# Discs in pseudoconvex domains.

# Discs in pseudoconvex domains

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## 1. Introduction

Let $D \subset \mathbf{C}^N$ be a domain in the complex Euclidean space $\mathbf{C}^N$ ($N > 1$), and let $y$ be a point in $D$. There exist many closed complex one-dimensional subvarieties (curves) $V \subset D$ passing through $y$. For instance, it suffices to take the common zero set $f_1 = f_2 = \cdots = f_s = 0$ of suitably chosen holomorphic functions on $D$ that vanish at $y$.

A special class of closed complex curves in $D$ are the *proper analytic discs*, i.e., the images $F(\Delta)$ of proper holomorphic maps $F : \Delta \to D$ from the open unit disc $\Delta \subset \mathbf{C}$ into $D$. A natural question appears [6]: *Given a point $y \in D$, can we find a proper analytic disc in $D$ passing through $y$?*

In this article we give a positive answer to this question for all bounded pseudoconvex domains in $\mathbf{C}^N$ with $\mathscr{C}^2$ boundary, and a counterexample for non-pseudoconvex domains with disconnected boundary. More precisely, we prove the following results:

THEOREM 1. *Let $D \subset\subset \mathbf{C}^N$ be a strongly pseudoconvex domain with boundary of class $\mathscr{C}^k$, with $N, k \geq 2$. Given a point $y \in D$ and a vector $X \in \mathbf{C}^N$, there is a mapping $F : \bar{\Delta} \to \bar{D}$ of class $\mathscr{C}^{k-0}(\bar{\Delta})$ that is holomorphic on the open disc $\Delta$ and satisfies $F(b\Delta) \subset bD$, $F(0) = y$, and $F'(0) = \lambda X$ for some $\lambda > 0$.*

Stated informally, the theorem asserts that through each point of a strongly pseudoconvex domain in any given direction there passes a proper analytic disc that is smooth up to the boundary. Here, as usual, $\mathscr{C}^{k-0} = \mathscr{C}^k$ if $k$ is not an integer, and $\mathscr{C}^{k-0} = \bigcup_{0 < \alpha < 1} \mathscr{C}^{k-1,\alpha}$ if $k$ is an integer.

We have a similar result for smoothly bounded weakly pseudoconvex domains, except that we are not able to get smoothness up to the boundary:

THEOREM 2. *Let $D \subset\subset \mathbf{C}^N$ ($N \geq 2$) be a pseudoconvex domain with boundary of class $\mathscr{C}^2$. Given a point $y \in D$ and a vector $X \in \mathbf{C}^N$ there is a proper holomorphic map $F : \Delta \to \bar{D}$ satisfying $F(0) = y$ and $F'(0) = \lambda X$ for some $\lambda > 0$.*

If $D$ admits a defining function that is plurisubharmonic near $bD$, then one can of course apply Theorem 1 to get a proper analytic disc in $D$ through $y$ that is of class $\mathscr{C}^{2-0}$ up to the boundary.

By technical modifications of our method one can construct proper analytic discs as above satisfying various additional properties. For instance, if $N \geq 3$, there exist proper holomorphic *embeddings* $F : \Delta \to D \subset \mathbf{C}^N$ satisfying $F(0) = y$ and $F'(0) = \lambda X$, and for $N = 2$ there exist holomorphic *immersions* with the same properties. Moreover, we shall see from the construction that one can prescribe, up to a positive scalar, any finite number of derivatives $F'(0)$, $F''(0), \ldots$ of the map $F$ at the origin. We leave out the details.

From Theorems 1 and 2 and from our Main Lemma in section two it follows immediately that there exist proper analytic discs in $D$ containing a given finite subset of $D$:

COROLLARY 3. *Let* $D \subset\subset \mathbf{C}^N$ $(N > 1)$ *be a pseudoconvex domain with* $\mathscr{C}^2$ *boundary. For each finite set of points* $y_1, y_2, \ldots, y_n \in D$ *and vectors* $X_1$, $X_2, \ldots, X_n \in \mathbf{C}^N$ *there are a proper holomorphic map* $F : \Delta \to D$ *and points* $\zeta_1$, $\zeta_2, \ldots, \zeta_n \in \Delta$ *such that for each $j$, $1 \leq j \leq n$, we have* $F(\zeta_j) = y_j$ *and* $F'(\zeta_j) = \lambda_j X_j$ *for some* $\lambda_j > 0$. *If $D$ is strictly pseudoconvex with* $\mathscr{C}^k$ *boundary, there is an $F$ as above that is of class* $\mathscr{C}^{k-0}$ *on* $\bar{\Delta}$.

If $D \subset\subset \mathbf{C}^N$ is a convex domain, then according to [6] there exist proper analytic discs in $D$ passing through any given *discrete* subset of $D$. It is very likely that by combining the techniques of this paper with those in [6] one can prove the same result for all bounded pseudoconvex domains with $\mathscr{C}^2$ boundary.

Virtually the same technique can be used to prove Theorems 1 and 2 for relatively compact pseudoconvex domains with $\mathscr{C}^2$ boundary in an $N$-dimensional Stein manifold. For strongly pseudoconvex domains one can use the embedding theorem of Fornæss and Henkin [9]. Again we shall not go into details of this.

We show by an example that pseudoconvexity cannot be entirely deleted from our hypothesis:

THEOREM 4. *For each $N \geq 2$ there exist a smoothly bounded domain $D \subset\subset \mathbf{C}^N$ and a point $x \in D$ such that there is no proper holomorphic map $F : \Delta \to D$ with $x \in F(\Delta)$.*

Here are some related open problems:

1. When $D$ is a weakly pseudoconvex domain with smooth boundary that does not admit a plurisubharmonic defining function, can we find discs as in Theorem 2 that are smooth up to the boundary?

2. Does Theorem 2 still hold if we assume no boundary regularity of $D$?

3. Let $\mathcal{M}$ be a Stein manifold of dimension $n \geq 2$. Given a point $p \in \mathcal{M}$, does there exist a proper holomorphic map $F : \Delta \to \mathcal{M}$ with $F(0) = p$? If so, can one also prescribe the direction of $F'(0)$ as above? Can one find analytic discs in $\mathcal{M}$ that contain any given finite (or discrete) subset of $\mathcal{M}$?

Another related problem is the following. Suppose that for each $\zeta \in b\Delta$ we are given a strongly pseudoconvex domain $D_\zeta \subset\subset \mathbf{C}^N$ containing the origin. Suppose also that the boundaries $bD_\zeta$ depend continuously or even smoothly on $\zeta \in b\Delta$. The problem is to construct continuous maps $F : \bar{\Delta} \to \mathbf{C}^N$, holomorphic on $\Delta$, such that $F(\zeta) \in bD_\zeta$ ($\zeta \in b\Delta$). Such maps are known to exist when all $D_\zeta$ are convex [1], [4], [7], [11], and in this case their graphs fill up the interior of the entire polynomially convex hull of the set $K = \bigcup_{\zeta \in b\Delta} \bar{D}_\zeta$. In the non-convex case the problem is well understood only for $N = 1$, see [3], [8], and [12]. Using the methods of this paper one can solve this problem under suitable additional assumptions on a defining function $P : b\Delta \times \mathbf{C}^N \to \mathbf{R}$ satisfying $D_\zeta = \{z \in \mathbf{C}^N : P(\zeta, z) < 0\}$.

