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The length of a set in the sphere whose polynomial hull contains the origin*

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ABSTRACT

Let X be a compact subset of the unit sphere in the complex Euclidean space $\mathbb{C}^n$ such that the origin $0\in\mathbb{C}^n$ belongs to the polynomial hull of X. Assuming that X is rectifiable in the Hausdorff $(\mathscr{H}^1, 1)$-sense, it is shown that the length of X is at least 2π.

INTRODUCTION

It is well known that every pure one-dimensional complex variety $A\subset\mathbb{C}^n$ with reasonably "nice" boundary bA satisfies the *isoperimetric inequality*

$$(\text{Length}(bA))^2\geq 4\pi\,\text{Area}(A).$$

Moreover, if A contains the origin $0\in\mathbb{C}^n$ while bA lies outside the ball $\mathbb{B}(r)=\{z\in\mathbb{C}^n:|z|<r\}$ of radius r, then $\text{Area}(A)\geq\pi r^2$ and therefore

$$\text{Length}(bA)\geq 2\pi r.$$

(See Chirka [6], p. 180 and p. 195, and Bishop [4].) The same inequalities hold if A is an immersed minimal surface with connected boundary; see Section 7.3 in [5]. The constants in these inequalities are the best possible.

H. Alexander [3] extended the isoperimetric inequality to closed Jordan curves $X\subset\mathbb{C}^n$: If X is not polynomially convex, then the set $A=\hat{X}\setminus X$ is an irreducible one-dimensional subvariety of $\mathbb{C}^n\setminus X$, and the isoperimetric ine-

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quality holds when the length and the area are computed using the Hausdorff measures $\mathcal{H}^1$ resp. $\mathcal{H}^2$ on $\mathbb{C}^n$. (For Hausdorff measures see Federer [7, p. 171].) In particular, if X is a closed Jordan curve in the unit sphere $S=\{z\in\mathbb{C}^n:|z|=1\}$ whose polynomial hull $\hat{X}$ contains the origin $0\in\mathbb{C}^n$, then we have

$$\mathcal{H}^1(X)\geq 2\sqrt{\pi}(\mathcal{H}^2(\hat{X}))^{1/2}\geq 2\pi.$$

Recall that the polynomial hull of X is the set

$$\hat{X}=\{z\in\mathbb{C}^n:|f(z)|\leq\sup_X|f| \text{ for all } f\in\mathcal{O}(\mathbb{C}^n)\}.$$

In this article we shall consider the following question that was raised by Stout [8, problem 4.2.2]:

If $X\subset S$ is a compact set in the unit sphere whose polynomially convex hull $\hat{X}$ contains the origin $0\in\mathbb{C}^n$, must the length of X be at least 2π?

In [11] Stout proved a weaker inequality $\mathcal{H}^1(X)\geq\sqrt{2}\pi$ that improved the previously known result $\mathcal{H}^1(X)\geq 2$ by Sibony [9]. (The result of Sibony applies also to sets that are not contained in a sphere.)

Here we give a very simple proof of the inequality

(∗) $\qquad \mathcal{H}^1(X)\geq 2\pi$ when $X\subset S$ and $0\in\hat{X}$

for $(\mathcal{H}^1,1)$-rectifiable compact sets $X\subset S$. Together with a new result of Mark Lawrence this settles the general case as well.

THE RESULT

DEFINITION. (*Federer* [7, p.251].)

(a) *A set $X\subset\mathbb{R}^k$ is 1-rectifiable if it is the image of a bounded subset $U\subset\mathbb{R}$ under a Lipschitz continuous mapping $f: U\to\mathbb{R}^k$.*
(b) *X is $(\mathcal{H}^1,1)$-rectifiable if $\mathcal{H}^1(X)<\infty$ and almost all of X (with respect to the length $\mathcal{H}^1$) can be covered by a countable union of 1-rectifiable sets.*

Our main result is

THEOREM. *If X is a compact $(\mathcal{H}^1,1)$-rectifiable subset of the unit sphere $S\subset\mathbb{C}^n$ such that the origin $0\in\mathbb{C}^n$ belongs to the polynomial hull $\hat{X}$, then $\mathcal{H}^1(X)\geq 2\pi$.*

Clearly the result can be stated for the ball $r\mathbb{B}$ of radius r: *If $X\subset rS$ is compact and $(\mathcal{H}^1,1)$-rectifiable, and if $0\in\hat{X}$, then $\mathcal{H}^1(X)\geq 2\pi r$.*

The isoperimetric inequality does not hold in the context of our Theorem. Namely, Alexander constructed in [2] a compact disconnected set $X\subset\mathbb{C}^2$ of finite length whose polynomial hull $\hat{X}$ has infinite area. His set X is not highly pathological, it consists of a countable disjoint union of real-analytic simple closed curves, and $A=\hat{X}\setminus X$ is countable union of analytic subsets of $\mathbb{C}^2\setminus X$. Obviously X is $(\mathcal{H}^1,1)$-rectifiable.

