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Mappings of quadric Cauchy-Riemann manifolds

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Introduction

Let $\mathbf{B}^n=\{z\in\mathbf{C}^n:|z|<1\}$ be the open unit ball of the complex euclidean space $\mathbf{C}^n$, and let $b\mathbf{B}^n=\{z\in\mathbf{C}^n:|z|=1\}$ denote its boundary, the unit sphere. The motivation for the present paper is the following result of the author [For, Theorem 1.4]:

Theorem. *If Ω is an open connected subset of $b\mathbf{B}^n$ and*

$$F=(F_1,\ldots,F_{n'}):\Omega\to b\mathbf{B}^{n'}\qquad (n'\geqq n>1)$$

is a mapping of class $n'-n+1$ that satisfies the tangential Cauchy-Riemann equations on Ω (in short, a CR *mapping), then F extends to a complex rational mapping on $\mathbf{C}^n$.*

Thus every proper holomorphic mapping $F:\mathbf{B}^n\to\mathbf{B}^{n'}$ that is smooth of class $n'-n+1$ on the closed ball $\bar{\mathbf{B}}^n$ is rational. This result is useful in the problem of classification of proper holomorphic mappings between balls, see the papers by D'Angelo [DA1, DA2, DA3]. In the equidimensional case $n=n'>1$ the result is due to Alexander [Al] (see also [Ru, p. 316]): If the mapping is not constant, it extends to an automorphism of the ball. Recently Pinčuk and Tsyganov [PT] proved the same result for all continuous nonconstant CR mappings $F:\Omega\subset b\mathbf{B}^n\to b\mathbf{B}^n$.

The punctured sphere $b\mathbf{B}^{n+1}\backslash\{p\}$ is equivalent via the Cayley transformation [Ru, p. 31] to the quadric hypersurface (hyperquadric) defined by

$$\Im w=|z_1|^2+\ldots+|z_n|^2, \tag{0.1}$$

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so the above result bears on CR mappings of such hyperquadrics, possibly of different dimensions. In fact the result applies to all hyperquadrics

$$\Im w = \sum_{j,k=1}^{n} a_{jk} z_j \bar{z}_k \tag{0.2}$$

with a positive definite hermitian matrix $A=(a_{jk})$, since every such quadratic form can be diagonalized by a unitary change of z coordinates.

In the present paper we use elementary complexification arguments, similar to those in [For], to extend this phenomenon to mappings of *quadric Cauchy-Riemann* (CR) *manifolds* in $\mathbf{C}^n$, of arbitrary dimensions and codimensions.

We denote the coordinates on $\mathbf{C}^n$ by (w,z), where $w \in \mathbf{C}^d$, $z \in \mathbf{C}^m$, $d+m=n$, $d,m>0$. Let $\langle z,\zeta \rangle = \sum z_j \zeta_j$. We associate to each d-tuple $A=(A_1,\ldots,A_d)$ of hermitian $m \times m$ matrices the *quadric* CR *manifold* $M=M_A \subset \mathbf{C}^n$ defined by

$$\Im w_k = \langle A_k z, \bar{z} \rangle, \quad 1 \leqq k \leqq d. \tag{0.3}$$

Clearly M is a Cauchy-Riemann submanifold of $\mathbf{C}^n$ of real codimension d and of CR dimension m; we shall say that M is of type (d,m). The hermitian quadratic form

$$z \to \langle Az, \bar{z} \rangle = (\langle A_1 z, \bar{z} \rangle, \ldots, \langle A_d z, \bar{z} \rangle)$$

with values in $\mathbf{R}^d$ is called the *Levi form* of M. To simplify the notation we shall drop the indices and write

$$\Im w = \langle Az, \bar{z} \rangle .$$

Each quadric of this form is affinely homogeneous, i.e., there is a group of complex affine transformations of $\mathbf{C}^n$ acting transitively on M [PS, p. 15].

Under rather mild assumptions on their Levi forms we prove that every CR mapping of quadric CR manifolds with some initial amount of regularity extends to a complex rational mapping on $\mathbf{C}^n$, and the space of all such mappings between a given pair of quadrics is finite dimensional (Theorems 1.1 and 1.5). Theorem 1.1 generalizes a result of Tumanov (see Corollary 1.3). We wish to point out that our method is different from the method of Tumanov, which seems to apply only to CR diffeomorphisms between quadrics of the same type (d,m).

We also prove an extendability result for CR mappings from certain real-analytic CR manifolds into quadrics (Theorem 1.6). Our method provides a unified approach to a variety of mapping problems of CR manifolds.

The idea of our proof goes back to the reflection principle of Lewy [Le] and Pinčuk [Pin]. Their technique was further developed by Webster [We1, We2, We3], Diedrich and Webster [DW], Diederich and Fornæss [DF1, DF2], and the author [For]. Somewhat different approach to mapping problems was developed by Baouendi, Jacobowitz, and Treves [BJT], Baouendi, Bell, and Rothschild [BBR], and Baouendi and Rothschild [BR1, BR2].

Another important ingredient are the results on holomorphic extendability of CR functions due to Boggess and Polking [BP], Baouendi, Chang, and Treves [BCT], Tumanov [Tu1], and Baouendi and Rothschild [BR3]. Finally, our result in [For] for mappings of hyperquadrics depended on a lemma of Cima and Suffridge [CS] to the effect that holomorphic functions of certain kind must be rational. In the present paper we give a simple proof (Proposition 3.1) that applies in the more general setting we are dealing with.

Quadric CR manifolds are important for several reasons. First, they are local second order models of general CR manifolds. From the analysis on hypersurfaces it is well known that such local models play a very important role, especially in the case when their Levi form is nondegenerate. For instance, there is a simple proof of Fefferman's theorem on smooth extendability of proper holomorphic mappings between strongly pseudoconvex domains, based on the scaling method and the knowledge of the automorphism group of the ball (Pinčuk and Hasanov [PH]). Second, the quadric CR manifolds (0.3) are Shilov boundaries of the associated Siegel domains of the second kind (Piatetsky-Shapiro [PS, p. 13]). These domains are wedges with edge M, defined by (2.3) below. As such, the quadrics are important for the function theory on Siegel domains and in questions concerning holomorphic mappings between these domains [TH1]. Since every classical Cartan domain (bounded symmetric domain) admits representations as a Siegel domain of the second kind [PS], the mappings of quadric CR manifolds are related to proper holomorphic mappings between Cartan domains; see Henkin and Novikov [HN], Tumanov and Henkin [TH2], and Tumanov [Tu2].

For these reasons it is desirable to have as much information as possible on the CR mappings between a given pair of quadric CR manifolds. We hope that our method can be applied to the computation of the holomorphic automorphism group of quadric CR manifolds. Presently these groups are known only for certain special classes of quadrics (see [CM] for hyperquadrics).

When the quadrics have different CR dimensions, the classification problem for CR mappings between them is rather difficult already in the case of hyperquadrics (spheres); see the papers [DA2, DA3]. The study of such mappings is simplified substantially if we know apriori that they are rational.