The paper is organized as follows. In section two we state our Main Lemma, and based on this Lemma we prove Theorems 1 and 2 and Corollary 3. Section three contains technical results required in the proof of the Main Lemma in section four. In section five we construct the example claimed by Theorem 4.

## 2. Proof of Theorems 1 and 2

We first show that it suffices to prove Theorems 1 and 2 and Corollary 3 for $N = 2$.

Let $D \subset\subset \mathbf{C}^N$ be a pseudoconvex domain with $\mathscr{C}^2$ boundary, $y \in D$ and $X \in \mathbf{C}^N \backslash \{0\}$. Choose a complex basis $X_1 = X, X_2, \ldots, X_N$ of $\mathbf{C}^N$. On the space $\mathbf{C}^2 \times \mathbf{C}^{2N}$ we use the coordinates $z = (z_1, z_2)$, $\lambda = (\lambda_1, \ldots, \lambda_N)$, $v = (v_1, \ldots, v_N)$. Let $\Phi : \mathbf{C}^2 \times \mathbf{C}^{2N} \to \mathbf{C}^N$ be the entire map defined by

$$\Phi(z, \lambda, v) = y + z_1 X_1 + z_2 X_2 + \sum_{j=1}^{N} (\lambda_j z_1^2 + v_j z_2^2) X_j.$$

Notice that $\Phi(0, \cdot, \cdot) \equiv y$ and $\partial\Phi/\partial z_j(0, \cdot, \cdot) = X_j$ for $j = 1, 2$.

If $\Phi(z, \lambda, v) = p \in bD$, then at least one of the variables $z_1, z_2$ is nonzero, hence $\Phi$ is a submersion here. It follows that $\Phi$ is transverse to $bD$. By the transversality theorem we conclude that for almost all values $\lambda_0, v_0 \in \mathbf{C}^N$ the map

$\Phi( \cdot , \lambda_0, v_0) : \mathbf{C}^2 \to \mathbf{C}^N$ is transverse to $bD$. Choosing $\lambda_0$ and $v_0$ sufficiently small we insure that the connected component $\Omega_0$ of the preimage $\{z \in \mathbf{C}^2 : \Phi(z, \lambda_0, v_0) \in D\}$ containing the origin is a bounded pseudoconvex domain in $\mathbf{C}^2$ with $\mathscr{C}^2$ boundary. If $D$ is strongly pseudoconvex then so is $\Omega_0$.

Suppose that Theorems 1 and 2 and Corollary 3 hold in dimension two. If $F_0 : \Delta \to \Omega_0$ is a proper holomorphic map satisfying $F_0(0) = 0$ and $F_0'(0) = (\lambda, 0)$, then

$$F(\zeta) = \Phi(F_0(\zeta), \lambda_0, v_0) \qquad (\zeta \in \Delta)$$

is a proper holomorphic map of $\Delta$ into $D$ satisfying Theorem 1 resp. 2. Similarly one proves Corollary 3.

From now on we shall only consider the case $N = 2$. Let $D \subset\subset \mathbf{C}^2$ be a pseudoconvex domain with $\mathscr{C}^2$ boundary. According to [2] there is a $\mathscr{C}^2$ defining function $\tau$ for $D$ such that $D = \{\tau < 0\}$, $\nabla\tau \neq 0$ near $bD$, and there is a negative strictly plurisubharmonic function $\rho$ on $D$ such that near $bD$ we have $\rho = -(-\tau)^\epsilon$. In particular, there is a $T < 0$ such that the gradient $(\nabla\rho)(z)$ is nonvanishing for $T < \rho(z) < 0$, and the domains

$$D(t) = \{z \in D : \rho(z) < t\} \qquad (T < t < 0)$$

are strongly pseudoconvex with $\mathscr{C}^2$ boundary. When $D$ itself is strongly pseudoconvex we can of course choose a defining function $\rho$ for $D$ satisfying these properties. For each $t_1, t_2, T < t_1 < t_2 < 0$, let

$$\mathscr{V}(t_1, t_2) = \{z \in D : t_1 < \rho(z) < t_2\}.$$

MAIN LEMMA. *Let $T < t_1 < t_2 < 0$. There is a $\mu_0 > 0$, depending only on $t_1, t_2$, with the following property: Let $0 < r < 1$ and let $F : \bar{\Delta}\backslash r\Delta \to \mathscr{V}(t_1, t_2)$ be a continuous map satisfying $\rho(F(\zeta)) > c$ $(\zeta \in \bar{\Delta}\backslash r\Delta)$. Suppose that $\mu$ is a positive continuous function on $b\Delta$ such that $\mu(\zeta) \le \mu_0$ $(\zeta \in b\Delta)$, let $\epsilon > 0$, and let $0 < R < 1$. Then there exists a continuous map $G : \Delta \to \mathbf{C}^2$ that is holomorphic on $\Delta$ and satisfies*
  (i) $F(\zeta) + G(\zeta) \in D$ $(\zeta \in \bar{\Delta}\backslash r\Delta)$,
  (ii) $\rho(F(\zeta) + G(\zeta)) > c$ $(\zeta \in \bar{\Delta}\backslash r\Delta)$,
  (iii) $|\rho(F(\zeta) + G(\zeta)) - \rho(F(\zeta)) - \mu(\zeta)| < \epsilon(\zeta \in b\Delta)$,
  (iv) $|G(\zeta)| < \epsilon(|\zeta| \le R)$, *and*
  (v) $G(0) = 0, G'(0) = 0.$

*Remark.* The Lemma also holds if we choose finitely many points $\zeta_1$, $\zeta_2, \ldots, \zeta_n \subset \Delta$ and replace (v) by the following stronger condition:

(v') $G(\zeta_i) = 0$ and $G'(\zeta_j) = 0$ for $1 \le j \le n$.

We defer the proof of the Main Lemma to section four below.

*Proof of Theorem* 1. It suffices to prove the following: If $T < t_0 < 0$, $y \in D(t_0)$, and $X \in \mathbb{C}^2$, there is a map $F: \bar{\Delta} \to \overline{D(t_0)}$ of class $\mathscr{C}^{2-0}$ that is holomorphic on $\Delta$, $F(0) = y$, and $F'(0) = \lambda X$ for some $\lambda > 0$. (If $\rho$ is of class $\mathscr{C}^k$, the same proof will give $F \in \mathscr{C}^{k-0}(\bar{\Delta})$.)

