

COMPLEX TANGENTS OF REAL SURFACES IN COMPLEX SURFACES

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Introduction. In this paper we study the complex tangents of real surfaces in complex surfaces. More precisely, let M be a closed real surface, i.e., a smooth, compact, two-dimensional manifold without boundary. Given an immersion resp. embedding $\pi: M \rightarrow \mathcal{M}$ of M into a complex surface \mathcal{M} (a complex manifold of dimension two), we consider the question to what extent can one simplify the structure of the set of complex tangents of π by a regular homotopy (resp. isotopy) of immersions (resp. embeddings).

Recall that a point $p \in M$ is called a *complex tangent* of π if the tangent space $\pi_*(T_p M)$ is a complex linear subspace (a complex line) in $T_{\pi(p)} \mathcal{M}$. The immersion is *totally real* at every point that is not a complex tangent. An immersion without complex tangents is said to be totally real. When M is orientable and we choose an orientation on M , then every complex tangent p of π is either *positive* or *negative*, depending on whether the orientation on $\pi_*(T_p M)$ induced from $T_p M$ by π_* agrees or disagrees with the canonical orientation of $\pi_*(T_p M)$ as a complex line.

Recall that a *regular homotopy* is a family of immersions $\pi_t: M \rightarrow \mathcal{M}$, $t \in [0, 1]$, such that π_t and all its derivatives depend continuously on the parameter t . Immersions π_0 and π_1 are *regularly homotopic* if there exists a regular homotopy connecting π_0 to π_1 . If all immersions in the family π_t are embeddings, we call π_t an *isotopy* of embeddings.

Thom's transversality theorem (see [1] or [2]) implies that a generic immersion $\pi: M \rightarrow \mathcal{M}$ only has isolated complex tangents, and its double points are transverse self-intersections (normal crossings) that avoid the complex tangents of π . In this paper we shall only study immersions satisfying these properties, and we will not mention this again.

It is well known that one cannot change the complex tangents arbitrarily by a regular homotopy since their number, counted with suitable algebraic multiplicities, is an invariant $I(\pi)$ of the regular homotopy class of the immersion, called the *index* of π (Chern and Spanier [10], Eliashberg and Harlamov [24], Webster [33], and Forstnerič [16]). Before proceeding, we must recall the definition of $I(\pi)$.

First, we recall from [16] and [33] the *index* $I(p; \pi) \in \mathbb{Z}$ of an isolated complex tangent of π . Let U be a small disc neighborhood of p in M . In suitable local holomorphic coordinates (z, w) on \mathcal{M} near $\pi(p)$, the surface $\pi(U)$ is a graph $w = f(z)$ of a smooth complex function f defined near the origin in \mathbb{C} , with $\pi(p)$ corresponding

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to $z = w = 0$. Since the graph of f is totally real at a point $(z, f(z))$ if and only if $\bar{\partial}f(z) = \partial f / \partial \bar{z}(z) \neq 0$ (see Section 2), the origin $z = 0$ is an isolated zero of the function $\bar{\partial}f(z)$.

Definition 1. (a) With the notation as above, the index $I(p; \pi)$ is defined to be the winding number of the function $\bar{\partial}f$ around the origin $z = 0$. If π is totally real at p , we set $I(p; \pi) = 0$.

(b) The index $I(\pi)$ is the integer

$$I(\pi) = \sum_{p \in M} I(p; \pi).$$

(Thus, if π is totally real, we have $I(\pi) = 0$.)

(c) If M is oriented, we define the positive (resp. negative) index $I_{\pm}(\pi)$ of π by summing the indices $I(p, \pi)$ over positive (resp. negative) complex tangents of π .

According to Lemma 2.3 in Section two, the definition of $I(p; \pi)$ is independent of the choice of local coordinates on \mathcal{M} . The index $I(p; \pi)$ can be interpreted as a local intersection number and also as the index of a suitable normal vector field of the immersion at p .

When π is an embedding, we shall identify M with its image in \mathcal{M} and denote its index by $I(M; \mathcal{M})$, or by $I(M)$ when it is clear from the context what \mathcal{M} is.

Recall from Bishop [9] that a generic complex tangent is either *elliptic* or *hyperbolic*; a simple calculation (see [16]) shows that the elliptic points have index $+1$ and the hyperbolic points index -1 . Thus, if π only has e elliptic and h hyperbolic complex tangents, we have $I(\pi) = e - h$.

Next, we recall the relevant properties of the index $I(\pi)$. It is related to the Euler number $\chi(M)$ of M and the normal Euler number $\chi(\pi; \mathcal{M})$ of the immersion by the formula

$$I(\pi) = \chi(M) + \chi(\pi; \mathcal{M}) \quad (1)$$

that was proved by Eliashberg and Harlamov [24] and Webster [33]. The special case when M is an embedded orientable surface in \mathbb{C}^2 was proved earlier by Bishop [9]; in that case the normal Euler number vanishes, and hence

$$I(M) = \chi(M) = 2 - 2g$$

where g is the genus of M . See also [12] and [16]. When $\mathcal{M} = \mathbb{C}^2$, we also have

$$I(\pi) = \chi(M) - 2d(\pi) \quad (\text{mod } 2 \text{ if } M \text{ is nonorientable}) \quad (2)$$

where $d(\pi)$ is the number of double points of the immersion π , counted algebraically as in Whitney [36]. This follows from (1) and from the elementary observation $\chi(\pi; \mathbb{C}^2) + 2d(\pi) = 0 \pmod{2}$ if M is nonorientable; see [6, p. 597].

The formulas (1) and (2) provide obstructions to the existence of totally real embeddings or immersions. For instance, if M is orientable and embedded totally real into \mathbb{C}^2 , we get $\chi(M) = I(M) = 0$, and hence M is the torus. Wells proved in [35] that every closed real n -manifold M that embeds totally real into \mathbb{C}^n has Euler number zero. Another consequence of (2) is that a surface with odd Euler number admits no totally real immersions into \mathbb{C}^2 (Corollary 1.5(a) below).

If M is oriented and $\pi: M \rightarrow \mathbb{C}^2$ is an immersion, it is known that the positive and the negative indices are equal:

$$I_+(\pi) = I_-(\pi) = I(\pi)/2. \quad (3)$$

(See [10], [12], or [24].) We shall give a very simple proof of this fact in Proposition 2.4 below. If π is an embedding into \mathbb{C}^2 , it follows that

$$I_+(\pi) = I_-(\pi) = \chi(M)/2 = 1 - g(M).$$

For embeddings of an oriented surface into an arbitrary complex surface, we have

$$I_{\pm}(\pi) = 1/2(I(\pi) \pm c(M)) \quad (4)$$

where $c(M)$ is the value of the first Chern class of \mathcal{M} on $\pi(M) \subset \mathcal{M}$ (see [12], Proposition 1.4.1).

It is known through the work of Gromov and others that the index $I(\pi)$ (or $I_{\pm}(\pi)$ when M is orientable) is *the only obstruction to deforming π into a totally real embedding*. In Gromov's terminology, *for totally real immersions and embeddings of real n -manifolds into complex n -manifolds ($n \geq 2$), the h -principle holds in the absolute and the relative form*; see Section 2.4.5 in [23]. This allows us, in particular, to construct embeddings of real surfaces into complex surfaces with prescribed number and behavior of the complex tangents, provided that there are no homotopical obstructions.

Gromov's method of 'convex integration' that is used in the proof of his h -principle is quite splendid as it yields very many seemingly unrelated results of analysis and geometry at the same time. However, almost every theory that is so general has the disadvantage that, when one applies the machine to a given simple minded problem, one does not see very well why it works and what is behind it. In our particular problem, we do not get any geometric picture by reading Section 2.4.5 of [23]. Moreover, Gromov's methods, powerful as they are, do not seem widely known, nor are they easily knowable.

We hope that this justifies at least somewhat the present paper in which we provide a more direct and elementary proof of the fact that one can replace a pair of complex tangents (of the same sign) of a given embedding or immersion by a single complex tangent, provided that the index is preserved. In particular, if the index equals zero, one can cancel all complex tangents and obtain a totally real embedding (resp. immersion). This is explained in Theorem 1.1 and its corollaries

in Section 1 below. It should be stressed that Theorem 1.1 is itself a corollary of results in [23], but the present proof is new and, hopefully, more illuminating.

A large part of the paper is devoted to construction of totally real embeddings (or embeddings with minimal number of complex tangents) of real surfaces into \mathbf{C}^2 or \mathbf{CP}^2 . These results, summarized in Corollaries 1.2–1.5, follow from Theorem 1.1, but they can also be deduced from results known earlier. In [23] and [24] this problem was reduced to the question about possible normal bundles of real 2-surfaces in a complex 2-dimensional manifold. This is a classical problem which was solved in the case $\mathcal{M} = \mathbf{C}^2$ by Massey [29] who proved a conjecture by Whitney. It seems that the explicit computations were never published anywhere (except some special cases); so it seems useful to summarize these facts in one paper. We also find new totally real embeddings of some unorientable surfaces into the complex projective plane (Theorem 1.6).

We apply our technique to construct holomorphically convex embeddings of every closed surface M other than the two-sphere into \mathbf{C}^2 (Theorem 1.8). This yields examples of Stein domains in \mathbf{C}^2 that are homotopically equivalent to M (Corollary 1.9).

The paper is organized as follows. In Section 1 we state the main results. In Section 2 we recall the relevant properties of the Maslov index and its connection with the index $I(\pi)$ of Definition 1. In Section 3 we consider the behavior of index under the operation “connected sum”, and we construct some specific embedding and immersions used in the proof of results in Section 1. In Section 4 we explain a modification lemma for the $\bar{\partial}$ operator in the disc that is used in the proof of Theorem 1.1 given in Section 5. In Section 6 we construct a regular Stein neighborhood basis of surfaces in \mathbf{C}^2 with complex tangents of negative type. In Section 7 we consider embeddings of surfaces into the projective plane \mathbf{CP}^2 .

