1 hold. Suppose that the deficiency function v defined Sect. 4 is constant in a neighborhood of a point $z^0 \in M$ in M (i.e., $z^0 \in M_0$). Then f extends holomorphically to a neighborhood of z^0 in $\mathbf{C}^n$.*

Since M_0 is open and dense in M, this will also prove Theorem 1.1. We may assume that $z^0 = 0, f(0) = 0$, and $v(0) = m$. Let X be the set defined by (5.1). Choose a neighborhood $V = V' \times V''$ of $(0,0) \in \mathbf{C}^n \times \mathbf{C}^N$ satisfying Proposition 5.1 (d). Since the fiber $X_0 \cap V''$ is an m-dimensional subvariety of V'', there is an $(N-m)$-dimensional complex subspace L of $\mathbf{C}^N$ such that 0 is an isolated point of $X_0 \cap L$. After a unitary change of coordinates in $\mathbf{C}^N$ we may assume that $L = \{w'_1 = 0, \ldots, w'_m = 0\}$.

We can find a polydisc $P = P' \times P'' \subset V''$ centered at 0, with $P' \subset \mathbf{C}^m$ and $P'' \subset \mathbf{C}^{N-m}$, such that

$$X_0 \cap (\bar{P}' \times \partial P'') = \emptyset.$$

Since X is a closed subset of $(U_1 \backslash D) \times U'$, there is a neighborhood W of 0 in $\mathbf{C}^n$, $W \backslash D \subset V'$, satisfying

$$X \cap ((W \backslash D) \times P' \times \partial P'') = \emptyset .$$

Hence the restriction of the coordinate projection

$$\sigma : (W \backslash D) \times P' \times P'' \to (W \backslash D) \times P'$$

to the subvariety $X' = X \cap ((W \backslash \bar{D}) \times P)$ is a proper holomorphic mapping of X' into $(W \backslash \bar{D}) \times P'$.

Denote by Y the top dimensional part of the variety X'. Since each fiber $X_{w'} \cap V''$ is m-dimensional for $w \in \mathbf{C}^n$ close to 0 according to Proposition 5.1 (d), we have $\dim Y = n+m$. It follows that

$$\sigma : Y \to (W \backslash \bar{D}) \times P'$$

is a branched analytic covering onto $(W \backslash \bar{D}) \times P'$ [31, p. 108]. Let

$$\Phi_\alpha(z, w') = \sum_\beta a_{\alpha,\beta}(z, w'_1, \ldots, w'_m)(w'_{m+1}, \ldots, w'_N)^\beta$$

be the canonical defining functions of this analytic cover, see [50, p. 369] or [14, p. 40]. The coefficients $a_{\alpha,\beta}$ are holomorphic functions on $(V \backslash \bar{D}) \times P'$, so they extend across the Levi pseudoconcave hypersurface $(M \cap V) \times P' \subset \mathbf{C}^{n+m}$ to holomorphic functions in a neighborhood W_1 of the origin in C^{n+m} [3, 45]. We denote by Ψ_α the corresponding extension of Φ_α. The set

$$\tilde{Y} = \{(z, w') : \Psi_\alpha(z, w') = 0 \text{ for all } \alpha\}$$

is a purely $(n+m)$-dimensional analytic subvariety of a neighborhood of $(0,0) \in \mathbf{C}^{n+N}$ which we again denote by $W \times P$. By construction $\tilde{Y}$ extends Y, i.e.,

$$\tilde{Y} \cap ((W \backslash \bar{D}) \times P) = Y .$$

We claim that for each $w \in M \cap W$ we have

$$f(w) \in \tilde{Y}_w \quad \text{and} \quad \tilde{Y}_w \subset Q'_{f(w)} . \tag{6.1}$$

The first property holds since $(w, f(w))$ lies in the closure $\bar{Y}$ of Y in $W \times P$ by Proposition (5.1) (d), and certainly $\bar{Y} \subset \tilde{Y}$. To prove the second inclusion we first note that $\tilde{Y}_w \subset \bar{Y}$ for $w \in M \cap W$ since $\tilde{Y}$ is a branched covering extending Y. The set X is closed in $W \times P$, so $\bar{Y} \subset X$, and therefore $\tilde{Y}_w \subset X_w$. By Proposition 5.1 (c) we also have $X_w \subset Q'_{f(w)}$, so (6.1) is proved.

Suppose now that the hypersurface $M' = bD'$ satisfies Condition (A) defined by (2.4). Denote by

$$\Gamma(f|_M) = \{(z, f(z)) : z \in M\}$$

the graph of f over M. Since f is smooth on $D \cup M$, $\Gamma(f|_M)$ is a smooth submanifold of $\mathbf{C}^{n+N}$. Condition (A) and (6.1) together imply that for sufficiently small neighborhoods U of 0 in $\mathbf{C}^{n+N}$ we have

$$\tilde{Y} \cap (M \times M') \cap U = \Gamma(f|_M) \cap U .$$

Since $\tilde{Y}$ is a complex subvariety and M and M' are real-analytic subsets, the above identity exhibits the smooth manifold $\Gamma(f|_M)\cap U$ as a real-analytic subset of U. It follows that $\Gamma(f|_M)\cap U$ is a real-analytic submanifold of $\mathbf{C}^{n+N}$ according to [36, p. 96], so $f|_{M\cap U}$ is real-analytic by the implicit function theorem [10]. Consequently f extends holomorphically to a neighborhood of 0 in $\mathbf{C}^n$. This concludes the proof of Theorem 6.1.

