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An elementary proof of Fefferman's theorem

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1. Introduction

One of the classical problems of several complex variables is to understand the boundary behavior of biholomorphic mappings $f : D \to D'$ between bounded domains $D, D' \subset \mathbf{C}^n$ with C^∞-smooth boundaries. In one variable there is a classical theorem, due to Kellog [21], to the effect that every such mapping extends to a C^∞-smooth diffeomorphism $f : \overline{D} \to \overline{D}'$ of their closures. If the boundaries bD, bD' are smooth real-analytic curves, then f extends holomorphically to a neighborhood of $\overline{D}$ according to the much more elementary Schwarz reflection principle.

It is not a simple task to prove the corresponding statements for domains in $\mathbf{C}^n$ for $n > 1$, and the problem is still open on arbitrary domains. The first general result in this direction was the following theorem of C. Fefferman [14] in 1974. By smooth we always mean $\mathscr{C}^\infty$ unless otherwise specified.

1.1 Theorem *If $f : D \to D'$ is a biholomorphic mapping between bounded domains D, D' in $\mathbf{C}^n$ with smooth strongly pseudoconvex boundaries, then f extends to a smooth diffeomorphism of $\overline{D}$ onto $\overline{D}'$.*

Recall that a domain $D \subset \mathbf{C}^n$ is *strongly pseudoconvex* if there is a defining function $r(z)$ on a neighborhood of $\overline{D}$ satisfying $D = \{r < 0\}, dr \neq 0$ on $bD = \{r = 0\}$, so that for each $z \in bD$ the *Levi form*

$$\mathscr{L}(z; w) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(z) w_j \overline{w}_k \tag{1.1}$$

is positive definite on the maximal complex tangent space

$$\mathrm{H}_z bD = \left\{ w \in \mathbf{C}^n : \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(z) w_j = 0 \right\}. \tag{1.2}$$

Fefferman's proof involved deep and rather difficult analysis of the Bergman kernel function and the associated Bergman metric of a smooth strongly pseudoconvex

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domain in $\mathbf{C}^n$. On the other hand, if the mapping f is assumed to be continuously differentiable up to the boundary and the boundaries are real-analytic, it is not difficult to show that f extends holomorphically past the boundary. This 'reflection principle' was first proved by Lewy [23] and Pinčuk [27] by a clever application of the implicit function theorem. Webster [35] provided another proof, using the classical edge-of-the-wedge theorem.

Today there exist several proofs of Theorem 1.1. The result has been localized and extended to proper holomorphic mappings between pseudoconvex domains of finite type in $\mathbf{C}^n$. One approach relies on the biholomorphic invariance of the Bergman kernel and the regularity of the associated $\bar{\partial}$-Neumann problem. See the papers by Bell and Ligocka [9], Bell [4,5,6], Bell and Catlin [7,8], and Diederich and Fornæss [11,12]. Independent proofs using more elementary techniques were given by Nirenberg, Webster, and Yang [25], Lempert [22], and more recently by Pinčuk and Hasanov [30]. See also the recent survey [16] by the author.

The purpose of this paper is to give yet another proof of Theorem 1.1. It is based on two classical results of complex analysis: the edge-of-the-wedge theorem and its generalization to $\mathscr{C}^\infty$ edges (Theorem 3.1 below), and the theorem of Julia-Carathéodory in several variables (Rudin [33, p.174]) that follows from the invariant Schwarz lemma.

A similar approach was developed in the papers by Webster [35], Nirenberg, Webster and Yang [25], and Pinčuk and Hasanov [30]. Julia's theorem has been used in this context in [25], but our proof seems shorter and simpler. The difference between our proof and that of Pinčuk and Hasanov is that we replace their method of scaling by the theorem of Julia-Carathéodory. This way we can avoid some of the delicate points in the normal families arguments of [30] and [28]. We believe that our proof is accessible also to beginners in complex analysis. It is especially elementary in the case of real-analytic boundaries.

The proof is completely local and gives the following local theorem.

1.2 Theorem *Let M and M' be smooth strongly pseudoconvex hypersurfaces in $\mathbf{C}^n$ (not necessarily closed), and let $D \subset \mathbf{C}^n$ be a domain in the pseudoconvex side of M, containing M in its boundary. Let $f : D \cup M \to \mathbf{C}^n$ be a continuous mapping that is holomorphic in D and takes M to M'. If for some $p \in M$ the fiber $K = f^{-1}(f(p))$ is a compact subset contained in the (relative) interior of M, then f is smooth in a neighborhood of p in $D \cup M$.*

Remarks.

1. This theorem has been proved for mappings between pseudoconvex hypersurfaces of finite type by Bell and Catlin [8] via the Bergman kernel method.