I wish to thank Jože Vrabec for very helpful conversations and suggestions concerning the topological part of the paper. He pointed out to me the result of Massey [29] and showed me the embedding $\mathbf{RP}^2 \rightarrow \mathbf{C}^2$ given in Lemma 3.2. I thank Hans Sterk from the University of Amsterdam, who explained to me the connection of (1) with the genus formulas for curves in \mathbf{CP}^2 , and Jan Wiegerinck with whom I had stimulating conversations on the subject. I thank Edgar Lee Stout who raised (for the second time) my interest in this subject by asking the question answered by Proposition 1.7. Last but not least, I wish to thank the referee for his useful criticism of the first version of this paper.

1. Results. Let $\Sigma(k) \subset \mathbf{C}^2$ be the graph of the function

$$w = z^k \bar{z} \quad \text{if } k \geq 0, \quad w = \bar{z}^{|k|+1} \quad \text{if } k < 0.$$

Clearly, 0 is the only complex tangent of $\Sigma(k)$ with index equal to k .

Definition 2. A complex tangent $p \in M$ of an immersion $\pi: M \rightarrow \mathcal{M}$ is said to be of type $k \in \mathbf{Z}$ if there exists a neighborhood U of p in M and a local holomorphic chart on \mathcal{M} , centered at $\pi(p)$, that carries $\pi(U)$ onto a neighborhood of 0 in $\Sigma(k)$. If

$\mathcal{M} = \mathbb{C}^2$, we require in addition that this can be done by a complex affine change of coordinates on \mathbb{C}^2 .

According to this definition, a complex tangent of type 0 is in fact a totally real point.

We shall now explain our main result on joining pairs of complex tangents. For nondegenerate complex tangents this was done earlier by Eliashberg and Harlamov [24], but that paper is not easily accessible.

Let $p_0, p_1 \in M$ be isolated complex tangents of an immersion $\pi: M \rightarrow \mathcal{M}$. We choose on $T_{p_j}M$ the canonical orientation induced by π_* from the natural orientation of $\pi_*(T_{p_j}M)$ as a complex line. If $\gamma \subset M$ is a curve connecting p_0 to p_1 , we can carry the canonical orientation of $T_{p_0}M$ along γ to an orientation of $T_{p_1}M$. We say that γ is *orientation-preserving* (resp. *orientation-reversing*) if this orientation agrees (resp. disagrees) with the canonical orientation of $T_{p_1}M$.

Our main technical result is the following theorem.

THEOREM 1.1. *Let $p_0, p_1 \in M$ be isolated complex tangents of an immersion $\pi_0: M \rightarrow \mathcal{M}$ of a real surface M into a complex surface \mathcal{M} and let $\gamma \subset M$ be an orientation-preserving simple smooth curve connecting p_0 to p_1 . Let U be a neighborhood of γ in M that does not contain any other complex tangents or double points of π_0 . Given a point $q \in \gamma$ and an $\varepsilon > 0$, there is a regular homotopy of immersions $\pi_t: M \rightarrow \mathcal{M}$, $t \in [0, 1]$, satisfying*

- (a) $\pi_t = \pi_0$ on $M \setminus U$ for all t ,
- (b) $\sup_M \delta(\pi_t, \pi_0) < \varepsilon$ for all t , where δ is any fixed metric on \mathcal{M} ,
- (c) every π_t has the same number of double points as π_0 , and
- (d) $\pi_t|_U$ has a single complex tangent q of type $k = I(p_0; \pi_0) + I(p_1; \pi_0)$. In particular, if $k = 0$, π_t is totally real on U ,
- (e) if π_0 is an embedding, then $\{\pi_t\}$ is an isotopy of embeddings.

Theorem 1.1 is proved in Sections 4 and 5 below. We have already mentioned in the introduction that it also follows from results of Gromov [23].

Theorem 1.1 has several corollaries. We first consider the case when M is orientable and $\pi: M \rightarrow \mathbb{C}^2$ is an immersion with isolated complex tangents. Then every curve in M joining a pair of complex tangents of the same sign is orientation-preserving. Using Theorem 1.1 repeatedly, we can join all complex tangents of the same sign into a single complex tangent of type $I_{\pm}(\pi)$. It follows from (3) and (4) that we cannot join complex tangents of opposite signs.

On the other hand, if M is unorientable, we can join every pair of complex tangents by a curve with the required properties. Namely, if a chosen curve from p_0 to p_1 is orientation-reversing, we follow it by another orientation-reversing simple loop at p_1 that exists since M is unorientable. The composed curve is orientation-preserving, and we can replace it by a simple smooth orientation-preserving curve from p_0 to p_1 . Using Theorem 1.1, we can join all complex tangents of π into one complex tangent of type $I(\pi)$.

We summarize these results in the following corollary.

COROLLARY 1.2. (a) *Every embedding $\pi: M \rightarrow \mathcal{M}$ can be changed by an isotopy to an embedding with two complex tangents of types $I_{\pm}(\pi)$ (with one complex tangent of type $I(\pi)$ if M is unorientable). The same holds for regular homotopy of immersions.*

(b) *An embedding $\pi: M \rightarrow \mathcal{M}$ is isotopic to a totally real embedding if and only if $I(\pi) = 0$ and $c(M) = 0$. The last condition $c(M) = 0$ always holds when $\mathcal{M} = \mathbb{C}^2$ or when M is unorientable.*

(c) *An immersion $\pi: M \rightarrow \mathbb{C}^2$ is regularly homotopic to a totally real immersion if and only if $I(\pi) = 0$.*

Remark. The proof of Theorem 1.1 can also be used to create complex tangents. More precisely, we can split an isolated complex tangent p of index k into any number m of isolated complex tangents p_1, p_2, \dots, p_m (of the same sign as p if M is orientable), with p_j of type k_j , provided that $\sum_{j=1}^m k_j = k$. For instance, one can isotope a given initial embedding to an embedding that only has elliptic or hyperbolic points (depending on the index) of types $+1$ (resp. -1).

COROLLARY 1.3. *A closed surface M of genus g admits a totally real embedding into \mathbb{C}^2 if and only if M is orientable and $g = 1$ (M is the torus), or M is unorientable and $g = 2 \pmod{4}$. (M is the connected sum of an odd number $g/2$ of Klein bottles.)*

Proof. Let $\pi: M \rightarrow \mathbb{C}^2$ be an embedding; so $d(\pi) = 0$. If M is orientable, (2) implies $I(\pi) = \chi(M) = 2 - 2g$, and hence $I(\pi) = 0$ if and only if $g = 1$, i.e., M is the torus. The standard embedding $T = \{(e^{i\theta}, e^{i\eta}); \theta, \eta \in \mathbb{R}\}$ is totally real (even Lagrangian).

If M is unorientable, with Euler number $h = 2 - g$, Whitney proved that for every embedding into \mathbb{C}^2 we have $\chi(\pi; \mathbb{C}^2) = 2h \pmod{4}$ (see Massey [29]). From (1) we get $I(\pi) = 3h \pmod{4}$, and hence $I(\pi) = 0$ implies $h = 0 \pmod{4}$ whence $g = 2 \pmod{4}$. Thus, M must be a connected sum of an odd number $g/2$ of Klein bottles.

An explicit totally real embedding of the Klein bottle K into \mathbb{C}^2 was given by Rudin [31]. Givental [19] constructed Lagrangian (whence totally real) embeddings of the unorientable surfaces with Euler characteristic $h \leq -4$, $h = 0 \pmod{4}$. It is still an open problem whether the Klein bottle itself admits a Lagrangian embedding into \mathbb{C}^2 .

Alternatively, we can use Corollary 1.2 to construct the required totally real embeddings as follows. Using the operation “connected sum”, we obtain in Section 3 a set of embeddings of an unorientable surface M into \mathbb{C}^2 with the corresponding set of normal Euler numbers equal to

$$\{2h - 4, 2h, 2h + 4, \dots, 4 - 2h\}, \quad h = \chi(M).$$

Massey proved in [29] that this is precisely the set of all possible normal Euler numbers of embeddings into \mathbb{C}^2 . Thus, the set of possible indices of embeddings into \mathbb{C}^2 equals

$$\{3h - 4, 3h, 3h + 4, \dots, 4 - h\}. \quad (5)$$

This set contains zero if and only if $h = 0 \pmod{4}$; hence Corollary 1.2(b) gives a totally real embedding in these cases. Corollary 1.3 is proved.

Remark 1. Corollary 1.3 is a special case of the following result of M. Audin [6], based on the work of Gromov [21], [23] (see also [15]): *A connected closed $2n$ -dimensional manifold M admits a totally real embedding into \mathbb{C}^{2n} if and only if its complexified tangent bundle is trivial and $\chi(M) = 0 \pmod{4}$ if M is unorientable.* For surfaces our approach in the present paper is simpler than the one in [23] or [15] since we do not use Whitney's method [36] of removing pairs of double points of immersions $M^n \rightarrow \mathbb{R}^{2n}$ that is problematic in dimension four. Corollary 1.3 also follows from the results of [24].

Remark 2. It is known that every totally real immersion $M^n \rightarrow \mathbb{C}^n$ is regularly homotopic to a Lagrangian immersion (Gromov [21], [23] and Lees [28]). Moreover, if $n + 1$ is not a power of two, all totally real immersions of an n -manifold into \mathbb{C}^n are regularly homotopic to each other [6].

We now summarize the results on complex tangents of closed surfaces in \mathbb{C}^2 . Some of them have been known before, but we take this opportunity to present them coherently.