7. Extension of proper holomorphic maps to balls

In this section we shall prove Theorem 1.3 in which the hypersurface M' is the unit sphere $\{z'\in\mathbf{C}^N : |z'|=1\}$, the boundary of the unit ball $\mathbf{B}^N\subset\mathbf{C}^N$. Our assumption is that the holomorphic map $f: D\to\mathbf{B}^N$ extends to a map of class $\mathscr{C}^s(D\cup M)$, with $s=N-n+1$, and $f(M)\subset M'=b\mathbf{B}^N$. At the same time we shall prove Theorem 1.4 in which $M=b\mathbf{B}^n$.

Fix a point $z^0\in M$ and assume that $z^0=0$. Choose polydisc neighborhoods $U\subset U_0$ of 0 of the form (2.2) as in Sect. 2.

We denote by $\tilde{M}$ the set

$$\tilde{M}=\{(z,w): z\in U_0\cap(D\cup M),\ w\in U,\ r(z,w)=0\}.$$

The interior of $\tilde{M}$ is a complex manifold of dimension $2n-1$, an open subset of the complexification M_C. The part of the boundary of $\tilde{M}$ contained in $(U_0\cap M)\times U$ is smooth real-analytic and contains the totally real submanifold

$$T=\{(z,\bar{z}): z\in M\cap U\}$$

of real dimension $2n-1$. By choosing U and U_0 sufficiently small we may assume that $\tilde{M}$ is connected.

We denote by $A(\tilde{M})$ the algebra of all continuous functions on $\tilde{M}$ that are holomorphic in the interior of $\tilde{M}$. By a theorem of Pinčuk [39] every nonempty open subset of T is a uniqueness set for functions in $A(\tilde{M})$.

Let Q_w be the complex hypersurface in $\mathbf{C}^n$ defined by (1.3). Recall that for each point $(z,w)\in M_C$ we have $z\in Q_{\bar{w}}$. Let $x=(x_2,\dots,x_n)\in\mathbf{C}^{n-1}$. According to (2.8) we can parametrize the germ of the complex hypersurface $Q_{\bar{w}}$ at the point z by a mapping

$$\zeta_1(x)=z_1+\sum_{|\alpha|>0} c_\alpha(z,w)x^\alpha,$$

$$\zeta_j(x)=z_j+x_j,\quad 2\leqq j\leqq n. \tag{7.1}$$

Each coefficient $c_\alpha(z,w)$ is a holomorphic function of $(z,w)\in M_C$.

Let $(z,w)\in\tilde{M}$, so $z\in D\cup M$. We extend f to a $\mathscr{C}^s$ map on a neighborhood of $D\cup M$ in $\mathbf{C}^n$. Consider the restriction of f to $Q_{\bar{w}}$ in the coordinates x, i.e., the function $f(\zeta(x))$. Its Taylor expansion of order s at the point $x=0$ is of the form

$$f(\zeta(x))=\sum_{|\beta|\leqq s} a_\beta(z,w)x^\beta+o(|x|^s). \tag{7.2}$$

7.1. **Lemma.** *For each $\beta\in\mathbf{Z}_+^{n-1}$, $|\beta|\leqq s$, we have $a_\beta\in A(\tilde{M})$.*

Proof. We only need to observe that the coefficient $a_\beta(z, w)$ is a linear combination of the derivatives $\partial^{|\gamma|}f/\partial z^\gamma(z)$ of order $|\gamma| \leqq |\beta|$ whose coefficients are polynomials in the terms $c_\alpha(z, w)$ for $|\alpha| \leqq |\beta|$. The lemma follows.

In the case when $M = b\mathbf{B}^n$ is the unit sphere, the associated complex hypersurface Q_w is the hyperplane

$$Q_w = \left\{ z \in \mathbf{C}^n : \sum_{j=1}^{n} z_j \bar{w}_j = 1 \right\} = w/|w|^2 + w^\perp, \quad w \neq 0.$$

By a rotation we may assume that the initial point $z^0 \in b\mathbf{B}^n$ is $(1, 0, \dots, 0)$, and $w_1 \neq 0$ on the neighborhood U of z^0. For each $w \in U$ the vectors

$$t_1(w) = (w_2, -w_1, 0, \dots, 0)$$
$$t_2(w) = (w_3, 0, -w_1, \dots, 0)$$
$$\dots\dots\dots$$
$$t_{n-1}(w) = (w_n, 0, 0, \dots, -w_1)$$

form a complex basis of $\bar{w}^\perp$. For a fixed $z \in Q_{\bar{w}}$ we can parametrize $Q_{\bar{w}}$ by the map

$$\zeta(x) = z + \sum_{j=1}^{n-1} x_j t_j(w), \quad x = (x_1, \dots, x_{n-1}) \in \mathbf{C}^{n-1}.$$

In this case the coefficients $a_\beta(z, w)$ in the Taylor expansion (7.2) are homogeneous polynomials of order $|\beta| \leqq s$ in the second variable w.