1 Results

Let M_A be the quadric (0.3), associated to the quadratic hermitian form

$$\langle Az,\bar z\rangle=(\langle A_1 z,\bar z\rangle,\dots,\langle A_d z,\bar z\rangle).$$

We shall use the following terminology.

Definition 1. (a) *The form* $\langle Az,\bar z\rangle$ *is nondegenerate if* $\langle Az,\zeta\rangle=0$ *for all* $\zeta\in\mathbf{C}^m$ *implies* $z=0$. *In this case we say that the quadric* M_A *is Levi nondegenerate.*

(b) *The quadric* M_A *is strongly pseudoconvex if there is a vector* $\eta=(\eta_1,\dots,\eta_d)\in\mathbf{R}^d$ *such that the matrix* $\eta\cdot A=\sum\eta_j A_j$ *is strongly positive definite.*

(c) M_A *is strongly* 1*-pseudoconcave if for each* $\eta\in\mathbf{R}^d\setminus\{0\}$ *the matrix* $\eta\cdot A$ *has at least one negative eigenvalue.*

We associate to M_A the convex cone in $\mathbf{R}^d$,

$$\Gamma=\Gamma_A=\mathrm{Co}\{\langle Az,\bar z\rangle\in\mathbf{R}^d:z\in\mathbf{C}^m\}, \tag{1.1}$$

where Co stands for the linearly convex hull. Notice that M_A is strongly pseudoconvex if and only if it is Levi nondegenerate and $\Gamma_A\setminus\{0\}$ is contained in an open half-space of $\mathbf{R}^d$.

Let $M'\subset\mathbf{C}^{n'}$ be another quadric of type (d',m'), given by

$$\Im w'=\langle Bz',\bar z'\rangle. \tag{1.2}$$

For each $p\in M$ we denote by $\mathrm{T}_p^{\mathbf{C}}M$ the maximal complex subspace of the real tangent space T_pM.

1.1 Theorem. *Assume that*

(a) $M = M_A$ *is the quadric* (0.3) *of type* (d, m) *such that the associated cone* Γ_A *has nonempty interior,*

(b) $M' = M'_B$ *is a Levi nondegenerate quadric* (1.2) *of type* (d', m),

(c) $F: \omega \subset M \to M'$ *is a* CR *mapping of class* $\mathscr{C}^1$, *defined on an open connected subset* ω *of* M, *such that the differential* $dF(p)$ *is a linear isomorphism of* $T_p^{\mathbf{C}} M$ *onto* $T_{F(p)}^{\mathbf{C}} M'$ *at some point* $p \in \omega$.

Then F *extends to a complex rational mapping on* $\mathbf{C}^n$. *Moreover, the degree of every such mapping* F *is bounded from above by a constant* $C = C(d, m)$ *depending only on the type* (d, m) *of* M.

1.2 Corollary. *Every local* CR *diffeomorphism* $F: \omega \subset M \to \omega' \subset M'$ *between Levi nondegenerate hyperquadrics* $M, M' \subset \mathbf{C}^{n+1}$ *of the form* (0.2) *extends to a birational mapping on* $\mathbf{C}^{n+1}$.

Corollary 1.2 follows immediately from Theorem 1.1 by observing that the corresponding cone $\Gamma_A \subset \mathbf{R}$ has nonempty interior whenever $A \neq 0$. Similarly we have

1.3 Corollary. *If* $M = M_A \subset \mathbf{C}^n$ *is a Levi nondegenerate quadric* (0.3) *whose cone* Γ_A *has nonempty interior, then every local* CR *diffeomorphism* $F: \omega \subset M \to \omega' \subset M$ *extends to a birational mapping on* $\mathbf{C}^n$. *The group of all rational* CR *automorphisms of* M *is finite dimensional.*

Corollary 1.3 was first proved by Tumanov [Tu2], using different methods.

In the case when the quadric $M = M_A$ is strongly pseudoconvex and the cone Γ (1.1) has nonempty interior, Henkin and Tumanov [TH1] proved that every local CR homeomorphism $F: \omega \subset M \to \omega' \subset M$ extends to a holomorphic automorphism of the associated *Siegel domain of the second kind*, defined by

$$\mathscr{W} = \{(w, z) \in \mathbf{C}^n : \Im w - \langle Az, \bar{z} \rangle \in \operatorname{int} \Gamma\}.$$

In other terms, $\mathscr{W}$ is a wedge with edge M and cone Γ.

Thus the group of CR automorphisms of M is isomorphic to the group of holomorphic automorphisms of $\mathscr{W}$. Since $\mathscr{W}$ is equivalent to a bounded domain in $\mathbf{C}^n$ [PS], it follows that $\operatorname{Aut} M$ is a finite dimensional Lie group. In addition to this, Tumanov proved in [Tu2] that every proper holomorphic self-mapping $F: \mathscr{W} \to \mathscr{W}$ is an automorphism, thus generalizing Alexander's theorem [Al] to certain Siegel domains of the second kind.

Opposite to this is the case when M is strongly 1-pseudoconcave. Then the convex cone Γ_A has no supporting hyperplane at 0, so it equals $\mathbf{R}^d$. This implies that every CR function on $\omega \subset M$ extends holomorphically to a neighborhood of each point $p \in \omega$ in $\mathbf{C}^n$ (Naruki [Na]), hence every local CR homeomorphism $F: \omega \subset M \to \omega' \subset M$ extends locally to a biholomorphic map. If, in addition, M is Levi nondegenerate, F extends to a birational mapping on $\mathbf{C}^n$ according to Corollary 1.3.

We denote by $\operatorname{Aut}_0(M)$ the group of local CR automorphisms of M preserving the origin $0 \in M$.

1.4 Corollary. *The group* $\operatorname{Aut}_0 M$ *of a quadric is finite dimensional if and only if* M *is Levi nondegenerate and its Levi cone* Γ (1.1) *has nonempty interior.*

Proof. Corollary 1.3 implies that the last two conditions are sufficient for the finite dimensionality of $\operatorname{Aut}_0 M$.

If M is Levi degenerate, it is isomorphic to $M_1 \times \mathbf{C}^k$ for some integer $k>0$, where M_1 is a Levi nondegenerate quadric in $\mathbf{C}^{n-k}$. For each $f \in \mathrm{Aut}_0 M_1$ and for every $g \in \mathrm{Aut}_0 \mathbf{C}^k$ we obtain a local automorphism of M by setting $F(z,\zeta)=(f(z),g(\zeta))$. Thus $\mathrm{Aut}_0 M$ is infinite dimensional.

Similarly, if the cone Γ has empty interior, it lies in a real hyperplane of $\mathbf{R}^d$, so there is a real-linear combination $A'=\sum_{j=1}^{d} c_j A_j$ such that $\langle A'z, \bar{z}\rangle = 0$ for all $z \in \mathbf{C}^m$. Changing coordinates on w-space we may assume that $A'=A_1$, so the first equation in (0.3) is $\Im w_1 = 0$. Thus $M=M_1 \times \mathbf{R}^k$ for some positive integer k and quadric $M_1 \subset \mathbf{C}^{n-k}$. For each $f \in \mathrm{Aut}_0 M_1$ and each local automorphism g of $\mathbf{C}^k$ satisfying $g(\mathbf{R}^k) \subset \mathbf{R}^k$, $g(0)=0$, we get a local automorphism of M as above. Thus $\mathrm{Aut}_0 M$ is infinite dimensional. This proves Corollary 1.4.

