Mappings of Strongly Pseudoconvex Cauchy-Riemann Manifolds

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0. Introduction. In this paper we extend the theorem due to C. Fefferman [12] on boundary regularity of biholomorphic mappings $f : \mathscr{D} \to \mathscr{D}'$ between smoothly bounded, strongly pseudoconvex domains $\mathscr{D}, \mathscr{D}' \subset \mathbb{C}^n$ to certain generic smooth Cauchy–Riemann manifolds in $\mathbb{C}^n$ with nondegenerate Levi form.

The local version of Fefferman's theorem can be stated as follows: Let M and M' be smooth strongly pseudoconvex hypersurfaces in $\mathbb{C}^n$ ($n > 1$) and $f : M \to M'$ a homeomorphic mapping so that both f and f^{-1} satisfy the tangential Cauchy–Riemann equations (in the weak sense). Then f is necessarily a *smooth diffeomorphism*. The point is that, under these conditions, f extends to a biholomorphic mapping from a domain $\mathscr{D} \subset \mathbb{C}^n$ bounded in part by M to a similar domain $\mathscr{D}'$ bounded in part by M', so we are back in the usual setting.

This theorem has been generalized to much wider classes of hypersurfaces and was proved under weaker hypotheses on f. For this development we refer the reader to the survey papers by Bedford [6], Bell [7], and the author [13], and to the references therein.

In this paper we consider another kind of generalization of Fefferman's theorem. Let M and M' be local smooth, generic, Cauchy–Riemann (CR) manifolds in $\mathbb{C}^n$, of real codimension $d > 1$ and of Cauchy–Riemann (CR) dimension $m > 0$ ($m + d = n$). A CR-homeomorphism $f : M \to M'$ is a topological homeomorphism such that both f and f^{-1} are CR mappings, i.e., they satisfy the tangential Cauchy–Riemann equations in the weak sense. Our main result is that, under certain geometric assumptions on M and M', every such mapping is a smooth diffeomorphism.

Our hypotheses are of two kinds. First, we require that M and M' are

strongly pseudoconvex (Definition 1, §1). This condition is a natural generalization of the strong pseudoconvexity of hypersurfaces. The second condition, we call it *over-extendability*, concerns the holomorphic extendability of CR functions on M respectively M' to wedges. We require that, at the chosen point $p \in M$, every CR function h defined on M near p extends near p to a wedge $\mathscr{W} = \mathscr{W}(\Gamma)$ with edge M so that the cone Γ determining the wedge is strictly larger than the Levi cone of M at p (Definition 2).

The Main Theorem (§1) states that whenever M and M' are smooth, strongly pseudoconvex and over-extendable at $p \in M$ respectively $p' \in M'$, then every local CR homeomorphism $f : M \to M'$ with $f(p) = p'$ is a smooth CR diffeomorphism near p.

Mappings of this kind arise in the following situation. Suppose $\mathscr{D} \subset \mathbb{C}^n$ is a domain containing a smooth generic CR manifold M in its boundary $\partial \mathscr{D}$, and such that $\mathscr{D}$ is wedge-like near M (i.e., it contains a wedge with edge M). If $M' \subset \partial \mathscr{D}'$ satisfies a similar condition, and if $f : \mathscr{D} \cup M \to \mathscr{D}' \cup M'$ is a homeomorphic map that is holomorphic on $\mathscr{D}$, with $f(M) = M'$, then $f : M \to M'$ is a CR homeomorphism.

This formulation also makes sense when M and M' have CR dimension zero, i.e., they are maximal totally real submanifolds of $\mathbb{C}^n$. The smoothness of f on M then follows by reflection on M and M' and applying the (smooth version of) edge of the wedge theorem, see Pinchuk and Hasanov [23].

It seems that the intermediate case when M, M' are not hypersurfaces but have positive CR dimension has not been treated, except in the papers [28, 29] by Webster in which he assumed from the outset that the map f is of class $\mathscr{C}^1$ on M. However, as is well-known from the hypersurface case, the hard problem is exactly to obtain some initial regularity of f.

The interesting point is that there is a deep connection between the mapping problem for strongly pseudoconvex CR manifolds of positive CR dimension and the mapping problem for wedges with totally real edges. This has been discovered (in the hypersurfaces case) by Lewy [21] and Pinchuk [22] and, in a more explicit form, by Webster [27]. Other important ingredients are certain estimates of the derivative of f, and these require most of the work. Among other things we use the generalized theorem of Julia-Carathéodory for f on certain families of osculating balls. In the hypersurface case this approach has been explained in the recent paper [15] by the author. The present proof uses similar ideas, but is technically more involved.

In §2 we use results on microlocal hypoanaliticity due to Baouendi, Chang, Rothschild, and Treves [1–4] in order to obtain some sufficient conditions for over-extentability. In §§3–5 we do the preparatory work concerning wedges and mappings between them. Among other things, we prove the Hopf lemma on wedges (Corollary 3.4), obtain information on the local polynomial hull of M (Proposition 4.2), and prove the boundary distance preserving property of f (Proposition 5.2). In §6 we prove the Main Theorem.

1. The main theorem. In the space $\mathbb{C}^n$ we shall use the coordinates (z, w), $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$, $w = u + iv = (w_1, \ldots, w_d) \in \mathbb{C}^d$. Let $M \subset \mathbb{C}^n$ be a smooth manifold of real codimension d defined near the origin by a set of d real equations

$$(1) \qquad v_j = \varphi_j(z, \bar{z}, u), \qquad 1 \le j \le d,$$

where $\varphi = (\varphi_1, \ldots, \varphi_d)$ is a smooth map with $\varphi(0) = 0$ and $d\varphi(0) = 0$. We shall use the vector notation

$$v = \varphi(z, \bar{z}, u).$$

For each $p \in M$, the maximal complex tangent space $T_p^{\mathbb{C}} M = T_p M \cap i T_p M$ has complex dimension m, so M is a generic *Cauchy-Riemann* (CR) *manifold of CR dimension* m. Conversely, every such manifold is locally of the form (1).

Recall that a $\mathscr{C}^1$ function f on M is called CR if $df(p)$ is $\mathbb{C}$-linear on the maximal complex tangent space $T_p^{\mathbb{C}} M$ for each $p \in M$. Equivalently, $\int_M f \bar{\partial} \alpha = 0$ for all smooth forms α of type $(n, m-1)$ on $\mathbb{C}^n$ such that $\operatorname{supp} \alpha \cap M$ is compact; this is used as the definition of CR when f is merely continuous on M.

We shall now define the *Levi form* of M at 0. According to [8], Proposition 3.1, we can find local holomorphic coordinates near the origin in $\mathbb{C}^n$ such that M is given by

$$(2) \qquad v = Q(z, \bar{z}) + R(z, \bar{z}, u),$$

where $Q = (Q_1, \ldots, Q_d)$ is a hermitian quadratic form on $\mathbb{C}^m$ with values in $\mathbb{R}^d$ and $|R(z, \bar{z}, u)| = o(|z|^2 + |u|^2)$. The form Q is uniquely determined by M up to transformations of the form

$$(3) \qquad Q'(Az, \overline{Az}) = B \cdot Q(z, \bar{z}), \qquad z \in \mathbb{C}^m,$$

where $A \in \operatorname{GL}(m, \mathbb{C})$ and $B \in \operatorname{GL}(d, \mathbb{R})$. We call Q the *Levi form* of M at 0. For an intrinsic definition in terms of commutators of complex tangential vector fields see [8].

We associate to Q the *Levi cone* $C(Q)$ and its *dual cone* $C^*(Q)$ (also called the *polar*) by

$$C(Q) = \mathscr{C}o\{Q(z, \bar{z}) : z \in \mathbb{C}^m \setminus \{0\}\} \subset \mathbb{R}^d,$$

$$(4) \qquad C^*(Q) = \{\sigma \in \mathbb{R}^d : \sigma \cdot x \ge 0 \text{ for all } x \in C(Q)\}.$$

Here, $\sigma \cdot x = \sum_{j=1}^{d} \sigma_j x_j$ is the usual real inner product, and $\mathscr{C}o$ denotes the (linearly) convex hull. Clearly $C(Q) \cup \{0\}$ and $C^*(Q)$ are closed convex cones in $\mathbb{R}^d$. When $C(Q)$ is all of $\mathbb{R}^d$, $C^*(Q)$ is the trivial cone $\{0\}$. However, when $C(Q) \neq \mathbb{R}^d$, the Hahn–Banach theorem implies that $C^*(Q)$ is nontrivial, but it may still have empty interior. This will happen whenever $C(Q)$ contains a complete straight line through the origin, since $C^*(Q)$ is then contained in the orthogonal complement of that line.

DEFINITION 1. The manifold M given by (2) is said to be *strongly pseudo-convex* at the origin if there exists a vector $\sigma \in \mathbb{R}^d$ such that $\sigma \cdot Q$ is strongly positive definite on $\mathbb{C}^m$, that is,

$$(5) \qquad \sum_{j=1}^{d} \sigma_j Q_j(z, \bar{z}) > 0 \quad \text{for all } z \in \mathbb{C}^m \backslash \{0\}.$$

Clearly this property is preserved by the transformations (3).

It is not hard to see that the strong pseudoconvexity of M at 0 is equivalent to any of the following conditions:

(i) $C(Q)$ is contained in an open half space of $\mathbb{R}^d$ determined by a real hyperplane through the origin.

(ii) $C(Q)$ does not contain the origin.

(iii) Q is nondegenerate in the sense that $Q(z, \cdot) \equiv 0$ for some $z \in \mathbb{C}^m$ implies $z = 0$, and $C(Q)$ contains no complete straight line.

(iv) Q is nondegenerate and $C^*(Q)$ has nonempty interior.

(v) Locally near 0, M is contained in a strongly pseudoconvex hyper-surface (Proposition 4.4).

A propos (iv), we remark that the set of vectors $\sigma \in \mathbb{R}^d$ satisfying (5) is precisely the interior of the dual Levi cone $\operatorname{Int} C^*(Q)$, as follows immediately from the definition of $C^*(Q)$. We leave out the simple proof of these equivalences since we will not need them in the sequel. The same condition has been used by Khenkin and Tumanov in [18] and [26] where they proved that local CR homeomorphisms of strongly pseudoconvex quadrics whose Levi cones have nonempty interior extend to birational mappings on $\mathbb{C}^n$. See also [16] for related results.

We remark that when Q has the property that for each $\sigma \in \mathbb{R}^d \backslash \{0\}$, $\sigma \cdot Q$ has at least one *negative* eigenvalue, then $C(Q) = \mathbb{R}^d$, so every CR function (or distribution) on M extends holomorphically to an open neighborhood of 0 in $\mathbb{C}^n$ [1, 8]. In this case our mapping problem is not interesting, so we do not lose much generality by restricting our attention to the strongly pseudoconvex case.

To every open connected cone $\Gamma \subset \mathbb{R}^d$ with vertex 0 and a neighborhood U of 0 in $\mathbb{C}^n$ we associate the *wedge* $\mathscr{W}(\Gamma, U)$ with edge M by

$$(6) \qquad \mathscr{W}(\Gamma, U) = \{(z, w) \in U : \operatorname{Im} w - \varphi(z, \bar{z}, \operatorname{Re} w) \in \Gamma\}.$$

We say that a continuous CR function f on M extends holomorphically to $\mathscr{W}(\Gamma, U)$ if there is a holomorphic function on $\mathscr{W}(\Gamma, U)$ that is continuous up to $M \cap U$ and matches with f on $M \cap U$.

Let $S \subset \mathbb{R}^d$ be the unit sphere. We say that a cone $\Gamma \subset \mathbb{R}^d$ is *finer* than $\Gamma' \subset \mathbb{R}^d$ if $\Gamma \cap S$ is relatively compact in $\operatorname{Int} \Gamma' \cap S$. We denote this by $\Gamma < \Gamma'$ or $\Gamma' > \Gamma$. A wedge $\mathscr{W} = \mathscr{W}(\Gamma, U)$ is finer than $\mathscr{W}' = \mathscr{W}(\Gamma', U')(\mathscr{W} < \mathscr{W}')$ if $\Gamma < \Gamma'$ and $U \subset\subset U'$.

DEFINITION 2. The manifold M defined by (2) is *over-extendable* at the origin if every CR function h defined in a neighborhood of 0 in M can be extended holomorphically to a wedge $\mathscr{W}(\Gamma, U)$ with $\Gamma > C(Q)$ (= the Levi cone of M at 0).

REMARK. For every cone $\Gamma < C(Q)$, h can be extended holomorphically to $\mathscr{W}(\Gamma, U)$ for a sufficiently small U [1], [8].

When M is the quadric

$$(7) \qquad \operatorname{Im} w = Q(z, \bar{z}),$$

every h can be extended near 0 to a wedge $\mathscr{W}(\Gamma, U)$ with $\Gamma = \operatorname{Int} C(Q)$, and in general to no larger wedge. We are requiring the extendability to wedges of cones which are slightly larger than the Levi cone. We shall give some sufficient conditions for over-extendability in Section 2. Our use of over-extendability will become clear in Proposition 5.1 in Section 5.

DEFINITION 3. Let M and M' be manifolds of the form (2), of the same type (m, d). A *local CR-homeomorphism* $f : M \to M'$ at the origin is a homeomorphism $f : \omega \to \omega'$ of open neighborhood $0 \in \omega \subset M$, $0 \in \omega' \subset M'$, with $f(0) = 0$, such that both f and f^{-1} are CR mappings (i.e., they satisfy the tangential Cauchy–Riemann equations in the weak sense).

MAIN THEOREM. *Let M and M' be manifolds of the form (2) in $\mathbb{C}^n$, of the same type (m, d), and smooth of order $k > 3$. If M and M' are strongly pseudoconvex and over-extendable at the origin, then every local CR homeomorphism $f : M \to M'$ with $f(0) = 0$ is a smooth diffeomorphism of class $\mathscr{C}^{k-1-0}$ near the origin. When M and M' are real-analytic, then f extends to a biholomorphic mapping in a neighborhood of 0 in $\mathbb{C}^n$.*

Here, k need not be integer; if $k = [k] + \alpha$ with $0 < \alpha < 1$, then $\mathscr{C}^k = \mathscr{C}^{[k], \alpha}$ is the usual Hölder class. As usual, $\mathscr{C}^{k-0}$ means $\mathscr{C}^k$ if $k \notin \mathbb{Z}_+$, and $\mathscr{C}^{k-0} = \bigcup_{0 < \alpha < 1} \mathscr{C}^{k-1, \alpha}$ if $k \in \mathbb{Z}_+$.

As in the classical Fefferman's theorem for hypersurfaces, the hard problem is only to show that f is of class $\mathscr{C}^1$ (see Webster [27, 28]). Our theorem contains the hypersurface situation as a very special case, since the condition of over-extendability is then vacuously satisfied.