We return to the general case. Denote by $\langle z, w \rangle$ the complex bilinear form

$$\langle z, w \rangle = \sum z_j w_j.$$

7.2. **Lemma.** *For each $z \in M \cap U$ we have*

$$\langle a_\beta(z, \bar{z}), \overline{f(z)} \rangle = \begin{cases} 1, & \text{if } \beta = 0; \\ 0, & \text{if } 1 \leqq |\beta| \leqq s. \end{cases} \tag{7.3}$$

Proof. By translation of coordinates we may assume that $z = 0$. Recall that f is a map of class $\mathscr{C}^s$ near $0 \in \mathbf{C}^n$ that satisfies $\langle f(\zeta), \overline{f(\zeta)} \rangle = 1$ for $\zeta \in M$. Hence there is a function $p(\zeta)$ of class $\mathscr{C}^{s-1}$ near 0 such that

$$\langle f(\zeta), \overline{f(\zeta)} \rangle - 1 = p(\zeta) r(\zeta, \bar{\zeta})$$

for all ζ near 0.

We write

$$f(\zeta) = f^s(\zeta) + o(|\zeta|^s),$$
$$p(\zeta) = p^{s-1}(\zeta, \bar{\zeta}) + o(|\zeta|^{s-1}),$$

where f^s and p^{s-1} are Taylor polynomials of order s resp. $s-1$ at 0. Since r vanishes at 0, we have

$$\langle f^s(\zeta), \overline{f^s(\zeta)} \rangle - 1 = p^{s-1}(\zeta, \bar{\zeta}) r(\zeta, \bar{\zeta}) + o(|\zeta|^s).$$

As usually we complexify this identity by varying ζ and $\bar{\zeta}$ independently. The error term remains small of order s even in the complexified identity.

We now set $\bar{\zeta}=0$ and $\zeta=\zeta(x)\in Q_0$ (7.1). Then $r(\zeta(x),0)=0$ and $|\zeta(x)|=O(|x|)$, so we have

$$\langle f^s(\zeta(x)), \overline{f^s(0)}\rangle - 1 = o(|x|^s).$$

Recall that

$$f^s(\zeta(x)) = \sum_{|\beta|\leqq s} a_\beta(0,0)x^\beta + o(|x|^s).$$

If we insert this into the above identity and compare the coefficients of the powers x^β, we obtain the Eqs. (7.3) at the point $z=0$. This proves Lemma 7.2.

We now define an integer-valued function $k:\tilde{M}\to\mathbf{Z}_+$ by

$$k(z,w)=\operatorname{rank}\{a_\beta(z,w):1\leqq|\beta|\leqq s\}. \tag{7.4}$$

Clearly k is lower semicontinuous, so its restriction to T is locally constant on an open, everywhere dense subset $T_0\subset T$. Let

$$k_0=\min\{k(z,\bar{z}):(z,\bar{z})\in T_0\}.$$

7.3. **Lemma.** (a) $k_0\leqq N-1$.

(b) $k(z,w)\leqq k_0$ *for each* $(z,w)\in\tilde{M}$.

(c) $k\equiv k_0$ *on* T_0.

Proof. From (7.3) it follows that $k(z,\bar{z})\leqq N-1$ for each $z\in M\cap U$, so $k_0\leqq N-1$. This proves (a). Part (c) follows from (b) and the lower semicontinuity of k. It remains to prove (b).

Denote by $A(z,w)$ the matrix with entries in the algebra $A(\tilde{M})$ whose columns are the functions $\{a_\beta:1\leqq|\beta|\leqq s\}$ given by (7.2). Let $\delta(z,w)$ be the determinant of any (k_0+1) minor of $A(z,w)$; δ is a function in $A(\tilde{M})$ that vanishes on a nonempty open subset of T, according to the definition of k_0. The uniqueness theorem of Pinčuk [39] implies that $\delta\equiv0$ on $\tilde{M}$. This shows that the rank of the matrix $A(z,w)$ does not exceed k_0 and the lemma is proved.

Remark. We shall see that k is related to the deficiency function v defined in Sect. 4 above by

$$v(z)=N-(k(z,\bar{z})+1).$$

The function k is not necessarilly constant on $\tilde{M}$. The subset of the interior of $\tilde{M}$ defined by $k<k_0$ is a complex subvariety.

Let $z=z(w)$ be the mapping $U_1\subset U\to U$ defined at the beginning of Sect. 5. Recall that $z(w)$ satisfies the properties (i)–(iii) in Sect. 5, and it is anti-holomorphic in the variable w_1. In the case when $M=b\mathbf{B}^n$ we take $z(w)=w/|w|^2$. For $\zeta\in\mathbf{C}\backslash\{0\}$ we have

$$z(\zeta w)=\zeta w/\zeta\bar{\zeta}|w|^2=z(w)/\bar{\zeta},$$

so in this case $z(w)$ is anti-holomorphic on each complex line through 0 in $\mathbf{C}^n$.

Let $a_\beta(z,w)$ be as in (7.2) above. We define a subset $X\subset(U_1\backslash D)\times\mathbf{C}^N$ by the equations

$$\langle a_\beta(z(w),\bar{w}),\bar{w}'\rangle=\begin{cases}1, & \text{for } \beta=0;\\ 0, & \text{for } 1\leqq|\beta|\leqq s.\end{cases} \tag{7.5}$$

Here, $w\in U_1\backslash D$ and $w'\in\mathbf{C}^N$.