Example. Let $M \subset \mathbf{C}^3$ be defined by

$$\Im w_1 = z\bar{z}, \qquad \Im w_2 = 0,$$

so M is the product of the hyperquadric $S \subset \mathbf{C}^2$ with $\mathbf{R}$. Every CR mapping of M to itself is of the form $F(z,w_1,t)=(f_t(z,w_1), g(t))$, where $f_t : S \to S$ is a CR mapping of S to itself. Thus f_t is rational for each fixed t, but the CR condition on M imposes no regularity of f and g in the t variable. In this case the cone Γ is a ray in $\mathbf{R}^2$ without interior.

So far we have assumed that both quadrics have the same CR dimension, only their codimensions could differ. We obtain a similar result for mappings between quadrics of different CR dimensions, provided that the target is strongly pseudoconvex and the mapping is sufficiently regular to begin with.

1.5 Theorem. *Let $M=M_A$ be the quadric* (0.3) *of type (d,m) such that the cone Γ_A* (1.1) *has nonempty interior, and let $F:\omega \subset M \to M'$ be a* CR *mapping from an open connected subset $\omega \subset M$ to a strongly pseudoconvex quadric $M'=M'_B$* (1.2) *of type (d',m'). Let $r(p)$ be the rank of the linear map $dF(p): T_p^{\mathbf{C}} M \to T_{F(p)}^{\mathbf{C}} M'$.*

If F is smooth of class $s=m'-r(p)+1$ in a neighborhood of some point $p \in \omega$, then F extends to a complex rational mapping on $\mathbf{C}^n$. The degree of F is bounded from above by a constant $C(d,m,m')$ depending only on the indicated quantities.

This generalizes Theorem 1.4 in [For] where both $M \subset \mathbf{C}^n$ and $M' \subset \mathbf{C}^{n'}$ are hyperquadrics (0.1). In that case the differential $dF(p)$ is injective on $T_p^{\mathbf{C}} M$ (unless F is constant), so the required smoothness of F is $s=m'-m+1=n'-n+1$.

Most likely our smoothness assumption is not optimal. However, it is known that for each $n'>n>1$ there exist proper holomorphic mappings $F:\mathbf{B}^n \to \mathbf{B}^{n'}$ that are *continuous* on the closed ball but are *not rational* [Dor, Ha]. It is an open problem whether such a map can be continuously differentiable on the closed ball without being rational.

Notice that we may have $m'<m$ even when F is not a constant mapping. This happens for instance when M is the product $M=M_1 \times M_2$ of lower dimensional quadrics and the mapping $F: M_1 \times M_2 \to M'$ is constant on one of the factors.

Finally we prove an extension result for CR mappings of more general real-analytic CR manifolds into quadrics. Let $\phi(z,\bar{z},\Re w)$ be a real-analytic function in a neighborhood of 0 in $\mathbf{C}^m \times \mathbf{R}^d$, with values in $\mathbf{R}^d$, satisfying $\phi(0)=0$ and $d\phi(0)=0$. Let $M \subset \mathbf{C}^n$ be a local real-analytic submanifold of real codimension d, defined by

$$\Im w = \phi(z,\bar{z},\Re w). \tag{1.3}$$

Then M is a generic CR submanifold near 0 of CR dimension $m=n-d$.

After a local biholomorphic change of coordinates near the origin we may eliminate all pluriharmonic quadratic terms in ϕ and assume that the equation of M is

$$\Im w = \langle Az, \bar{z} \rangle + \phi'(z, \bar{z}, \Re w), \tag{1.4}$$

where $A=(A_1, \ldots, A_d)$ is hermitian and ϕ' contains terms of degree at least three [BP]. The quadratic part is the *Levi form* of M at the origin. The corresponding quadric M_A (0.3) osculates M to second order at the origin. We shall say that M is *Levi nondegenerate* or *strongly pseudoconvex* at the origin if the quadric M_A satisfies these properties.

We shall need a result of Tumanov [Tu1] on extendability of CR functions on M to holomorphic functions on wedges with edge M. For this purpose we recall a definition from [Tu1].

Definition 2. *A* CR *manifold M of the form* (1.3) *is said to be minimal at a point $p \in M$ if there exists no local* CR *manifold $N \subset M$ passing through p, of the same* CR *dimension m but of strictly smaller real dimension.*

It is easily seen that a quadric $M=M_A$ (0.3) is minimal at any point if and only if the cone Γ_A has nonempty interior. Equivalently, the components of the Levi form must be linearly independent. Thus M can be strongly pseudoconvex but not minimal. If M is a more general CR manifold (1.4) and the cone Γ_A has nonempty interior, then M is minimal at 0, but the converse need not hold.

1.6 Theorem. *Let $M \subset \mathbf{C}^n$ be a real-analytic* CR *manifold* (1.3) *of type* (d, m) *that is minimal at each point $p \in M$, and let $F: M \to M'$ be a* CR *mapping into a strongly pseudoconvex quadric $M' \subset \mathbf{C}^{n'}$* (1.2) *of type* (m', d'). *Denote by $r(p)$ the rank of $dF(p): \mathrm{T}_p^{\mathbf{C}} M \to \mathrm{T}_{F(p)}^{\mathbf{C}} M'$ and set $r_0 = \min\{r(p): p \in M\}$.*

If F is smooth of class $m'-r_0+1$ on M, there is a closed subset $E=E_F \subset M$ of surface measure zero in M such that F extends holomorphically to a neighborhood of each point $p \in M \backslash E$.

Contrary to the previous results the map F in Theorem 1.6 need not be rational. It is an open problem whether such a map extends holomorphically to a neighborhood of *every* point $p \in M$.

By a more careful argument one can also replace the target M' by arbitrary real-analytic strongly pseudoconvex CR manifold and still obtain the same conclusion for CR mappings $F: M \to M'$ of class $\mathscr{C}^\infty$. A theorem of this type for mappings between hypersurfaces of different dimensions has been proved in [For, Theorem 1.1]. Also, there are extension results of Webster [We2, We3] and Baouendi, Jacobowitz, and Treves [BJT] for mappings between real-analytic CR manifolds of the same type (d, m). See also Tumanov [Tu1].

In Sect. 2 we explain our method that relies on the Segre Q-varieties. We first extend F to a holomorphic mapping on a wedge $\mathscr{W}^+$ with edge M. Using the condition $F(M) \subset M'$ we associate to F its *characteristic variety* X_F as a solution set of the system of Eqs. (2.9)–(2.10). The variety X_F that is initially defined over the opposite wedge $\mathscr{W}^-$ can be extended across most points of M so that it contains the graph of F.

The announced results follow from Theorem 2.5 in Sect. 2. In Sect. 3 we prove Theorem 2.5 in the nondegenerate case when the system (2.10) has maximal rank. In Sect. 4 we deal with the degenerate case.