When M and M' have CR dimension $m = 1$, it suffices to assume that they are Levi nondegenerate (and over-extendable), since the Levi cone is then a ray in $\mathbb{R}^d \backslash \{0\}$ whence they are strongly pseudoconvex. This case

is just the opposite to the hypersurface case when the CR dimension is the maximal possible. Of course we have the whole range of intermediate cases where both $m \geq 2$ and $d \geq 2$.

Since a strongly pseudoconvex quadric (7) is never over-extendable yet the analogous result for CR homeomorphisms $f : M \to M'$ holds whenever the Levi cones of M and M' have nonempty interior according to [18], our condition on over-extendability is certainly not the best possible one. However, as we will see in Section 2, it holds in many cases, especially when the third order part in the Taylor expansion of M at 0 is sufficiently independent of the Levi form Q (Corollary 2.3). Over-extendability even holds in certain cases when the Levi cone $C(Q)$ has empty interior. For instance, if the CR dimension of M equals one, and if M is semirigid at 0 with all the higher Hörmander numbers being odd, then M is over-extendable (Corollary 2.4). For smooth *rigid* CR manifolds: $\mathrm{Im}\, w = \varphi(z, \bar{z})$, we give a rather specific procedure for checking the over-extendability by using the line sector property from [2] and [3]. (See Theorem 2.2.)

Before making any guesses as to what the optimal condition in our problem might be, we consider the following example. Let $M = M_1 \times \mathbb{R}$, where $M_1 \subset \mathbb{C}^{n-1}$ is a strongly pseudoconvex CR manifold (2). Every CR mapping $f : M \to M$ is of the form $f(z, t) = (g_t(z), h(t))$, where $g_t : M_1 \to M_1$ is CR, but the dependence of g_t and h on t is completely arbitrary, in spite of the strong pseudoconvexity of M. This is essentially the only known counterexample to the regularity problem within the class of strongly pseudoconvex CR manifolds.

In this case CR functions on M do not extend to any nontrivial wedge in $\mathbb{C}^n$. The necessary and sufficient condition for extendability to wedges near $0 \in M$ is that M is *minimal* at 0, in the sense that there exists no CR manifold $N \subset M$ passing through 0, of the same CR dimension as M but of smaller real dimension (Tumanov [25], Baouendi and Rothschild [5]). The following conjecture seems plausible:

CONJECTURE. If M and M' are smooth strongly pseudoconvex CR manifolds (2) that are minimal at the origin, then every local CR homeomorphisms $f : M \to M'$, $f(0) = 0$, is a smooth diffeomorphism near 0.

Every manifold M (2) whose Levi cone $C(Q)$ at the origin has nonempty interior is minimal at 0. We expect that the conjecture may be easier to prove in this case, perhaps by a reduction to the hypersurface situation as in Khenkin and Tumanov [18].

Another remark concerning the loss of smoothness in the Main Theorem is appropriate. Just as in Lempert [20] one can obtain a more precise result by introducing a different smoothness class that measures the smoothness of both M and the associated manifold $\widetilde{M}$ (6.1). If $M \in \mathscr{C}^k$, then $\widetilde{M} \in \mathscr{C}^s$ for some s between $k - 1$ and k. If both $\widetilde{M}$ and $\widetilde{M}'$ are of class $\mathscr{C}^s$,

$k - 1 \leq s \leq k$, then $f \in \mathscr{C}^{s-0}$. In the hypersurface case, the loss of smoothness is no more than $1/2 + 0$, and the loss of $1/2$ can actually occur.

2. Sufficient conditions for over-extendability. Let $M \subset \mathbb{C}^{m+d}$ be a generic smooth CR manifold of real codimension d given by (1.2), i.e.,

$$(1) \qquad v = Q(z, \bar{z}) + R(z, \bar{z}, u) = \varphi(z, \bar{z}, u),$$

where $w = u + iv \in \mathbb{C}^d$, the Levi form $Q = (Q_1, \ldots, Q_d)$ is strongly pseudoconvex in the sense of Definition 1 (§1), and R contains only terms of order ≥ 3. Let $C(Q)$ and $C^*(Q)$ be the Levi cone and its dual cone as defined by (1.4).

We will show how the microlocal results of Baouendi, Chang, Rothschild, and Tréves [1–4] can be used to get some sufficient conditions for over-extendability of the manifold (1) at the origin. For this purpose we must recall the notion of the *mini-FBI transformation* and the *hypoanalytic wave front set* from [3].

To every CR function (or distribution) h on M one associates its mini-FBI transformation $F_h(z, w, \sigma)$ as in [3], (6.3). The explicit form of this transformation will not be important for our purposes. Recall that this is an analogue of the Fourier transform, but with an additional factor in the kernel that is essentially the complex Gaussian kernel, whose purpose is to improve the convergence of the transform. It has been invented by Bros and Iagolnitzer and was subsequently used, with certain modifications, by the authors named above and by others in problems concerning the approximation and extension of CR functions. (See the references in [1] and [3].)

One says that a CR function h on M is *hypoanalytic* at a vector $\sigma_0 \in \mathbb{R}^d \backslash \{0\}$ if F_h has the exponential decay

$$(2) \qquad |F_h(z, w, \sigma)| \leq C \cdot e^{-|\sigma|/C},$$

uniformly for (z, w) in a neighborhood of 0 in $\mathbb{C}^n$ and for σ in a conical neighborhood of σ_0 in $\mathbb{C}^d$. The set of all directions $\sigma_0 \in \mathbb{R}^d \backslash \{0\}$ at which h is *not* hypoanalytic is called the *hypoanalytic wave front set* of h at 0, and is denoted by $WF_0(h)$. This is a closed cone in $\mathbb{R}^d \backslash \{0\}$. For related notions of the wave front set see Hörmander [30] and Tréves [31].

The importance of this notion is evident from the following result of Baouendi and Rothschild [3] (see also [1]). Let $\Gamma \subset \mathbb{R}^d$ be a strictly convex closed cone and h a CR distribution on M. The following are equivalent ([3, Theorem 7]):

(a) $WF_0(h) \subset \Gamma$.

(b) For every open cone $\Lambda < \Gamma^*$ (where Γ^* is the polar of Γ) there is an open neighborhood U of 0 in $\mathbb{C}^n$ such that h extends holomorphically to the wedge $\mathscr{W}(\Lambda, U)$ with edge M.

The following is a microlocal characterization of over-extendability (see Definition 2 in Section 1).

2.1 **Theorem.** *Let $M \subset \mathbb{C}^n$ be a strongly pseudoconvex CR manifold* (1). *The following are equivalent:*

(i) *M is over-extendable at the origin.*

(ii) *Every CR function h on M is hypoanalytic at every vector $\sigma \in \partial C^*(Q) \backslash \{0\}$.* (*Here, $C^*(Q)$ is the dual Levi cone, and $\partial C^*(Q)$ is its boundary.*)

Proof. Recall from [2] or [3] that a real-valued homogeneous polynomial $q_k(\zeta, \bar{\zeta})$ ($\zeta \in \mathbb{C}$) of degree $k \geq 2$ satisfies the *sector property* if we can find a $\mu \in \mathbb{C}$ and a sector (cone) $\mathscr{S}$ in the complex plane satisfying

$$(3) \qquad q_k(\zeta, \bar{\zeta}) + \operatorname{Re}(\mu \zeta^k) < 0 \text{ on } \mathscr{S}, \qquad \text{angle } (\mathscr{S}) > \pi/k.$$

Notice that when q_k is harmonic, $q_k = \operatorname{Re}(a\zeta^k)$, it does not have the sector property, since in this case we only have sectors $\mathscr{S}$ as above with angle $(\mathscr{S}) = \pi/k$. On the other hand, if q_k is nonharmonic and of odd degree, it always satisfies the sector property.

If $\sigma \in \mathbb{R}^d$ is any vector not in $C^*(Q)$, then by definition of $C^*(Q)$ we can find a $z^0 \in \mathbb{C}^m$ such that $a = \sigma \cdot Q(z^0, \bar{z}^0) < 0$. Hence for $\zeta \in \mathbb{C}$ we have

$$\sigma \cdot \varphi(\zeta z^0, \overline{\zeta z}^0, 0) = a \cdot \zeta \bar{\zeta} + O(|\zeta|^3),$$

and $a \cdot \zeta \bar{\zeta}$ clearly has the sector property since $a < 0$. Corollary 8.3 in [3] implies that every CR function h on M is hypoanalytic at such a vector σ at $0 \in M$. Thus $WF_0(h) \subset C^*(Q)$, for all CR functions h on M.

Recall that $WF_0(h)$ is a closed cone in $\mathbb{R}^d \backslash \{0\}$, and $C^*(Q)$ is a closed convex cone contained in a closed half-space in $\mathbb{R}^d$. If (ii) holds, then $WF_0(h) \subset \operatorname{Int} C^*(Q)$, so we can find a strongly convex closed cone Γ with $WF_0(h) \subset \Gamma < C^*(Q)$. Then $\Gamma^* > (C^*(Q))^* = C(Q)$, and h extends near 0 to a wedge $\mathscr{W}(\Lambda)$ for some cone Λ satisfying $C(Q) < \Lambda < \Gamma^*$, according to the implication (a)$\Rightarrow$(b). Thus (i) holds.

Clearly we can turn this around: if h over-extends at 0, say to a wedge $\mathscr{W}(\Lambda, U)$ for some open convex cone $\Lambda > C(Q)$, then $WF_0(h)$ must be contained in $\Lambda^* < C^*(Q)$, so (ii) holds. Theorem 2.1 is proved.

In certain cases one can test the hypoanalyticity of h at a given vector $\sigma \in \partial C^*(Q) \backslash \{0\}$ by using the sector property as in [2] or [3]. We shall assume that the smooth CR manifold M (1) is rigid, i.e., it can be represented in the form

$$(4) \qquad \operatorname{Im} w = \varphi(z, \bar{z}) = Q(z, \bar{z}) + R(z, \bar{z})$$

that does not depend on $\operatorname{Re} w$. The power series $R(z, \bar{z})$ has a unique decomposition $R(z, \bar{z}) = R_{(p)}(z, \bar{z}) + R_{(n)}(z, \bar{z})$, where $R_{(p)}$ contains all the pure (pluriharmonic) terms $\operatorname{Re}(a_\alpha z^\alpha)$, and

$$R_{(n)}(z, \bar{z}) = \sum_{|\alpha|, |\beta| \geq 1} a_{\alpha, \beta} z^\alpha \bar{z}^\beta \qquad (\bar{a}_{\beta, \alpha} = a_{\alpha, \beta}).$$

Recall from [3] that a real-valued homogeneous polynomial $q_k(\zeta, \bar{\zeta})$ $(\zeta \in \mathbb{C})$ of degree k is said to have the *extension property* if every CR function defined near the origin on the hypersurface

$$\Sigma = \{(\zeta, \eta) \in \mathbb{C}^2 : \operatorname{Im} \eta = q_k(\zeta, \bar{\zeta})\},$$

extends holomorphically to the side of Σ defined by $\operatorname{Im} \eta < q_k(\zeta, \bar{\zeta})$. If q_k has the sector property (3), then it also has the extension property [2].

Using Theorem III.4 from [2] and the Theorem 2.1 above we get the following sufficient condition for over-extendability on rigid CR manifolds.

2.2 THEOREM. *Let M be a smooth rigid CR manifold (4) that is strongly pseudoconvex at the origin. Suppose that for every vector $\sigma \in \partial C^*(Q) \backslash \{0\}$ we can find $z^0 \in \mathbb{C}^m \backslash \{0\}$ such that*
(i) $\sigma \cdot Q(z^0, \bar{z}^0) = 0$, *and*
(ii) $\sigma \cdot R_{(n)}(\zeta z^0, \overline{\zeta z}^0) = q_k(\zeta, \bar{\zeta}) + O(|\zeta|^{k+1})$,
where q_k is a homogeneous polynomial of degree $k \geq 3$ that has the extension property (or the sector property). Then M is over-extendable at the origin. In particular, if we can choose z^0 so that q_k is of odd degree, then q_k has the sector property.

REMARK. By definition of $C^*(Q)$ we know that for each $\sigma \in \partial C^*(Q) \backslash \{0\}$, $\sigma \cdot Q(z, \bar{z}) \geq 0$ on $\mathbb{C}^m$, and there is at least one direction $z^0 \in \mathbb{C}^m \backslash \{0\}$ such that $\sigma \cdot Q(z^0, \bar{z}^0) = 0$. Thus, what is required is that the lowest order homogeneous part in $\sigma \cdot R_{(n)}(\zeta z^0, \zeta z^0)$ has the extension (or the sector) property.

As a very special case we obtain the following Corollary. Denote by $Q^{(k)}(z, \bar{z})$ the nonpure homogeneous terms of degree k in R, so $Q^{(2)} \equiv Q$ is the Levi form and $R_{(n)} = \sum_{k=3}^{\infty} Q^{(k)}$.

2.3 COROLLARY. *(Same hypotheses as in Theorem 2.2). Suppose that there is an odd number $k \geq 3$ such that $Q^{(s)} \equiv 0$ for $3 \leq s < k$, and the polynomial*

$$\mathbb{C} \ni \zeta \mapsto \sigma \cdot (Q^{(2)}(\zeta z, \overline{\zeta z}) + Q^{(k)}(\zeta z, \overline{\zeta z})),$$

does not vanish identically for any $\sigma \in \partial C^*(Q)$ and $z \in \mathbb{C}^m \backslash \{0\}$. Then M is over-extendable at the origin.

EXAMPLE 1. We take $m = 2$, $d = 2$, $M \subset \mathbb{C}^4$ a rigid strongly pseudoconvex CR manifold (4). By a linear change of coordinates we can normalize its Levi form $Q = (Q_1, Q_2)$ so that one of the following two cases holds:

(A) $Q_1 = |z_1|^2 + |z_2|^2$, $\quad Q_2 = 0$.
(B) $Q_1 = |z_1|^2$, $\quad Q_2 = |z_2|^2$.

In the first case we have:

$$C(Q) = \{(\sigma_1, 0) \in \mathbb{R}^2 : \sigma_1 > 0\},$$
$$C^*(Q) = \{(\sigma_1, \sigma_2) \in \mathbb{R}^2 : \sigma_1 \geq 0\},$$

so $\partial C^*(Q) = \{(0, \sigma_2) : \sigma_2 \in \mathbb{R}\}$.

Thus we must check the hypoanalyticity at the two vector $\sigma^1 = (0, 1)$, $\sigma^2 = (0, -1)$. If the equations of M are

$$\text{Im}\, w_1 = |z_1|^2 + |z_2|^2 + O(|z|^3),$$
$$\text{Im}\, w_2 = Q^{(k)}(z, \bar{z}) + O(|z|^{k+1}) + (\text{pure terms}),$$

then the hypoanalyticity at σ^1 holds when $Q^{(k)}$ restricted to some complex line through 0 in $\mathbb{C}^2$ has the extension property; for σ^2 we must check $-Q^{(k)}$. In particular, if k is odd, then both $\pm Q^{(k)}$ satisfy the sector property (whence the extension property) along any line $\mathbb{C} \cdot z$ for which $Q^{(k)}(z, \bar{z}) \neq 0$, so M is over-extendable.