It is unknown whether the isoperimetric inequality holds for compact *connected* sets $X \subset \mathbb{C}^n$ of finite length. Recall that $A = \hat{X} \setminus X$ is then a pure one-dimensional analytic subvariety of $\mathbb{C}^n \setminus X$ (if not empty) according to Alexander [1]. For smooth curves X this had been proved by Stolzenberg [10].

REMARK ADDED TO THE PROOF. It suffices to prove the inequality $(*)$ for sets $X \subset S$ of finite length ($\mathscr{H}^1(X) < \infty$) that are minimal, in the sense that no proper compact subset of X contains 0 in its polynomial hull. Recently Mark Lawrence (private communication) informed me of his new result that such a set is necessarily ($\mathscr{H}^1$, 1)-rectifiable. Together with our theorem this implies

COROLLARY. *If X is a compact subset of S and $0 \in \hat{X}$ then $\mathscr{H}^1(X) \geq 2\pi$.*

Also, after the completion of the first version of this article, H. Alexander [12] and N. Poletski (private communication) informed me that they had independently proved the estimate $(*)$ by different methods.

PROOF OF THE THEOREM

The result will follow from the following Lemma and an integral geometric formula (Crofton formula) from Federer [7, p. 284].

LEMMA. *If X is a compact subset of $\mathbb{C}^n \setminus \{0\}$ of finite length such that $0 \in \hat{X}$, then almost every real hyperplane $\Sigma \subset \mathbb{C}^n$ passing through the origin intersects X at least at two points.*

Here, "almost every" is meant with respect to the volume measure on the Grassman manifold of real hypersurfaces $0 \in \Sigma \subset \mathbb{C}^n$.

PROOF. Since X has finite length and $0 \notin X$, Fubini's theorem implies that almost every complex hyperplane $L \subset \mathbb{C}^n$ passing through 0 misses X. Here, "almost every" refers to the volume measure on the Grassman manifold of complex $(n-1)$-dimensional subspaces of $\mathbb{C}^n$.

Fix such a hyperplane L, and let Σ be any real hyperplane in $\mathbb{C}^n$ containing L. Then L splits Σ in two open half-planes Σ_+ and Σ_-.

We claim that both Σ_+ and Σ_- intersect X. To see this, let $\pi : \mathbb{C}^n \to L^\perp$ be the orthogonal projection onto the complex line $L^\perp$ orthogonal to L. If X is disjoint from Σ_+ then $\pi(X) \subset L^\perp$ is disjoint from the real half-line $\pi(\Sigma_+)$, hence $0 \in L^\perp$ lies in the unbounded component of $L^\perp \setminus \pi(X)$. Thus 0 is not in the polynomial hull of $\pi(X)$ in $L^\perp$ and hence 0 is not in the hull of X in $\mathbb{C}^n$, a contradiction. (This also follows from Oka's criterion for polynomial convexity [10, p. 263]: since 0 belongs to $\hat{X}$, we can not move L continuously to infinity without hitting X.)

This shows that every such real hyperplane Σ intersects X in at least two points. Since the remaining set of real hyperplanes through the origin in $\mathbb{C}^n$ has measure zero, the lemma is proved.

We now apply Theorem 3.2.48 in Federer [7, p. 284] as follows. Let B be the intersection of a real hyperplane through the origin in $\mathbb{C}^n$ with the sphere S. By our hypothesis X is $(\mathcal{H}^1, 1)$ rectifiable, it is $\mathcal{H}^1$ measurable since Hausdorff measures are Borel regular, and $\mathcal{H}^1(X) < \infty$. Clearly B is $m = (2n-2)$-rectifiable and $\mathcal{H}^m$-measurable since it is an m-manifold. Applying Theorem 3.2.48 in [7] to the constant functions $\alpha = 1$ on $A = X$ and $\beta = 1$ on B we get

$$\int_{g \in \mathbb{O}(2n)} \int_{X \cap g(B)} d\mathcal{H}^0 \, d\theta_{2n}(g) = C \cdot \mathcal{H}^1(X) \cdot \mathcal{H}^m(B)$$

for some constant C depending only on m and n.

Recall that $\mathcal{H}^0$ is just the counting measure. The lemma implies $\mathcal{H}^0(X \cap g(B)) \geq 2$ for almost all $g \in \mathbb{O}(2n)$ with respect to the volume measure θ_{2n}. Hence we get

$$\mathcal{H}^1(X) \geq 2\theta_{2n}(\mathbb{O}(2n))/C\mathcal{H}^m(B).$$

To calculate the constant on the right hand side we choose X to be the intersection of S with a complex line through the origin, hence $\mathcal{H}^1(X) = 2\pi$. In this case $X \cap g(B)$ contains exactly two points for most $g \in \mathbb{O}(2n)$, hence the inequality above is actually an equality. Thus the value of the right hand side equals $\mathcal{H}^1(X) = 2\pi$.

This completes the proof of the Theorem.

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