2 The characteristic variety

We are to study a CR mapping $F: M \to M'$ of class $\mathscr{C}^s$. The value of s depends on the theorem we are proving: $s=1$ in Theorem 1.1, $s=m'-r(p)+1$ in Theorem 1.5, and $s=m'-r_0+1$ in Theorem 1.6.

Since our theorems are local, we fix a point $p \in M$ and change coordinates so that $p=0$ and $F(0)=0$. This is possible since each quadric (0.3) is affinely homogeneous [PS, p. 15]. We also assume that $dF(p): T_p^{\mathbf{C}}M \to T_{F(p)}^{\mathbf{C}}M'$ is of rank at least r_0 at each point $p \in M$ and F is smooth of class $s=m'-r_0+1$ on M. In Theorem 1.1 we have $r_0=m=m'$, so $s=1$.

To begin with, let M be a general real-analytic CR manifold (1.3), where ϕ is a convergent power series without constant and linear terms,

$$\phi(z,\bar{z},u)=\sum \phi_{\alpha,\beta,\gamma} z^\alpha \bar{z}^\beta u^\gamma,$$

the summation over $\alpha, \beta \in \mathbf{Z}_+^m$, $\gamma \in \mathbf{Z}_+^d$.

We polarize the equation of M by considering $(\bar{w},\bar{z})$ as complex variables independent of (w,z). The power series $\phi(\zeta,z,(\eta+w)/2)$ converges in a smaller polydisc neighborhood $U_0 \times U_0$ of the origin in $\mathbf{C}^n \times \mathbf{C}^n$. We shall choose U_0 small enough so that F is defined in a neighborhood of $M \cap \bar{U}_0$. The resulting equations define a local complex submanifold $M^o \subset U_0 \times U_0$ of complex codimension d, called the *polar* of M:

$$(\eta-w)/2i=\phi(\zeta,z,(\eta+w)/2). \tag{2.1}$$

For each (w,z) in a smaller polydisc neighborhood $U \subset U_0$ of 0 we define a complex submanifold of U_0 of complex codimension d by

$$Q(w,z)=\{(\eta,\zeta) \in U_0 : (\eta,\zeta;\bar{w},\bar{z}) \in M^o\}.$$

When M is the quadric $\Im w=\langle Az,\bar{z}\rangle$, $Q(w,z)$ is the affine complex subspace of $\mathbf{C}^n$ given by

$$Q(w,z)=\{(\eta,\zeta) \in \mathbf{C}^n : (\eta-\bar{w})/2i=\langle A\zeta,\bar{z}\rangle\}. \tag{2.2}$$

These varieties were first introduced by Segre [Seg] in the case when M is a real hypersurface. They have been used in mapping problems by several authors [We1, DW, DF1, For]. We recall some of their important properties (see [DW] or [DF1]):

(i) $(w,z) \in Q(\eta,\zeta)$ if and only if $(\eta,\zeta) \in Q(w,z)$,
(ii) $(w,z) \in Q(w,z)$ if and only if $(w,z) \in M$,
(iii) $Q(0,0)=\{w=0\}=\{0\}^d \times \mathbf{C}^m$.

Let $M' \subset \mathbf{C}^{n'}$ be another CR submanifold of the form (1.3), and let $Q'(w',z')$ be the corresponding variety (2.2). Our proof is based on the observation, due originally to Webster [We1], that every holomorphic mapping F, defined on a neighborhood of 0 in $\mathbf{C}^n$, that maps M to M', also maps the variety $Q(p)$ into $Q'(F(p))$ for all $p \in \mathbf{C}^n$ near 0. In particular, if F is biholomorphic near the origin, the varieties attached to M are in one-to-one correspondence with those attached to M'. The proof of this is very similar to the one given in [We1] or [For] in the hypersurface case.

This suggests that we associate to F its *characteristic variety*

$$X_F=\{(p,p') \in U_0 \times \mathbf{C}^{n'} : F(Q(p)) \subset Q'(F(p))\}.$$

We shall see that X_F is indeed a complex subvariety. The importance of X_F in the mapping problem is evident from the previous works [DF1] and [For].

Unfortunately we do not know that F is holomorphic in a full neighborhood of 0 in $\mathbf{C}^n$, so we need to find a different way of defining the characteristic variety.

We begin by extending F to a holomorphic mapping on a *wedge domain* with edge M. If $\Lambda \subset \mathbf{R}^d$ is an open convex cone with vertex 0 and U_0 is a neighborhood of 0 in $\mathbf{C}^n$, we define the corresponding wedge by

$$\mathscr{W}^+ = \mathscr{W}(\Lambda, U_0) = \{(w,z) \in U_0 : \Im w - \phi(z,\bar{z},\Re w) \in \Lambda\}. \tag{2.3}$$

If M is minimal at 0 in the sense of Definition 2, then according to Tumanov [Tu1] there is some wedge of this kind to which every CR function on $\omega \subset M$ extends as a holomorphic function. Moreover, if the function is of class $\mathscr{C}^s$ on ω, its holomorphic extension is of class $\mathscr{C}^s$ on $(\mathscr{W}' \cup M) \cap U_0$ for every strictly finer wedge $\mathscr{W}' \subset \mathscr{W}^+$ [Co2]. This also follows from the approximation theorem for CR functions due to Baouendi and Treves [BT].

If M is of the form (1.4) and the cone Γ_A (1.1) has nonempty interior, then the above holds for every cone Λ that is strictly smaller than the interior of Γ_A (Boggess and Polking [BP], Baouendi, Chang, and Treves [BCT]). If $M = M_A$ is the quadric (0.3), we may take $\Lambda = \operatorname{Int}\Gamma_A$ according to Naruki [Na]. See also Baouendi and Rothschild [BR3].

Thus we can extend F to a holomorphic mapping on a wedge $\mathscr{W}^+$ (2.3) that is smooth of class $\mathscr{C}^s$ on $(\mathscr{W}^+ \cup M) \cap U_0$.

Let $h(w,z) = (h'(w,z), z)$ be the real-analytic mapping in a neighborhood of $0 \in \mathbf{C}^n$ determined by the equation

$$(h' - \bar{w})/2i = \phi(z, \bar{z}, (h' + \bar{w})/2),$$

so $h(w,z) \in Q(w,z)$. Such h exists by the implicit function theorem. For each fixed $z = z^0$, the map $w \to h'(w,z^0)$ is an antiholomorphic reflection within the subspace $z = z^0$, fixing the totally real submanifold $M \cap \{z = z^0\}$. When M is the quadric (0.3) we get the mapping

$$h(w,z) = (\bar{w} + 2i\langle Az, \bar{z}\rangle, z). \tag{2.4}$$

Shrinking U if necessary we may assume that $h(w,z) \in U_0$ whenever $(w,z) \in U$. Let

$$\mathscr{W}^- = \{(w,z) \in U : h(w,z) \in \mathscr{W}^+\} \tag{2.5}$$

be the wedge that is opposite to $\mathscr{W}^+$ with respect to h. In the quadric case we get $\mathscr{W}^- = \mathscr{W}(-\operatorname{Int}\Gamma_A, U)$. In general, $\mathscr{W}^-$ contains a wedge $\mathscr{W}(-\Lambda', U)$ for some open convex cone $\Lambda' \subset \Lambda$.