Recently Pinčuk and Tsyganov have proved the same result for strongly pseudoconvex hypersurfaces without assuming that $f^{-1}(f(p))$ is a compact subset of M. (This was announced by Pinčuk at the conference at Santa Cruz, July 1989.)

2. The continuity of f on $D \cup M$ follows from the assumption that the cluster set of f at M is contained in M'. See for instance the papers [15] and [17].
3. We shall always restrict ourselves to the $\mathscr{C}^\infty$ case, although the same proof can be applied in the case of finite smoothness of M and M' (see [30], [22]). The following sharp regularity result has been obtained recently by Hurumov (to appear): If M and M' are of class $\mathscr{C}^k, k > 2$ then f is of class $\mathscr{C}^{k-1/2-0}$.

The paper is organized as follows. In section 2 we localize the problem near a given boundary point $p \in M$. In section 3 we recall some regularity results for mappings on wedge domains with a generic totally real edge ($\mathscr{C}^\infty$ version of the edge-of-the-wedge theorem). In section 4 we explain the connection between the two problems. This part follows closely the approach by Webster [35] and Pinčuk and Hasanov [30]. Everything up to this point is well-known.

The crucial part of our proof that differs from the existing ones is given in section 5 where we complete the reduction step (Proposition 5.1).

I wish to thank J.P. Rosay who called my attention to the work [30] and who shared with me many of his ideas on the subject during my visit to the University of Wisconsin in September 1988. I also thank S. Bell and J. Globevnik for several discussions on this subject. Last but not least, I thank S.I. Pinčuk who kindly explained me certain points concerning the normal families argument in the paper [30].

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2. Localization

We shall first prove that a nonconstant holomorphic mapping satisfying the conditions of Theorem 1.2 is biholomorphic locally near p. A similar localization can be found in the papers by Bell [6] and Bell and Catlin [8]. If the reader is only interested in the global case (Theorem 1.1), he may skip this section.

Let D' be a domain in $\mathbf{C}^n$ bounded in part by M' and lying in the pseudoconvex side of M'. Recall that $D \subset \mathbf{C}^n$ is a similar domain bounded in part by M.

Definition. *We say that a set $G \subset D$ is a one-sided neighborhood of a point $p \in M$ if G contains $U \cap D$ for some open neighborhood U of p in $\mathbf{C}^n$.*

2.1. Proposition. *Under the hypotheses of Theorem 1.2 there exist arbitrarily small one-sided neighborhoods G of $p \in M$ and G' of $f(p) \in M'$ such that the restriction $f : G \to G'$ is biholomorphic.*

Proof. If $V \subset \mathbf{C}^n$ is a sufficiently small open neighborhood of the compact set $K = f^{-1}(f(p))$ in $\mathbf{C}^n$, then $V \cap M \subset\subset M, V \cap D \subset\subset D \cup M$, and $f(V \cap D) \subset D'$ according to the maximum principle. For such V, the compact set $E = f(bV \cap \overline{D})$ does not contain $f(p)$, so there is a ball U' centered at $f(p)$ that does not meet E. We choose U' sufficiently small such that the domain $G' = U' \cap D'$ is compactly contained in $D' \cup M'$.

Let $G = f^{-1}(G') \cap V \cap D$. By continuity of f, G is a one sided neighborhood of each point in K. We claim that the restriction $f : G \to G'$ is proper holomorphic. To prove this, take an arbitrary compact subset $L' \subset G'$ and let $L = f^{-1}(L') \cap G$. If L is not compact, then its closure in $\mathbf{C}^n$ intersects the boundary of G, so there is a point $z \in bG$ for which $f(z) \in L'$. Since z cannot be in M, it must be in $bG \cap D$. Also, z cannot be in $bV \cap D$ by the construction of G'. Hence z is an interior point of $V \cap D$. By the continuity of f it follows that a neighborhood U of z in $V \cap D$ is mapped into G'. Hence $U \subset G$, which contradicts the assumption that z is a boundary point of G.

Thus, $f : G \to G'$ is proper holomorphic. Let $A = \{z \in G : \det Df(z) = 0\}$ be the branch locus of f in G. Pinčuk proved in [28] that A does not approach the strongly pseudoconvex boundary points of G when $n > 1$. (For self-mappings of the ball this is just the well known result of Alexander [1], [33, p.316].) If we now shrink the neighborhood V of K, we may assume that f is locally biholomorphic in $V \cap D$. Repeating the above procedure we find smaller domains, still called G and G', such that $f : G \to G'$ is proper, whence a holomorphic covering. If U' is a sufficiently small ball at $f(p)$, then $G' = U' \cap D'$ is simply connected. If we now replace G by the connected component of G whose closure contains p, then $f : G \to G'$ is biholomorphic.

A standard argument (see for instance [2] or [26]) shows that the inverse $f^{-1} : G' \to G$ extends continuously to a neighborhood of $f(p)$ in $\overline{G}'$, so p is an isolated point of the fiber K. Thus we may apply the above proof with V an arbitrary small neighborhood of p. This concludes the proof of Proposition 2.1.