In case (B) we have

$$C(Q) = \{(\sigma_1, \sigma_2) \in \mathbb{R}^2 : \sigma_1 \geq 0, \sigma_2 \geq 0\} \setminus \{(0, 0)\},$$

so $C^*(Q) = C(Q)$, and we must check hypoanalyticity at the vectors $\sigma^1 = (1, 0)$, $\sigma^2 = (0, 1)$. Suppose the equations of M are

$$\text{Im}\, w_1 = |z_1|^2 + Q^{(k)}(z, \bar{z}) + O(|z|^{k+1}) + (\text{pure terms}),$$
$$\text{Im}\, w_2 = |z_2|^2 + P^{(s)}(z, \bar{z}) + O(|z|^{s+1}) + (\text{pure terms}).$$

The hypoanalyticity at σ^1 holds when $\mathbb{C} \ni z_2 \to Q^{(k)}(0, z_2, 0, \overline{z_2})$ has the extension property (which is true if k is odd). The hypoanalyticity at σ^2 holds when $\mathbb{C} \ni z_1 \to P^{(s)}(z_1, 0, \overline{z_1}, 0)$ has the extension property.

Thus, if both k and s are odd, M is over-extendable at 0.

A similar analysis can be carried out whenever $d = 2$ and m is arbitrary. When $d \geq 3$, the analysis is more difficult since we must check hypoanalyticity at a set of vectors of positive dimension.

EXAMPLE 2. We consider CR manifolds of *CR dimension one* and of arbitrary codimension. We shall assume in addition that the origin M is strongly pseudoconvex, of finite type, and semirigid (see [3]). This means, that in suitable local holomorphic coordinates, we can represent M by

$$(5) \quad \begin{aligned} \text{Im}\, w_1 &= z\bar{z} + O(3), \\ \text{Im}\, w_k &= p_{m_k}(z, \bar{z}) + O(m_k + 1), \qquad k = 2, \ldots, r, \end{aligned}$$

where $w_k \in \mathbb{C}^{l_k}$, p_{m_k} is a homogeneous polynomial in $z \in \mathbb{C}$ with values in $\mathbb{R}^{l_k}$, m_k's are the higher Hörmander numbers of M at 0 of multiplicity l_k (the first number is $m_1 = 2$ with $l_1 = 1$), and the components of p_{m_k} are independent in the sense that for any $\eta \in \mathbb{R}^{l_k} \setminus \{0\}$, $\eta \cdot p_{m_k}$ is not M-harmonic of degree m (see [3, p. 435]). Semirigidity means that the variables $\text{Re}\, w_j$ do not enter the leading order terms p_{m_k}. We have $1 + l_2 + l_3 + \cdots + l_r = d$.

For every such manifold we have:

$$Q(z, \bar{z}) = (z\bar{z}, 0, \dots, 0),$$
$$C(Q) = \{(\sigma_1, 0, \dots, 0) \in \mathbb{R}^d : \sigma_1 > 0\},$$
$$\partial C^*(Q) = \{0\} \times \mathbb{R}^{d-1}.$$

From Theorem 2.1 we see that M is over-extendable at the origin when for each $k = 2, \dots, r$ and each $\eta \in \mathbb{R}^{l_k} \setminus \{0\}$, the polynomial $\eta \cdot p_{m_k}(z, \bar{z})$ satisfies the extension property (or the sector property). In particular, we have:

2.4 COROLLARY. *If $M \subset \mathbb{C}^{1+d}$ is a semirigid CR manifold (5) of CR dimension one, with the first Hörmander number at the origin $m_1 = 2$ and with all the higher Hörmander numbers being odd, then M is over-extendable at the origin.*

Analogous result holds whenever the first Hörmander number $m_1 = 2$ has multiplicity one. (See the case A in Example 1 above.)

EXAMPLE 3. Here is a very simple example of a manifold $M \subset \mathbb{C}^3$ of CR dim $M = 1$ that is strongly pseudoconvex and minimal at the origin, but is *not* over-extendable at 0:

$$\operatorname{Im} w_1 = |z|^2, \qquad \operatorname{Im} w_2 = |z|^4.$$

The Levi cone is $C = \{(\sigma_1, 0) : \sigma_1 > 0\} = \mathbb{R}_+ \times \{0\}$. If $0 \in \omega \subset M$ and $\mathcal{W}(\Gamma, U)$ is any wedge to which all CR functions on ω extend holomorphically, then we must have $\Gamma \subset \mathbb{R}_+ \times \mathbb{R}_+$, so Γ can not contain C.

To get a slightly more general example we replace the second equation by

$$\operatorname{Im} w_2 = A z^2 \bar{z}^2 + \operatorname{Re}(B z^3 \bar{z} + C z^4) + O(|z|^5).$$

To get over-extendability it suffices to check the sector property of $\pm$ the polynomial above. The term involving Cz^4 is irrelevant. Also, by rotation in the z coordinate we may assume $B \in \mathbb{R}$. Setting $z = e^{i\sigma}$, we must consider the longest interval for σ on which the expression $A + B \cdot \operatorname{Re} e^{2i\sigma} = A + B \cdot \cos 2\sigma$ is negative respectively positive. A simple calculation shows that the longest such interval has length $> \pi/4$ if and only if $|A/B| < \sqrt{2}/2$. In this case M is over-extendable. On the other hand, when $|A/B| > 1$, we can see just as before that M is not over-extendable at 0, since $A + B \cdot \cos 2\sigma$ is then always of the same sign.

The sufficient conditions for over-extendability presented above are far from satisfactory. Most of them only hold for smooth rigid manifolds, and they depend in a rather complicated way on higher order terms in the Taylor expansion of the defining function. Our feeling is that this condition is related to the behavior of the Levi cone $C_p(M)$ of M at points $p \in M$ near the origin. Intuitively speaking, if $C_p(M)$ turns rather generically in all directions in $\mathbb{R}^d$ as we pass through points $p \in M$ near 0, we expect to get

over-extendability at the origin. At the moment we do not know how to make this observation precise, but we hope to return to this question in a future publication.

Before concluding this section we note that the over-extendability is equivalent to the following, apparently stronger condition that will be used in Proposition 5.1 below.

2.5 PROPOSITION. *Let M be a strongly pseudoconvex CR manifold* (1), *with the Levi cone $C(Q)$ at the origin. Then M is over-extendable at 0 if and only if for every neighborhood $0 \in \omega \subset M$ of 0 we can find a cone $\Gamma > C(Q)$ and a neighborhood U of 0 in $\mathbb{C}^n$ such that every CR function on ω extends holomorphically to the wedge $\mathscr{W}(\Gamma, U)$.*

PROOF. The proof follows the same lines as the proof of Theorem 7 in [4]. It is an application of the extendability criteria by the mini-FBI transformation and a Baire category argument.

3. Geometry of wedges. In this section we shall first obtain some geometric information about the wedges (1.6) whose edge is an arbitrary CR manifold (1.1). This will enable us to prove a version of the Hopf lemma on wedges (Corollary 3.4) and a distance estimate for holomorphic mappings of wedges (Corollary 3.5).

In wedges (1.6) the origin $0 \in M$ has a special role since the cone Γ determining the wedge lies in the normal space $N_0 M$. The wedge is obtained by parallel translations of Γ along M. If $\zeta \in M$ is different from the origin, the translated cone $\zeta + \Gamma$ lies in a real d-plane that is tilted with respect to the normal space $N_\zeta M$ to M at ζ. Thus, a unitary change of coordinates that brings $T_\zeta M$ into $\mathbb{C}^m \times \mathbb{R}^d$ $(= T_0 M)$ transforms our wedge into a wedge-like domain that is no longer of the form (1.6).

Thus we must also consider the "tilted" wedges with edge M. Let $\Sigma \subset \mathbb{C}^n$ be a real d-dimensional subspace that is transverse to $T_0 M$. For each open connected cone $\Gamma \subset \Sigma$ with vertex 0 and each sufficiently small neighborhood U of 0 in $\mathbb{C}^n$ we define the tilted wedge with edge M by

(1) $$\mathscr{W}_\Sigma(\Gamma, U) = \{\zeta + t \in U : \zeta \in M, t \in \Gamma\}.$$

When $\Sigma = N_0 M$ we shall delete the index Σ and write $\mathscr{W}(\Gamma, U)$ as before. In this case the new definition (1) agrees with the old one (1.6), provided that we make the obvious identification of $N_0 M = \{0\}^m \times i\mathbb{R}^d$ with $\mathbb{R}^d$, which we shall freely do in the sequel.

Let $\Lambda \subset N_0 M = \{0\}^m \times i\mathbb{R}^d$ be the orthogonal projection of the cone Γ onto $N_0 M$. The following lemma shows that it suffices to consider the "straight" wedges (1.6), provided that we have some freedom in choosing the cones.

3.1 LEMMA. *For each pair of cones* Λ_1, $\Lambda_2 \subset N_0 M$ *satisfying* $\Lambda_1 < \Lambda <$ Λ_2 *there is a neighborhood* V *of the origin in* $\mathbb{C}^n$ *such that*

$$(2) \qquad \mathscr{W}(\Lambda_1, V) \subset \mathscr{W}_{\Sigma}(\Gamma, U) \cap V \subset \mathscr{W}(\Lambda_2, V).$$

The size of the largest such V *depends on* U, *on the angle between* Σ *and* $N_0 M$, *on the size of second derivatives of the defining function of* M, *and on the number*

$$d(\Lambda_1, \Lambda_2) = \sup\{d > 0 : \forall t \in \Lambda_1, B(t, d|t|) \subset \Lambda_2\}.$$

Here, $B(t, c)$ denotes the Euclidean ball in $N_0 M = \mathbb{R}^d$ with center t and radius c. (Clearly $d(\Lambda_1, \Lambda_2) > 0$ if and only if $\Lambda_1 < \Lambda_2$.)

PROOF. Fix a point $\zeta = (z, w)$ and a vector $t \in N_0 M$ such that $\zeta + t \in \mathscr{W}_{\Sigma}(\Gamma, U)$. Then there is a point $\zeta' = (z', w') \in M \cap U$ and a vector $t' \in \Gamma$ so that $\zeta + t = \zeta' + t'$. Thus

$$t' = (z - z', \operatorname{Re} w - \operatorname{Re} w') + [i(0, \operatorname{Im} w - \operatorname{Im} w') + t].$$

The first vector on the right hand side is in $T_0 M$, the second in $N_0 M$, so the second vector lies in the cone $\Lambda \subset N_0 M$. Hence

$$(3) \qquad t = t_0 + i(0, \operatorname{Im} w' - \operatorname{Im} w),$$

for some $t_0 \in \Lambda$. We would like to show that $t \in \Lambda_2$, provided that $|\zeta| = \varepsilon$ is sufficiently small. To do this we must estimate $|\operatorname{Im} w' - \operatorname{Im} w|$ in terms of $|t| = \delta$.

First we have $|\zeta - \zeta'| \le C_1 |t| = C_1 \delta$ for some $C_1 < \infty$ depending on the curvature of M and on the angle between Σ and $N_0 M$. Also,

$$|\operatorname{Im} w - \operatorname{Im} w'| \le \sup |\nabla \varphi| \cdot |\zeta - \zeta'|,$$

where the $\sup |\nabla \varphi|$ is taken on the interval from $(z, \operatorname{Re} w)$ to $(z', \operatorname{Re} w')$ in $\mathbb{C}^m \times \mathbb{R}^d$. This can be estimated by $C_2(|\zeta| + |\zeta - \zeta'|) \le C_3(\varepsilon + \delta)$ for some constant C_3 independent of ε and δ, so $|\operatorname{Im} w - \operatorname{Im} w'| \le C_3(\varepsilon + \delta)\delta < C_4 \delta$. We can make C_4 arbitrary small by requiring that $\zeta + t$ lies in a sufficiently small neighborhood V of the origin in $\mathbb{C}^n$ (so $\varepsilon + \delta$ is small). We determine V so that $2C_4 = \min\{1, d(\Lambda, \Lambda_2)\}$. Then (3) implies

$$|t_0| \ge |t| - |\operatorname{Im} w - \operatorname{Im} w'| \ge \delta - C_4 \delta \ge \delta/2,$$

and

$$|\operatorname{Im} w' - \operatorname{Im} w| \le C_4 \delta \le 2C_4 |t_0| \le d(\Lambda, \Lambda_2)|t_0|.$$

Hence t lies in the ball $B(t_0, d(\Lambda, \Lambda_2)|t_0|) \subset \Lambda_2$, so $\zeta + t \in \mathscr{W}(\Lambda_2, V)$. This proves the right inclusion in (2).

The proof of the left inclusion in (2) is obtained by reversing the roles of Σ and $N_0 M$ in the proof given above; we shall omit the details. Lemma 3.1 is proved.

3.2 PROPOSITION. *Let* $\mathscr{W}' < \mathscr{W}$ *be wedges* (1.6) *with edge* M *and with cones* $\Gamma' < \Gamma$. *For any pair of cones* $\Lambda' < \Lambda$ *in* $\mathbb{R}^d$ *satisfying*

$$\Gamma' < \Lambda' < \Lambda < \Gamma,$$

there is a neighborhood U of 0 in $\mathbb{C}^n$ and a family of unitary maps $U_p \in \mathscr{U}(n)$, depending continuously on $p \in M \cap U$, so that the associated affine transformations $\psi_p(\zeta) = U_p \cdot (\zeta - p)$ satisfy:

(a) $\psi_p(p) = 0$.

(b) $D\psi_p(p) = U_p$ *maps* $T_p M$ *onto* $\mathbb{C}^m \times \mathbb{R}^d (= T_0 M)$ *and* $N_p M$ *onto* $\{0\}^m \times i\mathbb{R}^d \ (= N_0 M)$.

(c) *The wedges* $\mathscr{W}_p = \psi_p(\mathscr{W}) \cap U$, $\mathscr{W}_p' = \psi_p(\mathscr{W}') \cap U$ *satisfy*

$$\mathscr{W}_p' \subset \mathscr{W}(\Lambda', U) \subset \mathscr{W}(\Lambda, U) \subset \mathscr{W}_p, \qquad p \in U \cap M.$$

Here, $\mathscr{W}(\Lambda', U)$ *and* $\mathscr{W}(\Lambda, U)$ *are wedges* (1.6) *with edge* $M_p = \psi_p(M) \cap U$.

PROOF. Choose any continuous family $p \in M \to \mathscr{U}_p \in U(n)$ of unitary maps satisfying (a) and (b). For $p \in M$ close to 0, $\psi_p(\mathscr{W})$ is a tilted wedge with edge $\psi_p(M)$ and with cone $U_p(\{0\}^d \times i\Gamma) \subset U_p(N_0 M)$. The orthogonal projection Γ_p of this cone onto $N_0 M$ is very close to the original cone $\Gamma = \Gamma_0$ if p is close to 0. Similar property holds for $\psi_p(\mathscr{W}')$. Thus, if U is chosen sufficiently small, we have

$$\Gamma_p' < \Lambda' < \Lambda < \Gamma_p, \qquad p \in M \cap U.$$

The property (c) now follows from Lemma 3.1, provided that we shrink U further if necessary. This proves Proposition 3.2.