Note that $a_0(z(w), \bar{w}) = f(z(w)) \neq 0$ provided that U is sufficiently small. Also, the vectors $\{a_\beta : 1 \leqq |\beta| \leqq s\}$ have rank $k(z(w), \bar{w}) \leqq k_0 \leqq N-1$ according to Lemma 7.3. Hence, for each fixed $w \in U_1 \backslash D$, the fiber

$$X_w = \{w' \in \mathbf{C}^N : (w, w') \in X\}$$

is a nonempty affine complex subspace of $\mathbf{C}^N$ of complex dimension

$$\dim X_w = N - k(z(w), \bar{w}) - 1 \geqq N - k_0 - 1 .$$

The intersection $X \cap ((U_1 \backslash \bar{D}) \times \mathbf{C}^N)$ is a real-analytic subset of $(U_1 \backslash \bar{D}) \times \mathbf{C}^N$.

If $w \in M \cap U_1$, then $z(w) = w$. Thus Lemma 7.2 implies

$$f(w) \in X_w \quad \text{and} \quad X_w \subset f(w) + f(w)^\perp \tag{7.6}$$

We now restrict our attention to a point $z^0 \in M$ such that $(z^0, \bar{z}^0) \in T_0$, so that $k \equiv k_0$ on a neighborhood of $(z^0, \bar{z}^0)$ in $\tilde{M}$. Such points form an open dense subset $M_0 \subset M$. We shall prove that f extends holomorphically to a neighborhood of z^0. This will prove Theorem 1.3. We may assume that $z^0 = 0$. In the case when $M = b\mathbf{B}^n$ we shall assume that $z^0 = (1, 0, \dots, 0)$.

Choose a neighborhood V_0 of 0, $V_0 \subset U_1$, such that $k(z(w), \bar{w}) = k_0$ for all $w \in V_0 \backslash D$. Because of (7.4) the system of linear equations (7.5) in w' has rank $k_0 + 1$ for all points $w \in V \backslash D$ in a smaller neighborhood $V \subset V_0$ of 0.

We shall distinguish two cases.

Case 1. $k_0 + 1 = N$.

In this case the system (7.5) of linear equations in the variables w' has maximal rank N for all $w \in V \backslash D$. Solving the system for w' by Cramer's formula we obtain a continuous mapping $w' = \tilde{f}(w) : V \backslash D \to \mathbf{C}^N$, so that

$$X = \{(w, \tilde{f}(w)) : w \in V \backslash D\} .$$

Because of (7.6) we have $f(w) = \tilde{f}(w)$ for all $w \in V \cap M$, so the mapping $F : V \to \mathbf{C}^N$ defined by

$$F(w) = \begin{cases} f(w), & \text{if } w \in V \cap \bar{D}; \\ \tilde{f}(w), & \text{if } w \in V \backslash D \end{cases} \tag{7.7}$$

is continuous in V and holomorphic in $V \cap D$.

We claim that the restriction of F to each complex line $L = \{w_2 = c_2, \dots, w_n = c_n\}$ for sufficiently small $c_2, \dots, c_n \in \mathbf{C}$ is holomorphic in $L \cap V$. A theorem of Hartogs [9, p. 139] then implies that F is holomorphic in all variables in a neighborhood of 0.

To prove the claim, recall that the mapping $z(w)$ is anti-holomorphic in w_1, so the functions $a_\beta(z(w), \bar{w})$ are anti-holomorphic in $L \backslash \bar{D}$. Hence the solution $w' = \tilde{f}(w)$ of (7.5) is holomorphic in $L \backslash \bar{D}$. Thus the restriction $F|_L$ is continuous on $L \cap V$ and holomorphic in $L \cap V \backslash M$. Morera's theorem implies that F is holomorphic in $L \cap V$ which establishes the claim.

This completes the proof of Theorem 1.3 in Case 1. The proof is similar to the original reflection principle in $\mathbf{C}^n$ discovered by Lewy [34] and Pinčuk [38].

We now look at the case $M = b\mathbf{B}^n$ and $z(w) = w/|w|^2$, assuming still that $k_0 + 1 = N$. Using Morera's theorem on the family of complex lines passing through the origin in $\mathbf{C}^n$ we see as above that the extended mapping F (7.7) is holomorphic. The solution $\tilde{f}(w)$ of the system (7.5) obtained by Cramer's formula is of the form

$$\tilde{f}(w)=g(\bar{w}/|w|^2, w)/h(\bar{w}/|w|^2, w),$$

where $g=(g_1, \dots, g_N)$ and h are sums of products of functions $a_\beta(\bar{w}/|w|^2, w)$. Hence g and h depend holomorphically on the first variable $\bar{w}/|w|^2$ and are polynomials of degree at most $K=Ns=N(N-n+1)$ in the second variable w.

Cima and Suffridge proved in [11] that holomorphic functions of this type are rational:

$$\tilde{f}(w)=(p_1(w), \dots, p_N(w))/q(w),$$

where the p_j's and q are holomorphic polynomials of degree at most K. Since $\tilde{f}$ is an analytic continuation of f, it follows that $f=p/q$ is a rational mapping.

Case 2. $k_0+1<N$.