3. Regularity of mappings on wedge domains.

In this section, we recall some results due to Pinčuk and Hasanov [30] and, independently, to Coupet [10], concerning mappings between wedge domains.

Perhaps the main reason why it is much easier to prove the boundary regularity theorem in one variable than in several variables is that the boundary of a domain in $\mathbf{C}^1$ is a real curve with no complex structure, while in $\mathbf{C}^n$ it is a real hypersurface with plenty of complex structure. This point of view suggests that the most natural generalization of the one-variable mapping problem to several variables is obtained by considering the behavior of holomorphic mappings near totally real submanifolds in $\mathbf{C}^n$.

Recall that a real submanifold $\Sigma \subset \mathbf{C}^n$ is said to be *totally real* if for each $z \in \Sigma$ the real tangent space $T_z\Sigma$ contains no nontrivial complex subspaces, i.e., $T_z\Sigma \cap iT_z\Sigma = \{0\}$. Clearly this requires $\dim_R \Sigma \leq n$. Σ is said to be *maximal real* if, in addition, $\dim_R \Sigma = n$. Thus, every smooth real curve in $\mathbf{C}^1$ is maximal real.

If $\Sigma \subset \mathbf{C}^n$ is a maximal real submanifold that is also real-analytic, we can find locally near each point $z \in \Sigma$ a biholomorphic change of coordinates so that, in the new coordinates, Σ corresponds to a piece of $\mathbf{R}^n \subset \mathbf{C}^n$. Thus Σ is the fixed point set of an anti-holomorphic reflection Φ, defined on a neighborhood of Σ in $\mathbf{C}^n$.

If Σ is merely smooth, we can find a smooth change of coordinates Ψ thats maps Σ to $\mathbf{R}^n$ and that is $\bar{\partial}$-flat on Σ, i.e., $\bar{\partial}\Psi$ vanishes to infinite order on Σ [24]. This gives us a reflection Φ that fixes Σ pointwise and that is *almost antiholomorphic*, in the sense that its holomorphic derivative $\partial\Phi$ vanishes to infinite order at every point of Σ.

A natural type of domains associated to a maximal real submanifold $\Sigma \subset \mathbf{C}^n$ are the *wedges with edge* Σ. Locally near $p \in \Sigma$ we can find n smooth real-valued functions $r_1, \ldots, r_n$ so that $\Sigma = \{z : r_1(z) = \cdots = r_n(z) = 0\}$, and the complex gradients

$$\partial r_j = \sum_{k=1}^{n} \frac{\partial r_j}{\partial z_k} dz_k$$

are $\mathbf{C}$-linearly independent on Σ. If U is a neighborhood of p in $\mathbf{C}^n$ and $\Gamma \subset \mathbf{R}^n$ is an open convex cone with vertex zero, we define the wedge with edge Σ:

$$\mathscr{W} = \mathscr{W}(U, \Gamma) = \{z \in U : r(z) \in \Gamma\}. \tag{3.1}$$

We recall the *edge-of-the-wedge theorem*: If $\mathscr{W}^+ = \mathscr{W}(U, \Gamma)$, $\mathscr{W}^- = \mathscr{W}(U, -\Gamma)$, and f is a continuous function on $\mathscr{W}^+ \cup \Sigma \cup \mathscr{W}^-$ that is holomorphic on $\mathscr{W}^+ \cup \mathscr{W}^-$, then f extends holomorphically to a neighborhood of $\Sigma \cap U$ in $\mathbf{C}^n$. For real-analytic edge Σ this follows from the classical result [34] where Σ is an open subset of $\mathbf{R}^n$ by an change of coordinates. For smooth edges see [3], [29], or [32].

There is a version of this result for *asymptotically holomorphic functions.* A smooth function f defined on a wedge $\mathscr{W}$ is said to be asymptotically holomorphic at Σ if each derivative $D^\alpha \bar{D}^\beta(\bar{\partial} f)$ of $\bar{\partial} f$ extends continuously to Σ and equals zero there. Here is the $\mathscr{C}^\infty$ version of the edge-of-the-wedge theorem: If Σ is smooth, f is continuous on $\mathscr{W}^+ \cup M \cup \mathscr{W}^-$ and smooth on $\mathscr{W}^+ \cup \mathscr{W}^-$, and if f is asymptotically holomorphic at Σ, then the restriction of f to Σ is also smooth. More precise results in this direction have been proved by Pinčuk and Hasanov [30, Theorem 1] and by Coupet [10]. In the case when $\Sigma = \mathbf{R}^n$ this also follows from a more general result concerning the $\mathscr{C}^\infty$ wave front set of a function on $\mathbf{R}^n$; see Hörmander [20, p.257]. The general case follows by applying a local $\mathscr{C}^\infty$ change of coordinates near $p \in \Sigma$ that takes Σ to $\mathbf{R}^n$ and is $\bar\partial$-flat on Σ.