Next we will show that, given any pair of wedges $\mathscr{W}' < \mathscr{W}$ (1.6) with edge M, we can exhaust the finer wedge $\mathscr{W}'$ in a suitably small neighborhood of the origin by linearly embedded $(m+1)$-dimensional complex balls of uniform radius $R > 0$, contained entirely in the larger wedge $\mathscr{W}$.

For each point ζ outside M but close to M there is a unique closest point $\pi(\zeta) \in M$ so that $\zeta - \pi(\zeta)$ belongs to the normal space $N_{\pi(\zeta)} M$. Since $T_{\pi(\zeta)} M$ is a generating subspace of $\mathbb{C}^n$, it follows that $i(\zeta - \pi(\zeta)) \in T_{\pi(\zeta)} M$. Let $d(\zeta) = |\zeta - \pi(\zeta)|$.

For each such ζ we denote by $\Pi_\zeta \subset \mathbb{C}^n$ the complex affine subspace

$$\Pi_\zeta = \pi(\zeta) + T_{\pi(\zeta)}^{\mathbb{C}} M + \mathbb{C} \cdot (\zeta - \pi(\zeta)),$$

of complex dimension $m+1$, passing through $\pi(\zeta) \in M$. Let $l(\zeta) = (\zeta - \pi(\zeta))/d(\zeta)$ be the unit vector in direction $\zeta - \pi(\zeta)$. We choose a $\mathbb{C}$-orthonormal frame $X_1, \ldots, X_m$ in $T_{\pi(\zeta)}^{\mathbb{C}} M$ and let $\psi_\zeta : \mathbb{C}^{m+1} \to \Pi_\zeta$ be the affine parametrization of Π_ζ given by

$$\psi_\zeta(z, \eta) = \pi(\zeta) + \sum_1^m z_j X_j + \eta \cdot l(\zeta).$$

Let $B(R) = \{(z, \eta) \in \mathbb{C}^{m+1} : |z|^2 + |\eta - R|^2 < R^2\}$, $R > 0$, and let

$$B_\zeta(R) = \psi_\zeta(B(R)) \subset \Pi_\zeta,$$

be the image ball of radius R contained in Π_ζ, with $\pi(\zeta) \in \partial B_\zeta(R)$. Notice that the vector $2Rl(\zeta)$ is the diameter of $B_\zeta(R)$, passing through $\pi(\zeta)$ and the center $\pi(\zeta) + Rl(\zeta)$ of $B_\zeta(R)$.

3.3 PROPOSITION (Balls in wedges). *Given wedges $\mathscr{W}' < \mathscr{W}$ of the form (1.6) with edge M, there is a neighborhood U of $0 \in \mathbb{C}^n$ and an $R > 0$ so that $B_\zeta(R) \subset \mathscr{W}$ for each $\zeta \in \mathscr{W}' \cap U$. The number R can be chosen so that it only depends on the curvature of M and on the number $d(\Gamma', \Gamma)$ associated with the cones $\Gamma', \Gamma \subset N_0 M$ determining $\mathscr{W}'$ respectively $\mathscr{W}$.*

PROOF. By Proposition 3.2 it suffices to consider the points $\zeta = (0, it) \in N_0 M$ with $t \in \Gamma'$, $|t| = 1$. Then $\pi(\zeta) = 0$, we may take $\{X_j\}$ to be the standard basis in $\mathbb{C}^m$, and we have

$$\Pi_\zeta = \Pi_t = \{(z, \eta it) : z \in \mathbb{C}^m, \eta \in \mathbb{C}\}.$$

Writing $\eta = x + iy$, $w = \eta it = u + iv$, we have $u = -yt$, $v = xt$. On the ball $B(R)$ we have $x = (|z|^2 + y^2)/R + x'$, $x' > 0$. To prove that the image belongs to the wedge $\mathscr{W}$ we must consider the expression

$$\operatorname{Im} w - \varphi(z, \bar{z}, \operatorname{Re} w) = v - \varphi(z, \bar{z}, u) = xt - \varphi(z, \bar{z}, -yt).$$

We can estimate the second term by

$$|\varphi(z, \bar{z}, -yt)| \leq C_1(|z|^2 + y^2) \quad \text{for} \quad |z|^2 + y^2 < R^2,$$

and write

$$\varphi(z, \bar{z}, -yt) = C_1(|z|^2 + y^2) \cdot \tilde{\varphi}(z, \bar{z}, -yt),$$

where $|\tilde{\varphi}| \leq 1$. Thus,

$$v - \varphi(z, \bar{z}, u) = (1/R)(|z|^2 + y^2)t + x't + C_1(|z|^2 + y^2)\tilde{\varphi}(z, \bar{z}, -yt)$$
$$= (1/R)(|z|^2 + y^2)(t + C_1 R\tilde{\varphi}(z, \bar{z}, -yt)) + x't.$$

If we choose R so that $C_1 R < d(\Gamma', \Gamma)$, then the vector $(t + C_1 R\tilde{\varphi})$ belongs to Γ for all $t \in \Gamma' \cap S$, so $v - \varphi(z, \bar{z}, u) \in \Gamma$. This proves that for this choice of R we have $B_t(R) \subset \mathscr{W}$ for all $t \in \Gamma' \cap S$. Proposition 3.3 is proved.

REMARK. If we denote by $\Delta_\zeta(R) \subset B_\zeta(R)$ the complex disc of radius R passing through the center of $B_\zeta(R)$, with $\pi(\zeta) \in \partial\Delta_\zeta(R)$, then we obtain a family of complex discs in $\mathscr{W}$ that exhaust every finer wedge $\mathscr{W}' < \mathscr{W}$ near the origin, provided that $R > 0$ is chosen sufficiently small. This already suffices in several applications; we shall state some of them here.

3.4 COROLLARY (Hopf lemma on wedges). *Let ρ be a continuous function on $\mathscr{W} \cup M$ that is zero on M and negative plurisubharmonic on $\mathscr{W}$. Then for every finer wedge $\mathscr{W}' < \mathscr{W}$ and every sufficiently small neighborhood U of 0 in $\mathbb{C}^n$ there is a constant $C > 0$ such that*

$$\rho(\zeta) \leq -C \operatorname{dist}(\zeta, M), \qquad \zeta \in \mathscr{W}' \cap U.$$

PROOF. Apply the one variable Hopf lemma to ρ on each disc $\Delta_\zeta(R)$ (see the Remark above). We may assume that the union of discs $\Delta_\zeta(R)$ for $\zeta \in \mathscr{W}' \cap U$ is contained in a wedge finer than $\mathscr{W}$, so the standard proof of the Hopf lemma shows that the constant C above can be chosen independent of ζ.

REMARK. A similar result has been proved in [23].

3.5 COROLLARY. *Suppose that* $\Omega \cap \mathbb{C}^N$ *is a domain with a plurisubharmonic defining function near* $\partial\Omega$. *If* $\mathscr{W} \subset \mathbb{C}^n$ *is a wedge* (1.6) *with edge* M *and* $f : \mathscr{W} \to \Omega$ *is a holomorphic mapping that is continuous on* $\mathscr{W} \cup M$ *and maps* M *into* $\partial\Omega$, *then for every finer wedge* $\mathscr{W}' < \mathscr{W}$ *there are a constant* $C > 0$ *and a neighborhood* U *of* $0 \in \mathbb{C}^n$ *so that*

$$\text{dist}(f(\zeta), \partial\Omega) \geq C \, \text{dist}(\zeta, M), \qquad \zeta \in \mathscr{W}' \cap U.$$

PROOF. Apply the previous Corollary to the negative plurisubharmonic function $\rho \circ f$ on the wedge $\mathscr{W}$.

REMARK. If Ω is an arbitrary pseudoconvex domain with $\mathscr{C}^2$ boundary, we can find a $\mathscr{C}^2$ defining function ρ such that $\tau = -(-\rho)^\varepsilon$ is plurisubharmonic on Ω for $\varepsilon > 0$ sufficiently small [11]. Applying the Hopf lemma to $\tau \circ f$ we obtain the estimate

$$\text{dist}(f(\zeta), \partial\Omega) \geq C \, \text{dist}(\zeta, M)^{1/\varepsilon}, \qquad \zeta \in \mathscr{W}' \cap U.$$

These kinds of estimates are well-known when M is a hypersurface.

4. Convex barriers and estimates of the local hull of M. Let $\mathscr{W}$ be a wedge (1.6) with edge M and $U \subset \mathbb{C}^n$ a neighborhood of the origin. Every real-valued function $\rho \in \mathscr{C}^1(U)$ satisfying $\rho|_{M \cap U} \equiv 0$, $d\rho \neq 0$ on $M \cap U$, and $\rho < 0$ on $\mathscr{W} \cap U$ will be called a *barrier* for the wedge $\mathscr{W}$ in U. Clearly every wedge $\mathscr{W}$ with an acute cone has plenty of barriers.

In the rest of this section we assume that the manifold M defined by

$$(1) \qquad\qquad \text{Im} \, w = Q(z, \bar{z}) + R(z, \bar{z}, \text{Re} \, w)$$

is strongly pseudoconvex at the origin, and we shall be interested in strongly plurisubharmonic and strongly convex barriers. Let $C(Q)$ and $C^*(Q)$ be the Levi cone and its dual cone as defined by (1.4).

For vectors $\sigma, \tau \in \mathbb{R}^d$ we denote $\sigma \cdot \tau = \sum_{j=1}^d \sigma_j \tau_j$. For each vector $\sigma \in \mathbb{R}^d$, $|\sigma| = 1$, we define the function

$$(2) \qquad \begin{aligned} \rho_\sigma(z, w) =\ & -\sigma \cdot \text{Im} \, w + (\sigma \cdot Q(z, \bar{z}) + |\text{Im} \, w|^2) \\ & + (\sigma \cdot R(z, \bar{z}, \text{Re} \, w) - |Q(z, \bar{z}) + R(z, \bar{z}, \text{Re} \, w)|^2). \end{aligned}$$

Notice that ρ is obtained by taking the inner product of σ with the defining equation (1) (with $\text{Im} \, w$ moved to the right hand side) and adding the squares of the equations in (1). We have arranged the terms so that

$\sigma \cdot Q + |\operatorname{Im} w|^2$ is the quadratic part, and the terms in the last parentheses are small of order 2. Clearly ρ_σ is a barrier for each wedge $\mathscr{W}(\Gamma)$ whose cone Γ is contained in the half space $\sigma^+ = \{\tau \in \mathbb{R}^d : \sigma \cdot \tau > 0\}$, at least in some neighborhood of the origin. Moreover, if $\sigma \in \operatorname{Int} C^*(Q)$, then $\sigma \cdot Q$ is positive definite on $\mathbb{C}^m$, so ρ_σ is strongly plurisubharmonic near the origin.

Recall that for each cone $\Gamma \subset \mathbb{R}^d$ its dual cone Γ^* is defined by $\Gamma^* = \{\sigma \in \mathbb{R}^d : \sigma \cdot \tau \geq 0 \text{ for all } \tau \in \Gamma\}$. Let S denote the unit sphere in $\mathbb{R}^d$.

4.1 LEMMA. *Let* $\Gamma \subset \mathbb{R}^d$ *be an open convex cone whose closure* $\overline{\Gamma}$ *contains the Levi cone* $C(Q)$ *of* M *at* 0. *Then for every compact subset* $K \subset\subset \operatorname{Int} \Gamma^* \cap S$ *there is a neighborhood* U *of* 0 *in* $\mathbb{C}^n$ *such that every function in the family* $\{\rho_\sigma : \sigma \in K\}$ *is a strongly plurisubharmonic barrier for the wedge* $\mathscr{W}(\Gamma, U)$.

PROOF. The condition $\overline{\Gamma} \supset C(Q)$ implies $\Gamma^* \subset C^*(Q)$, so ρ_σ is a plurisubharmonic barrier for $\mathscr{W}(\Gamma)$ for every $\sigma \in \operatorname{Int} \Gamma^*$ in some neighborhood U_σ of the origin. Clearly U_σ can be chosen to be independent of $\sigma \in K \subset\subset \operatorname{Int} \Gamma^* \cap S$. This proves Lemma 4.1.

Often it will be useful to have *strongly convex barriers*. In fact, a quadratic change of w-variables turns every function in $\{\rho_\sigma : \sigma \in K\}$ into a strongly convex one in some smaller neighborhood U_1 of $0 \in \mathbb{C}^n$. By a rotation in $\mathbb{R}^d$ we may assume that $\overline{\Gamma} \subset \{\sigma_1 > 0\} \cup \{0\}$ (otherwise Γ^* has no interior!). We introduce new w-coordinates $w^* = u^* + iv^*$ by

$$
(3) \qquad w_1 = w_1^* - \frac{i}{2} \sum_{j=1}^{d} (w_j^*)^2 ,
$$

$$
w_k = w_k^*, \qquad 2 \leq k \leq d .
$$

In the new coordinates ρ_σ is given by

$$
(4) \quad \rho_\sigma^*(z, w^*) = -\sigma \cdot v^* + \left(\sigma \cdot Q(z, \bar{z}) + \frac{\sigma_1}{2}|u^*|^2 + \left(1 - \frac{\sigma_1}{2}\right)|v^*|^2 \right) + o(2),
$$

so the quadratic part is strongly positive definite whenever $\sigma \in (\operatorname{Int} \Gamma^*) \cap S$ (since then $0 < \sigma_1 \leq 1$). On each compact set $\sigma \in K \subset\subset (\operatorname{Int} \Gamma^*) \cap S$ we have $0 < c \leq \sigma_1 \leq 1$, so the functions in $\{\rho_\sigma^* : \sigma \in K\}$ are strongly convex on a fixed neighborhood U_1 of $0 \in \mathbb{C}^n$.

Let M^* denote M in the new coordinates. The above implies that the polynomially convex hull of $M^* \cap \overline{U}_1$ is contained in $\bigcap \{\rho_\sigma^* \leq 0 : \sigma \in K\}$. Since a polynomial change of coordinates maps polynomial hulls to polynomial hulls, it follows that for all sufficiently small balls $0 \in U \subset \mathbb{C}^n$, the hull of $M \cap \overline{U}$ is contained in the set $\{\zeta \in \overline{U} : \rho_\sigma(\zeta) \leq 0 \text{ for all } \sigma \in K\}$.