Choose $t_1$ and $t_2$ such that $T < t_1 < t_0 < t_2 < 0$, and let $\mu_0$ be as in the Main Lemma, chosen small enough such that $t_0 - \mu_0 \ge t_1$. Denote by $\mathbf{B}^2$ the open unit ball in $\mathbb{C}^2$. There is an $\epsilon > 0$ such that

$$\overline{D(t_0)} + 2\epsilon \mathbf{B}^2 \subset D(t_2), \qquad bD(t_0) + 2\epsilon \mathbf{B}^2 \subset \mathscr{V}(t_0 - \mu_0, t_2). \tag{1}$$

We show the following:

LEMMA 1. *Suppose that* $f: \bar{\Delta} \to D(t_2)$ *is a continuous map that is holomorphic on* $\Delta$ *and satisfies* $f(0) \in D(t_0)$ *and* $f(b\Delta) \subset \mathscr{V}(t_0, t_2)$. *Given* $x \in D(t_0)$, $|x - f(0)| < \epsilon$, *there is a continuous map* $f_1: \bar{\Delta} \to D(t_2)$, *holomorphic on* $\Delta$, *satisfying* $f_1(0) = x, f'_1(0) = \lambda f'(0)$ *for some* $\lambda > 0$, *and* $f_1(b\Delta) \subset \mathscr{V}(t_0, t_2)$.

*Proof.* Let $t'_0, t_0 < t'_0 < t_2$, be so close to $t_0$ that $f(b\Delta) \subset \mathscr{V}(t'_0, t_2)$ and

$$D(t) \subset D(t_0) + \epsilon \mathbf{B}^2, \qquad bD(t) \subset bD(t_0) + \epsilon \mathbf{B}^2 \qquad (t_0 \le t \le t'_0). \tag{2}$$

By Sard's theorem there is a $t, t_0 < t < t'_0$, such that $\Omega = \{\zeta \in \Delta : \rho(f(\zeta)) < t\}$ is a relatively compact domain in $\Delta$ with $\mathscr{C}^2$ boundary. By the maximum principle, applied to the subharmonic function $\rho \circ f$, each connected component of $\Omega$ is simply connected, thus conformally equivalent to the disc. Let $\phi : \Delta \to \Omega_0$ be the conformal map onto the connected component $\Omega_0$ of $\Omega$ containing the origin, chosen such that $\phi(0) = 0$ and $\phi'(0) > 0$. Since the boundary of $\Omega_0$ is of class $\mathscr{C}^2$, $\phi$ extends to be of class $\mathscr{C}^{2-0}$ on $\bar{\Delta}$. The composition $\Phi = f \circ \phi: \bar{\Delta} \to \overline{D(t)}$ is of class $\mathscr{C}^{2-0}$, holomorphic on $\Delta$, and satisfies $\Phi(0) = f(0)$, $\Phi'(0) = \phi'(0)f'(0)$, and $\Phi(b\Delta) \subset bD(t)$.

Let $x \in D(t_0)$, $|x - f(0)| < \epsilon$. Then by (1) and (2),

$$\zeta \in \bar{\Delta} \to g(\zeta) = \Phi(\zeta) + (x - \Phi(0))$$

is a continuous map from $\bar{\Delta}$ into $\overline{D(t_0)} + 2\epsilon\mathbf{B}^2 \subset D(t_2)$ that is holomorphic on $\Delta$ and satisfies $g(b\Delta) \subset \mathscr{V}(t_0 - \mu_0, t_2)$. Choose an $r, 0 < r < 1$, such that $g(\bar{\Delta}\backslash r\Delta) \subset \mathscr{V}(t_0 - \mu_0, t_2)$. Choose $\eta > 0$ so small that $g(r\Delta) + \eta\mathbf{B}^2 \subset D(t_2)$. By the Main Lemma applied to $g$ there is a continuous map $h : \bar{\Delta} \to \mathbf{C}^2$, holomorphic on $\Delta$, satisfying $h(0) = 0, h'(0) = 0, |h(\zeta)| < \eta$ for $|\zeta| < r$, such that the map $f_1 = g + h$ satisfies $f_1(\bar{\Delta}\backslash r\Delta) \subset D(t_2)$ and $f_1(b\Delta) \subset \mathscr{V}(t_0, t_2)$. If $\zeta \in r\Delta$, then $|h(\zeta)| < \eta$, hence $f_1(\bar{\Delta}) \subset D(t_2)$. This proves Lemma 1.

We can now complete the proof of Theorem 1. Without loss of generality we may assume that the domain $D(t_0)$ is connected. There is a point $y_0 \in bD(t_0)$ such that the complex tangent space to $bD(t_0)$ at $y_0$ is spanned by the given vector $X \in \mathbf{C}^2\backslash\{0\}$.

We claim that there are a point $y_1 \in D(t_0)$ close to $y_0$ and a map $f_0 : \bar{\Delta} \to D(t_2)$, holomorphic on $\Delta$, satisfying $f_0(0) = y_1, f_0'(0) = \lambda X$ for some $\lambda > 0$, and $f_0(b\Delta) \subset \mathscr{V}(t_0, t_2)$. This can be seen immediately from the proof of Narasimhan's lemma [9, p. 111]: locally near $y_0$ we convexify the domain $D(t_0)$ by a local biholomorphic change of coordinates, we take a suitable linear disc in the convexified domain, and then pull it back to a disc in $D(t_2)$ satisfying the required properties. Of course it is essential that $X$ is complex tangent to $bD(t_0)$ at $y_0$!

Using Lemma 1 a finite number of times we can slide the initial disc $f_0$ to a disc $f_1 : \bar{\Delta} \to D(t_2)$ satisfying the same conditions, except that the new center is $f_1(0) = y$. By a generic perturbation of $f_1$ we insure that $f_1$ intersects the boundary of $D(t_0)$ transversely. Replacing $f_1$ by $f_1 \circ \phi$, where $\phi$ is a suitable conformal map of $\Delta$ onto the connected component of $\{\zeta \in \Delta : f_1(\zeta) \in D(t_0)\}$ containing 0 (see the proof of Lemma 1), we obtain the final map $F$ satisfying Theorem 1.

*Proof of Theorem 2.* Choose $t_0$, $T < t_0 < 0$, such that $y \in D(t_0)$. We choose sequences $t_0 < t_1 < t_2 < \cdots < 0$, $\lim_{j \to \infty} t_j = 0$, and $\epsilon_0 > \epsilon_1 > \epsilon_2 > \cdots > 0$, $\lim_{j \to \infty} \epsilon_j = 0$, such that

$$\overline{D(t_n)} + \epsilon_{n-1}\mathbf{B}^2 \subset D(t_{n+1}) \qquad (n = 1, 2, \dots). \tag{3}$$

We show that there are an increasing sequence of radii $r_0 < r_1 < r_2 < \cdots < 1$ with $\lim_{j \to \infty} r_j = 1$ and a sequence of continuous mappings $f_n : \bar{\Delta} \to D$ $(n = 1, 2, \dots)$, holomorphic on $\Delta$, such that for each $n = 1, 2, \dots$ the following hold:

(i) $f_n(\bar{\Delta}) \subset D(t_{n+1})$,

(ii) $\rho(f_n(\zeta)) > t_{n-1}$ $(\zeta \in \bar{\Delta}\backslash r_{n-1}\Delta)$,

(iii) $t_n < \rho(f_n(\zeta)) < t_{n+1}$ $(\zeta \in \bar{\Delta}\backslash r_n\Delta)$,

(iv) $|f_{n+1}(\zeta) - f_n(\zeta)| < \epsilon_n/2^n$ $(\zeta \in r_n\Delta)$, and

(v) $f_n(0) = y, f_n'(0) = \lambda X$ for some $\lambda > 0$ independent of $n$.