The above implies the following regularity results for mappings on wedges; this is a special case of results due to Pinčuk and Hasanov [30] and Coupet [10]:

3.1 Theorem *Let $\mathscr{W} = \mathscr{W}(U,\Gamma)$ be a wedge* (3.1) *with a smooth maximal real edge $\Sigma \subset \mathbf{C}^n$, let $\Sigma' \subset \mathbf{C}^{n'}$ be a smooth totally real submanifold, and let $F : \mathscr{W} \cup \Sigma \to \mathbf{C}^{n'}$ be a continuous mapping that is smooth on $\mathscr{W}$, asymptotically holomorphic at Σ, and $F(\Sigma) \subset \Sigma'$. Then the restriction of F to $\Sigma \cap U$ is smooth. If both Σ and Σ' are real-analytic and F is holomorphic on $\mathscr{W}$, then F extends holomorphically to a neighborhood of $\Sigma \cap U$ in $\mathbf{C}^n$.*

In the real-analytic case the result follows from the edge-of-the-wedge theorem as follows. Without loss of generality we can assume that Σ' is maximal real in $\mathbf{C}^{n'}$. Let Φ' be the anti-holomorphic reflection that fixes Σ' pointwise, and let Φ be the corresponding reflection on Σ. We can extend F to a wedge $\mathscr{W}^-$ that is essentially the opposite of $\mathscr{W}^+ = \mathscr{W}$ by taking $\Phi' \circ F \circ \Phi$. If F is holomorphic on $\mathscr{W}^+$ and Φ resp. Φ' are anti-holomorphic, the extended map is holomorphic also on $\mathscr{W}^-$, so the classical edge-of-the-wedge theorem shows that F extends holomorphically to a neighborhood of p in $\mathbf{C}^n$.

In the smooth case we use almost anti-holomorphic reflections Φ resp. Φ', together with a distance estimate

$$\operatorname{dist}\left(F(z), \Sigma'\right) \le C \operatorname{dist}(z, \Sigma),$$

valid on any finer wedge $\mathscr{W}_1^+ \subset \mathscr{W}^+$ with some $C < \infty$ (see [30] and [31]), to see that the extended map is asymptotically holomorphic at Σ also from the wedge $\mathscr{W}_1^-$. Hence the smoothness of F of Σ follows from the $\mathscr{C}^\infty$ edge-of-the-wedge theorem quoted above.

4. Reduction to the edge-of-the-wedge theorem.

We shall now explain the connection between Theorem 3.1 and the mapping problem. This is due to S. Webster [35], although the idea appeared implicitly already in the papers by H. Lewy [23] and S. Pinčuk [27].

In view of Proposition 2.1 we may assume the following situation. Let M and M' be local strongly convex smooth hypersurfaces containing the origin in $\mathbf{C}^n$ ($n > 1$). Let D be a local domain in $\mathbf{C}^n$ near the origin that is smoothly bounded and pseudoconvex along M, and let D' be a similar such domain bounded in part by M'. We have a biholomorphic mapping $f : D \to D'$ that extends to a homeomorphism of $\overline{D}$ onto $\overline{D}'$ and takes the origin to the origin, and we must prove that f is smooth on $D \cup M$ near the origin. Since the problem is a local one, we may shrink our sets towards the origin, which we shall freely do in the sequel.

Let $\langle z, w \rangle = \sum z_j w_j$. For each nonzero vector $a \in \mathbf{C}^n \backslash \{0\}$ we denote by [a] the complex hyperplane

$$[a] = \Lambda_a = \{w \in \mathbf{C}^n : \langle w, a \rangle = 0\} \in \mathbf{CP}^{n-1}.$$

Two vectors in $\mathbf{C}^n$ determine the same hyperplane precisely when they are multiples of each other, and every complex hyperplane through 0 is of this form. Hence the set of all such hyperplanes is the complex projective space $\mathbf{CP}^{n-1}$, and a is the homogeneous coordinate of Λ_a.

Following Webster [35] we associate to f the holomorphic mapping

$$F(z, \Lambda) = (f(z), \mathrm{D}f(z)\Lambda),$$

where $z \in D$ and $\Lambda \in \mathbf{CP}^{n-1}$ is a complex hyperplane through 0. Here, $\mathrm{D}f(z)\Lambda$ is the image of Λ by the derivative $\mathrm{D}f(z)$. Clearly F maps the domain $\tilde{D} = D \times \mathbf{CP}^{n-1}$ in the complex manifold $X = \mathbf{C}^n \times \mathbf{CP}^{n-1}$ biholomorphically onto the domain $\tilde{D}' = D' \times \mathbf{CP}^{n-1} \subset X$.