COROLLARY 1.4. *Let M be a closed orientable surface of genus $g = g(M)$.*

(a) *If $g = 1$ (M is the torus), then every embedding of M into \mathbb{C}^2 is isotopic to a totally real embedding.*

(b) *If $g \neq 1$, then every embedding of M into \mathbb{C}^2 has a nonempty disconnected set of complex tangents, and it is isotopic to an embedding with two complex tangents (one positive and one negative), both of type $1 - g$.*

(c) *Every totally real immersion $\pi: M \rightarrow \mathbb{C}^2$ satisfies $d(\pi) = 1 - g$. There exists a totally real immersion of M into \mathbb{C}^2 with precisely $|1 - g|$ double points (normal crossings).*

Proof. Parts (a) and (b) follow from Corollary 1.2 and the formula $I_{\pm}(\pi) = 1/2\chi(M) = 1 - g(3)$. If the set of complex tangents of an embedding is connected, all complex tangents are of the same sign, and the same holds after a generic perturbation. Thus, at least one of the numbers I_{\pm} equals zero whence $g = 1$.

From (2) we see that $I(\pi) = 0$ implies $d(\pi) = \chi(M)/2 = 1 - g$. We shall construct a totally real immersion with $|1 - g|$ double points in Section 3 (following the proof of Proposition 3.1). This will prove (c).

Remark. Apropos (a), T. Fiedler [13] proved that there exist embedded totally real tori $T_1, T_2 \subset \mathbb{C}^2$ that are isotopic, but not within the class of totally real tori.

COROLLARY 1.5. *Let M be a closed unorientable surface of genus g and Euler characteristic $h = 2 - g$. For each number $k \in \{3h - 4, 3h, 3h + 4, \dots, 4 - h\}$, there exists an embedding $M \rightarrow \mathbb{C}^2$ with a single complex tangent of type k . Conversely, the index of every embedding belongs to this set. Specifically, we have the following.*

- (a) If g is odd, M does not admit any totally real immersions into \mathbb{C}^2 . It admits an embedding with a single complex tangent of type $-k$, where $k \in \{1, 3\}$ and $g = k \pmod{4}$.
- (b) If $g = 2 \pmod{4}$, M admits a totally real embedding into \mathbb{C}^2 .
- (c) If $g = 0 \pmod{4}$, M admits no totally real embeddings into \mathbb{C}^2 . It admits an embedding with a single complex tangent of type -2 , and it also admits a totally real immersion with one double point.

Proof. Everything except the last assertion in (c) follows from Corollary 1.2 and from the existence of embeddings with indices (5). A totally real immersion with one double point is constructed in Section 3 (following the proof of Lemma 3.2). This proves Corollary 1.5.

In Section 7 we shall consider embeddings into the complex projective plane \mathbb{CP}^2 . Using a genus formula for an embedded complex projective curve $C \subset \mathbb{CP}^2$ of degree d , we obtain $I(C; \mathbb{CP}^2) = 3d$. For instance, we have embedded spheres $C_1, C_2 \subset \mathbb{CP}^2$ with indices 3 (resp. 6). By taking the connected sum of C_1 and C_2 with unorientable surfaces embedded into the finite part of \mathbb{CP}^2 , we prove the following theorem.

THEOREM 1.6. *Every unorientable surface of genus $g = 1, 2 \pmod{4}$ admits a totally real embedding into \mathbb{CP}^2 that is not regularly homotopic to an embedding into \mathbb{C}^2 .*

Thus, Corollary 1.3 and Theorem 1.6 provide nonequivalent totally real embeddings of the unorientable surfaces of genera $g = 2 \pmod{4}$ into \mathbb{CP}^2 .

Let M be an orientable surface of genus g . We have seen in the proof of Corollary 1.4 that an immersion $\pi: M \rightarrow \mathbb{C}^2$ with the connected set of complex tangents satisfies $d(\pi) = 1 - g$. When M is the sphere ($g = 0$), we obtain as a consequence the following result concerning the intersection of analytic and totally real discs in \mathbb{C}^2 .

PROPOSITION 1.7. *Suppose that A and E are smoothly embedded closed discs in \mathbb{C}^2 that are glued along their boundaries so as to give an immersed sphere. If the disc A is complex and the disc E is totally real at every interior point, then A and E also intersect in the interior. If we orient A and E so that the two orientations agree on their joint boundary, then their intersection index $A \cdot E$ equals one.*

This answers a question raised by E. L. Stout (personal communication). More general results on intersections of analytic and smooth discs have been proved recently in [11] and [37].

Using the methods of this paper, one can construct an embedded torus or embedded unorientable surfaces in \mathbb{C}^2 that contain a closed analytic disc and are totally real outside this disc. It suffices to stretch a single complex tangent into an analytic disc.

Corollaries 1.4 and 1.5 show that every closed surface except the sphere admits an embedding $M \subset \mathbb{C}^2$ with complex tangents of type $k < 0$. In Section 6 we

construct a smooth plurisubharmonic function $\rho \geq 0$ in an open neighborhood Ω of M satisfying $\rho^{-1}(0) = M$ and $d\rho \neq 0$ on $\Omega \setminus M$ (Proposition 6.1). The sublevel sets

$$\Omega_c = \{(z, w) \in \Omega: \rho(z, w) < c\}$$

for sufficiently small $c > 0$ are smoothly bounded pseudoconvex domains with the homotopy type of M . We call such a family a *regular Stein neighborhood basis* of the surface $M \subset \mathbb{C}^2$. It follows from [26, Theorem 4.3.4] that M is *holomorphically convex* in each Ω_c ; that is, for each point $p \in \Omega_c \setminus M$ there is a holomorphic function f on Ω_c with $f(p) = 1 > \sup_M |f|$. Thus, we have the following theorem.

THEOREM 1.8. *Every closed surface M other than the sphere admits a holomorphically convex embedding into \mathbb{C}^2 with a regular Stein neighborhood basis. If M is orientable, then every embedding of M into \mathbb{C}^2 is isotopic to an embedding satisfying this property.*

COROLLARY 1.9. *For every closed surface M other than the sphere, there exist Stein domains in \mathbb{C}^2 with the homotopy type of M .*

One may ask why is the two-sphere an exception. It is known that the envelope of holomorphy of every ‘nicely embedded’ two-sphere S in \mathbb{C}^2 is an embedded real three-ball \tilde{S} foliated by analytic discs. (It suffices to assume that S is generically embedded into a smooth closed strongly pseudoconvex hypersurface in \mathbb{C}^2 .) Results in this direction were obtained by Bedford and Gaveau [7], Bedford and Klingenberg [8], and Gromov [22]. If S is contained in a Stein domain $\Omega \subset \mathbb{C}^2$, then the envelope \tilde{S} is also contained in Ω , and hence S is zero-homotopic within Ω . This justifies the following conjecture.

CONJECTURE. *There exists no Stein domain (i.e., domain of holomorphy) in \mathbb{C}^2 with the homotopy type of the two-sphere.*

Corollary 1.9 raises the question: *For which real n -dimensional manifold M is there a Stein domain Ω in \mathbb{C}^n with the homotopy type of M ?*

Of course, the question is nontrivial only for manifolds that do not admit totally real embeddings into \mathbb{C}^n .

2. Index. We first recall the definition of the *Maslov index* of totally real n -dimensional manifolds in \mathbb{C}^n (see [5], [38], [14], [16]).

Let $G = G(n, 2n)$ denote the Grassmann manifold of oriented real n -dimensional subspaces of \mathbb{C}^n . For each $g \in G$ we choose n vectors X_1, \dots, X_n in \mathbb{C}^n that form a positively oriented real orthonormal basis of g and denote by X the complex $n \times n$ matrix with the j th column X_j . We set $\tau(g) = \det(X)$. If $Y = (Y_1, \dots, Y_n)$ is another such basis of g , then $Y = XA$ for some real orthogonal matrix $A \in \text{SO}(n, \mathbb{R})$ whence $\det Y = \det X$. Thus, we have a well-defined function $\tau: G \rightarrow \mathbb{C}$ such that $G_r = G \setminus \tau^{-1}(0)$ is precisely the set of all oriented totally real n -dimensional subspaces of \mathbb{C}^n .

The function τ induces a homomorphism of the first homology groups

$$H_{1(\tau)}: H_1(G_{tr}) \rightarrow H_1(\mathbb{C} \setminus \{0\}) = \mathbb{Z}.$$

Let $\pi: M \rightarrow \mathbb{C}^n$ be an immersion of an oriented n -dimensional manifold M into \mathbb{C}^n and let $\pi_*: M \rightarrow G$ be its Gauss map that takes each point $p \in M$ to the oriented tangent plane $\pi_*(T_p M) \subset \mathbb{C}^n$. If the immersion is totally real, the image $\pi_*(M)$ is contained in G_{tr} , and we may compose the map $H_1(\pi_*)$ with $H_1(\tau)$ to obtain a homomorphism

$$I_\pi = H_1(\tau) \circ H_1(\pi_*): H_1(M) \rightarrow \mathbb{Z}, \quad (6)$$

called the *index homomorphism* of the immersion π . If π is an embedding and we identify M with its image in \mathbb{C}^n , we shall also write $I_\pi = I_M$.

For each closed path $\gamma: S^1 = \mathbb{R}/\mathbb{Z} \rightarrow M$ we denote by $I_\pi(\gamma)$ the index of the 1-cycle in $H_1(M)$ defined by the path γ . Note that $I_\pi(\gamma)$ only depends on the restriction of π to an arbitrary neighborhood of $\gamma(S^1)$ in M .

Here is the simplest way to compute the index $I_\pi(\gamma)$. Choose continuous vector fields $X_1, \dots, X_n: S^1 \rightarrow \mathbb{C}^n$ such that for each $t \in S^1$ the vectors $X_1(t), \dots, X_n(t)$ form a real basis of the tangent space $\pi_*(T_{\gamma(t)} M)$. Such vector fields exist when M is orientable along the path γ . Then $I_\pi(\gamma)$ equals the winding number of the determinant function $\det(X_1(t), \dots, X_n(t)) \in \mathbb{C} \setminus \{0\}$ around the origin. Note that the determinant is without zeros since π is totally real.