Our goal is to extend the mapping F, holomorphic on the wedge $\mathscr{W}^+$, to the opposite wedge $\mathscr{W}^-$ as a complex subvariety X_F defined as above. For each $(w,z) \in \mathscr{W}^-$ we have a point $h(w,z) \in Q(w,z) \cap \mathscr{W}^+$ in the region where F is holomorphic, so the condition defining X_F makes sense. In order to use the assumption that F maps M to M' we shall replace the defining condition $F(Q(w,z)) \subset Q'(F(w,z))$ by the corresponding infinitesimal condition on the Taylor series of $F|_{Q(w,z)}$ at the reference point $h(w,z)$.

Let $\mathscr{W}^0 \subset M^0$ be the preimage of $\mathscr{W}^+$ under the first coordinate projection:

$$\mathscr{W}^0 = \{(\eta, \zeta; w, z) \in M^0 : (\eta, \zeta) \in \mathscr{W}^+\}.$$

This is a wedge-like domain whose edge contains the maximal real submanifold

$$T = \{(w, z; \bar{w}, \bar{z}) : (w,z) \in M \cap U_0\} \subset M^0.$$

We may assume that $\mathscr{W}^+$, $\mathscr{W}^-$, and $\mathscr{W}^0$ are connected.

For each point $(\eta, \zeta; \bar{w}, \bar{z}) \in \mathscr{W}^0$ we consider the Taylor expansion of the restriction $F|_{Q(w,z)}$, centered at (η, ζ). From the equation of the polar (2.1) we see that $Q(w,z)$ can be parametrized locally near (η, ζ) by

$$x \in \mathbf{C}^m \to \Theta(x) = \left(\eta + \sum_{|\alpha|>0} c_\alpha(\eta, \zeta; \bar{w}, \bar{z}) x^\alpha, \zeta + x\right), \tag{2.6}$$

where the function $\theta(x) = \sum_{|\alpha|>0} c_\alpha(\eta, \zeta; \bar{w}, \bar{z}) x^\alpha$ is a solution of the equation

$$\theta(x)/2i = \phi(\zeta + x, \bar{z}, (\eta + \theta(x) + \bar{w})/2) - \phi(\zeta, \bar{z}, (\eta + \bar{w})/2).$$

By the implicit function theorem a solution exists and is holomorphic in all indicated variables. In particular, the functions c_α are holomorphic in $M^0 \cap (U_0 \times U_0)$, provided that U_0 is sufficiently small. This means that $c_\alpha(\eta, \zeta; \bar{w}, \bar{z})$ is holomorphic in (η, ζ) and antiholomorphic in (w, z).

When (η, ζ) belongs to the wedge $\mathscr{W}^+$ where F is holomorphic, we insert (2.6) into the Taylor expansion of F at (η, ζ) to obtain

$$F(\Theta(x)) = F(\eta, \zeta) + \sum_{1 \leq |\beta| \leq s} a_\beta(\eta, \zeta; \bar{w}, \bar{z}) x^\beta + o(|x|^s). \tag{2.7}$$

2.1 Lemma. *Each function a_β for $|\beta| \leq s$ is holomorphic in the wedge $\mathscr{W}^0$ and extends continuously to $\mathscr{W}^0 \cup T$. Here s is the order of smoothness of F on $\omega \subset M$.*

Proof. It suffices to observe that a_β is a linear combination of the derivatives of F at (η, ζ) of order $\leq |\beta|$, with coefficients that are holomorphic polynomials of the functions c_α for $|\alpha| \leq |\beta|$.

We get more precise information when M is the quadric (0.3). Let $e_k \in \mathbf{C}^m$ be the k-th standard unit vector. Then the vectors

$$b_k(\bar{z}) = (2i\langle Ae_k, \bar{z}\rangle, e_k) \in \mathbf{C}^n, \quad 1 \leq k \leq m \tag{2.8}$$

form a basis of the affine subspace $Q(w,z)$, so $Q(w,z)$ is parametrized by

$$\Theta(x) = (\eta, \zeta) + \sum_{k=1}^m x_k b_k(\bar{z}) = (\eta + 2i\langle x, \bar{A}\bar{z}\rangle, \zeta + x)$$

for each $(\eta, \zeta) \in Q(w,z)$. This proves

2.2 Lemma. *When M is the quadric* (0.3) *and h is defined by* (2.4), *then each coefficient $a_\beta(\eta, \zeta; \bar{z})$ in the expansion* (2.7) *for* $1 \leq |\beta| \leq s$ *is a polynomial of degree $|\beta|$ in the variable $\bar{z}$ and is independent of w.*

Let $M' \subset \mathbf{C}^{n'}$ be the quadric (1.2). For each $(w,z) \in \mathscr{W}^-$ we have $h(w,z) \in Q(w,z) \cap \mathscr{W}^+$, so we can substitute the expansion (2.7) for $F|_{Q(w,z)}$ at the reference point $(\eta, \zeta) = h(w,z)$ into the defining equation of the affine variety $Q'(w', z')$. It will be convenient to use the notation

$$F = (f, g), \quad a_\beta = (f_\beta, g_\beta),$$

where the first component corresponds to the variable $w' \in \mathbf{C}^{d'}$ and the second to $z' \in \mathbf{C}^{m'}$. Thus

$$f(\Theta(x)) = f(h(w,z)) + \sum_{1 \leq |\beta| \leq s} f_\beta x^\beta + o(|x|^s),$$

and similarly for g.

The equations of X are

$$(f(\Theta(x))-\bar{w}')/2i=\langle Bg(\Theta(x)),\bar{z}'\rangle.$$

Comparing the coefficients of terms involving x^β we obtain the following system of equations:

$$\bar{w}'=f(h(w,z))-2i\langle Bg(h(w,z)),\bar{z}'\rangle, \tag{2.9}$$

$$\langle Bg_\beta(h(w,z);\ \bar{w},\bar{z}),\bar{z}'\rangle=f_\beta(h(w,z);\ \bar{w},\bar{z})/2i, \quad 1\leqq|\beta|\leqq s. \tag{2.10}$$

The equations are defined for all $(w,z)\in(\mathscr{W}^-\cup M)\cap U$ and are linear in $(\bar{w}',\bar{z}')\in\mathbf{C}^{n'}$. We define the *characteristic variety* X_F as the set of all solutions of this system.

The following proposition shows that X_F contains the graph of $F|_{M\cap U}$.

2.3 Proposition. *For each* $(w,z)\in M\cap U$ *the point* $(w',z')=F(w,z)\in M'$ *satisfies the system* (2.9)–(2.10).

Proof. Fix a point $p^0=(w^0,z^0)\in M\cap U$ and extend F as a function of class $\mathscr{C}^s$ in a neighborhood of p^0 in $\mathbf{C}^n$. Since $F(M)\subset M'$, the function $\Im f-\langle Bg,\bar{g}\rangle$ of class $\mathscr{C}^s$, with values in $\mathbf{R}^d$, vanishes on M. Hence we can find functions $q_1,q_2,\dots,q_d$ of class $\mathscr{C}^{s-1}$ near p^0, with values in $\mathbf{R}^d$, such that

$$\Im f(w,z)-\langle Bg(w,z),\overline{g(w,z)}\rangle=\sum_{j=1}^{d} q_j(w,z)(\Im w_j-\phi_j(z,\bar{z},\Re w))$$

is an identity near p^0 in $\mathbf{C}^n$.