The construction is by induction on $n$. By Theorem 1 there is a continuous map $f_1 : \bar{\Delta} \to D(t_2)$, holomorphic on $\Delta$, such that $f_1(0) = y$, $f_1'(0) = \lambda X$, and $t_1 < \rho(f_1(\zeta)) < t_2$ for $\zeta \in b\Delta$. Choose $r_0, r_1, 0 < r_0 < r_1 < 1$, such that (i), (ii), (iii), and (v) are satisfied for $n = 1$.

Suppose that $f_j$ and $r_j$ have been constructed for $1 \le j \le n$ so that (i), (ii), (iii), and (v) are satisfied. Using the Main Lemma a finite number of times we get a continuous map $f_{n+1} = f_n + g_n : \bar{\Delta} \to D$, holomorphic on $\Delta$, and a number $r_{n+1}, r_n < r_{n+1} < 1$, such that (iv) holds and (i), (ii), (iii), and (v) hold with $n$ replaced by $n + 1$.

Now, (iv) shows that the sequence $f_n$ converges uniformly on compact sets in $\Delta$ to a holomorphic map $F$. By (v) we have $F(0) = y$ and $F'(0) = \lambda X$. For $\zeta \in r_n \Delta$ we have $|F(\zeta) - f_n(\zeta)| < \epsilon_n$ by (iv), hence (i) and (3) imply

$$F(r_n \Delta) \subset D(t_{n+1}) + \epsilon_n \mathbf{B}^2 \subset D.$$

Thus $F(\Delta) \subset D$.

It remains to show that $F$ is a proper map into $D$. Let $\zeta \in r_{n+1}\Delta \setminus r_n\Delta$. By (ii) we have $\rho(f_{n+1}(\zeta)) > t_n$, and by (iv) $|F(\zeta) - f_{n+1}(\zeta)| < \epsilon_{n+1}$. Since $\epsilon_{n+1} < \epsilon_{n-2}$, (3) implies $\rho(F(\zeta)) \ge t_{n-1}$. This proves that for each $n, \rho(F(\zeta)) \ge t_{n-1}$ ($r_n < |\zeta| < 1$), which shows that $F : \Delta \to D$ is a proper map. Theorem 2 is proved.

*Proof of Corollary* 3. Choose $t_0, T < t_0 < 0$, such that $y_j \in D(t_0)$ $(1 \le j \le n)$. Let $\Delta_j \subset \mathbf{C}$ be the open disc of radius one with center at $3j \in \mathbf{C}$. By Theorem 1 there exist continuous maps $F_j : \bar{\Delta}_j \to \overline{D(t_0)}$, holomorphic on $\Delta_j$, satisfying $F_j(b\Delta) \subset bD(t_0)$, $F_j(3j) = y_j$, and $F_j'(3j) = \lambda_j X_j$ $(1 \le j \le n)$. Let $K$ be the union of the closed discs $\bar{\Delta}_j$ $(1 \le j \le n)$ and the interval $[3, 3n] \subset \mathbf{C}$. Let $\tilde{F} : K \to \overline{D(t_0)}$ be a continuous map that equals $F_j$ on $\bar{\Delta}_j$ and satisfies $\tilde{F}(bK) \subset bD(t_0)$. Here, $bK$ is the topological boundary of $K$ in $\mathbf{C}$.

Since the complement of $K$ in $\mathbf{C}$ is connected and $\tilde{F}$ is holomorphic in the interior of $K$, we can apply Mergelyan's theorem to approximate $\tilde{F}$ uniformly on $K$ by a polynomial mapping $F_0 : \mathbf{C} \to \mathbf{C}^2$ satisfying $F_0(3j) = y_j$, $F_0'(3j) = \lambda_j X_j$ $(1 \le j \le n)$. Let $U$ be a small simply connected neighborhood of $K$ with smooth boundary. If the approximation is close enough on $K$ and if $U$ is chosen sufficiently small, then $F_0(\bar{U}) \subset D$ and $F_0(bU) \subset \mathcal{V}(T, 0)$.

Since $U$ is conformally equivalent to the disc $\Delta$, we can now proceed as in the proof of Theorem 2 to modify the given map $F_0$ to a proper map $F : U \to D$, without changing the values of $F_0$ and its first derivative at the points $3j, 1 \le j \le n$. (See the remark following the Main Lemma). If $D$ is strictly pseudoconvex, we can make $F$ smooth up to the boundary as in the proof of Theorem 1. This proves Corollary 3.

Sections three and four are devoted to the proof of the Main Lemma.

## 3. Technical lemmas

Recall that the disc algebra $\mathscr{A}(\Delta)$ is the set of all continuous functions on $\bar{\Delta}$ that are holomorphic on $\Delta$.

**LEMMA 2.** *Let $V$ be a compact set and let $F : \bar{\Delta} \times V \to \mathbf{C}$ be a continuous function such that for each $v \in V$ the function $\zeta \to F(\zeta, v)$ belongs to the disc algebra. Given $\epsilon > 0$ there are $n \in \mathbf{Z}_+$ and a continuous map $G : \bar{\Delta} \times V \to \mathbf{C}$ such that for each $v \in V, \zeta \to G(\zeta, v)$ is a polynomial of degree $\leq n$ satisfying $|G(\zeta, v) - F(\zeta, v)| < \epsilon$ for all $(\zeta, v) \in \bar{\Delta} \times V$.*

*Proof.* There is an $r, 0 < r < 1$, such that $|F(r\zeta, v) - F(\zeta, v)| < \epsilon/2$ for all $(\zeta, v) \in \bar{\Delta} \times V$. By the Cauchy formula we have

$$
F(z, v) = \frac{1}{2\pi i} \int_{b\Delta} \frac{F(\zeta, v)}{\zeta - z} d\zeta
$$

$$
= \frac{1}{2\pi i} \int_{b\Delta} \left[ \frac{1}{\zeta} + \frac{z}{\zeta^2} + \cdots + \frac{z^n}{\zeta^{n+1}} \right] F(\zeta, v) \, d\zeta + z^{n+1} \frac{1}{2\pi i} \int_{b\Delta} \frac{F(\zeta, v)}{\zeta^{n+1}(\zeta - z)} d\zeta
$$

$$
= G_n(z) + R_n(z).
$$

Since $F$ is bounded on $b\Delta \times V$, the remainder $R_n(z)$ tends to zero uniformly on $r\bar{\Delta} \times V$ as $n \to \infty$, hence $|F - G_n| < \epsilon/2$ ($|z| \leq r, v \in V$) if $n$ is sufficiently large. Since $G_n$ is a polynomial of degree at most $n$ in $z$, Lemma 2 is proved.

*Remark.* If $F(0, v) = 0$ for all $v \in V$ then we may take $G(0, v) = 0$ for all $v \in V$. If $V \subset \mathbf{R}^N$ and $F$ is smooth on $\Delta \times \text{Int } V$, then $G$ will be smooth on $\Delta \times \text{Int } V$.