We collect the basic properties of the index that we shall use in the sequel.

PROPOSITION 2.1. (a) I_π is independent of the choice of orientation on M .

(b) If $\pi_t: M \rightarrow \mathbb{C}^n$ ($t \in [0, 1]$) is a regular homotopy through totally real immersions, then $I_{\pi_0} = I_{\pi_1}$.

(c) Let $\gamma: S^1 \rightarrow M$ be a closed path, $U \subset \mathbb{C}^n$ an open neighborhood of $\gamma(S^1)$, and $\Phi: U \rightarrow \Phi(U) \subset \mathbb{C}^n$ a biholomorphic mapping. Denote by k the winding number of the function $t \in S^1 \rightarrow \det D\Phi(\gamma(t)) \in \mathbb{C} \setminus \{0\}$. Then

$$I_{\Phi \circ \pi}(\gamma) = I_\pi(\gamma) + k.$$

A global biholomorphic change of coordinates on \mathbb{C}^2 does not affect the index.

Proof. Properties (a) and (b) follow from the definition. For a proof of (c) see Lemma 7 in [16].

Suppose now that U is an open set in the complex plane, $g: U \rightarrow \mathbb{C}$ is a smooth complex function on U , and $M \subset \mathbb{C}^2$ is the graph of g :

$$M = \{(z, g(z)) \in \mathbb{C}^2: z \in U\}. \quad (7)$$

Let $z = x + iy$ and w be the variables on \mathbb{C}^2 . The vectors

$$X(z) = (1, \partial g / \partial x(z)), \quad Y(z) = (i, \partial g / \partial y(z)) \quad (8)$$

form a real basis of the tangent space $T_{(z, g(z))}M$ for all $z \in U$. Since the determinant of the matrix with columns $X(z)$ and $Y(z)$ equals

$$d(z) = \frac{\partial g}{\partial y}(z) - i \frac{\partial g}{\partial x}(z) = -2i \frac{\partial g}{\partial \bar{z}}(z),$$

the surface M is totally real at $(z, g(z))$ if and only if $\partial g / \partial \bar{z}(z) \neq 0$. The formula for computing the index of a closed path shows the following proposition.

PROPOSITION 2.2. *Let M be as above. If $\gamma = (\gamma_1, \gamma_2): S^1 \rightarrow M$ is any closed path contained in the set of totally real points of M , then $I_M(\gamma)$ equals the winding number of the function $t \in S^1 \rightarrow \partial g / \partial \bar{z}(\gamma_1(t)) \in \mathbb{C} \setminus \{0\}$.*

Using the Maslov index, we can compute the index $I(p; \pi)$ of an isolated complex tangent of an immersion $\pi: M \rightarrow \mathcal{M}$ as follows. Choose a biholomorphic map $\Phi: V \rightarrow \mathbb{B}^2$ from a neighborhood V of $\pi(p)$ in \mathcal{M} onto the ball $\mathbb{B}^2 \subset \mathbb{C}^2$. Let U be a neighborhood of p in M , U homeomorphic to the disc $\Delta \subset \mathbb{C}$, such that $\pi(U) \subset V$ and π is totally real at every point of $U \setminus \{p\}$. We pull back the canonical orientation of the complex line $\pi_*(T_p M)$ by π_* to an orientation of $T_p M$ and orient U accordingly.

LEMMA 2.3. *If $\gamma: S^1 \rightarrow U \setminus \{p\}$ is any closed path with winding number one (in the chosen orientation on U), then $I(p; \pi) = I_{\Phi \circ \pi}(\gamma)$.*

Proof. $I_{\Phi \circ \pi}(\gamma)$ is independent of the choice of γ , and by Proposition 2.1(c) it is also independent of (V, Φ) . With suitable choice of U and (V, Φ) , the surface $\Phi \circ \pi(U) \subset \mathbb{C}^2$ is a graph of the form (7), and the required equality follows from Proposition 2.2. Lemma 2.3 is proved.

The index $I(p; \pi)$ of an isolated complex tangent can also be defined as the local intersection number of the Gauss map associated to π with the submanifold H of the Grassmannian $G(2, 4)$ of real two-planes in \mathbb{C}^2 consisting of complex lines (see Chern and Spanier [10], Bishop [9], and Forstnerič [14, p. 94]).

Yet another way of defining $I(p; \pi)$ is due to Webster [33]. Assume for simplicity that $M \subset \mathcal{M}$ is embedded. Let $N \rightarrow M$ be the normal bundle of M in \mathcal{M} so that $TM \oplus N = T\mathcal{M}|_M$. If we choose a local orientation on TM , we can coorient the normal bundle N locally by the requirement that the two orientations add up to the standard orientation of the complex bundle $T\mathcal{M}|_M$.

Let $\tau: T\mathcal{M}|_M \rightarrow N$ be the projection with kernel TM . Denote by J the almost-complex structure operator on $T\mathcal{M}$ induced by the complex structure on \mathcal{M} . Every tangent vector field X on M gives rise to a normal vector field $\tilde{X} = \tau(JX)$. Let p_0 be an isolated complex tangent of π . Choose a tangent vector field X on M that has no critical points in a neighborhood U of p_0 in M . Then \tilde{X} is a normal vector field with an isolated zero at p_0 . We let $I(p_0; M)$ be the index of \tilde{X} at p_0 , i.e., the winding number of the fiber coordinate of the map $p \rightarrow \tilde{X}_p$ at $p = p_0$. It is easily seen that the definition does not depend on the choice of the normal bundle N and the vector field X . The definition is local, and hence it also applies to immersions.

To see that this definition coincides with Definition 1 we may assume that M is locally near p_0 given by (7), with p_0 corresponding to the point $0 \in \mathbb{C}^2$. We may choose the normal bundle with constant fibers $N_p = \{0\} \times \mathbb{C}$. Let X and Y be the tangent vector fields to M defined by (8) above. We then have

$$\tilde{X}(z) = iX(z) - Y(z) = (0, i\partial g/\partial \bar{x}(z) - \partial g/\partial y(z)) = 2i(0, \partial g/\partial \bar{z}(z))$$

and similarly $\tilde{Y}(z) = 2(0, \partial g/\partial \bar{z}(z))$; so their index at 0 equals the winding number of $\partial g/\partial \bar{z}$ around 0. This shows that the two definitions are equivalent.

Let $\pi: M \rightarrow \mathcal{M}$ be an immersion with isolated complex tangents $p_1, p_2, \dots, p_m \in M$. Recall that the index of π is the integer $I(\pi) = \sum_{j=1}^m I(p_j; \pi)$. (This is to be distinguished from the index homomorphism I_π defined by (6)!)

The last definition of the index $I(p; \pi)$ gives a simple proof of the formula (1), due to Webster [33]: Let X be a tangent vector field on M with isolated zeros that avoid the set of complex tangents of π . Then \tilde{X} is a normal vector field of π with isolated zeros at zeros of X and also at each complex tangent of π . If $p \in M$ is a zero of X (so p is not a complex tangent of π), then we have $\text{Ind}_p X + \text{Ind}_p \tilde{X} = 0$. Hence, the sum of indices of X and \tilde{X} at all critical points equals $I(\pi)$. However, the total index of X equals $\chi(M)$, and that of \tilde{X} equals $\chi(M; \mathcal{M})$; so we obtain (1).

Recall that for an oriented M we have defined $I_+(\pi)$ and $I_-(\pi)$ by summing up the indices of positive (resp. negative) complex tangents of π .

PROPOSITION 2.4. *For an immersion $\pi: M \rightarrow \mathbb{C}^2$ of a closed oriented surface into \mathbb{C}^2 , we have $I_+(\pi) = I_-(\pi) = I(\pi)/2$. More generally, if M is an oriented surface with boundary bM and if $\pi: M \rightarrow \mathbb{C}^2$ is an immersion that is totally real near bM , then*

$$I_+(\pi) - I_-(\pi) = I_\pi(bM).$$

Proof. Choose an oriented triangulation $\{\Delta_j\}$ of M such that each Δ_j contains at most one complex tangent of π in its interior, and no complex tangent lies on the boundary of any cell Δ_j . If M has boundary bM , we further require that each Δ_j is either entirely contained in the interior of M or else one of its sides lies in bM . We orient the boundary curve $b\Delta_j$ coherently with Δ_j .

The index $I_\pi(b\Delta_j)$ equals zero if Δ_j contains no complex tangents, and by Lemma 2.3 it equals $\pm I(p; \pi)$ if Δ_j contains the complex tangent p , where the sign depends on the signature of p .

Denote by M_* the surface obtained by removing from M all the complex tangents of π . Since $\sum [b\Delta_j] = [bM]$ in the first homology group $H_1(M_*)$ and since $I_\pi: H_1(M_*) \rightarrow \mathbb{Z}$ is a group homomorphism it follows that

$$I_\pi(bM) = \sum I_\pi(b\Delta_j) = I_+(\pi) - I_-(\pi).$$

If M is closed, we get $I_+(\pi) - I_-(\pi) = 0$. Since $I_+(\pi) + I_-(\pi) = I(\pi)$, the proposition follows.

3. The connected sum. Recall that the connected sum $M_1 \# M_2$ of two surfaces is a surface obtained by removing a disc Δ_j from each M_j and connecting the two punched surfaces by a tube $\Sigma = S^1 \times [0, 1]$ glued to M_j along the curve $b\Delta_j$.

If $\pi_j: M_j \rightarrow \mathcal{M}$ are embeddings (or immersions) into a complex surface \mathcal{M} , we can perform the connected sum of $\pi_1(M_1)$ and $\pi_2(M_2)$ within \mathcal{M} . The result is an embedding (or immersion) of the connected sum $M_1 \# M_2$ into \mathcal{M} that we shall denote by $\pi_1 \# \pi_2$. (This is determined only up to a regular homotopy.)