If we now fix a cone $\Gamma > C(Q)$, we can find finitely many vectors $\sigma_1, \dots,$ $\sigma_k \in \operatorname{Int} C^*(Q) \cap S$ such that

$$
(5) \qquad C(Q) \subset \bigcap_{j=1}^{k} \sigma_j^+ < \Gamma .
$$

If $\rho_j = \rho_{\sigma_j}$ $(1 \leq j \leq k)$ are the corresponding functions (2) and B is a sufficiently small closed ball in $\mathbb{C}^n$ centered at the origin, then the polynomial hull $M \cap B$ is contained in

$$\{\zeta \in B : \rho_j(\zeta) \leq 0,\, 1 \leq j \leq k\} \subset \mathscr{W}(\Gamma, B) \cup (M \cap B).$$

The last inclusion is elementary and follows from (5). This proves

4.2 PROPOSITION. *If M is a strongly pseudoconvex manifold* (1) *with the Levi cone $C(Q)$ at the origin, then for every cone $\Gamma > C(Q)$ there is a closed ball $B \subset \mathbb{C}^n$ centered at the origin such that the polynomial hull of $M \cap B$ is contained in the wedge $\mathscr{W}(\Gamma, B) \cup (M \cap B)$.*

The same proof shows the following

4.3 PROPOSITION. *Let M be a strongly pseudoconvex $\mathscr{C}^2$ manifold* (1) *with the Levi cone $C(Q)$ at 0. For every strongly convex cone $\Gamma \subset \mathbb{R}^d$ satisfying $\overline{\Gamma} \supset C(Q)$ we can find a small neighborhood U of $0 \in \mathbb{C}^n$ so that the image $\mathscr{W}^*$ of the wedge $\mathscr{W} = \mathscr{W}(\Gamma, U)$ (1.6) in the coordinates (z, w^*) defined by (3) satisfies*

$$(6) \qquad\qquad \mathscr{W}^* \cap U^* \subset \mathscr{D}_1 \cap \mathscr{D}_2 \cap \cdots \cap \mathscr{D}_d,$$

in some neighborhood U^* of $0 \in \mathbb{C}^n$, where each $\mathscr{D}_j$ is a strongly convex domain

$$\mathscr{D}_j = \{\zeta \in U^* : \rho^*_{\sigma_j}(\zeta) < 0,\; 1 \leq j \leq d\},$$

every $\rho^*_{\sigma_j}$ is of the form (4), and the vectors $\sigma_1, \ldots, \sigma_d \in (\mathrm{Int}\,\Gamma^*) \cap S$ are linearly independent.

If we denote the image of M in coordinates (z, w^*) by M^*, the condition on $\sigma_1, \ldots, \sigma_d$ implies that

$$M^* \cap U^* = \partial\mathscr{D}_1 \cap \cdots \cap \partial\mathscr{D}_d \cap U^*,$$

and the intersection is transverse, provided that U^* is sufficiently small.

We have seen that a strongly pseudoconvex manifold (1) lies locally near 0 in many strongly pseudoconvex hypersurfaces. This property characterizes strongly pseudoconvex CR manifolds:

4.4 PROPOSITION. *Let $M \subset \mathbb{C}^n$ be a $\mathscr{C}^2$ manifold of the form* (1.1) *near the origin, of real codimension $d > 1$. Then M is strongly pseudoconvex at 0 if and only if it is locally near 0 contained in a strongly pseudoconvex hypersurface.*

PROOF. It remains to prove the "if" part. Write M in the form (1). After a linear change of w-coordinates the strongly pseudoconvex hypersurface Σ containing M is given locally by an equation

$$(7) \qquad\qquad \mathrm{Im}\, w_d = A(z, \bar{z}) + B(z, w', \mathrm{Re}\, w_d),$$

where A contains quadratic terms involving z and $\bar{z}$, and B contains the remaining quadratic terms and terms of higher order. We substitute the first $d-1$ equations (1) for M into the right hand side of (7). Clearly this does not affect the quadratic part A of (7). The condition $M \subset \Sigma$ implies that the last equation of M now agrees with the new equation for Σ. Comparing the quadratic parts involving z or $\bar{z}$ we conclude $A(z, \bar{z}) = Q_d(z, \bar{z})$, so A is hermitian. Since A is the restriction of the Levi form of Σ to $\{w = 0\} \subset T_0^{\mathbb{C}} \Sigma$, A must be positive definite (we adjust the sign of w_d if necessary), so $Q_d(z, \bar{z})$ is positive definite. Thus M is strongly pseudoconvex.

5. Estimates of the mapping. In this section we assume that $f : M \to M'$ is a local CR homeomorphism of CR manifolds (1.2) that are strongly pseudoconvex and over-extendable at the origin, and $f(0) = 0$. We denote by $C(Q)$ respectively $C(Q')$ the Levi cone of M resp. M' at the origin.

5.1 PROPOSITION. *There exist open, strongly convex cones in $\mathbb{R}^d$ satisfying $C(Q) < \Gamma_1 < \Gamma_2 < \Gamma_3$ and $C(Q') < \Gamma_1' < \Gamma_2'$, and there exist neighborhoods $U_1 \subset U_2 \subset U_3$ and $U_1' \subset U_2'$ of the origin in $\mathbb{C}^n$ so that the following hold:*

(i) *f extends holomorphically to the wedge $\mathscr{W}_2 = \mathscr{W}_M(\Gamma_2, U_2)$ and maps it into $\mathscr{W}_2' = \mathscr{W}_{M'}(\Gamma_2', U_2')$.*
(ii) *f maps $\mathscr{W}_1 = \mathscr{W}_M(\Gamma_1, U_1)$ into $\mathscr{W}_1' = \mathscr{W}_{M'}(\Gamma_1', U_1')$.*
(iii) *f^{-1} extends holomorphically to $\mathscr{W}_2' = \mathscr{W}_{M'}(\Gamma_2', U_2')$ and maps it into $\mathscr{W}_3 = \mathscr{W}_M(\Gamma_3, U_3)$.*
(iv) *f^{-1} maps $\mathscr{W}_1'$ to $\mathscr{W}_2$.*

Moreover, we can choose Γ_3 respectively Γ_2' to be contained in a prescribed cone $\Gamma_0 > C(Q)$ respectively $\Gamma_0' > C(Q')$.

The index in $\mathscr{W}_M$ indicates that we have a wedge with edge M, and similarly for M'.

PROOF. Choose $\Gamma_3 > C(Q)$ and $U_3 \subset \mathbb{C}^n$, and consider the inverse map $f^{-1} : M' \to M$. If $\omega' \subset M'$ is a sufficiently small neighborhood of $0 \in M'$, then the polynomial hull of $f^{-1}(\omega') \subset M$ is contained in the wedge $\mathscr{W}_3 \cup M = \mathscr{W}_M(\Gamma_3, U_3) \cup M$ according to Proposition 4.2. Since M' is over-extendable at 0, there is a cone $\Gamma_2' > C(Q')$ and a neighborhood U_2' of 0 in $\mathbb{C}^n$ so that every CR function on ω' extends holomorphically to the wedge $\mathscr{W}_2' = \mathscr{W}_{M'}(\Gamma_2', U_2')$ (see Proposition 2.5). Hence this wedge is contained in the polynomial hull of $\bar{\omega}'$, so f^{-1} maps $\mathscr{W}_2'$ to $\mathscr{W}_3$. Thus (iii) holds.

We repeat the same argument with f instead of f^{-1}. We use the over-extendability of M at 0 to find a cone Γ_2 satisfying $C(Q) < \Gamma_2 < \Gamma_3$ and a neighborhood $U_2 \subset\subset U_3$ of 0 such that f extends to the wedge $\mathscr{W}_2 = \mathscr{W}_M(\Gamma_2, U_2)$ and maps $\mathscr{W}_2$ into $\mathscr{W}_2'$. Thus (i) holds.

Consider again f^{-1}. Using the over-extendability of M' at 0 and Propo-

sition 4.2 we find a wedge $\mathscr{W}_1' < \mathscr{W}_2'$ with cone Γ_1', $C(Q') < \Gamma_1' < \Gamma_2'$, so that $f^{-1}(\mathscr{W}_1') \subset \mathscr{W}_2$. Thus (iv) holds. Finally, we repeat the same with f to find $\mathscr{W}_1 \subset \mathscr{W}_2$ such that $f(\mathscr{W}_1) \subset \mathscr{W}_1'$. This proves Proposition 5.1.

5.2 PROPOSITION. *Let $f : M \to M'$ be as in the Main Theorem. Then there is a wedge $\mathscr{W} = \mathscr{W}(\Gamma, U)$ with edge M whose cone Γ satisfies $C(Q) < \Gamma$, such that f extends holomorphically to $\mathscr{W}$, it satisfies the distance estimate*

$$(1) \qquad \frac{1}{C}\,\mathrm{dist}(\zeta, M) \le \mathrm{dist}(f(\zeta), M') \le C\,\mathrm{dist}(\zeta, M), \qquad \zeta \in \mathscr{W},$$

for some $C > 0$, and is uniformly Hölder continuous with the exponent $1/2$ on $\mathscr{W} \cup (M \cap U)$.

PROOF. Choose wedges $\mathscr{W} = \mathscr{W}_1 < \mathscr{W}_2 < \mathscr{W}_3$ and $\mathscr{W}_1' < \mathscr{W}_2'$ as in Proposition 5.1. Assume also that $\mathscr{W}_3$ and $\mathscr{W}_2'$ are sufficiently small so that they admit strongly plurisubharmonic barriers ρ respectively ρ'. The estimate (1) is obtained by applying the Hopf lemma (Corollary 3.4) to the negative plurisubharmonic function $\rho' \circ f$ on $\mathscr{W}_2$ and to $\rho \circ f^{-1}$ on $\mathscr{W}_2'$.

To get the Hölder estimate we first apply the change of coordinates (4.3) on the target side so that $\mathscr{W}_2'$ is contained in a strongly convex domain $\mathscr{D}$ with $M' \subset \partial\mathscr{D}$. For each point $\zeta \in \mathscr{W}$ and each vector $X \in \mathbb{C}^n\backslash\{0\}$ we can find a linear complex disc $\Delta(\zeta; X)$ in $\mathscr{W}_2$, centered at ζ, in direction X, of radius comparable to $\mathrm{dist}(\zeta, M)$. On the target side, the largest such disc in $\mathscr{D}$, centered at $f(\zeta)$, in any direction, has radius $\le C_1\,\mathrm{dist}(f(\zeta), M')^{1/2}$ for some constant C_1. Since $\mathscr{D}$ is convex, the result of [17] and (1) imply the following estimate on the derivative of f at ζ:

$$|Df(\zeta)X|/|X| \le C_2\,\mathrm{dist}(f(\zeta), M')^{1/2}/\,\mathrm{dist}(\zeta, M)$$
$$\le C_3\,\mathrm{dist}(\zeta, M)^{-1/2}, \qquad \zeta \in \mathscr{W}.$$

A standard argument shows that f is Hölder continuous on $\mathscr{W} \cup (M \cap U)$ with the Hölder exponent $1/2$. This proves Proposition 5.2.

In order to obtain more precise information on $Df(\zeta)$ for $\zeta \in \mathscr{W}$ we shall introduce certain affine coordinate changes on the domain and the target. In the domain we fix a pair of wedges $\mathscr{W} = \mathscr{W}_1$ and $\mathscr{W}_2$ with cones Γ, Γ_2 satisfying $C(Q) < \Gamma < \Gamma_2$. On the target side we use the coordinates in which Proposition 4.3 holds, i.e., we have d strongly convex domains $\mathscr{D}_1, \ldots, \mathscr{D}_d \subset \mathbb{C}^n$ so that $M' \cap U' = \partial\mathscr{D}_1 \cap \cdots \cap \partial\mathscr{D}_d \cap U'$ for a suitably small neighborhood U' of $0 \in \mathbb{C}^n$, the boundaries $\partial\mathscr{D}_j$ intersect transversely along M', and f maps $\mathscr{W}_2$ holomorphically into $\mathscr{D}_1 \cap \cdots \cap \mathscr{D}_d$.

Moreover, we may assume that the distance estimate (1) holds on $\mathscr{W}_2$.

For each point $p \in M$ we choose a unitary matrix $U_p \in \mathscr{U}(n)$ satisfying

$$(2) \qquad \begin{aligned} U_p(T_pM) &= \mathbb{C}^m \times \mathbb{R}^d = T_0M, \\ U_p(N_pM) &= \{0\}^m \times i\mathbb{R}^d = N_0M. \end{aligned}$$

Let $\psi_p : \mathbb{C}^n \to \mathbb{C}^n$ be the associated affine map

$$\psi_p(\zeta) = U_p(\zeta - p). \tag{3}$$

We denote by U_p' resp. ψ_p' similar maps associated to $f(p) \in M'$. Then we have

$$f = (\psi_p')^{-1} \circ f_p \circ \psi_p, \qquad p \in M, \tag{4}$$

where f_p is the expression for f in the new coordinates. Notice that f_p maps the manifold $\psi_p(M)$ of the form (1.1) to the manifold $\psi_p'(M')$, and $f_p(0) = 0$.

We will assume that $\mathscr{W}_2$ is sufficiently small such that each point $\zeta \in \mathscr{W}_2$ has the unique closest point $p = \pi(\zeta) \in M$, and $(\zeta - p) \in N_p M$. We then have $\psi_p(\zeta) = (0, it(\zeta))$, for some $t(\zeta) \in \mathbb{R}^d$ with $|t(\zeta)| = |\zeta - p| = \mathrm{dist}(\zeta, M)$.

From (2)–(4) we get

$$Df(\zeta) = (U_p')^{-1} \cdot Df_p(0, it(\zeta)) \cdot U_p. \tag{5}$$

For each $t \in \mathbb{R}^d$ such that f_p is defined at $(0, it)$ we shall write its derivative in the block notation

$$Df_p(0, it) = \begin{pmatrix} A_p(t) & B_p(t) \\ C_p(t) & D_p(t) \end{pmatrix}, \tag{6}$$

with blocks of sizes $A \in \mathbb{C}^{m \times m}$, $B \in \mathbb{C}^{m \times d}$, $C \in \mathbb{C}^{d \times m}$, $D \in \mathbb{C}^{d \times d}$. If we choose U_p to depend continuously on $p \in M$, then for all $p \in M$ sufficiently close to 0 we may take t to be an arbitrary vector of sufficiently small length in cone $\Gamma_0 \subset \mathbb{R}^d$ satisfying $C(Q) < \Gamma < \Gamma_0 < \Gamma_2$. (Here, Γ is the cone determining the wedge $\mathscr{W} = \mathscr{W}_1$, and Γ_2 determines $\mathscr{W}_2$.)

5.3 PROPOSITION. *For each point $p \in M$ sufficiently close to the origin and each vector $t \in \mathbb{R}^d$ such that the point $\zeta = p + U_p^{-1}(0, it)^t = \psi_p^{-1}(0, it)$ is contained in $\mathscr{W}$, the blocks in (6) satisfy the following estimates:*

(a) $A_p(t) = O(1)$, $D_p(t) = O(1)$,
  $B_p(t) = O(|t|^{-1/2})$, $C_p(t) = O(|t|^{1/2})$.

(b) $C_p(t) = o(|t|^{1/2})$.