Now the system (7.5) has rank $k_0+1<N$. Let $m=N-(k_0+1)$. Renumerating the variables w' we may assume that this systm has rank k_0+1 in $w'_{m+1}, \dots, w'_N$, so the solution is of the form

$$w'_{m+k}=b_{k,0}(w)+\sum_{j=1}^{m} b_{k,j}(w)w'_j, \qquad w\in V\backslash D,\ 1\leqq k\leqq N-m, \tag{7.8}$$

provided that we shrink V if necessary. These equations represent X as a graph over $(V\backslash D)\times\mathbf{C}^m$. The functions $b_{k,j}$ are continuous in $V\backslash D$.

7.4 **Proposition.** *The functions $b_{k,j}$ in (7.8) are holomorphic in $V\backslash\bar{D}$, so $X\cap(U_1\backslash\bar{D})$ is a complex subvariety of dimension $n+m$.*

Proof. This is not obvious since the functions $a_\beta(z(w), \bar{w})$ contain both holomorphic and anti-holomorphic terms. All we know at this point is that the $b_{k,j}$'s are holomorphic in w_1, for the reason explained in the proof of Case 1.

The proof depends on the following two lemmas.

7.5. **Lemma.** *Let $h: U\to\mathbf{C}^N$ be a holomorphic map defined on an open connected subset U of $\mathbf{C}^n$. For each $x\in U$ and $j\in\mathbf{Z}_+$ we set*

$$O_x^j=\operatorname{span}\{\partial^{|\alpha|}h/\partial x^\alpha(x): 1\leqq|\alpha|\leqq j\}.$$

Assume that for some $j\in\mathbf{Z}_+$ we have $O_x^j=O_x^{j+1}$ for all $x\in U$. Then there is a point $x_0\in U$ such that the image of h is contained in the affine subspace $h(x_0)+O_{x_0}^j$ of $\mathbf{C}^N$.

Proof. Choose a point $x_0\in U$ such that $\dim O_x^j$ is constant for all x in an open neighborhood U_0 of x_0. This is possible since the function $x\to\dim O_x^j$ is lower semi-continuous. Shrinking U_0 if necessary we can find a collection of vectors of the form

$$X_r(x)=\partial^{\alpha_r}h/\partial x^{\alpha_r}(x), \qquad 1\leqq|\alpha_r|\leqq j,\ 1\leqq r\leqq\dim O_x^j$$

that form a basis of O_x^j for all $x\in U_0$. Let $X(x)=\partial^\alpha h/\partial x^\alpha(x)$ for some multiindex α with $|\alpha|=j+1$. Our hypothesis $O_x^j=O_x^{j+1}$ implies that $X(x)=\sum\alpha_r(x)X_r(x)$ for some analytic functions a_r. Differentiating with respect to the coordinate x_i we have

$$\partial X/\partial x_i(x)=\sum_r \partial a_r/\partial x_i(x)X_r(x)+\sum_r a_r(x)\partial X_r/\partial x_i(x).$$

Since $O_x^j=O_x^{j+1}$, each term on the right lies in O_x^j whence $\partial X/\partial x_i(x)\in O_x^j$. Since α and i were arbitrary, it follows that $O_x^{j+2}=O_x^j$. By induction we have $O_x^k=O_x^j$ for each $k\geqq j$. The lemma now follows from the Taylor expansion of h at x_0.

7.6. **Lemma.** *For each $w\in V\backslash\bar{D}$ we denote by $\tilde{Q}_w$ the connected component of $Q_w\cup U$ containing $z(w)$. Then $f(\tilde{Q}_w)$ is contained in the affine complex subspace*

$$A_w=f(z(w))+\operatorname{span}\{a_\beta(z(w),\bar{w}):1\leqq|\beta|\leqq s\}\subset\mathbf{C}^N \tag{7.9}$$

of dimension k_0.

Proof. Note that $f(\tilde{Q}_w)$ can not be contained in any proper subspace of A_w.

Choose a connected open set $\Omega\subset\mathbf{C}^{n-1}$ with coordinates x and a holomorphic mapping $\xi:\Omega\to Q_w$ that parametrizes Q_w in a neighborhood of $z(w)$. Set $h(x)=f(\xi(x))$. Denote by O_x^j the subspace of $\mathbf{C}^N$ defined in the preceeding lemma. In view of (7.2) we also have

$$O_x^j=\operatorname{span}\{a_\beta(\xi(x),\bar{w}):1\leqq|\beta|\leqq j\},$$

so $\dim O_x^s=k_0$. Since f was assumed to be finite holomorphic on D, h has rank $n-1$ at a generic point of Ω, so we have $\dim O_x^1=n-1$ for all x in an open dense subset Ω_0 of Ω.

Consider the increasing flag $O_x^1\subset O_x^2\subset\ldots O_x^s$ of $s=N-n+1$ subspaces of C^N. We have

$$\dim O_x^s-\dim O_x^1=k_0-(n-1)<(N-1)-(n-1)=s-1.$$

Hence there is an integer $j=j(x)$ such that $O_x^j=O_x^{j+1}$. Since $x\to\dim O_x^j$ is a lower semicontinuous function of x, we can shrink Ω_0 and assume that $j(x)$ is constant in Ω_0.