If A is an invertible $n \times n$ matrix and $a \in \mathbf{C}^n \backslash \{0\}$ is a row-vector, a simple calculation shows that $A(\Lambda_a) = \Lambda_b$ for $b = a\Lambda^{-1}$. Here, $a\Lambda^{-1}$ is the matrix product of the row a with the matrix A^{-1}. Thus F is given in the homogeneous coordinates on $\mathbf{CP}^{n-1}$ by

$$F(z, [a]) = \left(f(z), [a\mathrm{D}f(z)^{-1}]\right). \tag{4.1}$$

Recall that $\mathrm{H}_z M \subset \mathrm{T}_z M$ is the maximal complex subspace of the real tangent space $\mathrm{T}_z M$ (1.2). We associate to M the smooth submanifold $\tilde{M}$ of X defined by

$$\tilde{M} = \{(z, \mathrm{H}_z M) \in X : z \in M\}. \tag{4.2}$$

Let $\tilde{M}' =\subset X$ be the analogous manifold associated to the hypersurface M'.

If f is continuously differentiable up to $D \cup M$, then for each $z \in M$ the derivative $Df(z)$ maps $\mathrm{H}_z M$ isomorphically onto $\mathrm{H}_{f(z)} M'$. Hence the associated mapping F extends continuously from the domain $\tilde{D}$ to $\tilde{D} \cup \tilde{M}$ and maps $\tilde{M}$ to $\tilde{M}'$. Its restriction to $\tilde{M}$ is given by

$$\tilde{f}(z, \mathrm{H}_z M) = \left(f(z), \mathrm{H}_{f(z)} M'\right).$$

Fefferman's theorem now follows immediately from Theorem 3.1 and the following

Lemma. (Webster [35].) *If $M \subset \mathbf{C}^n$ is strongly pseudoconvex, the associated manifold $\tilde{M} \subset \mathbf{C}^n \times \mathbf{CP}^{n-1}$ is totally real (whence maximal real).*

Namely, the domain $\tilde{D}$ clearly contains a wedge $\mathscr{W}$ of type (3.1) with the edge $\tilde{M}$, so Theorem 3.1 implies that $\tilde{f}$ is smooth on $\tilde{M}$ whence f is smooth on M.

Suppose now that f is merely continuous up to M. In order to use Theorem 3.1 one has to prove

4.1 Theorem *There is a wedge $\mathscr{W} \subset \tilde{D}$ with edge $\tilde{M}$ such that the restriction of F to $\mathscr{W}$ extends continuously to $\tilde{M}$ and equals $\tilde{f}$ there.*

Once this is established, the smoothness of f on M follows from Theorem 3.1 as above. We shall prove Theorem 4.1 in the following section. A different proof had been given before by Pinčuk and Hasanov [30], using the scaling method.

5. An application of Julia-Carathéodory's Theorem.

Let $M = \{r(z) = 0\}$ and $M' = \{r'(z) = 0\}$ by strongly convex hypersurfaces, with defining functions whose Taylor expansion at the origin is of the form

$$r(z) = \Re z_n + Q(z', \Im z_n) + o(|z'|^2 + |\Im z_n|^2), \tag{5.1}$$

where **Q** is a strongly positive definite quadratic form in the indicated variables. Also let $D = \{r(z) < 0\}, D' = \{r'(z) < 0\}$. We shall use the notation

$$\partial_j r = \frac{\partial r}{\partial z_j}, \quad \bar{\partial}_j r = \frac{\partial r}{\partial \bar{z}_j}, \quad \partial r = (\partial_1 r, \ldots, \partial_n r), \quad \bar{\partial} r = (\bar{\partial}_1 r, \ldots, \bar{\partial}_n r).$$

From (1.2) we see that the maximal complex tangent space $\mathrm{H}_z M$ has the homogeneous coordinate $\partial r(z)$. Since $\partial_n r(z) \neq 0$ in a neighborhood of the origin, we can restrict our considerations to the coordinate chart $\mathbf{C}^{n-1} \subset \mathbf{CP}^{n-1}$ on which the last coordinate is nonzero. Let $p = (p_1, \ldots, p_{n-1})$ be the affine coordinate of the point $[p_1, \ldots, p_{n-1}, 1] \in \mathbf{CP}^{n-1}$. Then $\tilde{M}$ is given by

$$r(z) = 0, \quad p_j = \frac{\partial_j r(z)}{\partial_n r(z)}, \quad 1 \le j \le n-1,$$

and similarly for $\tilde{M}'$.