**COROLLARY.** *Let $\Lambda : b\Delta \times b\Delta \to \mathbf{C}$ be a continuous map such that for each $\zeta \in b\Delta$, $L_\zeta = \{\Lambda(\eta, \zeta) : \eta \in b\Delta\}$ is a Jordan curve with the origin contained in the bounded part of its complement. Given $\epsilon > 0$ there is a function $f \in \mathscr{A}(\Delta)$ satisfying $f(\zeta) \in L_\zeta + \epsilon\Delta$ for each $\zeta \in b\Delta$, $f(0) = 0$, and $f'(0) = 0$.*

*Remark.* The Corollary gives an approximate solution of the Riemann-Hilbert boundary value problem with the data $L_\zeta$ ($\zeta \in b\Delta$). The exact solution, i.e., the existence of functions $f \in \mathscr{A}(\Delta)$ satisfying $f(\zeta) \in L_\zeta$ ($\zeta \in b\Delta$), is a much deeper result; see the papers [3] and [12].

*Proof.* For each $\zeta \in b\Delta$ let $D_\zeta$ be the domain bounded by $L_\zeta$, and let $\Phi_\zeta : \Delta \to D_\zeta$ be the conformal map that satisfies $\Phi_\zeta(0) = 0, \Phi'_\zeta(0) > 0$. Then the map

$F(\eta, \zeta) = \Phi_\zeta(\eta)$ is continuous on $\bar{\Delta} \times b\Delta$, and $\eta \to F(\eta, \zeta)$ is in the disc algebra for each $\zeta \in b\Delta$. By Lemma 2 there are $n \in \mathbf{Z}_+$ and a continuous map $G: \bar{\Delta} \times b\Delta \to \mathbf{C}$ such that for each $\zeta \in b\Delta$, $\eta \to G(\eta, \zeta)$ is a polynomial of degree at most $n$ without constant term, satisfying

$$|G(\eta, \zeta) - F(\eta, \zeta)| < \epsilon/2, \qquad (\eta, \zeta) \in \bar{\Delta} \times b\, \Delta.$$

Write

$$G(\eta, \zeta) = \sum_{j=1}^{n} a_j(\zeta)\eta^j \qquad (\zeta \in b\Delta, \eta \in \bar{\Delta}).$$

For each $j$ there are polynomials $P_j$ and $Q_j$ satisfying

$$|a_j(\zeta) - P_j(\zeta) - Q_j(1/\zeta)| < \epsilon/2n \qquad (\zeta \in b\Delta).$$

Let $m \in \mathbf{Z}_+$ be greater than the degree of each polynomial $Q_j$, $1 \le j \le n$, and set

$$f(\zeta) = \sum_{j=1}^{n} [P_j(\zeta) + Q_j(1/\zeta)](\zeta^m)^j.$$

Then $f$ is a polynomial in $\zeta$ that vanishes at 0 to arbitrary finite order (by choosing $m$ sufficiently large). If $\zeta \in b\Delta$ then $|f(\zeta) - G(\zeta^m, \zeta)| < \epsilon/2$ which implies that $|f(\zeta) - F(\zeta^m, \zeta)| < \epsilon$. In particular, $f_\zeta \in L_\zeta + \epsilon\Delta$. This completes the proof of the Corollary.

*Remark.* If $0 < R < 1$ then, by choosing $m$ large enough, we can get $f$ as above with the additional property $|f(\zeta)| < \epsilon$ ($|\zeta| \le R$).

As before we denote by $\mathbf{B}^2$ the open unit ball in $\mathbf{C}^2$.

LEMMA 3. *Let $T < t_1 < t_2 < 0$ and let $L = \{(w_1, w_2) \in \mathbf{B}^2 : w_1 = 0\}$. There is a $v_0 > 0$ and for each $z \in \mathscr{V}(t_1, t_2)$ there is a biholomorphic map $\Psi_z : \mathbf{B}^2 \to \Psi_z(\mathbf{B}^2) \subset \mathbf{C}^2$ satisfying*
   (i) $\Psi_z(0) = 0$ ($z \in \mathscr{V}(t_1, t_2)$),
   (ii) $z + \Psi_z(\mathbf{B}^2) \subset D$ ($z \in \mathscr{V}(t_1, t_2)$),
   (iii) $(z, w) \to \Psi_z(w)$ *is smooth on* $\mathscr{V}(t_1, t_2) \times \mathbf{B}^2$,
   (iv) *for each* $z \in \mathscr{V}(t_1, t_2)$ *and for each* $v$, $-v_0 \le v \le v_0$, *the set*

$$P(z, v) = \{w \in \mathbf{B}^2 : \rho(z + \Psi_z(w)) < \rho(z) + v\}$$

*is a convex domain and*

$$S(z, v) = \{w \in \mathbf{B}^2 : \rho(z + \Psi_z(w)) = \rho(z) + v\}$$

*is a smooth surface,*

(v)  *for each* $z \in \mathscr{V}(t_1, t_2)$ *and for each* $v, -v_0 \le v < 0$, *we have* $\overline{P(z, v)} \cap L = \varnothing$,

(vi)  *for each* $z \in \mathscr{V}(t_1, t_2)$ *and for each* $v, 0 < v \le v_0$, $S(z, v) \cap L$ *is a simple closed curve.*

*Remark.* The convexity of $P(z, v)$ implies that if $0 < v \le v_0$ then $L$ intersects $S(z, v)$ transversely.

*Proof of Lemma* 3. The proof will be split into three parts.

*Part* 1. Let $e_1', e_2'$ be the standard basis of $\mathbf{C}^2$. Fix a point $z \in \mathscr{V}(T, 0) = \{z \in D : T < \rho(z) < 0\}$, and choose a new coordinate system in $\mathbf{C}^2$ by putting $e_1(z) = \nabla\rho(z)/|\nabla\rho(z)|$ and letting $e_2(z)$ be canonically orthogonal to $e_1(z)$, that is, if $e_1(z) = \alpha e_1' + \beta e_2'$, then $e_2(z) = -\bar{\beta}e_1' + \bar{\alpha}e_2'$. The Taylor formula gives

$$\rho(z + u_1 e_1(z) + u_2 e_2(z)) = \rho(z) + 2\Re\left[|\nabla\rho(z)|u_1 + 1/2 \sum_{j, k = 1}^{2} (D_j D_k \rho)(z)u_j u_k\right]$$

$$+ \sum_{j, k = 1}^{2} (D_j \bar{D}_k \rho)(z)u_j \bar{u}_k + o(z, |u|^2). \tag{4}$$

Since $\rho$ is of class $\mathscr{C}^2$ we have

$$\lim_{|u| \to 0} \frac{o(z, |u|^2)}{|u|^2} = 0,$$

uniformly with respect to $z \in \mathscr{V}(t_1, t_2)$ (since this set is relatively compact in $\mathscr{V}(T, 0)$).