Lemma 7.5 implies that $h(\Omega_0)$ and consequently $h(\Omega)$ is contained in the affine subspace $h(x)+O_x^j$ of $\mathbf{C}^N$ for some $x\in\Omega_0$. Fix any such x, and set $A=h(x)+O_x^j$. The definition of h implies that $f(\tilde{Q}_w)\subset A$, so $A_w\subset A$. Since $\dim A_w=k_0$ and $\dim A\leqq k_0$, we have $A=A_w$, and $f(\tilde{Q}_w)\subset A_w$. This proves Lemma 7.6.

We continue with the proof of Proposition 7.4. Fix a point $w^1\in V\backslash\bar{D}$. We can find a neighborhood V_1 of w^1, $V_1\subset V\backslash\bar{D}$, and an anti-holomorphic map $\xi:V_1\to D\cap U$ satisfying

$$\xi(w)\in\tilde{Q}_w \text{ for } w\in V_1, \qquad \xi(w^1)=z(w^1).$$

(See the proof of part (a) of Proposition 5.1.)

We define a subset $X'\subset V_1\times\mathbf{C}^N$ by the Eqs. (7.5), except that we replace $z(w)$ by $\xi(w)$. With this replacement the Eqs. (7.5) become anti-holomorphic in $w\in V_1$ and $w'\in\mathbf{C}^N$, so X' is a complex subvariety (in fact, a complex submanifold) of $V_1\times\mathbf{C}^N$. The fibers of X' are affine subspaces of dimension $m=N-(k_0+1)$.

From Lemma 7.6 it follows that $X\cap(V_1\times\mathbf{C}^N)\subset X'$. Since both sets have affine fibers of dimension m, they are equal. Therefore $X\cap(V_1\times\mathbf{C}^N)$ is complex analytic.

Since w^1 was an arbitrary point of $V\backslash\bar{D}$, we proved that $X\cap(V\backslash\bar{D}\times\mathbf{C}^N)$ is a complex variety. Thus the functions $b_{k,j}$ in (7.8) are holomorphic in $V\backslash\bar{D}$. This proves Proposition 7.4.

By the theorem of Hartogs the functions $b_{k,j}$ extend across M to holomorphic functions in a neighborhood of $0\in\mathbf{C}^n$ which we still denote by V. Using the Eqs. (7.8) we extend X to a complex submanifold of $V\times\mathbf{C}^N$ of dimension $n+m$.

The last step in the proof is as in Sect. 6. From (7.6) we conclude $X_w\cap b\mathbf{B}^N=\{f(w)\}$, so

$$X\cap((M\cap V)\times b\mathbf{B}^N)=\{(w,f(w)):w\in M\cap V\}.$$

On the left hand side we have a real-analytic subset of $V\times\mathbf{C}^N$. Hence $f|_M$ is real-analytic in an open dense subset $M_0\subset M\cap V$, and it extends holomorphically to a neighborhood of M_0. This completes the proof of Theorem 1.3.

It remains to prove Theorem 1.4 in Case 2. The same argument as given in Case 1 above proves that the holomorphic functions $b_{k,j}(w)$ in (7.8) are rational,

$$b_{k,j}(w)=P_{k,j}(w)/Q_j(w),$$

where $P_{k,j}(w)$ and $Q_j(w)$ are holomorphic polynomials of degree at most $(k_0+1)(N-n+1)$. Thus the set X extends to a rational subvariety of $\mathbf{C}^{n+N}$ defined by the equations

$$Q_j(w)w'_{m+j}=P_{j,0}(w)+\sum_{r=1}^{m}P_{j,r}(w)w'_r,\qquad 1\leqq j\leqq N-m. \tag{7.9}$$

Recall from (7.6) that for each $w\in b\mathbf{B}^n\cap V$ we have $X_w\cap b\mathbf{B}^N=\{f(w)\}$. This implies that the restriction of the function $|w'|^2=\sum w'_j\bar{w}'_j$ to X_w has precisely one critical point $w'=f(w)$.

If we use the variables $w'_1,\ldots,w'_m$ as the coordinates on X_w and express the remaining variables from (7.9), this restriction equals

$$\sum_{j=1}^{m}w'_j\bar{w}'_j+\sum_{j=1}^{N-m}\left(P_{j,0}+\sum_{r=1}^{m}P_{j,r}w'_r\right)\left(\bar{P}_{j,0}+\sum_{r=1}^{m}\bar{P}_{j,r}\bar{w}'_r\right)\Big/q_j\bar{Q}_j.$$

The point $w'_j=f_j(w)$, $1\leqq j\leqq m$, is the unique solution of the system of equations that we obtain by differentiating the above expression with respect to the variables $\bar{w}'_l$, $1\leqq l\leqq m$, and setting the derivative equal to zero:

$$w'_l+\sum_{j=1}^{N-m}\bar{P}_{j,l}\left(P_{j,0}+\sum_{r=1}^{m}P_{j,r}w'_r\right)\Big/Q_j\bar{Q}_j=0,\qquad 1\leqq l\leqq m. \tag{7.10}$$

Cramer's formula shows that the solution of the above system is of the form

$$f_j(w)=w'_j=a_j(w,\bar{w})/b(w,\bar{w}),\qquad 1\leqq j\leqq m,\ w\in V\cap b\mathbf{B}^n, \tag{7.11}$$

where V is a sufficiently small neighborhood of $(1,0,\ldots,0)$, and the functions a_j and b are polynomials in the variables w and $\bar{w}$.