We shall give an explicit construction of the connected sum $\pi_1 \# \pi_2$ within \mathcal{M} that adds two hyperbolic complex tangents of index -1 , one positive and one negative in case that the surfaces M_1, M_2 are oriented.

To simplify the notation we identify M_j with $\pi_j(M_j) \subset \mathcal{M}$. Choose a pair of totally real points $p_j \in M_j$ ($j = 1, 2$) that are not double points of M_1 (resp. M_2) or in $M_1 \cap M_2$, and choose a simple real-analytic arc $\gamma \in \mathcal{M}$ with endpoints p_1 and p_2 that does not intersect $M_1 \cup M_2$ elsewhere. Let X_1 be a real-analytic vector field of type $(1, 0)$, defined on a neighborhood of γ in \mathcal{M} and tangent to γ . By choosing γ correctly we may assume that JX_1 is tangent to M_j at p_j for $j = 1, 2$. We can find another real-analytic vector field X_2 on \mathcal{M} near γ that is \mathbb{C} -independent of X_1 and is tangent to M_j at p_j for $j = 1, 2$.

Let $z = x + iy$ and $w = u + iv$ be coordinates on \mathbb{C}^2 . There is a tubular neighborhood Ω of γ in \mathcal{M} and a biholomorphic map $\Phi: \Omega \rightarrow \Omega' \subset \mathbb{C}^2$ satisfying

- (a) $\Phi(\gamma) = [-1, 1] \times \{0\}$, $\Phi(p_1) = (-1, 0)$, $\Phi(p_2) = (1, 0)$, and
- (b) $\Phi_*(X_1) = c\partial/\partial z$ ($c > 0$) and $\Phi_*(X_2) = \partial/\partial w$ at each point of γ .

It follows that $\Phi_*(T_{p_j}M_j) = \{(iy, u): y, u \in \mathbb{R}\}$ for $j = 1, 2$. After a small deformation of the surface M_1 at p_1 , the image $\Phi(\Omega \cap M_1) = \tilde{M}_1 \subset \Omega'$ will contain the disc

$$\Delta_1 = \{(-1 + iy, u): y, u \in \mathbb{R}, y^2 + u^2 < \varepsilon^2\}$$

for some $\varepsilon > 0$. Similarly, we insert into $\tilde{M}_2 = \Phi(\Omega \cap M_2)$ the disc $\Delta_2 = \{(1 + iy, u): y^2 + u^2 < \varepsilon^2\}$.

Let Σ be the smoothly embedded tube in $\mathbb{C} \times \mathbb{R}$ defined by

$$\Sigma = \{(x + iy, u): -1 \leq x \leq 1, y^2 + u^2 = g(x^2)\},$$

where g is any smooth increasing function on $[0, 1)$ that is continuous on $[0, 1]$ and satisfies the conditions $g(0) = \varepsilon^2/4$, $g(1) = \varepsilon^2$, $g'(t) > 0$ for all $t \in [0, 1)$, and $\lim_{t \rightarrow 1} g^{(k)}(t) = \infty$ for all $k = 1, 2, \dots$.

If $q \in \Sigma$ is a complex tangent, we must have $T_q\Sigma = \{w = 0\}$. There are precisely two such points: $q_1 = (0, \varepsilon/2)$ and $q_2 = (0, -\varepsilon/2)$, both of them hyperbolic with index -1 . Thus, $I(\Sigma) = -2$.

We now remove Δ_j from \tilde{M}_j and glue the resulting punched surfaces together along $b\Delta_j$ with the tube Σ . Denote the new surface in Ω' by \tilde{M}_0 . The resulting surface

$$M = (M_1 \setminus \Omega) \cup (M_2 \setminus \Omega) \cup \Phi^{-1}(\tilde{M}_0)$$

is the connected sum $M_1 \# M_2$ within \mathcal{M} .

If M_1 and M_2 are disjointly embedded in \mathcal{M} , then M is also embedded. In general, the self-intersection number of M equals the sum of the corresponding numbers of M_1 and M_2 , plus the intersection number of M_1 and M_2 within \mathcal{M} .

The connected sum satisfies the following properties.

PROPOSITION 3.1. (a) $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2$.

(b) $\chi(\pi_1 \# \pi_2; \mathcal{M}) = \chi(\pi_1; \mathcal{M}) + \chi(\pi_2; \mathcal{M})$.

(c) $I(\pi_1 \# \pi_2) = I(\pi_1) + I(\pi_2) - 2$.

Proof. The property (a) is well known (see [32]), and (c) follows from the construction. Together with (1) we get (b), although this is also well known (see [30]).

We shall now consider some immersions and embeddings of the two-sphere S^2 , the torus \mathbf{T} , and the projective plane \mathbf{RP}^2 into \mathbf{C}^2 . To begin with we construct the totally real immersed orientable surfaces required in Corollary 1.4(d).

The standard embedding of the sphere is

$$S = \{(z, u) : z \in \mathbf{C}, u \in \mathbf{R}, |z|^2 + u^2 = 1\} \subset \mathbf{C}^2.$$

We orient S as the boundary of the three-ball $|z|^2 + u^2 < 1$. Clearly, S has two elliptic complex tangents $p_0 = (0, 1)$ and $p_1 = (0, -1)$, the first one positive and the second one negative, and $S_* = S \setminus \{p_0, p_1\}$ is totally real. Thus, $I_+(S) = I_-(S) = 1$ and $I(S) = 2 = \chi(S)$, as expected from (1).

An explicit Lagrangian (whence totally real) immersion of the n -sphere into \mathbf{C}^n with one double point can be found in Weinstein [34, p. 26]. For $n = 2$ it is given by

$$\pi(x_1, x_2, a) = (x_1(1 + 2ia), x_2(1 + 2ia)), \quad x_1^2 + x_2^2 + a^2 = 1.$$

Let $S_1 \subset \mathbf{C}^2$ be its image, a totally real immersed sphere with one double point. Its self-intersection number equals $d = 1$, and the normal Euler number is -2 .

For each immersion $\pi = (\pi_1, \pi_2) : M \rightarrow \mathbf{C}^2$ we set $\bar{\pi} = (\pi_1, \bar{\pi}_2)$. Since the reflection $(z, w) \rightarrow (z, \bar{w})$ changes the orientation on \mathbf{C}^2 , it changes the sign of the normal Euler number, and hence $\chi(\bar{\pi}; \mathbf{C}^2) = -\chi(\pi; \mathbf{C}^2)$. Thus, the sphere \bar{S}_1 has $\chi(\bar{S}_1; \mathbf{C}^2) = 2$.

Let T_0 be the standard embedded totally real (lagrangian) torus in \mathbf{C}^2 . The connected sum $T_1 = T_0 \# \bar{S}_1$ is an immersed torus with one double point and $\chi(T_1; \mathbf{C}^2) = 2$; hence $I(T_1) = 2$ according to (1). (We first translate T_0 or \bar{S}_1 so as to make them disjoint.) Thus, for each surface $M \subset \mathbf{C}^2$ we have $I(M \# T_1) = I(M)$ by Proposition 3.1(c). It follows that the disjoint connected sum

$$\overbrace{T_0 \# T_1 \# \cdots \# T_1}^{(g-1) \text{ times}}$$

is an immersed orientable surface of genus $g > 1$, with $|g - 1|$ double points and with index zero; hence it can be deformed to a totally real immersion.

This proves Corollary 1.4(d). We now proceed to construct embedded and immersed unorientable surfaces required in Corollary 1.5.

Recall that the projective plane \mathbf{RP}^2 is the quotient of the sphere obtained by identifying each pair of antipodal points. Its Euler number equals 1. Every mapping $F: \mathbf{C} \times \mathbf{R} \rightarrow \mathbf{C}^2$ that is a function of the quadratic terms z^2 , zu , and u^2 induces a mapping $\tilde{F}: \mathbf{RP}^2 \rightarrow \mathbf{C}^2$.

LEMMA 3.2. *The image of S under the mapping $F(z, u) = (z^2, zu)$ is an embedded projective plane $P_1 \subset \mathbf{C}^2$ with a single complex tangent of index three; hence $I(P_1) = 3$ and $\chi(P_1; \mathbf{C}^2) = 2$.*

Proof. F extends to the holomorphic map $F(z, w) = (z^2, zw)$. If $F(z_1, u_1) = F(z_2, u_2)$, then $z_1^2 = z_2^2$ and $z_1 u_1 = z_2 u_2$; hence $z_2 = \pm z_1$, and the second equation implies $u_2 = \pm u_1$ unless $z_1 = z_2 = 0$. In this last case the assumption that both points lie on S implies $u_1, u_2 = \pm 1$. Thus, F induces a one-to-one mapping $\tilde{F}: \mathbf{RP}^2 \rightarrow \mathbf{C}^2$.

To prove that \tilde{F} is an embedding it suffices to show that $F|_S$ is an immersion. Its derivative equals $DF(z, w) = \begin{pmatrix} 2z & 0 \\ w & z \end{pmatrix}$, and the Jacobian determinant is $d(z, w) = 2z^2$. The branch locus $\{z = 0\}$ intersects S only at the points $(0, \pm 1)$, whence F is locally biholomorphic at each point of S_* . Thus $F|_{S_*}$ is an immersion, and $F(S_*)$ is totally real in \mathbf{C}^2 . At the two exceptional points $(0, \pm 1)$ the derivative of F equals $\begin{pmatrix} 0 & 0 \\ \pm 1 & 0 \end{pmatrix}$. Since the intersection of its kernel $\{0\} \times \mathbf{C}$ with the tangent space $T_{(0, \pm 1)} S = \mathbf{C} \times \{0\}$ is trivial, $F|_S$ is an immersion also at the points $(0, \pm 1)$.