*Part* 2. For each $z \in \mathscr{V}(T, 0)$ we define the entire map $\Phi_z : \mathbf{C}^2 \to \mathbf{C}^2$ by

$$\Phi_z(u_1 e_1(z) + u_2 e_2(z)) = w_1 e_1' + w_2 e_2',$$

where

$$w_1 = |(\nabla\rho)(z)|u_1 + \frac{1}{2} \sum_{j, k = 1}^{2} (D_j D_k \rho)(z)u_j u_k,$$

$$w_2 = u_2.$$

Note that $\Phi_z(0) = 0$. Since everything in the definition of $\Phi_z$ depends smoothly on $z$, it follows that $(z, w) \to \Phi_z(w)$ is a smooth map on $\mathscr{V}(T, 0) \times \mathbf{C}^2$.

For each $z \in \mathscr{V}(T, 0)$ we get, using bases $\{e_1(z), e_2(z)\}$ and $\{e_1', e_2'\}$,

$$(D\Phi_z)(0) = \begin{pmatrix} |(\nabla\rho)(z)| & 0 \\ 0 & 1 \end{pmatrix},$$

which shows that $(D\Phi_z)(0)$ is invertible for each $z \in \mathscr{V}(T, 0)$. Let $T < s_1 < t_1 < t_2 < s_2 < 0$. By the Inverse Function Theorem there is a ball $B \subset \mathbf{C}^2$, centred at the origin, such that for each $z \in \mathscr{V}(s_1, s_2)$, $\Phi_z$ maps a neighborhood of 0 biholomorphi-cally onto a neighborhood of 0 that contains $B$ and such that $(z, w) \to \Phi_z^{-1}(w)$ is smooth on $\mathscr{V}(s_1, s_2) \times B$. Denote $\Psi_z = \Phi_z^{-1}|_B$. Passing to a smaller $B$ if necessary we have

(a) $\Psi_z(0) = 0$ $(z \in \mathscr{V}(s_1, s_2))$,
(b) $z + \Psi_z(B) \subset D$ $(z \in \mathscr{V}(s_1, s_2))$,
(c) $(z, w) \to \Psi_z(w)$ is smooth on $\mathscr{V}(s_1, s_2) \times B$.

*Part 3.* Let $s_1 < s_1' < t_1 < t_2 < s_2' < s_2$. For each $z = \mathscr{V}(s_1, s_2)$ we have

$$(D\Phi_z)(0)(e_1(z)) = |(\nabla\rho)(z)|e_1',$$

which implies that $L$ is tangent to $S(z, 0)$ at the origin. By (c) we have

$$\Psi_z(w) = (D\Psi_z)(0)(w) + o(z, |w|),$$

where $\lim_{|w| \to 0} o(z, |w|)/|w| = 0$, uniformly with respect to $z \in \mathscr{V}(s_1', s_2')$. In the bases $\{e_1', e_2'\}$ and $\{e_1(z), e_2(z)\}$ we have

$$(D\Psi_z)(0) = \begin{pmatrix} b(z) & 0 \\ 0 & 1 \end{pmatrix},$$

where $b(z) = 1/|(\nabla\rho)(z)|$. If $u = \Psi_z(w) = u_1 e_1(z) + u_2 e_2(z)$, we thus get

$$u_1 = b_1(z)w_1 + o(z, |w|),$$

$$u_2 = w_2.$$

It follows that

$$u_1 \bar{u}_1 = b(z)^2 w_1 \bar{w}_1 + o_1(z, |w|^2),$$

$$u_1 \bar{u}_2 = b(z) w_1 \bar{w}_2 + o_2(z, |w|^2),$$

$$\bar{u}_1 u_2 = b(z) \bar{w}_1 w_2 + o_3(z, |w|^2),$$

$$u_2 \bar{u}_2 = w_2 \bar{w}_2,$$

where $\lim_{|w| \to 0} o_j(z, |w|^2)/|w|^2 = 0$ $(1 \le j \le 3)$, uniformly with respect to $z \in \mathscr{V}(s_1', s_2')$. Using (4) we get

$$\rho(z + \Psi_z(w)) = \rho(z + u_1(w_1, w_2)e_1(z) + u_2(w_1, w_2)e_2(z))$$

$$= \rho(z) + 2\Re w_1 + \sum_{j,k=1}^{2} (D_j \bar{D}_k \rho)(z) u_j \bar{u}_k + o(z, |u|^2), \tag{5}$$

where $\lim_{|u| \to 0} o(z, |u|^2)/|u|^2 = 0$, uniformly with respect to $z \in \mathscr{V}(s_1', s_2')$. The ratio $|u(w)|/|w|$ is bounded from above and from below away from zero as $w \to 0$, uniformly with respect to $z \in \mathscr{V}(s_1', s_2')$. It follows that $o(z, |u|^2)$ in (5) is in fact $o(z, |w|^2)$, where $\lim_{|w| \to 0} o(z, |w|^2)/|w|^2 = 0$, uniformly with respect to $z \in \mathscr{V}(s_1', s_2')$. Thus

$$\rho(z + \Psi_z(w)) = \rho(z) + 2\Re w_1 + \sum_{j,k=1}^{2} b_{j,k}(z) w_j \bar{w}_k + o(z, |w|^2),$$

where

$$(b_{j,k}(z)) = \begin{pmatrix} b(z)^2 (D_1 \bar{D}_1 \rho)(z) & b(z)(D_1 \bar{D}_2 \rho)(z) \\ b(z)(\bar{D}_1 D_2 \rho)(z) & (D_2 \bar{D}_2 \rho)(z) \end{pmatrix}.$$

By strict plurisubharmonicity of $\rho$ its complex Hessian is strictly positive definite. Since $b(z) = 1/|(\nabla \rho)(z)| > 0$, the matrix $(b_{j,k}(z))$ is also strictly positive definite, and its eigenvalues are bounded from above and from below away from zero, uniformly for $z \in \mathscr{V}(s_1, s_2)$. The surface $S(z, v)$ in $B$ is given by the equation

$$2\Re w_1 + \sum_{j,k=1}^{2} b_{j,k}(z) w_j \bar{w}_k + o(z, |w|^2) = v,$$

where

$$\lim_{|w| \to 0} o(z, |w|^2)/|w|^2 = 0,$$

uniformly with respect to $z \in \mathscr{V}(s_1', s_2')$. This implies the existence of a number $v_0 > 0$, depending only on $t_1, t_2$, satisfying the properties (iv), (v), and (vi) of Lemma 3, with $B$ in place of the unit ball $\mathbf{B}^2$. To complete the proof we simply rescale $B$ to $\mathbf{B}^2$.

**LEMMA 4.** *Let* $T < t_1 < t_2 < 0$. *There is a* $v_0, 0 < v_0 < t_1 - T$, *and for each* $z \in \mathscr{V}(t_1, t_2)$ *there is a holomorphic map* $\psi_z : \Delta \to \mathbf{C}^2$ *such that*

   (i) $M_z = z + \psi_z(\Delta)$ *is a submanifold of an open neighborhood of* $z$ *contained in* $D$, *and* $\zeta \to z + \psi_z(\zeta)$ *maps* $\Delta$ *biholomorphically to* $M_z$,

   (ii) $\psi_z(0) = 0$,

   (iii) *for each* $v, -v_0 \le v < 0$, *we have* $\{p \in D : \rho(p) \le \rho(z) + v\} \cap M_z = \varnothing$,

   (iv) *for each* $v, 0 < v \le v_0$, $M_z$ *intersects* $\{p \in D : \rho(p) = \rho(z) + v\}$ *transversely in a simple closed curve, and*

   (v) *the map* $(z, \zeta) \to \psi_z(\zeta)$ *is smooth on* $\mathscr{V}(t_1, t_2) \times \Delta$.