(c) *If the limit $D_p^* = \lim_{t \to 0} D_p(t)$ exists as $t \to 0$ within some cone in $\Gamma \subset \mathbb{R}^d$ with nonempty interior, then it is a real-valued $d \times d$ matrix.*

Moreover, the estimates in (a) are uniform with respect to p and t.

REMARK. At this point we are not able to prove that the estimate in (b) holds uniformly with respect to p, so we stated it separately. We shall prove in section 6 that the limit in (c) exists for almost every $p \in M$ (with respect to the surface measure on M) as $t \to 0$ within some smaller cone contained in Γ.

PROOF. We shall give the proof for the point $p = 0 \in M$ since the proof for any other point is just the same.

Fix a point $\zeta = (0, it) \in \mathcal{W}$ and let $f(\zeta) = \zeta' = (z', u' + iv')$. If $X \in \mathbb{C}^n$ is any vector of length one, then by [17] we have $|Df(\zeta)X| \leq A \cdot R_2 / R_1$, where A is an absolute constant, R_1 is the radius of the largest linear complex disc in $\mathcal{W}_2$, centered at ζ, in direction of the vector X, and R_2 is the radius of the largest such disc in $\bigcap_{i=1}^{d} \mathcal{D}_j$, centered at ζ', in direction of the vector $Df(\zeta)X$.

For each X we can take R_1 proportional to $|t|$ or bigger. When $X \in \mathbb{C}^m \times \{0\}^d$, we can take R_1 proportional to $|t|^{1/2}$. On the other hand, we have $\mathrm{dist}(\zeta', M') \approx |t|$, so R_2 is at most $C|t|^{1/2}$ for some $C > 0$ because of the strong convexity of the domains $\mathcal{D}_j$. Thus $|Df(\zeta)X|$ is always $\leq C_1 |t|^{-1/2}$, and is bounded when X is complex tangential. This gives the estimates (a) for the blocks A and B.

The estimates for C and D require some additional work. By Hölder continuity we have $|\zeta'| = |f(\zeta)| \leq C_2 |t|^{1/2}$. Let $\zeta_1' = (z', u' + iv_1')$ be the uniquely determined point in M' that differs from ζ' only in the v' coordinate. From the defining equation of M' we have $v_1' = \varphi'(z', \bar{z}', u')$, so $|v_1'| = O(|t|)$ (since $\varphi'(0) = 0$, $d\varphi'(0) = 0$). Also, $|v_1' - v'| \approx \mathrm{dist}(\zeta', M') = O(|t|)$. This implies

$$(7) \qquad\qquad |v'| = O(|t|).$$

Hence the projection onto the w'-space $\{0\}^m \times \mathbb{C}^d$ of any linear complex disc $\Delta \subset \mathcal{D}_1 \cap \cdots \cap \mathcal{D}_d$ centered at ζ' has radius at most $C_3 |t|$. This implies the estimates for the blocks C and D corresponding to the components $f_{m+1}, \ldots, f_n$ of f.

Since all of these estimates only depend on the radii R_1 and R_2 and on the distance estimate for f, it is clear that the same holds uniformly for $p \in M$ sufficiently near 0.

To prove the estimate (b) we shall use the generalized theorem of Carathéodory on the angular derivative [24, Theorem 8.5.6]. Again we shall take $p = 0 \in M$.

We can find osculating balls $B_1, \ldots, B_d \subset \mathbb{C}^n$ such that $B_j \supset \mathcal{D}_j$, $\partial B_j \cap \partial \mathcal{D}_j = \{0\}$, and the normals to ∂B_j at 0 are linearly independent vectors $\sigma_j \in \{0\}^m \times i\mathbb{R}^d = N_0 M'$, $1 \leq j \leq d$. Fix a point $(0, it) \in \mathcal{W}$. In Section 3, Proposition 3.3, we have constructed an $(m + 1)$-dimensional complex plane Π_t, generated in the present context by $T_0^{\mathbb{C}} M = \mathbb{C}^m \times \{0\}^d$ and the vector $(0, it)$, containing a ball $B_t(R) \subset \Pi_t$ of radius $R > 0$ (R independent of t), such that $B_t(R) \subset \mathcal{W}_2$ and $\partial B_t(R) \cap \partial \mathcal{W}_2 = \{0\} \in M$.

For each $j \in \{1, \cdots, d\}$ we consider the restricted mapping $f : B_t(R) \to B_j$ between balls. Clearly $f(0) = 0$, and the estimate (7) implies that

$$(8) \qquad \liminf_{B_t(R) \ni \zeta \to 0} \mathrm{dist}(f(\zeta), \partial B_j) / \mathrm{dist}(\zeta, \partial B_t(R)) < \infty.$$

For a fixed j we write $f = (f_{(t)}, f_{(n)})$, where $f_{(n)}$ is the normal component of f with respect to ∂B_j at 0 (i.e., the projection of f onto the normal direction to ∂B_j at 0), and $f_{(t)}$ is the tangential component. The cited theorem [24, p. 177] implies

$$\partial f_{(n)}/\partial z_k(0, i\varepsilon t) = o(\varepsilon^{1/2}), \qquad 1 \le k \le m.$$

Since the normals to ∂B_j $(1 \le j \le d)$ at 0 span $\{0\}^m \times \mathbb{C}^d$ over $\mathbb{C}$, we get

$$\partial f_j/\partial z_k(0, it) = o(|t|), \qquad 1 \le k \le m, \ m+1 \le j \le n,$$

when $t \in \Gamma_0$, $|t| \to 0$. This is precisely the estimate (b) on $C_p(t)$ at $p = 0$.

It remains to prove (c). The theorem quoted above implies that for each fixed $t \in \Gamma_0$ and $j \in \{1, \ldots, d\}$, the derivative of $f_{(n)}$ in the direction of the vector $(0, it)$ (the "normal" direction in $B_t(R)$ at 0) converges to a real number:

$$(9) \qquad \lim_{\varepsilon \to 0} i \sum_{k=1}^{d} t_k \partial f_{(n)}/\partial w_k(0, i\varepsilon t) \in \mathbb{R}.$$

For each j, $f_{(n)}$ is a linear combination with purely imaginary coefficients of the components $\{f_l\}_{l=m+1}^n$. By the assumption (c) the limits

$$(10) \qquad \lim_{t \to 0} \partial f_l/\partial w_k(0, it), \qquad 1 \le k \le d, \ m+1 \le l \le n,$$

exists as $t \to 0$ within certain cone. Applying (9) in d linearly independent direction $t \in \mathbb{R}^d$ we conclude that the limits

$$\lim_{t \to 0} i \partial f_{(n)}/\partial w_k(0, it) \in \mathbb{R}, \qquad 1 \le k \le d,$$

are real-valued. Finally, as j runs from 1 to d, the normals to ∂B_j at 0 span $\{0\}^m \times i\mathbb{R}^d$, so the limits in (10) are also real-valued. This proves (c), and Proposition 5.3 is proved.

REMARK. In the proof of (c) we had to know in advance that the limits (9) exist and are independent of t. The problem is that on wedges there is no immediate Lindelöf's theorem: a bounded holomorphic function may have a limit along certain radial direction, but may fail to have the nontangential limit. For versions of Lindelöf's theorem in $\mathbb{C}^n$ see [9, 10, 19, 23].

5.4 COROLLARY (Notation as in Proposition 5.1). *There is a constant $C > 0$ such that*

$$1/C \le |\det Df(\zeta)| \le C, \qquad \zeta \in \mathcal{W}.$$

PROOF. The estimate on $|\det Df(\zeta)|$ from above follows immediately from (5) and the estimates in Proposition 5.3 (a). We use the fact that these estimates are uniform with respect to p and t.

The estimate on $|\det Df(\zeta)|$ from below follows by applying the first part of the Corollary to the mapping $f^{-1} : \mathcal{W}_2' \to \mathcal{W}_3$ that maps $\mathcal{W}_1'$ into $\mathcal{W}_2$ (see Proposition 5.1). This proves Corollary 5.4.

6. Proof of the main theorem. Let $f : M \to M'$ be a CR homeomorphism as in the Main Theorem. Let $\mathscr{W} = \mathscr{W}(\Gamma, U)$ be a wedge with edge M such that $\Gamma > C(Q)$, f extends holomorphically to $\mathscr{W}$ and satisfies the estimates of Proposition 5.3 and Corollary 5.4 there.

If, in addition, f is a CR *diffeomorphism* of class $\mathscr{C}^1$ on M, then for each point $\zeta \in M$ the derivative $Df(\zeta)$ maps $T^{\mathbb{C}}_{\zeta}M$ isomorphically onto $T^{\mathbb{C}}_{f(\zeta)}M'$.

If we think of the complex m-planes $T^{\mathbb{C}}_{\zeta}M$ and $T^{\mathbb{C}}_{f(\zeta)}M'$ as points in the complex Grassman manifold $\mathrm{Gr}(m, n)$ of complex m-dimensional subspaces of $\mathbb{C}^n$, it is natural to associate to M respectively M' the manifolds $\widetilde{M}$ respectively $\widetilde{M}'$ in $\mathbb{C}^n \times \mathrm{Gr}(m, n)$ by

$$(1) \qquad \widetilde{M} = \{(\zeta, T^{\mathbb{C}}_{\zeta}M) : \zeta \in M\} \subset \mathbb{C}^n \times \mathrm{Gr}(m, n),$$

and analogously for $\widetilde{M}'$. Then f lifts to a continuous mapping $\widetilde{f} : \widetilde{M} \to \widetilde{M}'$ defined by

$$(2) \qquad \widetilde{f}(\zeta, T^{\mathbb{C}}_{\zeta}M) = (f(\zeta), T^{\mathbb{C}}_{f(\zeta)}M'), \qquad \zeta \in M.$$

Notice that $\widetilde{f}$ can be defined even when f is merely continuous on M.

Over the wedge $\mathscr{W}$ we can lift f to the holomorphic mapping

$$(3) \qquad \begin{aligned} F : \mathscr{W} \times \mathrm{Gr}(m, n) &\to \mathbb{C}^n \times \mathrm{Gr}(m, n), \\ F(\zeta, \Lambda) &= (f(\zeta), Df(\zeta)\Lambda). \end{aligned}$$

Here, $Df(\zeta)\Lambda$ is the image of $\Lambda \in \mathrm{Gr}(m, n)$ under the linear map $Df(\zeta)$; this requires that $Df(\zeta)$ is nondegenerate for each $\zeta \in \mathscr{W}$, as is the case in our situation.

When $f \in \mathscr{C}^1(M)$ and $Df(\zeta)$ is nondegenerate for $\zeta \in M$, then F extends continuously from $\mathscr{W} \times \mathrm{Gr}(m, n)$ to $\widetilde{M}$ and coincides with $\widetilde{f}$ on $\widetilde{M}$. Moreover, Webster proved in [28] that the manifold $\widetilde{M}$ is *totally real* at a point $(\zeta, T^{\mathbb{C}}_{\zeta}M)$ if and only if the *Levi form of M is nondegenerate* at $\zeta \in M$. The proof of the Main Theorem now follows exactly as in Webster [28, 29], provided that we use the smooth version of the edge-of-the-wedge theorem given in [23]. This will be explained in more details below. In this case one does not need the over-extendability of M respectively M' at 0; instead it suffices to assume that M and M' are minimal at 0, so the result of Tumanov [25] can be applied to extend f respectively f^{-1} to some wedge. In this case we do not require any of the results of Sections 3–5.

We now drop the assumption $f \in \mathscr{C}^1(M)$. We will nevertheless find a suitable wedge $\widetilde{\mathscr{W}} \subset \mathscr{W} \times \mathrm{Gr}(m, n)$ with edge $\widetilde{M}$ so that the mapping F (3) extends continuously from $\widetilde{\mathscr{W}}$ to $\widetilde{\mathscr{W}} \cup \widetilde{M}$ and coincides with $\widetilde{f}$ on $\widetilde{M}$. This will suffice to conclude the proof of the Main Theorem along the same lines as before. In the hypersurface case this approach has been developed in the papers by Pinchuk and Hasanov [23] and the author [15]. Our present proof includes the hypersurface situation as a very special case.

Before proceeding, we must introduce homogeneous coordinates on the Grassmannian $\mathrm{Gr}(m, n)$ and express the map F (3) using these coordinates. To each matrix $P \in \mathbb{C}^{d \times (m+d)}$ of rank d we associate the complex m-plane

$$\Lambda_P = [P] = \{\zeta \in \mathbb{C}^{m+d} : P\zeta = 0\},$$

where $P\zeta$ is the matrix product. Clearly every $\Lambda \in \mathrm{Gr}(m, m+d)$ is of this form, and we have $[P_1] = [P_2]$ if and only if $P_2 = B \cdot P_1$ for some $B \in \mathrm{GL}(d, \mathbb{C})$.

If $A \in \mathrm{GL}(n, \mathbb{C})$, $n = m + d$, then A maps each m-plane onto an m-plane as follows:

$$
\begin{aligned}
A([P]) &= \{A\zeta \in \mathbb{C}^n : P\zeta = 0\} \\
&= \{\zeta' \in \mathbb{C}^n : PA^{-1}\zeta' = 0\} \\
&= [PA^{-1}].
\end{aligned}
$$

We shall say that P is the homogeneous coordinate of $[P] \in \mathrm{Gr}(m, m+d)$. When $d = 1$, we have $\mathrm{Gr}(m, m+1) = \mathbb{C}P^m$, the complex projective space.

In these coordinates the map F can be expressed by

(4)
$$F(\zeta, [P]) = (f(\zeta), [P \cdot Df(\zeta)^{-1}]).$$

We must also write the manifold $\widetilde{M}$ (1) in the coordinate notation. Let $M \subset \mathbb{C}^n$ be defined by

$$r(\zeta) = r(z, w) = -\operatorname{Im} w + \varphi(z, \bar{z}, \operatorname{Re} w) = 0,$$

where we think of $r = (r_1, \ldots, r_d)^t$ as a column vector in $\mathbb{R}^d$. We denote by $r_\zeta = (r_z, r_w)$ the Jacobian ∂-matrix of r of dimension $d \times n$, where $r_z = (\partial r_j / \partial z_k)$ and $r_w = (\partial r_j / \partial w_l)$ for $1 \leq j, l \leq d$, $1 \leq k \leq m$. Notice that

$$T_p^{\mathbb{C}} M = [r_\zeta(p)], \qquad p \in M.$$

Since $\varphi(0) = 0$, $d\varphi(0) = 0$, the matrix r_w is invertible in a neighborhood of the origin in $\mathbb{C}^n$, and we shall always work under this hypothesis.

This allows us to consider F only on the coordinate chart of $\mathrm{Gr}(m, n)$ consisting of points $[P] \in \mathrm{Gr}(m, n)$ for which $P = (P_1, P_2)$ and the matrix $P_2 \in \mathbb{C}^{d \times d}$ is invertible. On this chart we can use the affine coordinate $[P] = P_2^{-1} P_1 \in \mathbb{C}^{d \times m}$. Hence $T_p^{\mathbb{C}} M$ has the affine coordinate $r_w^{-1}(p) r_z(p)$ for $p \in M$.