Thus, $F(S) = P_1$ is an embedded projective plane in \mathbf{C}^2 with the single complex tangent at $F(0, \pm 1) = (0, 0)$. Choose an $\varepsilon \in (0, 1)$ and consider the path $\gamma(\theta) = (\varepsilon e^{i\theta}, \sqrt{1 - \varepsilon^2})$ in S . According to (2.4), we have $I_S(\gamma) = I(p_0; S) = 1$. The winding number of the jacobian determinant $d(z, w) = 2z^2$ along γ equals 2, and hence Proposition 2.1(c) implies $I((0, 0); P_1) = 1 + 2 = 3$. Lemma 3.2 is proved.

Let \bar{P}_1 be the reflection of P_1 as explained above, so that $\chi(\bar{P}_1; \mathbf{C}^2) = -2$ and $I(\bar{P}_1) = -1$.

Fix $g \in \mathbf{Z}_+$ and write $g = k + l$, $k, l \in \mathbf{Z}_+$. Proposition 3.1 implies that the embedded unorientable surface of genus g defined by

$$M_g^{k,l} = \overbrace{P_1 \# \cdots \# P_1}^{k \text{ times}} \# \overbrace{\bar{P}_1 \# \cdots \# \bar{P}_1}^{l \text{ times}}$$

has the normal Euler characteristic $2k - 2l$ and index $I = 2 - g + 2(k - l) = 3h + 4(k - 1)$, where $h = 2 - g$ is the Euler number. Thus, we obtain embeddings of the unorientable surface of genus g into \mathbf{C}^2 with the set of indices equal to $\{3h - 4, 3h, 3h + 4, \dots, 4 - h\}$. (See also Massey [29].) These embeddings were used in Corollary 1.5.

Notice that a totally real Klein bottle in \mathbf{C}^2 can be obtained by modification of the Klein bottle $K_1 = P_1 \# \bar{P}_1 \subset \mathbf{C}^2$ that has index zero. An explicit example was

constructed by Rudin [31]. The other two Klein bottles $K_2 = P_1 \# P_1$ (resp. $\bar{K}_2 = \bar{P}_1 \# \bar{P}_1$) have index 4 (resp. -4).

The connected sum $K^{(2)} = K_1 \# K_2$ of two Klein bottles has index 2. Thus, if M is any surface in \mathbb{C}^2 , Proposition 3.1(c) implies $I(M \# K^{(2)}) = I(M)$. By successively adding copies of $K^{(2)}$ to K_1 , we thus obtain a connected sum of any odd number of Klein bottles embedded into \mathbb{C}^2 with index zero; these can be made totally real.

Similarly, $K_1 \# K_1$ has index -2 . By adding copies of $K^{(2)}$ we obtain a connected sum of any even number of Klein bottles with index -2 (Corollary 1.5(c)).

The connected sum $P_2 = P_1 \# S_1 \subset \mathbb{C}^2$ of P_1 with the totally real immersed sphere S_1 is an immersed projective plane with index 1 and with one double point. The connected sum $K_3 = P_1 \# P_2$ is an immersed Klein bottle with one double point and index 2. Hence for every surface $M \subset \mathbb{C}^2$ we have $I(M \# K_3) = I(M)$.

The connected sum of two Klein bottles $K_1 \# K_3$ is immersed with one double point and with index zero; thus it can be deformed into a totally real immersion. Similarly, we can add K_3 to any totally real embedded connected sum of an odd number of Klein bottles to get an even number of Klein bottles immersed totally real into \mathbb{C}^2 with one double point. This proves Corollary 1.5(c). Similarly, one proves part (a) of Corollary 1.5.

4. A modification lemma for $\bar{\partial}$ on the disc. Let $f: \bar{\Delta} \rightarrow \mathbb{C}$ be a smooth complex function on the closed unit disc in the complex plane. We have seen in Section 2 that its graph

$$M = \{(\zeta, f(\zeta)): |\zeta| \leq 1\}$$

is totally real at a point $(\zeta, f(\zeta))$ if and only if $\bar{\partial}f(\zeta) = (\partial f / \partial \bar{\zeta})(\zeta) \neq 0$. Thus, the complex tangents of M lie over the zeros of $\bar{\partial}f$.

Suppose now that $\bar{\partial}f(\zeta) \neq 0$ when $|\zeta| = 1$; equivalently, suppose that M is totally real near its boundary bM that lies over the circle $T = \{|\zeta| = 1\}$. We would like to modify M in the interior, leaving it fixed near the boundary, so as to remove all the complex tangents of M . Alternatively, we want to modify f , keeping it fixed near T , such that the modified function has a nonvanishing $\bar{\partial}$ -derivative on $\bar{\Delta}$.

Clearly, there is a topological obstruction for doing this, namely, the winding number of the function $\bar{\partial}f$ around T . In order to find the required modification, the winding number must vanish. It turns out that this is the only obstruction.

LEMMA 4.1. *Let f be a smooth complex function on the closed unit disc $\bar{\Delta} \subset \mathbb{C}$ such that $\bar{\partial}f(\zeta) \neq 0$ for $|\zeta| = 1$, and the winding number of $\bar{\partial}f$ along the curve $|\zeta| = 1$ equals zero. Given an $\varepsilon > 0$, there exists a smooth function \tilde{f} on $\bar{\Delta}$ satisfying*

- (i) \tilde{f} equals f in a neighborhood of the circle $|\zeta| = 1$,
- (ii) $|\tilde{f} - f| < \varepsilon$ on $\bar{\Delta}$, and
- (iii) $\bar{\partial}\tilde{f}(\zeta) \neq 0$ for all $|\zeta| \leq 1$.

Remark. Clearly, Lemma 4.1 applies to simply connected domains in \mathbb{C} since a conformal transformation only affects the $\bar{\partial}$ -derivative by a nonvanishing factor.

Proof of Lemma 4.1. This lemma follows from the work of Gromov [21], [23]; a similar result was used in [17]. However, in order to make our proof self-contained, we will reduce Lemma 4.1 to an elementary lemma from control theory. The same result holds if we replace the operator $\bar{\partial}$ with any first-order linear complex differential operator on $\bar{\Delta}$ that does not vanish at any point of $\bar{\Delta}$.

After a small perturbation f near 0, we may assume that $\bar{\partial}f(0) \neq 0$. Choose $0 < a < 1$ such that $\bar{\partial}f(\zeta) \neq 0$ for $|\zeta| \leq a$. We shall only modify f on the annulus $A = \{\zeta \in \mathbb{C}: a < |\zeta| < 1\}$.

Let $B = \mathcal{C}^1(T)$ be the Banach space of all complex valued functions of class \mathcal{C}^1 on the circle $T = \mathbb{R}/2\pi\mathbb{Z}$. Write $\zeta = re^{i\theta}$. We associate to each $f \in \mathcal{C}^2(\bar{\Delta})$ the \mathcal{C}^1 path $r \in [a, 1] \rightarrow f_r \in B$ by $f_r(\theta) = f(re^{i\theta})$.

A simple computation shows that in polar coordinates

$$\frac{\partial f}{\partial \bar{\zeta}} = \frac{\zeta}{2r} \left(\frac{\partial f}{\partial r} + \frac{i}{r} \frac{\partial f}{\partial \theta} \right)$$

when $r \neq 0$; so the condition $\partial f / \partial \bar{\zeta} \neq 0$ is equivalent to the condition that the function

$$g_r = \frac{\partial f}{\partial r} + \frac{i}{r} \frac{\partial f_r}{\partial \theta} \in B$$

is nonvanishing.

Our hypothesis on f implies that g_r is nonvanishing for $r = 1$ and for $r = a$, and its winding number equals -1 for these two values of r . (This is because the winding number of $\partial f / \partial \bar{\zeta}$ equals zero on the circles $r = 1$ and $r = a$.) Choose $\eta > 0$ sufficiently small such that $|g_r(\theta)| > \eta$ for $r = 1, a$ and for all $\theta \in \mathbb{R}$.

For each $r \in [a, 1]$ we let $\Omega_r \subset B$ be the open set consisting of all $h \in B$ satisfying

$$(i) \quad \left| h(\theta) + \frac{i}{r} \frac{\partial f}{\partial \theta}(re^{i\theta}) \right| > \eta \text{ for all } \theta \in \mathbb{R}, \text{ and}$$

$$(ii) \quad \text{the winding number of the function } \theta \rightarrow h(\theta) + \frac{i}{r} \frac{\partial f}{\partial \theta}(re^{i\theta}) \text{ equals } -1.$$

Clearly, Ω_r is an open connected set in B , and the union

$$\Omega = \bigcup_{a \leq r \leq 1} \{r\} \times \Omega_r \subset [a, 1] \times B$$

is an open set in $[a, 1] \times B$. We say that a \mathcal{C}^1 path $\gamma: [a, 1] \rightarrow B$ is Ω -allowable if the graph of its derivative γ' is contained in Ω , i.e., $(\partial \gamma / \partial r)(r) \in \Omega_r$ for each r .

We claim that *the convex hull of each Ω_r equals B* . To prove this, choose $g \in B$ and write

$$e^{i\theta}g(\theta) = g_1(\theta) + ig_2(\theta)$$

with g_1 and g_2 real-valued. For every $R > 0$ we have

$$g = e^{-i\theta}(R + g_1) + ie^{-i\theta}(R + g_2) - Re^{-i\theta} - iRe^{-i\theta}.$$

If $R > 0$ is chosen sufficiently large, then all four functions on the right-hand side belong to Ω_r , and the claim is proved.

By construction of Ω_r the derivative $(\partial f / \partial r)(r \cdot)$ belongs to Ω_r for $r = 1$ and $r = a$. One easily finds a continuous function $\phi: [a, 1] \rightarrow B$ such that

- (i) $\phi(r) = (\partial f / \partial r)(r \cdot)$ for $r = a$ and $r = 1$, and
- (ii) $\phi(r) \in \Omega_r$ for each $r \in [a, 1]$.