*Proof.* The maps $\psi_z(\zeta) = \Psi_z(0, \zeta)$ $(\zeta \in \Delta)$, where $\Psi_z$ is given by Lemma 3, satisfy all the required properties.

## 4. Proof of the Main Lemma

Applying Lemma 4 to $\mathscr{V}(t_1', t_2')$ where $T < t_1' < t_1 < t_2 < t_2' < 0$ we get $v_0$ and the maps $\psi_z, z \in \mathscr{V}(t_1', t_2')$. Using the compactness of $\mathscr{V}(t_1', t_2')$ we see (after replacing $\Delta$ by a slightly smaller disc) that in Lemma 4 we may assume that each $\psi_z$ extends holomorphically across $b\Delta$ and that $(z, \psi) \to \psi_z(\zeta)$ is smooth on $\mathscr{V}(t_1', t_2') \times \bar{\Delta}$.

To approximate a holomorphic map on $\Delta$ smoothly on compact subsets of $\Delta$ it suffices to approximate it by holomorphic maps uniformly on $\Delta$. Thus, given $\alpha > 0$, there is $\delta > 0$ with the following property: If $\Theta : b\Delta \times \bar{\Delta} \to \mathbf{C}^2$ is a smooth map such that $\Theta(\zeta, \cdot)$ is holomorphic on $\Delta$ for each fixed $\zeta \in b\Delta$, and if

$$|\Theta(\zeta, \eta) - \psi_{F(\zeta)}(\eta)| < \delta \qquad (\zeta \in b\Delta, \eta \in \bar{\Delta}),$$

then we have

   (a) $F(\zeta) + \Theta(\zeta, \Delta) \subset D$ $(\zeta \in b\Delta)$,

   (b) $\rho(F(\zeta) + \Theta(\zeta, \lambda)) > \rho(F(z)) - \alpha$ $(\zeta \in b\Delta, \lambda \in \bar{\Delta})$,

   (c) *for each* $v, \alpha < v \le v_0$, *the set*

$$\Gamma_\zeta(v) = \{\lambda \in \Delta : \rho(F(\zeta) + \Theta(\zeta, \lambda)) = \rho(F(z)) + v\}$$

is a smooth simple closed curve containing 0 in its interior part, and

(d) the curves $\Gamma_\zeta(v)$ depend smoothly on $\zeta \in b\Delta$ and $v$, $\alpha < v \leq v_0$.

Set $\mu_0 = v_0$ and choose $\alpha > 0$ so small that $\mu(\zeta) > \alpha$ ($\zeta \in b\Delta$) and $\rho(F(\zeta)) > c + \alpha$ ($\zeta \in \bar{\Delta}\backslash r\Delta$). Further, choose $d > 0$ so small that

$$x \in F(\bar{\Delta}\backslash r\Delta), \quad |x - y| < d \qquad \text{implies } y \in D \text{ and } \rho(y) > c. \tag{6}$$

With no loss of generality we may assume that the function $\mu$ is smooth on $b\Delta$.

By Lemma 2 there are $n \in \mathbf{Z}_+$ and a function

$$\Omega(\zeta, \eta) = a_1(\zeta)\eta + \cdots + a_n(\zeta)\eta^n$$

such that

$$|\Omega(\zeta, \eta) - \psi_{F(\zeta)}(\eta)| < \delta/2 \qquad (\zeta \in b\Delta, \eta \in \bar{\Delta}).$$

For each $j$, $1 \leq j \leq n$, we choose holomorphic polynomials $P_j$ and $Q_j$ such that the function

$$\Theta(\zeta, \eta) = \sum_{j=1}^{n} [P_j(\zeta) + Q_j(1/\zeta)]\eta^j$$

satisfies

$$|\Omega(\zeta, \eta) - \Theta(\zeta, \eta)| < \delta/2 \qquad (\zeta \in b\Delta, \eta \in \bar{\Delta}).$$

Consequently $|F - \Theta| < \delta$ on $b\Delta \times \bar{\Delta}$, hence the properties (a)–(d) hold.

By (b) we have the inequality ·

$$\rho(F(\zeta) + \Theta(\zeta, \eta)) > \rho(F(\zeta)) - \alpha \qquad (\eta \in \bar{\Delta}) \tag{7}$$

that holds initially for all $\zeta \in b\Delta$, and after passing to a larger $R < 1$ it also holds for all $\zeta \in \bar{\Delta}\backslash R\Delta$.

Choose $m \in \mathbf{Z}_+$ greater than the degrees of all $Q_j$, $1 \leq j \leq n$. Since $\alpha < \mu(\zeta) < v_0$, the properties (c) and (d) imply that

$$\Lambda_\zeta = \{\lambda \in \Delta : \rho(F(\zeta) + \Theta(\zeta, \lambda)) = \rho(F(\zeta)) + \mu(\zeta)\}$$

is a continuously varying family of smooth simple closed curves enveloping 0. There is a $\gamma > 0$ such that for all $\zeta \in b\Delta$ and $\eta \in \Lambda_\zeta + \gamma\Delta$ we have

$$\left|\rho(F(\zeta) + \Theta(\zeta, \eta)) - \rho(F(\zeta)) - \mu(\zeta)\right| < \epsilon.$$

By the Corollary (Section 2) there is a function $\omega \in \mathscr{A}(\Delta)$ such that $\zeta^m\omega(\zeta) \in \Lambda_\zeta + \gamma\Delta$ for each $\zeta \in b\Delta$. Starting with an even larger $m$ we may assume that

$$\left|\Theta(\zeta, \zeta^m\omega(\zeta))\right| < \min\{\epsilon, d\} \qquad (|\zeta| \leq R).$$

Define

$$G(\zeta) = \Theta(\zeta, \zeta^m\omega(\zeta)) \qquad (\zeta \in \bar\Delta).$$

Then $G$ is continuous on $\bar\Delta$, holomorphic on $\Delta$, and by construction it satisfies the properties (i), (iii), (iv), and (v) in the Main Lemma. To prove (ii), observe that by (7) we have $\rho(F(\zeta) + G(\zeta)) > c$ ($\zeta \in \bar\Delta\backslash R\Delta$). If $\zeta \in \bar\Delta\backslash r\Delta$, $|\zeta| \leq R$, then $|G(\zeta)| < d$ so by (6), $\rho(F(\zeta) + G(\zeta)) > c$. This completes the proof of the Main Lemma.