We claim that, as a consequence, f_j is a rational function of the variable w for $1\leqq j\leqq m$. This follows from Proposition 7.7 below. Inserting the functions $f_1,\ldots,f_m$ into (7.9) we see that $f_{m+1},\ldots,f_N$ are rational as well, so f is a rational mapping.

7.7. **Proposition.** *Let $U \subset \mathbf{C}^n$ be an open ball centered at a point in $b\mathbf{B}^n$. If $P(z, \bar{z})$ and $Q(z, \bar{z})$ are polynomials in z and $\bar{z}$ such that $Q \neq 0$ on U and P/Q satisfies the tangential Cauchy-Riemann equations on $b\mathbf{B}^n \cap U$, then there exists a rational function $g(z) = p(z)/q(z)$ that agrees with P/Q on $b\mathbf{B}^n \cap U$.*

Proof. We many replace the sphere $b\mathbf{B}^n$ by the Heisenberg group

$$S = \left\{ \operatorname{Im} z_n = \sum_{j=1}^{n-1} z_j \bar{z}_j \right\} \subset \mathbf{C}^n$$

that is rationally equivalent to the punctured sphere by the Cayley map [42, p. 31].

In P and Q we substitute

$$\bar{z}_n = z_n - 2i \sum_{j=1}^{n-1} z_j \bar{z}_j. \tag{7.12}$$

The resulting polynomials P_1 and Q_1 do not contain $\bar{z}_n$, so they are holomorphic in the variable z_n. Notice that P_1 resp. Q_1 agree with P resp. Q on $S \cap U$ since the above is an identity on S.

Shrinking U if necessary we may assume that Q_1 has no zeros on U. We claim that the function $g = P_1/Q_1$ is holomorphic on U. To prove this we fix a point $z^0 \in S \cap U$ and let $h(z)$ be a holomorphic function on a neighborhood of z^0 that agrees with P/Q on S. Such a function exists since P/Q is real-analytic and satisfies the tangential $\bar{\partial}$-equations on $S \cap U$. The restrictions of g and h to complex lines $L = \{z_1 = a_1, \ldots, z_{n-1} = a_{n-1}\}$ are holomorphic in z_n, and they agree on $L \cap S$. It follows that $g = h$ near z^0, so g is holomorphic there. Since g is real-analytic, it is consequently holomorphic on all of U.

It remains to show that g is rational in z. This follows from the following elementary lemma whose proof will be ommited.

7.8. **Lemma.** *Suppose that a rational function*

$$f(x) = \left(\sum_{j=0}^{k} a_j x^j \right) \Bigg/ \left(\sum_{j=0}^{m} b_j x^j \right), \qquad a_k \neq 0,\ b_m \neq 0,$$

satisfies $\partial f/\partial x \equiv 0$ for x in an open connected subset of C. Then $k = m$ and $f \equiv a_k/b_k$.

To conclude the proof of Proposition 7.7 we write

$$P_1/Q_1 = \left(\sum_{j=0}^{k} \bar{z}_1^j P_{1,j} \right) \Bigg/ \left(\sum_{j=0}^{m} \bar{z}_1^j Q_{1,j} \right),$$

where $P_{1,j}$ and $Q_{1,j}$ are polynomials not involving $\bar{z}_1$. Since

$$(\partial/\partial \bar{z}_1)(P_1/Q_1) = 0,$$

the lemma with $x = \bar{z}_1$ implies $k = m$ and $P_1/Q_1 = P_{1,k}/Q_{1,k}$ on a suitably smaller set. We repeat the same argument with the variables $\bar{z}_2, \ldots, \bar{z}_{n-1}$. After $n-1$ steps we obtain holomorphic polynomials p and q such that $g = p/q$ almost everywhere on U. We may assume that p and q contain no common factors. Since g is holomorphic on U, q cannot have any zeros on U, so we have $P/Q = p/q$ on $U \cap S$. This proves Proposition 7.7.

We have proved that f extends to a rational mapping on $\mathbf{C}^n$. The fact that f is holomorphic on $\mathbf{B}^n$ and has no poles on $b\mathbf{B}^n$ was proved in [38] and [28].

It remains to estimate the degree of f. In Case 1 ($k_0+1=N$) we have already seen that the degree is bounded by $N(N-n+1)$. In Case 2, the degree of polynomials $P_{k,j}$ and Q_j appearing in the coefficients $b_{k,j}$ (7.8) is bounded by $C=(N-m)(N-n+1)$. The solution (7.11) of the system (7.10) contains at most $2mC$ holomorphic terms in each numerator and denominator. Finally, when we substitute (7.12) into the (7.11), the number of holomorphic terms is at most doubled to

$$4mC=4m(N-m)(N-n+1)\leqq N^2(N-n+1).$$

This completes the proof of Theorem 1.4.

8. Examples

In this section we shall illustrate our method by calculating the variety X defined by (7.5) for certain proper polynomial maps of balls.