Let $G : D \times \mathbf{C}^{n-1} \to \mathbf{C}^n \setminus \{0\}$ be the holomorphic mapping

$$G(z,p) = (p_1, \ldots, p_{n-1}, 1) \mathrm{D}f(z)^{-1}.$$

Then the mapping F (4.1) can be expressed by

$$F(z,p) = (f(z), [G(z,p)]), \quad z \in D, p \in \mathbf{C}^{n-1}.$$

Shrinking D if necessary we may assume that each point $z \in D$ has a unique closest point $\pi(z) \in M$. For $\alpha > 0$ and U a small neighborhood of 0 in $\mathbf{C}^n$ we denote by $\mathscr{W}_\alpha(U) \subset U \times \mathbf{C}^{n-1}$ the wedge

$$\mathscr{W}_\alpha(U) = \left\{ (z,p) : z \in U \cap D, \ \left| p_j - \frac{\partial_j r(\pi(z))}{\partial_n r(\pi(z))} \right| < \alpha \operatorname{dist}(z,M), \ 1 \le j \le n-1 \right\}$$

with edge $\tilde{M}$. Now Theorem 4.1 will follow immediately from

5.1. Proposition *For each $\alpha > 0$ there is a neighborhood U of the origin in $\mathbf{C}^n$ such that the functions $G_1, G_2, \dots, G_n$ and $1/G_n$ are bounded holomorphic on $\mathscr{W}_\alpha(U)$. Moreover, the quotients G_j/G_n for $1 \le j \le n-1$ extends continuously to $\mathscr{W}_\alpha(U) \cup \tilde{M}$ so that for each $\zeta \in M \cap U$ we have*

$$\lim_{z \to \zeta} \frac{G_j(z,p)}{G_n(z,p)} = \frac{\partial_j r'(f(\zeta))}{\partial_n r'(f(\zeta))}.$$

Notice that for points $(z,p) \in \mathscr{W}_\alpha$, $z \to \zeta \in M$ implies that p converges to the affine coordinate of $H_\zeta M$. Since the right hand side above is the affine coordinate of the point $\mathrm{H}_{f(\zeta)} M' \in \mathbf{CP}^{n-1}$, proposition implies that F extends continuously to $\mathscr{W}_\alpha(U) \cup \widetilde{M}$ and coincides with $\tilde{f}$ on $\widetilde{M}$, so Theorem 4.1 holds.

Proof of Proposition 5.1. Let $U \subset \mathbf{C}^n$ be a small ball centered at the origin. Fix a $\beta > 1$. For each point $\zeta \in U \cap M$ we let

$$\Gamma_\zeta = \{ z \in D : |z - \zeta| < \beta \operatorname{dist}(z,M) \}$$

be a nontangential approach region for ζ in D. We denote by $\tilde{\Gamma}_\zeta \subset \mathscr{W}_\alpha$ the preimage of Γ_ζ in the wedge $\mathscr{W}_\alpha$ under the coordinate projection $(z,p) \to z$.

We now fix $\zeta \in M \cap U$ and choose new coordinates $w = (w', w_n)$ on $\mathbf{C}^n$ so that ζ corresponds to the point $w = 0$, and the outer normal direction to M at ζ corresponds to the axis $\Re w_n > 0$. To find an explicit expression for w we choose a unitary matrix $U_\zeta \in \mathrm{U}(n)$ satisfying $U_\zeta \nabla r(\zeta) = t e_n$ for some $t > 0$, where $\nabla r(z) = 2\bar{\partial} r(z)$ is the gradient of r at ζ (considered as a column vector), and $e_n = (0', 1)$. We may assume that $|\nabla r(\zeta)| = 2$ for each $\zeta \in M$. By conjugating and transposing the above relation we obtain

$$\partial r(\zeta) U_\zeta^{-1} = e_n. \tag{5.2}$$

The relation between the old coordinates z and the new coordinates w is then

$$w = \phi_\zeta(z) = U_\zeta(z - \zeta).$$

Clearly we can choose U_ζ to depend continuously on $\zeta \in M$. We introduce a similar coordinate change

$$w' = \psi_{f(\zeta)}(z') = V_{f(\zeta)}\left(z' - f(\zeta)\right)$$

for the target hypersurface M' at the point $f(\zeta)$, with $V_{f(\zeta)} \in \mathrm{U}(n)$. To simplify the notation we shall drop ζ and write $\phi_\zeta = \phi$, $U = U_\zeta$, etc.

We now have $f = \psi^{-1} \circ f_\zeta \circ \phi$, where f_ζ is the expression for our mapping in the new coordinates w resp. w'. The chain rule gives

$$G(z,p) = (p_1, \dots, p_{n-1}, 1)U^{-1}\mathrm{D}f_\zeta\,(\phi(z))^{-1}\,V. \tag{5.3}$$

Let $z \in \Gamma_\zeta$, $w = \phi_\zeta(z)$, and $\epsilon = \epsilon(z) = \mathrm{dist}(z, M)$. Then we also have $\epsilon = \mathrm{dist}(w, M_\zeta)$, where $M_\zeta = \phi_\zeta(M)$. Write

$$\mathrm{D}f_\zeta(w) = \begin{pmatrix} A(w) & B(w) \\ C(w) & D(w) \end{pmatrix}, \tag{5.4}$$

where A is a square matrix of dimension $(n-1)$, B is a column $(n-1) \times 1$, C is a row $1 \times (n-1)$, and D is a scalar. Of course these entries also depend on ζ, but we shall suppress ζ to simplify our notation.