*Remark.* To prove the Main Lemma with the stronger condition (v′) we choose a Blaschke product $P(\zeta)$ that vanishes to second order at each point $\zeta_j \in \Delta$, $1 \leq j \leq n$, we choose $\omega \in \mathscr{A}(\Delta)$ satisfying $\zeta^m P(\zeta)\omega(\zeta) \in \Lambda_\zeta + \gamma\Delta$ for $\zeta \in b\Delta$, and we set

$$G(\zeta) = \Theta(\zeta, \zeta^m P(\zeta)\omega(\zeta)) \qquad (\zeta \in \bar\Delta).$$

## 5. An example

In this section we construct for each $N \geq 2$ a smoothly bounded domain $D \subset\subset \mathbf{C}^N$ with a point $x \in D$ that is not contained in the image of any proper holomorphic map $f : \Delta \to D$.

Let $\mathbf{B}^N$ be the unit ball in $\mathbf{C}^N$. For $x \in \mathbf{C}^N\backslash\{0\}$ we denote by $H(x)$ the real hyperplane through the origin that is perpendicular to $x$. If $\rho > 0$ write $K(x, \rho) = \{z \in b\mathbf{B}^N : |z - x| \leq \rho\}$.

There are $\delta$, $0 < \delta < 1/2$, and $\zeta$, $0 < \alpha < 1$, such that if $1 < R < 1 + \delta$, if $x \in b\mathbf{B}^N$, and if $\Omega$ is the connected component of $\mathbf{C}^N\backslash[RK(x, 1/3) \cup (\alpha x + H(x))]$ that contains $x$, then

$$b\Omega = [RK(x, 1/3) \cap \bar P] \cap [(\alpha x + H(x)) \cap R\mathbf{B}^N], \tag{8}$$

where $P$ is a half-space of $\mathbf{C}^N$ determined by the hyperplane $\alpha x + H(x)$.

There are $n \in \mathbf{Z}_+$ and points $x_j \in b\mathbf{B}^N$, $1 \le j \le n$, such that $\bigcup_{j=1}^{n} K(x_j, 1/3) = b\mathbf{B}^N$. Choose numbers $R_j$, $1 \le j \le n$, $1 < R_1 < R_2 < \cdots < R_n < 1 + \delta$. For each $j$ we fatten $R_j K(x_j, 2/3)$ to get a smoothly bounded domain $U_j \subset (3/2)\mathbf{B}^N$ that contains $R_j K(x_j, 2/3)$ and has connected boundary. We can choose the domain $U_j$ so small that their closures are pairwise disjoint and $\bar{U}_j \cap H(x_j) = \varnothing$ ($1 \le j \le n$). Define $D = 2\mathbf{B}^N \backslash \bigcup_{j=1}^{n} \bar{U}_j$.

Suppose that $f : \Delta \to D$ is a proper holomorphic map such that $f(0) = 0$. Its total cluster set $C(f) = \bigcap_{0 < r < 1} \overline{f(\Delta \backslash r\Delta)}$ is a connected compact set contained in $bD$. Since $bD = b(2\mathbf{B}^N) \cup [\bigcup_{j=1}^{n} bU_j]$ is a disjoint union of $n + 1$ compact connected sets, it follows that either $C(f) \subset bU_j$ for some $j$ or $C(f) \subset b(2\mathbf{B}^N)$. We will show that none of these is possible.

Suppose first that $C(f) \subset bU_j$ for some $j$, $1 \le j \le n$. As $f$ is bounded, the maximum principle implies that $f(\Delta)$ is contained in the closed convex hull of $C(f)$. However, $\bar{U}_j$ is a connected compact set that misses $H(x_j)$, so its convex hull does not contain the point $f(0) = 0 \in H(x_j)$, a contradiction.

Thus $C(f)$ must be contained in $b(2\mathbf{B}^N)$, hence $f$ is a proper map from $\Delta$ to $2\mathbf{B}^N$. Since $f(\Delta)$ is connected and since $f(0) = 0$, there is a $\zeta_0 \in \Delta$ such that $f(\zeta_0) = x \in b\mathbf{B}^N$. There is a $j, 1 \le j \le n$, such that $x \in K(x_j, 1/3)$, hence $K(x, 1/3) \subset K(x_j, 2/3)$, and it follows that $R_j K(x, 1/3) \subset U_j$. Recall that $1 < R_j < 1 + \delta$. Denote by $\Omega$ the connected component of $\mathbf{C}^N \backslash [R_j K(x, 1/3) \cup (\alpha x + H(x))]$ that contains $x$. Since $\Omega \subset (3/2)\mathbf{B}^N$, $f^{-1}(\Omega)$ is an open, relatively compact subset of $\Delta$. Let $G$ be the component of $f^{-1}(\Omega)$ that contains $\zeta_0$. Since $f(\Delta) \subset D$, it follows that $f(\Delta)$ misses $U_j$ whence it misses $R_j K(x, 1/3)$. Since $f(bG) \subset b\Omega$, (8) implies that $f(bG) \subset \alpha x + H(x)$ which, by the maximum principle, gives $f(G) \subset \alpha x + H(x)$. In particular, $f(\zeta_0) = x \in \alpha x + H(x)$ which contradicts the fact that $0 < \alpha < 1$. This shows that there is no proper holomorphic map $f : \Delta \to D$ satisfying $0 \in f(\Delta)$.

*Remark.* The domain $D$ constructed above has disconnected boundary. We do not know whether there exists a domain $D \subset \mathbf{C}^N$ with smooth *connected* boundary such that all proper holomorphic maps $f : \Delta \to D$ avoid certain point $x \in D$.

REFERENCES

[1] H. ALEXANDER and J. WERMER, *Polynomial hulls with convex fibers*, Math. Ann. *271* (1985), 99–109.

[2] K. DIEDERICH and J. E. FORNÆSS, *Pseudoconvex domains: Bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. *39* (1977), 129–147.

[3] F. FORSTNERIČ, *Polynomially convex hulls of sets fibered over the circle*, Indiana Univ. Math. J. *37* (1988), 869–889.

[4] F. FORSTNERIČ, *Polynomially convex hulls with piecewise smooth boundaries*, Math. Ann. *276* (1986), 97–104.

[5]  J. GLOBEVNIK, *Boundary interpolation and proper holomorphic maps from the disc to the ball*, Math. Z. *198* (1988), 143–150.

[6]  J. GLOBEVNIK, *Relative embeddings of discs into convex domains*, Invent. Math. *98* (1989), 331–350.

[7]  J. GLOBEVNIK, *On moduli of boundary values of holomorphic functions*, to appear in Math. Z.

[8]  J. W. HELTON and D. E. MARSHALL, *Frequency domain design and analytic selections*, Indiana Univ. Math. J. *39* (1990), 157–184.

[9]  S. G. KRANTZ, *Function Theory of Several Complex Variables*, Wiley-Interscience, New York, 1982.

[10]  Z. SLODKOWSKI, *An analytic set-valued selection and its applications to the corona theorem, to polynomial hulls and joint spectra*, Trans. Amer. Math. Soc. *294* (1986), 367–377.

[11]  Z. SLODKOWSKI, *Polynomial hulls with convex fibers and complex geodesics*, J. Funct. Anal. *94* (1990), 156–176.

[12]  Z. SLODKOWSKI, *Polynomial hulls in $C^2$ and quasicircles*, Ann. Sc. Norm. Sup. Pisa, cl. s.c., Ser. IV, *26* (1989), 367–391.

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