Let $f:\mathbf{C}^n\to\mathbf{C}^N$ be a polynomial mapping that takes $\mathbf{B}^n$ properly into $\mathbf{B}^N$. Recall that for each $w\in\mathbf{C}^n\backslash\{0\}$ we have

$$Q_w=\left\{z\in\mathbf{C}^n:\sum_{j=1}^{n}\bar{z}_j w_j=1\right\}$$

and

$$X_w=\left\{w'\in\mathbf{C}^N:\sum_{j=1}^{N}\overline{f_j(z)}\,w'_j=1 \text{ for all } z\in Q_w\right\}.$$

On the set $w_1\neq 0$ we express

$$\bar{z}_1=(1-\bar{z}_2w_2-\ldots-\bar{z}_nw_n)/w_1 \tag{8.1}$$

and substitute into the equation of X_w. The variables $z_2,\ldots,z_n$ are now free, so we can equate the coefficients of the terms $\bar{z}_2^{\alpha_2}\ldots\bar{z}_n^{\alpha_n}$. This leads to the system of defining equations of the form (2.4) for X_w.

Example 1. $f:\mathbf{C}^n\to\mathbf{C}^N$, $f(z)=(z,0)$. One finds easily that

$$X_w=\{w'\in\mathbf{C}^N: w_j=w'_j,\ 1\leqq j\leqq n\},\qquad w\in\mathbf{C}^n\backslash\{0\},$$

so $\dim X_w=N-n$.

Example 2. Faran [24] and Cima and Suffridge [12] listed all proper maps from $\mathbf{B}^2$ to $\mathbf{B}^3$ that are of class $\mathscr{C}^2$ on $\bar{\mathbf{B}}^2$. Up to the equivalence with respect to the automorphism groups of $\mathbf{B}^2$ and $\mathbf{B}^3$ there are four maps. One of them is

$$f(z_1,z_2)=(z_1^2,\sqrt{2}z_1z_2,z_2^2).$$

Fix $(w_1,w_2)\in\mathbf{C}^2$, $w_1\neq 0$, and express z_1 as in (5.1). The equation for X_w is

$$\left(\frac{1-\bar{z}_2w_2}{w_1}\right)^2 w'_1+\sqrt{2}\,\frac{1-\bar{z}_2w_2}{w_1}\,\bar{z}_2w'_2+\bar{z}_2^2w'_3=1.$$

Equating the coefficients of 1, $\bar{z}_2$, $\bar{z}_2^2$, we obtain three equations

$$\frac{1}{w_1^2} w_1' = 1$$

$$-\frac{2w_2}{w_1^2} w_1' + \frac{\sqrt{2}}{w_1} w_2' = 0$$

$$\frac{w_2^2}{w_1^2} w_1' - \frac{\sqrt{2} w_2}{w_1} w_2' + w_3' = 0 .$$

Their solution is unique:

$$w_1' = w_1^2 , \qquad w_2' = \sqrt{2} w_1 w_2 , \qquad w_3' = w_2^2 .$$

We obtain the same result when $w_2 \neq 0$. Hence X coincides with the graph of f in this case. The same holds for the other two nontrivial maps from $\mathbf{B}^2$ to $\mathbf{B}^3$.

Example 3. D'Angelo classified the proper monomial maps from $\mathbf{B}^2$ to $\mathbf{B}^4$ [16, 17]. The varieties corresponding to these maps have rather diverse behavior. Perhaps the most interesting example is the one parameter family of maps

$$f_\theta(z_1, z_2) = (z_1, \cos\theta \cdot z_2, \sin\theta \cdot z_1 z_2, \sin\theta \cdot z_2^2) , \qquad \theta \in \mathbf{R} .$$

Assume that θ is not a multiple of $\pi/2$. Then the variety X decomposes in two components X^1, X^2 of dimension three. X^1 projects onto $\mathbf{C}^2 \backslash \{0\}$ and has one dimensional fibers

$$X_w^1 = \{ f_\theta(w) + \lambda(0, -\sin\theta, \cos\theta \cdot w_1, \cos\theta \cdot w_2) , \lambda \in \mathbf{C} \} .$$

X^2 projects onto $\{w_1 = 0,\ w_2 \neq 0\}$ and has two dimensional fibers

$$X_{(0, w_2)}^2 = \{ f_\theta(0, w_2) + \lambda(0, -\sin\theta, \cos\theta \cdot w_1, \cos\theta \cdot w_2)$$
$$+ \mu(-\sin\theta, 0, w_2, 0),\ \lambda, \mu \in \mathbf{C} \} .$$

Example 4. Let $f: \mathbf{C}^n \to \mathbf{C}^{n(n+3)/2}$ be the map

$$f(z) = \frac{1}{\sqrt{2}} (z_1, \ldots, z_n, z_1^2, \ldots, z_n^2, \sqrt{2} z_1 z_2, \sqrt{2} z_1 z_3, \ldots, \sqrt{2} z_{n-1} z_n)$$

whose components are all linear and quadratic terms. (Deleting the linear terms we obtain the Veronese map.) f maps the unit n-ball properly into the unit $n(n+3)/2$-ball.

Consider the equation for X. After we express z_1 by (5.1) and substitute into the equation for X_w, we obtain $1 + (n+2)(n-1)/2$ linear equations for w', one for each constant, linear, and quadratic term in the variables z_j, $2 \leqq j \leqq n$. Since we have $(n+3)/2$ variables w_j', we conclude that

$$\dim X_w \geqq \frac{(n+3)n}{2} - \frac{(n+2)(n-1)}{2} - 1 = n , \qquad w \in \mathbf{C}^n \backslash \{0\} .$$

Thus we have a map whose image is not contained in any proper affine subspace of $\mathbf{C}^N$, and yet the dimension of X is at least twice the dimension of the graph of f.

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