We shall now estimate various entries in (5.3) in terms of the boundary distance ϵ. All constants in these estimates will be independent of $\zeta \in M \cap U$ and $z \in \Gamma_\zeta$, unless otherwise specified.

First of all we have a distance estimate

$$\mathrm{dist}\left(f(z), M'\right) \le c_1\epsilon \tag{5.5}$$

for some $c_1 > 0$ (see [2] or [26]). For any given complex direction in $\mathbf{C}^n$ we can find a linear complex disc in D in that direction, centered at z, of radius ϵ. In complex tangent directions we can find larger discs of radii proportional to $\epsilon^{1/2}$. On the other hand, the largest linear disc in D' centered at $f(z)$ has radius at most $c_2\epsilon^{1/2}$ for some $c_2 > 0$. Since the domains D resp. D' are convex along M resp. M', a theorem of Graham [18,19] gives an estimate

$$|\mathrm{D}f_\zeta(w)| = |\mathrm{D}f(z)| \le c_3\epsilon^{-1/2}.$$

Integrating this estimate along the straight path from 0 to w and using $f_\zeta(0) = 0$ we get $|f_\zeta(w)| \le 2c_3\epsilon^{1/2}$.

Set $w' = f_\zeta(w) = f(z)$, and let $\tilde{w}$ be the unique point in $M'_{f(\zeta)}$ that differs from w' only in the real part of the last coordinate. Since the hypersurface $M'_{f(\zeta)}$ has a defining function of the form (5.1) near the origin, the estimate $|w'| \le 2c_3\epsilon^{1/2}$ implies $|\Re\tilde{w}_n| \le c_4\epsilon$ for some $c_4 > 0$. Since $|\Re w'_n - \Re\tilde{w}_n|$ approximately equals $\mathrm{dist}(w', M_{f(\zeta)})$ which is $\le c_1\epsilon$ by (5.5), it follows that

$$|\Re w'_n| \le c_5\epsilon \tag{5.6}$$

for some $c_5 > 0$. Hence the projection of any linear disc in D', centered at $f(z) = w'$, onto the complex line through $f(\zeta)$ in the w'_n-direction has radius at most $c_5\epsilon$. A theorem of Graham [19] now implies the following bounds on the derivatives of f_ζ:

$$A(w) = O(1), \quad B(w) = O(\epsilon^{-1/2}), \quad C(w) = O(\epsilon^{1/2}), \quad D(w) = O(1) \tag{5.7}$$

as $\epsilon = \mathrm{dist}(z, M) \to 0$. These estimates are uniform for $\zeta \in M \cap U$ since they only depend on the geometry (curvature) of M resp. M' and on the constant in the distance estimate for f.

The estimates (5.7) imply $\det \mathrm{D}f(z) = \det \mathrm{D}f_\zeta(w) = O(1)$ as $\epsilon \to 0$, so $\det \mathrm{D}f$ is a bounded holomorphic function on $D \cap U$. Applying the same to the inverse f^{-1} we conclude that $1/\det \mathrm{D}f$ is bounded on $D \cap U$ as well. The formulas for computing the inverse matrix show that the entries in

$$\mathrm{D}f_\zeta(w)^{-1} = \begin{pmatrix} A'(w) & B'(w) \\ C'(w) & D'(w) \end{pmatrix}$$

satisfy the same estimates (5.7), uniformly in $\zeta \in M \cap U$.

We can refine these estimates by applying the Julia-Carathéodory's theorem for mappings of balls (Rudin [33]). At the point $\zeta \in M \cap U$ we osculate M from within by a ball $B_\zeta \in D$ that is tangent to M at the point ζ. At the image point $f(\zeta) \in M'$ we osculate D' from without by a ball $B'_{f(\zeta)}$ containing D' that is tangent to M' at $f(\zeta)$; this is possible since M' is strongly convex.