It now follows from Lemma 2.1.7 in [21] that we can approximate the path $f: [a, 1] \rightarrow B$ in the B -norm by an Ω -allowable \mathcal{C}^1 path $\tilde{f}: [a, 1] \rightarrow B$ that coincides with f near the endpoints of $[a, 1]$. More precisely, for any $\varepsilon > 0$ we can find a \mathcal{C}^1 path $\tilde{f}: [a, 1] \rightarrow B$ satisfying

- (a) \tilde{f} coincides with f near the endpoints of the segment $[a, 1]$,
- (b) $(\partial \tilde{f} / \partial r)(r \cdot) \in \Omega_r$ for all $r \in [a, 1]$, and
- (c) $\|f_r - \tilde{f}_r\|_B < \varepsilon$ for each $r \in [a, 1]$.

The cited Lemma 2.1.7 is proved in [21] and is completely elementary. Most likely, the lemma is older than the reference [21]. In fact, this lemma is the heart of the matter of Gromov's method of convex integration of differential relations.

Decrease ε in Lemma 4.1 so that $\varepsilon \leq a\eta/2$ and let \tilde{f} be chosen as above. We extend \tilde{f} to $[0, 1]$ by setting $\tilde{f} = f$ on $[0, a]$. This gives a \mathcal{C}^1 function on $\bar{\Delta}$ by setting $\tilde{f}(re^{i\theta}) = \tilde{f}_r(\theta)$. By (a), \tilde{f} coincides with f near $T = b\Delta$ (and also near the origin). It follows from (b) and (c) that for $r \in [a, 1]$ we have

$$\left| \frac{\partial \tilde{f}}{\partial r} + \frac{i}{r} \frac{\partial \tilde{f}}{\partial \theta} \right| \geq \left| \frac{\partial \tilde{f}}{\partial r} + \frac{i}{r} \frac{\partial f_r}{\partial \theta} \right| - \frac{1}{r} \left| \frac{\partial \tilde{f}_r}{\partial \theta} - \frac{\partial f_r}{\partial \theta} \right| \geq \eta - \frac{\varepsilon}{r} \geq \frac{\eta}{2} > 0.$$

This implies that the graph of \tilde{f} over $\bar{\Delta}$ is totally real, and hence \tilde{f} is the required modification of f . This proves Lemma 4.1.

5. Proof of Theorem 1.1. In this section we will reduce Theorem 1.1 to Lemma 4.1 proved in Section 4 above.

Shrinking $U \subset M$ if necessary, we may assume that U is homeomorphic to the disc, the points p_0 and p_1 are the only complex tangents of π_0 on U , and $\pi_0: U \rightarrow \mathcal{M}$ is an embedding onto a smooth surface $V = \pi_0(U) \subset \mathcal{M}$. Set $\lambda = \pi_0(\gamma)$, where $\gamma \subset M$ is the smooth simple arc connecting p_0 and p_1 .

Our goal is to show that we can holomorphically spread a neighborhood of λ in V as a graph of a complex function over a simply connected region in \mathbb{C} , provided of course that λ preserves the canonical orientation at the two endpoints. This will enable us to complete the proof using Lemma 4.1.

On λ we choose a pair of smooth \mathbb{R} -linearly independent $(1, 0)$ vector fields X and Y that are tangent to V and such that X is also tangent to λ . The assumptions imply that X and Y are \mathbb{C} -linearly independent except at the two endpoints $\pi_0(p_0) = z^0$ and $\pi_0(p_1) = z^1$ of λ .

Write $X = X^1$ and choose another vector field X^2 along λ that is \mathbb{C} -linearly independent of X^1 . Then $Y = aX^1 + bX^2$ for some smooth complex valued functions a and b . Let $Z = cX^1 + X^2$, where the smooth function c on λ will be deter-

mined later. Clearly, Z is independent of X^1 , and we have $Y = (a - bc)X^1 + bZ$. Write $a - bc = \alpha + i\beta$, with α, β real-valued. We wish to choose the function c such that $\beta(z) \neq 0$ for all $z \in \lambda$.

Since the only complex tangents of V on λ are its endpoints, the function b vanishes at the endpoints and is zero-free in the interior of λ . Moreover, since γ and therefore λ are both orientation-preserving, $\Im a$ is of the same sign at the endpoints of λ , say positive. If we take $c(z) = C\overline{ib(z)}$ for a sufficiently large $C > 0$, we have $\beta(z) = \Im a(z) + C|b(z)|^2 > 0$ ($z \in \lambda$), and the requirement is fulfilled.

We can find a neighborhood Ω of λ in \mathcal{M} and a smooth diffeomorphism $\Phi: \Omega \rightarrow \Omega' \subset \mathbb{C}^2$ satisfying

- (i) $\Phi(\lambda) = [0, T] \times \{0\}$ for some $T > 0$,
- (ii) the derivative Φ_* of Φ is \mathbb{C} -linear at each point of λ , and
- (iii) $\Phi_*X = (1, 0)$ and $\Phi_*Z = (0, 1)$ on λ .

It follows that $\Phi_*Y_z = (\alpha(z) + i\beta(z), b(z))$, $z \in \lambda$. Since $\beta(z) > 0$ for all $z \in \lambda$, the image tangent space Φ_*T_zV projects isomorphically onto the first coordinate axis z_1 for each $z \in \lambda$. The implicit function theorem implies that $\Phi(V \cap \Omega) \subset \mathbb{C}^2$ is a graph over a region in \mathbb{C} , provided that the neighborhood Ω of λ is chosen sufficiently small.

The smooth arc λ has a basis of Stein neighborhoods in \mathcal{M} . Since the first-order jet of Φ is \mathbb{C} -linear at each point $z \in \lambda$, we can approximate Φ on λ by a holomorphic mapping $\Psi = (\psi_1, \psi_2)$, defined on a neighborhood of λ in \mathcal{M} , such that Ψ_* approximates Φ_* on λ . Then Ψ is biholomorphic on a sufficiently small neighborhood Ω_1 of λ , and the image surface $\Psi(V \cap \Omega_1) = W \subset \mathbb{C}^2$ is a graph

$$W = \Psi(V \cap \Omega_1) = \{(\zeta, f(\zeta)): \zeta \in D \subset \mathbb{C}\}$$

of a smooth complex function f on a domain $D \subset \mathbb{C}$. This means that the map $\Psi \circ \pi_0$ spreads a neighborhood of the arc $\gamma \subset M$ as a graph over a domain in the first coordinate plane.

By construction the surface W is totally real except at the endpoints $w^0 = \Psi(z^0)$ and $w^1 = \Psi(z^1)$ of the curve $\Psi(\lambda)$, and the indices are preserved:

$$I(p_j; \pi_0) = I(z^j; V) = I(w^j; W), \quad j = 0, 1.$$

Write $w^j = (\zeta_j, f(\zeta_j))$ ($j = 0, 1$). The projection $\psi_1(\lambda) \subset D$ of the arc $\Psi(\lambda)$ into D is a simple smooth arc in D with the endpoints ζ_0, ζ_1 .

Recall that $q \in \gamma$ is the point at which we must produce a complex tangent of type k . Let $\pi_0(q) = z^2 \in \lambda$ and $\Psi(z^2) = w^2 = (\zeta_2, f(\zeta_2)) \in W$.

Choose a pair of smoothly bounded domains $D_1 \subset\subset D_0 \subset\subset D$ containing the arc $\psi_1(\lambda)$ and homeomorphic to the disc. Let $D_2 \subset\subset D_1$ be a small disc centered at $\zeta_2 \in \psi_1(\lambda)$ and let g be the real-analytic function on D_2 that defines a surface with an isolated complex tangent of type k at ζ_2 and satisfies $g(\zeta_2) = f(\zeta_2)$. When $\mathcal{M} = \mathbb{C}^2$, we choose g such that the preimage of its graph

$$\Psi^{-1}(\{(\zeta, g(\zeta)): \zeta \in D_2\}) \subset \mathbb{C}^2$$

is a surface with an isolated complex tangent of type k at z^2 ; i.e., its germ at z^2 is affinely equivalent to the germ of the model surface $\Sigma(k)$ at the origin.

Denote by τ_j the positively oriented boundary curve of D_j ($j = 1, 2$). Let G be a strip connecting D_2 with the annular region $D_0 \setminus \bar{D}_1$ and satisfying $G \cap \psi_1(\lambda) = \emptyset$.

We can find a function F , defined and smooth in a neighborhood of the annular region $A_* = (D_0 \setminus \bar{D}_1) \cup G \cup D_2$, such that $F = f$ on $D_0 \setminus \bar{D}_1$, $F = g$ on \bar{D}_2 , and $\bar{\partial}F = \partial F / \partial \bar{\zeta} \neq 0$ on $\bar{A}_* \setminus \{\zeta_2\}$. Moreover, we can make $|f - g|$ small on D_2 by choosing D_2 sufficiently small, and thus we can choose F satisfying $|F - f| < \eta$ on \bar{A}_* for any given $\eta > 0$. Notice that there is no problem in choosing F correctly on the strip G since the strip can be very thin.

The complement

$$D_* = D_0 \setminus \bar{A}_* = D_1 \setminus (\bar{G} \cup \bar{D}_2) \subset\subset D_0$$

is a region homeomorphic to the disc whose positively oriented boundary curve τ_* is homologous in $\bar{A}_* \setminus \{\zeta_2\}$ to the cycle $\tau_1 - \tau_2$.

Denote the winding number of a nonvanishing function h along a closed oriented curve $\tau \subset \mathbb{C}$ by $\mathcal{W}(h; \tau)$. By Proposition 2.2 and Lemma 2.3 in Section 2, we have

$$\begin{aligned} \mathcal{W}(\bar{\partial}F; \tau_*) &= \mathcal{W}(\bar{\partial}F; \tau_1) - \mathcal{W}(\bar{\partial}F; \tau_2) \\ &= \mathcal{W}(\bar{\partial}f; \tau_1) - \mathcal{W}(\bar{\partial}g; \tau_2) \\ &= I(w^0; W) + I(w^1; W) - k \\ &= 0. \end{aligned}$$

We can now apply Lemma 4.1 to extend F from the annular region A_* to the disc-region \bar{D}_* such that $\bar{\partial}F$ is nonvanishing on \bar{D}_* . Moreover, since $|F - f| < \eta$ on A_* , we can choose the extension such that the same holds on D_0 .