Write

$$f(w,z)=f^s(w,z)+o(s),$$

$$g(w,z)=g^s(w,z)+o(s),$$

$$q_j(w,z)=q_j^{s-1}(w,z)+o(s-1),$$

where f^s, g^s, and q^{s-1} are Taylor polynomials of indicated orders centered at p^0, and

$$o(s)=o((|z-z^0|+|w-w^0|)^s).$$

Note that f^s and g^s are holomorphic polynomials. Substituting into the above identity we obtain

$$\Im f^s-\langle Bg^s,\bar{g}^s\rangle=\sum_{j=1}^{d} q_j^{s-1}(\Im w_j-\phi_j(z,\bar{z},\Re w))+o(s).$$

We complexify this identity by varying the conjugate variables $(\bar{w},\bar{z})$ independently of (w,z). It is important to observe that the error term remains small of order s also in the complexified identity. This follows immediately from the fact that a holomorphic function $a(z,\zeta)$ whose restriction to the totally real subspace $\zeta=\bar{z}$ vanishes to order s at $(z^0,\bar{z}^0)$ also vanishes to order s as a function of (z,ζ) at this point.

Thus we may set $\bar{w}=\bar{w}^0$, $\bar{z}=\bar{z}^0$ and let (w,z) vary:

$$(f^s(w,z)-\overline{f^s(w^0,z^0)})/2i-\langle Bg^s(w,z),\overline{g^s(w^0,z^0)}\rangle$$
$$=\sum q_j^{s-1}(w,z;\bar{w}^0,\bar{z}^0)((w_j-\bar{w}_j^0)/2i-\phi_j(z,\bar{z}^0,(w+\bar{w}^0)/2))+o(s).$$

We now restrict (w, z) to the variety $Q(p^0)$ in the coordinates (2.6), so that the term in the right hand side vanishes up to order s at p^0. It follows that F^s maps $Q(p^0)$ into $Q'(F(p^0))$ up to order s, hence so does F. By the construction of the system (2.9)–(2.10) this means that $(p^0; F(p^0))$ is a solution. Proposition 2.3 is proved.

We extend the linear system (2.10) in $\bar{z}'$ to the polar M^0:

$$\langle Bg_\beta(\eta, \zeta; \bar{w}, \bar{z}), \bar{z}' \rangle = f_\beta(\eta, \zeta; \bar{w}, \bar{z})/2i, \quad 1 \leqq |\beta| \leqq s, \tag{2.11}$$

where $(\eta, \zeta; \bar{w}, \bar{z}) \in \mathscr{W}^0 \cup T \subset M^0$. The coefficients of the system are holomorphic on $\mathscr{W}^0$ and continuous up to the edge T.

Let $\mathscr{B}(\eta, \zeta; \bar{w}, \bar{z})$ be the coefficient matrix of the system (2.11). Its rank $\varrho(\eta, \zeta; \bar{w}, \bar{z})$ is an upper semicontinuous function on $\mathscr{W}^0 \cup T$. Let

$$\varrho_0 = \max\{\varrho(w, z; \bar{w}, \bar{z}) : (w, z) \in M \cap U\}. \tag{2.12}$$

2.4 Proposition. (a) *$\varrho \leqq \varrho_0$ everywhere on $\mathscr{W}^0$.*

(b) *There is a closed subset $E \subset M \cap U$ of surface measure zero in M such that $\varrho = \varrho_0$ on $M \cap U \backslash E$.*

Proof. To prove (a) we note that every minor (subdeterminant) of $\mathscr{B}$ of size $\varrho_0 + 1$ is a holomorphic function on $\mathscr{W}^0$, continuous up to T, that vanishes on T by definition of ϱ_0. Since T is a determining set for such functions according to Saddulaev [Sad] (see also Coupet [Co1]), this minor is zero on $\mathscr{W}^0$ as well. Since this holds for all minors of size $\varrho_0 + 1$, we conclude that $\varrho \leqq \varrho_0$.

The same argument shows that the set $\{\varrho < \varrho_0\} \subset \mathscr{W}^0 \cup T$ is defined by vanishing of functions, holomorphic on $\mathscr{W}^0$ and continuous up to T. From the identity theorem in [Co1] it follows that the closed set $E^0 = T \cap \{\varrho < \varrho_0\}$ has measure zero in T. Since the complex analytic subset $\{\varrho < \varrho_0\}$ can not disconnect $\mathscr{W}^0$, we have $\varrho = \varrho_0$ everywhere on $T \backslash E^0$. The closed subset $E \subset M$, obtained as the image of E^0 under the coordinate projection $(\eta, \zeta; w, z) \to (\eta, \zeta)$, satisfies the conclusion (b).

The results announced in Sect. 1 will follow from

2.5 Theorem. *Let $F: M \to M'$ be as in Theorem 1.5 or 1.6, and let $E_F \subset M$ be the set of measure zero given by Proposition 2.4. Then F extends holomorphically to a neighborhood of each point $p \in M \backslash E$. If M is the quadric as in Theorem 1.5, then F extends to a rational mapping on $\mathbf{C}^n$.*

We first show that this result immediately implies Theorem 1.1. Since each quadric is affinely homogeneous [PS, p. 15], we may assume that $p = 0$ and $F(0) = 0$. At the origin the vectors $b_k(0)$ (2.8) for $1 \leqq k \leqq m$ form the standard basis of the maximal complex tangent space $T_0^{\mathbf{C}} M = \{0\}^d \times \mathbf{C}^m$. Hence the coefficients $g_\beta(0)$ for $|\beta| = 1$ are the first order partial derivatives with respect to the variables $z = (z_1, \ldots, z_m)$ of the component g of F that is complex tangential to M' at 0. Therefore, if $dF(0)$ is a linear isomorphism of $T_0^{\mathbf{C}} M$ onto $T_0^{\mathbf{C}} M'$, the vectors $\{g_\beta(0) : |\beta| = 1\}$ are a basis of $\mathbf{C}^m$. Since the bilinear form associated to $B = (B_1, \ldots, B_{d'})$ is nondegenerate, we see immediately that the vectors $\{B_k g_\beta(0) : 1 \leqq k \leqq d', |\beta| = 1\}$ also span $\mathbf{C}^m$, so the system (2.10) has maximal rank $\varrho_0 = m$ at the origin. Thus Theorem 2.5 applies.

In Sect. 3 we shall prove Theorem 2.5 in the case when the rank $\varrho_0 = m'$ is maximal. The proof in the degenerate case $\varrho_0 < m'$ is more involved and will be given in Sect. 4.

3 The nondegenerate case

In this section we shall prove Theorem 2.5 in the special case when the system (2.9)–(2.10) defining the characteristic variety X_F is of maximal rank, i.e., $\varrho_0 = m'$. This part is similar to the original reflection principle of Lewy [Le] and Pinčuk [Pin].