It will be convenient to introduce the holomorphic mapping

(5)
$$
\begin{aligned}
G &: \mathscr{W} \times \mathbb{C}^{d \times m} \to \mathbb{C}^{d \times n}, \\
G(\zeta, P) &= (P, I^{d \times d}) \cdot Df(\zeta)^{-1}.
\end{aligned}
$$

In the chosen affine coordinate system on $\mathrm{Gr}(m, n)$ the map F is then given by

(6)
$$F(\zeta, P) = (f(\zeta), [G(\zeta, P)]), \qquad \zeta \in \mathscr{W}, \ P \in \mathbb{C}^{d \times m}.$$

Unfortunately we cannot pass to an affine coordinate system on the target yet.

We shall now define a special wedge $\widetilde{\mathscr{W}} \subset \mathbb{C}^n \times \mathbb{C}^{d \times m}$ with edge $\widetilde{M}$ as follows. For $\zeta \in \mathscr{W}$ we let $\pi(\zeta) \in M$ be its closest point in M. Recall that $T^{\mathbb{C}}_{\pi(\zeta)} M$ has the affine coordinate $(r_w^{-1} r_z)(\pi(\zeta))$. Fix $\alpha > 0$ sufficiently large and set

$$(7) \qquad \widetilde{\mathscr{W}} = \{(\zeta, P) \in \mathscr{W} \times \mathbb{C}^{d \times m} : |P - (r_w^{-1} r_z)(\pi(\zeta))| < \alpha \cdot \mathrm{dist}(\zeta, M)\}.$$

(Here we can use any matrix norm.) Our first goal is to prove:

6.1 LEMMA. *The holomorphic mapping G (5) is bounded on the wedge $\widetilde{\mathscr{W}}$.*

PROOF. In order to estimate $G(\zeta, P)$ we let $p = \pi(\zeta) \in M$ be the closest point to ζ on M. We introduce the affine change of coordinates (5.2)–(5.3), so f satisfies (5.4). We also let $t = t(\zeta) \in \mathbb{R}^d$ be such that $\psi_p(\zeta) = U_p(\zeta - p) = (0, it)$. Since the unitary map U_p takes $T^{\mathbb{C}}_p M = [r_\zeta(p)]$ isomorphically onto $\mathbb{C}^m \times \{0\}^d = [(0, I^{d \times d})]$, it follows that

$$(8) \qquad r_\zeta(p) \cdot U_p^{-1} = (0^{d \times m}, E(p)),$$

for some matrix $E(p) \in \mathrm{GL}(d, \mathbb{C})$. We may choose U_p to depend continuously on $p \in M$ and $U_0 = I^{n \times n}$, so $E(p)$ will also be continuous in $p \in M$ and $E(0) = r_w(0) = (i/2) I^{d \times d}$. We perform similar transformations on the target side with respect to the point $f(p) \in M'$; we denote the corresponding quantities by the same letters, only adding a prime.

From (5.4) we calculate by the chain rule

$$(9) \qquad Df(\zeta)^{-1} = U_p^{-1} \circ Df_p(0, it)^{-1} \circ U_p'.$$

Set

$$Df_p(0, it)^{-1} = \begin{pmatrix} \widetilde{A}_p(t) & \widetilde{B}_p(t) \\ \widetilde{C}_p(t) & \widetilde{D}_p(t) \end{pmatrix},$$

where the blocks have the same sizes as those in $Df_p(0, it)$ (5.6). Since $|\det Df_p(0, it)| = |\det Df(\zeta)|$ is bounded away from zero on $\mathscr{W}$ according to Corollary 5.4, Proposition 5.3 (a) and (b) implies that the blocks $\widetilde{A}_p$, $\widetilde{B}_p$, $\widetilde{C}_p$, $\widetilde{D}_p$ satisfy exactly the same estimates as the corresponding blocks A_p, B_p, C_p, D_p. In particular, as $t \to 0$,

$$(10) \quad Df_p(0, it)^{-1} = O(|t|^{-1/2}), \qquad \widetilde{C}_p(t) = O(|t|^{1/2}), \qquad \widetilde{D}_p(t) = O(1),$$

uniformly with respect to $p \in M$, and also $\widetilde{C}_p(t) = o(|t|^{1/2})$.

From the definition of the wedge $\widetilde{\mathscr{W}}$ (7) we see that for each $(\zeta, P) \in \widetilde{\mathscr{W}}$, with $\pi(\zeta) = p \in M$, we have

$$P = r_w^{-1}(p) r_z(p) + O(|t|),$$

so

$$(11) \qquad (P, I^{d \times d}) = r_w^{-1}(p) r_\zeta(p) + O(|t|).$$

We can now estimate $G(\zeta, p)$ as follows:

$$\begin{aligned}
G(\zeta, P) &= (P, I^{d \times d}) \cdot Df(\zeta)^{-1} \\
&= r_w^{-1}(p) \cdot r_\zeta(p) \cdot U_p^{-1} \cdot Df_p(0, it)^{-1} \cdot U_p' + O(|t|^{1/2}) \\
&= r_w^{-1}(p) \cdot (0^{d \times m}, E(p)) \cdot Df_p(0, it)^{-1} \cdot U_p' + O(|t|^{1/2}) \\
&= r_w^{-1}(p) \cdot E(p) \cdot (\widetilde{C}_p(t), \widetilde{D}_p(t)) \cdot U_p' + O(|t|^{1/2}).
\end{aligned}$$

We have used (8)–(11) in these calculations. If we take into account $\widetilde{C}_p(t) = O(|t|^{1/2})$, and

$$\begin{aligned}
(0^{d \times m}, \widetilde{D}_p(t)) \cdot U_p' &= \widetilde{D}_p(t) \cdot (0^{d \times m}, I^{d \times d}) \cdot U_p' \\
&= \widetilde{D}_p(t) \cdot E'(p)^{-1} \cdot r_{\zeta'}'(f(p)),
\end{aligned}$$

(we have used the analogue of (8) for the point $f(p)$), we finally get

$$G(\zeta, P) = [r_w^{-1}(p) E(p)] \cdot \widetilde{D}_p(t) \cdot [E'(p)^{-1} r_{\zeta'}'(f(p))] + O(|t|^{1/2}).$$

The expressions in the square brackets are continuous with respect to $p \in M$, $\widetilde{D}_p(t)$ is uniformly bounded, and the term $O(|t|^{1/2})$ is also uniform with respect to p. This implies that G is bounded on $\widetilde{\mathscr{W}}$ and Lemma 6.1 is proved.

To simplify the notation we introduce the function

$$H(\zeta) = r_w^{-1}(p) \cdot E(p) \cdot \widetilde{D}_p(t) \cdot E'(p)^{-1}, \qquad \zeta \in \mathscr{W},$$

with values in $\mathbb{C}^{d \times d}$, so

$$G(\zeta, P) = H(\zeta) \cdot r_{\zeta'}'(f(p)) + O(|t|^{1/2}), \qquad (\zeta, P) \in \widetilde{\mathscr{W}}.$$

We split G as

$$G = (G_1, G_2), \qquad G_1 \in \mathbb{C}^{d \times m}, \qquad G_2 \in \mathbb{C}^{d \times d},$$

where

$$(12) \qquad \begin{aligned}
G_1(\zeta, P) &= H(\zeta) \cdot r_{z'}'(f(p)) + O(|t|^{1/2}), \\
G_2(\zeta, P) &= H(\zeta) \cdot r_{w'}'(f(p)) + O(|t|^{1/2}).
\end{aligned}$$

Notice that the first term in $G(\zeta, P)$ does not depend on the second component P which only contributes a term $O(|t|^{1/2})$, provided of course that $(\zeta, P) \in \widetilde{\mathscr{W}}$.

6.2   PROPOSITION.   *There is a smaller wedge $\mathscr{W}_0 \subset \mathscr{W}$ with edge M and the corresponding wedge*

$$\widetilde{\mathscr{W}_0} = \{(\zeta, P) \in \widetilde{\mathscr{W}} : \zeta \in \mathscr{W}_0\},$$

with edge $\widetilde{M}$ so that the following hold:

(a) *For almost every $p \in M$ with respect to the surface measure on M, the function G has a limit*

$$\lim_{\zeta \to p} G(\zeta, P) = G^*(p) \in \mathbb{C}^{d \times n},$$

as $(\zeta, P) \in \widetilde{\mathscr{W}_0}$ and $\zeta \to p$ nontangentially within the wedge $\mathscr{W}_0$. (This means that $|\zeta - p|/\operatorname{dist}(\zeta, \partial \mathscr{W}_0)$ stays bounded.)

(b) *$|\det G_2|$ is bounded away from 0 on $\widetilde{\mathscr{W}_0}$ near $\widetilde{M}$, so $G_2^{-1}G_1$ is bounded holomorphic there.*

(c) *$G_2^{-1}G_1$ extends continuously from $\widetilde{\mathscr{W}_0}$ to M so that for each $p \in M$:*

(13) $$\lim_{\zeta \to p} G_2^{-1}G_1(\zeta, P) = r'_{w'}(f(p))^{-1} \cdot r'_{z'}(f(p)).$$

(d) *The mapping F defined by (3) (or (6)) extends continuously to $\widetilde{\mathscr{W}_0} \cup \widetilde{M}$ and coincides with $\widetilde{f}$ (2) on $\widetilde{M}$.*

PROOF.   Our final goal is to prove (d), from which the Main Theorem will follow by applying the smooth version of the edge-of-the-wedge theorem as in [23] or [15].

Clearly (d) follows from (c) since, in the affine coordinates on $\operatorname{Gr}(m, n)$, F equals

$$F(\zeta, P) = (f(\zeta), (G_2^{-1}G_1)(\zeta, P)).$$

Notice that the right hand side in (13) is just the affine coordinate of $T^{\mathbb{C}}_{f(p)}M'$, so $F|_{\widetilde{M}} = \widetilde{f}$ as required.

Also, if (b) holds, then $|\det H(\zeta)|$ is bounded away from zero when $|t|$ is sufficiently small (i.e. $\zeta \in \mathscr{W}_0$ is close to M), hence (12) implies

(14) $$(G_2^{-1}G_1)(\zeta, P) = r'_{w'}(f(p))^{-1} \cdot r'_{z'}(f(p)) + O(|t|^{1/2}).$$

As $\zeta \to p$, $t \to 0$ and we have (13).

The only factor in the matrix $H(\zeta)$ over which we have no a priori control is $\widetilde{D}_p(t)$. Clearly the property (b) is equivalent to having $|\det \widetilde{D}_p(t)|$ bounded away from zero. Unfortunately we are not able to derive such an estimate directly from Proposition 5.3.

In order to prove b) we will first show that G_2 has a.e. boundary values $G_2^*(p)$ on $\widetilde{M}$ such that $\det G_2^*(p)$ satisfies one of the estimates

$$\pm \operatorname{Re}(\det G_2^*(p)) \geq C > 0,$$

for some constant $C > 0$ and for $p \in M$ close to 0. Since G_2 is bounded holomorphic on $\widetilde{\mathscr{W}}$, the same estimate (with a smaller C) will hold on $\widetilde{\mathscr{W}_0}$

near $\widetilde{M}$, so (b) will be verified. At this point a technical problem appears: the manifold $\widetilde{M} \subset \mathbb{C}^n \times \mathrm{Gr}(m, n)$ is totally real but is not generating, unless $d = 1$ and $m = n - 1$. To avoid this problem we shall first define a manifold $\Sigma \subset \mathbb{C}^n \times \mathrm{Gr}(m, n)$ of real dimension $\dim_{\mathbb{R}} \Sigma = 2\dim_{\mathbb{R}} \widetilde{M}$, called the *approximate complexification* of $\widetilde{M}$, that contains $\widetilde{M}$ as a maximal totally real submanifold, and whose tangent bundle $T\Sigma$ is complex-linear to as higher degree as possible along $\widetilde{M}$.

When M is real-analytic, so is $\widetilde{M}$, and we let Σ be the usual complexification of $\widetilde{M}$, i.e., a complex submanifold of dimension equal to $\dim_{\mathbb{R}} \widetilde{M}$ containing $\widetilde{M}$. Such Σ is unique near $\widetilde{M}$.

When M is merely of class $\mathscr{C}^k$, so $\widetilde{M} \in \mathscr{C}^{k-1}$, we first parametrize $\widetilde{M}$ locally by a $\mathscr{C}^{k-1}$ map $\Phi : \tilde{\omega} \subset \mathbb{R}^{2m+d} \to \widetilde{M}$ defined as follows. Let $x, y \in \mathbb{R}^m$, $u \in \mathbb{R}^d$, $z = x + iy \in \mathbb{C}^m$. Define

$$P(x, y, u) = (r_w^{-1} . r_z)(z, u + i\varphi(z, \bar{z}, u))$$
$$= (\tfrac{1}{2}\varphi_u + \tfrac{i}{2}I^{d \times d})^{-1} \cdot \varphi_z(z, \bar{z}, u) \in \mathbb{C}^{d \times m},$$

and

(15) $\qquad \Phi(x, y, u) = (z, u + i\varphi(z, \bar{z}, u), P(x, y, u)) \in \mathbb{C}^n \times \mathbb{C}^{d \times m}.$

Here, $\varphi_u \in \mathbb{C}^{d \times d}$ and $\varphi_z \in \mathbb{C}^{d \times m}$ are the matrices of derivatives of $\varphi = (\varphi_1, \ldots, \varphi_d)$ (1.1) with respect to the indicated variables.

We now extend Φ to complex-valued (x, y, u) in a small neighborhood of the origin in $\mathbb{C}^{2m+d}$ so that the extension is $\mathscr{C}^{k-1}$, smooth away from $\mathbb{R}^{2m+d}$, and $\bar{\partial}\Phi$ and all its derivatives of order $\leq k - 2$ vanish on $\mathbb{R}^{2m+d}$. When M is real-analytic, we may take Φ to be holomorphic. In particular, $D\Phi$ is $\mathbb{C}$-linear at each point of $\mathbb{R}^{2m+d}$ near the origin. We let $\Sigma \subset \mathbb{C}^n \times \mathbb{C}^{d \times m}$ be the local image of Φ near $\Phi(0) = (0, T_0^{\mathbb{C}} M)$. In the real-analytic case Σ is the usual complexification of $\widetilde{M}$.