To summarize, we have found for any given $\eta > 0$ a smooth function F on D_0 satisfying

- (a) $F = f$ near the boundary of D_0 ,
- (b) $F = g$ near the point ζ_2 ,
- (c) $\bar{\partial}F \neq 0$ on $D_0 \setminus \{\zeta_2\}$, and
- (d) $\sup_{D_0} |F - f| < \eta$.

Let $F_t = (1 - t)f + tF$, $t \in [0, 1]$, let $W_t \subset D_0 \times \mathbb{C}$ be the graph of F_t over D_0 , and set $V_t = \Psi^{-1}(W_t) \subset \mathcal{M}$. By construction the family of surfaces $\{V_t; t \in [0, 1]\}$ defines an isotopy of the initial surface $V_0 = V = \pi_0(U)$ to the surface V_1 whose only complex tangent z^2 is of type k . The isotopy is fixed near the boundary of V , and the deformation can be made arbitrarily small in the \mathcal{C}^0 -sense by choosing $\eta > 0$ sufficiently small. This gives a regular homotopy of immersions π_t with the required properties. Theorem 1.1 is proved.

6. Stein neighborhood basis.

PROPOSITION 6.1. *Let $M \subset \mathbb{C}^2$ be a smoothly embedded surface with isolated complex tangents of negative type (Definition 2). Then there is a neighborhood Ω of M and a smooth plurisubharmonic function ρ on Ω satisfying*

- (a) $M = \{z \in \Omega: \rho(z) = 0\}$, and
- (b) $d\rho(z) \neq 0$ for $z \in \Omega \setminus M$.

Proof. Recall that $\Sigma(k)$ is given by $z_2 = \bar{z}_1^{|k|+1}$. The nonnegative function

$$\begin{aligned}\rho^{(k)}(z) &= |z_2 - \bar{z}_1^{|k|+1}|^2 \\ &= |z_1|^{2|k|+2} + |z_2|^2 - 2\Re(z_1^{|k|+1} z_2)\end{aligned}$$

is plurisubharmonic, strongly plurisubharmonic on $z_1 \neq 0$, and it vanishes precisely on $\Sigma(k)$.

Let the complex tangents of $M \subset \mathbb{C}^2$ be p_1, \dots, p_m , where p_j is of type k_j . Let U_j be a small ball centered at p_j such that $M \cap U_j = A_j(\Sigma(k_j)) \cap U_j$ for a suitable complex affine transformation A_j . Assume also that the balls U_j are pairwise disjoint. Define a function $\rho_j: U_j \rightarrow [0, \infty)$ by $\rho_j = \rho^{(k_j)} \circ A_j^{-1}$.

The function $\rho_0(z) = \text{dist}(z, M)^2$ is smooth in a neighborhood of M in \mathbb{C}^2 and is strongly plurisubharmonic in a neighborhood U_0 of the totally real part $M \setminus \{p_1, \dots, p_m\}$ of M .

We obtain the required function ρ by patching ρ_0 with ρ_j in U_j . Let χ_j be a smooth function on \mathbb{C}^2 with values in $[0, 1]$ that is identically one near p_j and is identically zero outside U_j . We set

$$\rho(z) = \prod_{j=1}^m (1 - \chi_j(z)) \rho_0(z) + \sum_{j=1}^m \chi_j(z) \rho_j(z).$$

Notice that $\rho = \rho_0$ outside $\bigcup U_j$, and $\rho = (1 - \chi_j)\rho_0 + \chi_j\rho_j$ on U_j . Since the patching of ρ_0 with ρ_j occurs only at points of M where both functions are strongly plurisubharmonic, the function ρ is also plurisubharmonic in a neighborhood of M .

We must see that $d\rho(z) \neq 0$ at points z not on M but sufficiently close to M . On U_j we have

$$d\rho = (1 - \chi_j)d\rho_0 + \chi_j d\rho_j + d\chi_j(\rho_j - \rho_0).$$

The last term vanishes to second order on $M \cap U_j$. The sum of the first two terms is a convex linear combination of the gradients $d\rho_0$ and $d\rho_j$ that vanishes to first order on $M \cap U_j$ and is nonvanishing in a punctured neighborhood of $M \cap U_j$. Thus, ρ has the required properties in a sufficiently small neighborhood Ω of M . This proves Proposition 6.1.

The sublevel sets $\Omega_c = \{z \in \Omega: \rho(z) < c\}$ for sufficiently small $c > 0$ are smoothly bounded pseudoconvex domains homotopic to M . In fact, the orthogonal projec-

tion onto M taking each point to its closest point in M is the deformation retraction of Ω_ϵ onto M . Thus, the family Ω_ϵ is a regular Stein neighborhood basis of M , and M is *holomorphically convex* in Ω (see [26, p. 91]).

7. Surfaces in the complex projective plane. Let $C \subset \mathbf{CP}^2$ be a smooth embedded complex curve of degree d . This means that in homogeneous coordinates $[z_1, z_2, z_3]$ on \mathbf{CP}^2 the curve C is given by an equation $F(z_1, z_2, z_3) = 0$, where F is a homogeneous polynomial of degree d . We choose on C the canonical orientation given by its complex structure. Such a curve C is a closed Riemann surface whose topological type is determined by its genus $g(C)$.

Recall the index formula (1)

$$I(C; \mathbf{CP}^2) = \chi(C) + \chi(C; \mathbf{CP}^2) = \chi(C) + C \cdot C.$$

By the theorem of Bezout [20, p. 670], we have $C \cdot C = d^2$. We also have the genus formula

$$g(C) = (d-1)(d-2)/2$$

[20, p. 220] relating the genus and the degree of C . Substituting this into the index formula we get $I(C; \mathbf{CP}^2) = 3d$. After a generic small perturbation of C , we obtain a surface in \mathbf{CP}^2 with isolated complex tangents, all of the same sign; hence Theorem 1.1 implies that they can be joined to a single complex tangent of type $3d$. This proves the following proposition.

PROPOSITION 7.1. *The index of a smooth embedded complex curve $C \subset \mathbf{CP}^2$ of degree d equals $I(C; \mathbf{CP}^2) = 3d$. There is an isotopy of C to a surface $C_1 \subset \mathbf{CP}^2$ with a single complex tangent of type $3d$.*

Example 1. There are two nonisomorphic embeddings of the Riemann sphere into \mathbf{CP}^2 : a degree-one embedding as a hyperplane $C_1 = H$, and a degree-two embedding onto the curve C_2 defined in the homogeneous coordinates $[z_1, z_2, z_3]$ by the equation $z_1 z_2 = z_3^2$. The index equals $I(C_1) = 3$ and $I(C_2) = 6$.

Example 2. Every compact Riemann surface of genus one (a topological torus) can be realized as a nonsingular cubic curve T in \mathbf{CP}^2 [20, p. 222]. Thus, $d = 3$ and $I(T; \mathbf{CP}^2) = 9$.

Choose a smooth complex curve $C \subset \mathbf{CP}^2$ and let $M \subset \mathbf{C}^2 \subset \mathbf{CP}^2$ be a closed surface embedded into the finite part of \mathbf{CP}^2 . After a suitable affine transformation applied to M , we may assume that $C \cap M = \emptyset$; hence we can make the connected sum $C \# M$ in \mathbf{CP}^2 as in Section 3. If C is of degree d , Propositions 7.1 and 3.1(c) together imply

$$I(C \# M; \mathbf{CP}^2) = I(C; \mathbf{CP}^2) + I(M; \mathbf{C}^2) - 2 = I(M) + 3d - 2.$$

If we take C to be one of the spheres C_1, C_2 of Example 1 above, then the surface $C_j \# M \subset \mathbf{CP}^2$ ($j = 1, 2$) is homeomorphic to M and

$$I(C_1 \# M; \mathbf{CP}^2) = I(M) + 1, \quad I(C_2 \# M; \mathbf{CP}^2) = I(M) + 4.$$

This gives two inequivalent embeddings of M into \mathbf{CP}^2 . None of these embeddings can be deformed into the finite part of \mathbf{CP}^2 since their intersection index with the line at infinity equals one (resp. two) by Bezout's theorem.

For orientable surfaces we cannot produce new totally real embeddings into \mathbf{CP}^2 in this way since the positive and the negative indices of $C \# M$ are different, due to the fact that all complex tangents of C are of the same sign. However, this can be done for certain unorientable surfaces as follows.

Proof of Theorem 1.6. Suppose that M is unorientable of genus g . If $g = 1 \pmod{4}$, then $3\chi(M) = 3 \pmod{4}$, and (5) shows that there is an embedding $M \subset \mathbf{C}^2$ with index -1 . The connected sum $C_1 \# M \subset \mathbf{CP}^2$ with the line in \mathbf{CP}^2 is an embedding of M into \mathbf{CP}^2 with index zero; hence by Corollary 1.2(b) it is isotopic to a totally real embedding.

If $g = 2 \pmod{4}$, then $3h = 0 \pmod{4}$. By (5) there is an embedding $M \subset \mathbf{C}^2$ with index -4 . The connected sum $C_2 \# M$ with the sphere of degree two in \mathbf{CP}^2 is an embedding of M into \mathbf{CP}^2 with index zero; hence it is isotopic to a totally real embedding. This proves Theorem 1.6.

In this way we cannot produce totally real embeddings of the unorientable surfaces with genera $g = 0, 3 \pmod{4}$ into \mathbf{CP}^2 . We do not know if any such embeddings exist.

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