Fix a point $p^0 \in M \backslash E$. To simplify the notation we shall choose the coordinates so that $p^0 = 0$ and $F(p^0) = 0$. From the system (2.10) at $(w, z) = (0, 0)$ we choose a suitable set of m' independent equations. The same equations remain independent if we vary (w, z) in $(\mathscr{W}^- \cup M) \cap U_1$ for a suitably small neighborhood $U_1 \subset U$ of 0, so we obtain a unique solution $z' = z^*(w, z)$ there. Inserting this solution into (2.9) we also obtain $w' = w^*(w, z)$.

Since the Eqs. (2.9)–(2.10) are antiholomorphic in w, the solution $F^* = (w^*, z^*)$ is holomorphic in w for each fixed z. Proposition 2.3 shows that for $(w, z) \in M \cap U_1$ we have $F^*(w, z) = F(w, z)$. Thus we may extend F to $(\mathscr{W}^- \cup M) \cap U_1$ by setting $F = F^*$ there. Then for each z near 0 the function $F(\cdot, z)$ satisfies the hypotheses of the edge-of-the-wedge theorem on the double wedge $(\mathscr{W}^+ \cup M \cup \mathscr{W}^-) \cap \{z = \text{const}\}$, so it extends holomorphically to an open set $V_z \subset \mathbf{C}^d \times \{z\}$. Since the edge $M \cap \{z = \text{const}\}$ depends analytically on z, we can take V_z to be independent of z.

We have obtained an extension of F to a neighborhood $U^* = U_1^* \times U_2^*$ of 0 such that $F(\cdot, z)$ is holomorphic in $w \in U_1^*$ for each $z \in U_2^*$. Since F is holomorphic on the open set $\mathscr{W}^+$, a theorem of Hartogs [BM, p. 141] implies that F is holomorphic near 0. This proves the first part of Theorem 2.5 in the case of maximal rank.

Assume now that $M = M_A$ is the quadric (0.3), and let h be defined by (2.4). According to Lemma 2.2, the coefficients of the system (2.9)–(2.10) are holomorphic in $(h(w, z); \bar{z})$ and polynomials in $\bar{z}$. By Cramer's formula each component of the solution (w^*, z^*) is of the form

$$H(q, z) = G(q, z)/g(q, z), \qquad q = \overline{h(w, z)}, \tag{3.1}$$

where G and g are holomorphic in the variables (q, z) and polynomials in z. The last part of Theorem 2.5 follows immediately from

3.1 Proposition. *If there is an open $V \subset \mathbf{C}^n$ such that the function H (3.1) is holomorphic as a function of $(w, z) \in V$, then H is a rational function of (w, z).*

A similar result has been proved by Cima and Suffridge [CS] for the special case when M is a sphere.

Proof. Let $V = V_1 \times V_2$, with $V_1 \subset \mathbf{C}^d$ and $V_2 \subset \mathbf{C}^m$. For each point $z \in V_2$ we let $\Lambda(z) \subset \mathbf{C}^n$ be the complex subspace spanned by the vectors $b_1(\bar{z}), \ldots, b_m(\bar{z})$ (2.8), i.e., $\Lambda(z)$ is the parallel translation of $Q(w, z)$ to 0.

Fix a point $z^0 \in V_2$ and introduce new coordinates $(\tau, \varrho) \in \mathbf{C}^n$, defined as functions of $(w, \zeta) \in \mathbf{C}^n$ by

$$(\tau, \varrho) = (w, z^0) + \sum_{k=1}^{m} \zeta_k b_k(\bar{z}^0).$$

Equivalently,

$$\tau = w + 2i\langle A\zeta, \bar{z}^0\rangle, \qquad \varrho = z^0 + \zeta. \tag{3.2}$$

In these coordinates we have

$$q = \overline{h(\tau, \varrho)} = (\tau - 2i\langle A\varrho, \bar{\varrho}\rangle, \bar{\varrho}). \tag{3.3}$$

3.2 Lemma. *Suppose that* $g(q, \varrho)$ *is a holomorphic function of* (q, ϱ) *that is polynomial of order* $\leqq K$ *in* ϱ. *If we substitute* q *by* (3.3) *and* (τ, ϱ) *by* (3.2), *then* $\partial^\alpha g/\partial\zeta^\alpha|_{\zeta=0}=0$ *whenever* $|\alpha|>K$.

Proof. The chain rule gives

$$\frac{\partial g}{\partial \zeta_j}=\sum_{k=1}^{d}\frac{\partial g}{\partial q_k}\frac{\partial q_k}{\partial \zeta_j}+\sum_{k=d+1}^{n}\frac{\partial g}{\partial q_k}\frac{\partial \bar{\varrho}_{k-d}}{\partial \zeta_j}+\sum_{k=1}^{m}\frac{\partial g}{\partial \varrho_k}\frac{\partial \varrho_k}{\partial \zeta_j}.$$

We have $\partial\varrho_k/\partial\zeta_j=\delta_{jk}$, $\partial\bar{\varrho}_k/\partial\zeta_j=0$, and for $1\leqq k\leqq d$

$$\begin{aligned}\frac{\partial q_k}{\partial \zeta_j}&=\frac{\partial \tau_k}{\partial \zeta_j}-2i\frac{\partial}{\partial \zeta_j}\langle A_k\varrho, \bar{\varrho}\rangle\\&=2i\langle A_k e_j, \bar{z}^0\rangle-2i\langle A_k e_j, \bar{z}^0+\bar{\zeta}\rangle\\&=-2i\langle A_k e_j, \bar{\zeta}\rangle.\end{aligned}$$

Inserting this into the expression for $\partial g/\partial\zeta_j$ we obtain

$$\frac{\partial g}{\partial \zeta_j}=-2i\sum_{k=1}^{d}\frac{\partial g}{\partial q_k}\langle A_k e_j, \bar{\zeta}\rangle+\frac{\partial g}{\partial \varrho_j}.$$

When we differentiate further with respect to ζ, the terms $\langle A_k e_j, \bar{\zeta}\rangle$ are preserved, so

$$\frac{\partial^\alpha g}{\partial \zeta^\alpha}=\frac{\partial^\alpha g}{\partial \varrho^\alpha}+\text{terms containing } \bar{\zeta}.$$

The lemma follows.

The lemma implies that we can write G and g, as functions of (w, ζ), in the form

$$G(w,\zeta)=G_0(w,\zeta)+\sum_{k=1}^{m}\bar{\zeta}_k G_k(w,\zeta),$$

where G_0 is holomorphic in (w,ζ) and polynomial of order $\leqq K$ in ζ (similarly for g). Recall that the quotient $H=G/g$ is holomorphic in (w,ζ) in a suitable domain. Comparing the holomorphic terms in the Taylor expansion of $Hg=G$ at the point $\zeta=0$ we conclude that H is the quotient of the holomorphic parts $H=G_0/g_0$. Since G_0 and g_0 are polynomials in ζ, it follows that $H(w,\zeta)$ is rational in ζ for each fixed $w\in V_1$.