Consider now the restricted mapping $f : B_\zeta \to B'_{f(\zeta)}$. From (5.6) it follows that the restriction satisfies

$$c_6^{-1} < \liminf_{z \to \zeta} \frac{\mathrm{dist}\left(f(z), bB'_{f(\zeta)}\right)}{\mathrm{dist}(z, bB_\zeta)} < c_6 \tag{5.8}$$

for some constant $c_6 > 1$ independent of ζ. Theorem 8.5.6 in Rudin [33] now gives the finer estimates

$$B(w) = o(\epsilon^{-1/2}), \quad C(w) = o(\epsilon^{1/2}), \quad \lim_{\epsilon \to 0} D(w) = D_\zeta, \tag{5.9}$$

where $D_\zeta > 0$ depends only on the lim inf in (5.8). Thus we have

$$c_7^{-1} < D_\zeta < c_7 \tag{5.10}$$

for some $c_7 > 1$ independent of ζ. Alternatively, the first two estimates in (5.9) show that $\det \mathrm{D}f_\zeta(w) \to \det A(w)D(w)$ as $\epsilon \to 0$. Since both factors are uniformly bounded and their product is bounded from 0, the factors are also bounded away from 0, so we get (5.10). However, the theorem we used does not give us uniformity in (5.9) with respect to ζ.

Since $1/\det \mathrm{D}f$ is bounded on D, (5.9) implies similar estimates for the entries of the inverse matrix $\mathrm{D}f_\zeta(w)^{-1}$:

$$B'(w) = o(\epsilon^{-1/2}), \quad C'(w) = o(\epsilon^{1/2}), \quad \lim_{\epsilon\to 0} D'(w) = \frac{1}{D_\zeta}. \tag{5.11}$$

Let $(z,p) \in \tilde{\Gamma}_\zeta$. Then

$$(p_1,\ldots,p_{n-1},1) = \frac{\partial r(\zeta)}{\partial_n r(\zeta)} + O(\epsilon),$$

so (5.2) gives

$$(p_1,\ldots,p_{n-1},1)U^{-1} = \frac{1}{\partial_n r(\zeta)} e_n + O(\epsilon),$$

From the expression (5.3) we further have

$$\begin{aligned} G(z,p) &= \frac{1}{\partial_n r(\zeta)} e_n \mathrm{D}f_\zeta(w)^{-1} V + O(\epsilon^{1/2}) \\ &= \frac{1}{\partial_n r(\zeta)} \left(C'(w), D'(w)\right) V + O(\epsilon^{1/2}). \end{aligned}$$

The term $O(\epsilon^{1/2})$ is uniform in ζ. From the estimates (5.7) for $C'(w)$ and $D'(w)$ we see that G is bounded on $\tilde{\Gamma}_\zeta$, and the bound is independent of ζ. Hence G is bounded on the wedge $\mathscr{W}_\alpha$. Moreover, as z tends to ζ within Γ_ζ, it follows from (5.11) that $G(z,p)$ has the limit within the region $\tilde{\Gamma}_\zeta$ equal to

$$\begin{aligned} G^*(\zeta) = \lim_{z\to\zeta} G(z,p) &= \frac{1}{\partial_n r(\zeta) D_\zeta} e_n V_{f(\zeta)} \\ &= \frac{1}{\partial_n r(\zeta) D_\zeta} \partial r'(f(\zeta)). \end{aligned}$$

The last equality follows from (5.2).

When $\zeta \in M$ is close to 0 then $\partial_n r(\zeta)$ is close to 1 and $V = V_{f(\zeta)}$ is close to the identity matrix. Hence the real part of the last component of $G^*(\zeta)$ is positive and bounded away from zero, uniformly in ζ.

We claim that, as a consequence, the real part of G_n is itself bounded away from zero on every finer wedge $\mathscr{W}_{\alpha'}$ $(\alpha' < \alpha)$ sufficiently close to the edge $\tilde{M}$. This will imply that $1/G_n$ is bounded as well. In the case of a straight edge $\tilde{M} \subset \mathbf{R}^{2n-1}$ this follows from the Poisson integral formula applied on linear discs that lie in the wedge and abut the edge. The same applies in the case when the edge is real-analytic via change of coordinates. In the smooth case we can either construct (non-linear) analytic discs in the wedge that abut the edge and whose images fill a finer wedge. Such constructions are well-known, see for instance [10], [13]. Alternatively, we can apply a change coordinates that is $\bar{\partial}$-flat on the edge and prove a similar result for asymptotically holomorphic functions. The details of this approach are very similar to those in Rosay [32].

It follows that each quotient G_j/G_n converges to $\partial_j r'(f(\zeta))/\partial_n r'(f(\zeta))$ as $(z,p) \in \tilde{\Gamma}_\zeta$ and $z \to \zeta$. Since the limit function is continuous on the edge $\tilde{M}$, G_j/G_n extends continuously to the edge from every finer wedge $\mathscr{W}_{\alpha'}$ $(\alpha' < \alpha)$ according to Rosay [32]. This completes the proof of Proposition 5.1.

Remark. An alternative proof of Theorem 4.1 can be found in the paper [30] by Pinčuk and Hasanov. Their method is based on special changes of coordinates and nonhomogeneous scaling; it gives an alternative way to obtain the crucial estimates (5.9).

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