Next we want to find a nonempty wedge.

$$\mathscr{V}_0 = (\mathbb{R}^{2m+d} + i\Gamma_0) \cap V_0 \subset \mathbb{C}^{2m+d},$$

with edge $\mathbb{R}^{2m+d} \cap V_0$ (V_0 being a small neighborhood of the origin) such that

(16) $\qquad\qquad\qquad \Phi(\mathscr{V}_0) \subset \widetilde{\mathscr{W}}.$

Let $\Gamma \subset \mathbb{R}^d$ be the cone determining the wedge $\mathscr{W}$. Choose an arbitrary finer cone $\Gamma' < \Gamma$ and let $\Gamma_0 \subset \mathbb{R}^{2m+d}$ be a cone contained in a small conical neighborhood of $\{0\}^{2m} \times \Gamma'$, satisfying $\Gamma_0 \cap (\{0\}^{2m} \times \mathbb{R}^d) = \Gamma'$. We claim that the inclusion (16) holds provided that Γ_0 and V_0 are chosen sufficiently small. To see this, notice that for each $t \in \mathbb{R}^d$ the vector

$$iD_t\Phi(0) = i\sum_{j=1}^{d} t_j \partial\Phi/\partial u_j(0),$$

belongs to $T_{\Phi(0)}\Sigma$. Since at the origin φ contains no quadratic terms except the Levi form, a simple calculation shows

$$iD_t\Phi(0) = (0, it, 0).$$

If $t \in \Gamma$, then $iD_t\Phi(0)$ points to the interior of the wedge $\widetilde{\mathscr{W}}$. The definition of $\mathscr{V}_0$ implies the inclusion (16).

Consider now the smooth bounded function $G \circ \Phi$ on $\mathscr{V}_0$. Denote by d the distance from the edge in $\mathscr{V}_0$. Since G is bounded holomorphic on $\widetilde{\mathscr{W}}$, the Cauchy estimates give $|\partial G| = O(d^{-1})$. Also, $|\bar\partial\Phi| = O(d^{k-2})$ by construction. Thus the chain rule gives

$$\bar\partial(G \circ \Phi) = O(d^{-1}) \cdot O(d^{k-2}) = O(d^{k-3}).$$

Since $k > 2$, Theorem 4 in [14] implies that $G \circ \Phi$ has a nontangential limit within $\mathscr{V}_0$ at almost every point of the edge $\mathbb{R}^{2m+d} \cap V_0$. The cited theorem is stated in [14] only for a special cone Γ^*, but since we have considerable freedom in choosing Γ_0, we may assume that Γ_0 can be covered by finitely many cones isomorphic to Γ^*. Thus the result applies in our situation.

This implies that at almost every point $(p, T_p^{\mathbb{C}}M) \in \widetilde{M}$, the function $G(\zeta, P)$ has a limit $G^*(p)$ as $(\zeta, P) \to (p, T_p^{\mathbb{C}}M)$ nontangentially within the wedge $\Phi(\mathscr{V}_0) \subset \Sigma \cap \widetilde{\mathscr{W}}$.

Clearly the first coordinate projection of $\Phi(\mathscr{V}_0)$ onto $\mathbb{C}^n$ contains a finer wedge $\mathscr{W}_0 < \mathscr{W}$ with edge M. Let $\widetilde{\mathscr{W}_0} \subset \widetilde{\mathscr{W}}$ be the corresponding wedge with edge $\widetilde{M}$ defined by

$$\widetilde{\mathscr{W}_0} = \{(\zeta, P) \in \widetilde{\mathscr{W}} : \zeta \in \mathscr{W}_0\}.$$

Since the second coordinate P only contributes a term $O(|t|^{1/2})$ in $G(\zeta, P)$ that vanishes as $\zeta \to M$ (see (12)), $G(\zeta, P)$ has the same nontangential limit

$$\lim_{\zeta \to p} G(\zeta, P) = G^*(p),$$

as $(\zeta, P) \in \widetilde{\mathscr{W}_0}$ and $\zeta \to p$ nontangentially in $\mathscr{W}_0$.

Fix a point $p \in M$ at which the limit exists. From (12) it follows that

$$\lim_{t \to 0} \widetilde{D}_p(t) = \widetilde{D}^*(p),$$

also exists as $t \to 0$ through certain cone in $\mathbb{R}^d$ so that $\zeta = \psi_p^{-1}(0, it) \in \mathscr{W}_0$.

We claim that the limit $\widetilde{D}^*(p) \in \mathrm{GL}(d, \mathbb{R})$ is real-valued and $|\det \widetilde{D}^*(p)|$ is bounded away from zero, uniformly with respect to $p \in M$. To prove this, note first that the estimates (a) and (b) in Proposition 5.3 imply

$$\det Df_p(0, it) = \det A_p(t) \det D_p(t) + o(1),$$

as $t \to 0$. Since this is bounded away from zero and $|\det A_p(t)|$, $|\det D_p(t)|$ are bounded from above, they are also bounded away from zero for $|t|$ small.

Thus $D_p(t)$ is invertible, and the formulas for calculating the inverse matrix show that $|D_p(t)^{-1} - \tilde{D}_p(t)| = o(1)$. Hence

$$\exists \lim_{t \to 0} D_p(t) = \tilde{D}^*(p)^{-1}.$$

Proposition 5.3 (c) implies that $\tilde{D}^*(p)$ is real-valued, and we have already seen above that $|\det \tilde{D}^*(p)|$ is bounded and bounded away from 0, uniformly in p.

Now we can see from (12) that the second component $G_2^*(p) \in \mathbb{C}^{d \times d}$ has $\det G_2^*(p)$ nearly real-valued for $p \in M$ sufficiently close to 0, and it is bounded away from 0. Thus there is a $C > 0$ such that for each $p \in M$ where the limit exists we have

$$\pm \operatorname{Re}[\det G_2^*(p)] \geq C > 0. \tag{17}$$

A priori the sign depends on p, but we claim that one of the two signs holds for almost all $p \in M$.

This follows from the following well-known fact. Let h be a bounded holomorphic function on the unit disc $\Delta \subset \mathbb{C}$, with a.e. boundary values h^* on $\partial \Delta$. Assume that we have an arc $I = I_1 \cup I_2 \subset \partial \Delta$ such that I_1, I_2 are measurable sets, $\operatorname{Re} h^* \geq 1$ a.e. on I_1, and $\operatorname{Re} h^* \leq -1$ a.e. on I_2. Then one of the sets I_1, I_2 has measure zero. Here is a sketch of the proof. Suppose both I_1 and I_2 have positive measure. By Runge's approximation theorem there is a polynomial P on $\mathbb{C}$ such that $g = P \circ h$ is arbitrarily close to 0 on I_1 and to 1 on I_2. Suppose for the sake of the argument that $I_1 \cup I_2 = \partial \Delta$. Since $\log |g|$ is subharmonic, we have

$$\log |g(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |g^*(e^{i\theta})| \, d\theta,$$

and this is very small since $|g^*|$ is close to 0 on I_1. Applying the same to $\log |1 - g|$ we get that $\log |1 - g(0)|$ is also very small, a contradiction. If $I_1 \cup I_2 = I$ is just a proper subarc of $\partial \Delta$ we apply a similar proof for a suitably chosen point $z \in \Delta$ close to I.

We apply this to the function $h = (\det G_2) \circ \Phi$ on the wedge $\mathscr{V}_0$. We have that $|\bar{\partial} h| = O(d^{k-3})$ is bounded since $k \geq 3$. On each linear disc $\Delta \subset \mathscr{V}_0$ abutting the edge $\mathbb{R}^{2m+d} \cap V_0$ along an arc I we first correct h to a holomorphic function $\tilde{h}$ by $\tilde{h} = h - T(\bar{\partial} h|_\Delta)$, where T is the Cauchy-Riemann operator on Δ solving the equation $\bar{\partial}(Tu) = u$, chosen so that $T(\bar{\partial} h|_\Delta)$ vanishes at a prescribed point $p_0 \in I$. Since T maps $L^\infty(\Delta)$ boundedly into each Hölder space $\mathscr{C}^\alpha(\Delta)$ $(\alpha < 1)$, the correction function is so small (provided that Δ is small) that $\pm \operatorname{Re} \tilde{h}^*(p) \geq C/2$ a.e. on $I_1 \cup I_2 = I$. As before we conclude that one of the two sets I_1, I_2 must have measure zero. Clearly we have enough discs in $\mathscr{V}_0$ to prove that either $\operatorname{Re} h^* \geq C$ or $-\operatorname{Re} h^* \geq C$ a.e. on the edge $\mathbb{R}^{2m+d} \cap V_0$.

Thus we may assume that (17) holds with the $+$ sign for almost all $p \in M$; the proof in the other case is analogous. The proof of Theorem 4 in [14], applied to the function $h = (\det G_2) \circ \Phi$ on $\mathscr{V}_0$, shows that

$$\mathrm{Re}[(\det G_2) \circ \Phi] \geq C/2,$$

on $\mathscr{V}_0$ sufficiently close to the edge $\mathbb{R}^{2m+d} \cap V_0$. Now (12) implies $\mathrm{Re}(\det G_2) \geq C/2$ on $\widetilde{\mathscr{W}_0}$ near $\widetilde{M}$. This proves Proposition 6.2 (b), so the Proposition is proved.

CONCLUSION OF THE PROOF OF MAIN THEOREM. Recall that we have totally real manifolds $\widetilde{M}$, $\widetilde{M}' \subset \mathbb{C}^n \times \mathbb{C}^{d \times m} = \mathbb{C}^N$ of class $\mathscr{C}^{k-1}$, a wedge domain $\widetilde{\mathscr{W}_0} \subset \mathbb{C}^N$ with edge $\widetilde{M}$, and a bounded holomorphic mapping $F : \widetilde{\mathscr{W}_0} \to \mathbb{C}^N$ that extends continuously to $\widetilde{\mathscr{W}_0} \cup \widetilde{M}$ and maps $\widetilde{M}$ into $\widetilde{M}'$.

If the manifolds $\widetilde{M}$ and $\widetilde{M}'$ were generating in $\mathbb{C}^N$ (i.e., of real dimension N), we could apply Theorem 3 in [23] to conclude that $F|_{\widetilde{M}}$ is of class $\mathscr{C}^{k-1-0}$, whence $f|_M \in \mathscr{C}^{k-1-0}$ as required. (For this we would only need $k \geq 3$.)

Unfortunately $\widetilde{M}$ and $\widetilde{M}'$ are not generating unless $d = 1$, and we must do some more work to reach the same conclusion. We shall assume $k > 3$ in the rest of the proof. (However, k need not be an integer.)

Let $s = 2m + d$, and let $\Phi : \tilde{\omega} \subset \mathbb{C}^s \to \mathbb{C}^N$ be the mapping as in (15) whose restriction to $\omega = \mathbb{R}^s \cap \tilde{\omega}$ locally parametrizes M, and such that $\bar{\partial}\Phi$ vanishes to order $k - 2$ on ω. In the terminology and notation of [23], Φ is asymptotically holomorphic of order $(k - 2, k - 1)$ on $\tilde{\omega}$ at the edge ω, $\Phi \in O_{(k-2,k-1)}(\tilde{\omega})$.

Let Φ' be the analogous mapping associated to $\widetilde{M}'$, except in this case we extend Φ' to a neighborhood of 0 in $\mathbb{C}^N$ as a local diffeomorphism onto its image in $\mathbb{C}^N$. Clearly its inverse $\Psi = (\Phi')^{-1}$ maps $\widetilde{M}'$ to $\mathbb{R}^s \times \{0\} \subset \mathbb{C}^N$ and is asymptotically holomorphic of the same order $(k - 2, k - 1)$ on $\widetilde{M}'$.

We now show that $\Psi \circ F \in \mathscr{C}^\infty(\widetilde{\mathscr{W}_0})$ is asymptotically holomorphic of order $((k - 3)/2, 0)$ at the edge $\widetilde{M}$. Let Z be the coordinate on $\mathbb{C}^N$. From the distance estimate for f (Proposition 5.2) and from (14) we obtain the distance estimate

(18) $\mathrm{dist}(F(Z), \widetilde{M}') \leq C \cdot \mathrm{dist}(Z, \widetilde{M})^{1/2},$

for $Z \in \widetilde{\mathscr{W}_0}$. The usual argument involving the Kobayashi metric then gives

$$|\partial F / \partial Z_j(Z)| = O(\mathrm{dist}(Z, \widetilde{M})^{-1/2}), \qquad Z \in \widetilde{\mathscr{W}_0}.$$

(See the proof of Lemma 2 in [23].) The chain rule gives

$$\partial(\Psi \circ F) / \partial \overline{Z}_j(Z) = \sum_{l=1}^N (\partial \Psi / \partial \overline{Z}'_l)(F(Z)) \cdot \overline{\partial F_l / \partial Z_j(Z)}.$$

Set $d = \text{dist}(Z, \widetilde{M})$. The first term in each product on the right is $O(d^{(k-2)/2})$ according to (18) and the construction of Ψ, and the second term is $O(d^{-1/2})$. Thus

$$|\overline{\partial}(\Psi \circ F)| = O(d^{(k-3)/2}).$$

Similarly we can obtain the approximate estimates for the higher order derivatives of $\Psi \circ F$, so $\Psi \circ F \in O_{((k-3)/2, 0)}(\widetilde{\mathscr{W}_0})$ as claimed.

Let $\mathscr{V}_0 = \mathscr{V}_0^+$ be the wedge (16) in $\mathbb{C}^s$, with the edge $\mathbb{R}^s \cap V_0$. Since $\Phi \in O_{k-2, k-1}(\mathscr{V}_0)$ and $k - 2 \geq (k-3)/2 + 1$, the composition

$$F^+ = \Psi \circ F \circ \Phi \in \mathscr{C}^\infty(\mathscr{V}_0)$$

is in $O_{((k-3)/2, 0)}(\mathscr{V}_0)$ according to Proposition 2 in [23]. Notice that F^+ extends continuously to the edge $\mathbb{R}^s \cap V_0$ and maps it into $\mathbb{R}^s \times \{0\} \subset \mathbb{C}^N$. Using the antiholomorphic reflection $Z \to \overline{Z}$ on both the domain and the target we extend F to a mapping F^-, defined on the opposite wedge

$$\mathscr{V}_0^- = (\mathbb{R}^s - i\Gamma_0) \cap V_0,$$

so that $F^- \in O_{((k-3)/2, 0)}(\mathscr{V}_0^-)$, and F^- matches with F^+ on the common edge $\mathbb{R}^s \cap V_0$. Theorem 1 in [23] implies that the restriction $F^+|_{\mathbb{R}^s \cap V_0}$ is smooth of class $\mathscr{C}^{(k-1)/2-0}$, so $F|_{\widetilde{M}}$ is of the same class.

Since $k > 3$, we get $F|_{\widetilde{M}} \in \mathscr{C}^1$. This implies that the distance estimate (18) holds without the power $1/2$ on the right, and $|\partial F|$ is bounded on $\widetilde{\mathscr{W}_0}$. Repeating the same proof with these improved estimate gives $F^\pm \in O_{(k-2, 0)}(\mathscr{V}_0^\pm)$, so Theorem 1 in [23] implies $F|_{\widetilde{M}} \in \mathscr{C}^{k-1-0}$. This proves $f|_M \in \mathscr{C}^{k-1-0}$ when $k > 3$. The same applies to f^{-1}, so the Main Theorem is proved.

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