This shows that for each $(w,z)\in V$, the restriction of $H=G/g$ to the affine subspace $(w,z)+\Lambda(z)$, parallel to $\Lambda(z)$ and passing through (w,z), is rational. In particular, if $\ell\subset\Lambda(z)$ is any complex line through 0, then H is rational on the translated line $p+\ell$ for every point p in a smaller neighborhood $V'\subset V$ of 0.

To conclude the proof it now suffices to show that the set of vectors

$$\mathscr{B}=\{b_k(\bar{z}): 1\leqq k\leqq m,\ z\in V_2\}$$

spans all of $\mathbf{C}^n$. This will give us n linearly independent directions $\ell_j\subset\Lambda(z^j)$ for various $z^j\in V_2$ such that H is rational on the translates of ℓ_j to points of V'. The theorem on separately rational functions [BM, p. 201] then implies that $H=G/g$ is rational.

To obtain a contradiction we suppose that $\mathscr{B}$ does not span $\mathbf{C}^n$. Then there is a vector

$$\beta=(\beta_1,\ldots,\beta_n)\neq 0$$

such that $\langle \beta, b_k(\bar{z})\rangle = 0$ for all $k \in \{1, \ldots, m\}$ and $z \in V_2$. This means

$$2i\left\langle\left(\sum_{j=1}^{d} \beta_j A_j\right) e_k, \bar{z}\right\rangle + \beta_{d+k} = 0, \qquad 1 \leqq k \leqq m,\ z \in V_2.$$

Hence $\beta_{d+k} = 0$, $1 \leqq k \leqq m$, and $\sum_{j=1}^{d} \beta_j A_j = 0$. Thus the components of the Levi form of M_A are linearly dependent. This contradicts the assumption that the cone Γ_A has nonempty interior. The contradiction shows that $\mathscr{B}$ spans $\mathbf{C}^n$ as claimed, and Proposition 3.1 is proved.

4 The degenerate case

In this section we shall prove Theorem 2.5 in the case when the system (2.9)–(2.10) has rank $\varrho_0 < m'$. Let $E \subset M$ be the set given by Proposition 2.4. Fix a point $p^0 \in M \cap U \backslash E$. As in Sect. 3 we may shrink our neighborhoods and assume that $p^0 = 0$, $F(p^0) = 0$, and $M \cap U_0 \cap E = \emptyset$.

We first show that the system (2.11) is compatible (solvable) in a neighborhood of $(0, 0)$ in $T \cup \mathscr{W}^0$. We can find ϱ_0 columns from the coefficient matrix $\mathscr{B}(\eta, \zeta; \bar{w}, \bar{z})$ that are linearly independent when $(\eta, \zeta; \bar{w}, \bar{z}) \in T \cup \mathscr{W}^0$ is close to q^0. Let $\mathscr{B}_0$ be the matrix consisting of these ϱ_0 columns, together with the column of the right hand sides $f_\beta/2i$ of (2.11). Consider any maximal minor Δ of $\mathscr{B}_0$. At points of T the system is compatible, so $\Delta = 0$ on T. Since Δ is holomorphic on $\mathscr{W}^0$, the uniqueness theorem of Sadullaev [Sad] implies that $\Delta \equiv 0$ on $\mathscr{W}^0$ as well. Since this holds for each minor of $\mathscr{B}_0$, the system (2.11) is compatible and of constant rank ϱ_0 near q^0.

Let $\delta = m' - \varrho_0$ be the *defficiency* of the system (2.10). We can relabel the variables z' so that the system (2.10) has rank ϱ_0 in $\{z'_k : \delta + 1 \leqq k \leqq m\}$. Hence the solution of (2.10) for points $(w, z) \in \mathscr{W}^- \cup M$ close to 0 is of the form

$$z'_k = c_{k,0}(w, z) + \sum_{j=1}^{\delta} c_{k,j}(w, z) z'_j, \qquad \delta + 1 \leqq k \leqq m', \tag{4.1}$$

where each coefficient $c_{k,j}(w, z)$ is a real-analytic function on $\mathscr{W}^-$ that extends continuously up to M. From (2.9) we obtain similar expressions for the variables w':

$$w'_k = d_{k,0}(w, z) + \sum_{j=1}^{\delta} d_{k,j}(w, z) z'_j, \qquad 1 \leqq k \leqq d'. \tag{4.2}$$

These equations define the variety X_F on $\mathscr{W}^- \cup M$ near 0.

The crucial point of the proof is

4.1 Proposition. *The functions $c_{k,j}$ and $d_{k,j}$ are holomorphic in $\mathscr{W}^- \cap U_1$ for a suitably small neighborhood U_1 of 0, so $X \cap (\mathscr{W}^- \cap U_1) \times \mathbf{C}^{n'}$ is a complex analytic subvariety of dimension $n + \delta$.*

Proof. We consider again the linear system (2.11). Our first goal is to prove that the set of solutions z' of the system is independent of the choice of the point $(\eta, \zeta) \in Q(w, z)$, provided that we stay in the same connected component of $Q(w, z) \cap \mathscr{W}^+$.

The arguments that were used in the proof of Proposition 2.4 show that there is a point $p^1 \in M \backslash E$ arbitrary close to 0 such that set of vectors

$$\mathscr{G}_s = \{g_\beta(\eta, \zeta; \bar{w}, \bar{z}) : 1 \leqq |\beta| \leqq s\} \subset \mathbf{C}^{m'} \tag{4.3}$$

is of constant rank ϱ_1 near the point $(p^1, \bar{p}^1)$ on $\mathcal{W}^0 \cup T$. Since the rank of the system (2.11) is $\varrho_0 < m'$ and the hermitian form $\langle B\zeta, z\rangle$ is nondegenerate, we also have $\varrho_1 < m'$.

To prove that the solution set of (2.11) is independent of $(\eta, \zeta) \in Q(w, z)$ we need to show that the linear span of the set $\mathcal{G}_s$ is independent of (η, ζ).

Fix $(w, z) \in \mathcal{W}^-$ near p^1 and consider the map $g: Q(w, z) \to \mathbf{C}^{m'}$. Let $O^j(\eta, \zeta)$ be its osculating space of order j at the point (η, ζ), i.e., the linear span of its derivatives of order $\leqq j$ at this point. If we use the coordinates on $Q(w, z)$ given by (2.6), the derivatives of g are precisely the functions g_β. Hence $O^j(\eta, \zeta)$ is the linear span of the set $\mathcal{G}_j$ (4.3).

When (w, z) is close to a point $p \in M$, the tangent space to $Q(w, z)$ at each point (η, ζ) close to p is close to $T_p^{\mathbf{C}}M$. From the assumption that $dF(p): T_p^{\mathbf{C}}M \to T_{F(p)}^{\mathbf{C}}$ has rank at least r_0 for each $p \in M$ it follows that g has rank at least r_0 at $(\eta, \zeta) \in Q(w, z)$, provided that both (w, z) and (η, ζ) are sufficiently close to M, so $\dim O^1(\eta, \zeta) \geqq r_0$. We also know that $\dim O^s(\eta, \zeta) = \varrho_1 < m'$.

We thus have an increasing sequence $O^1(w, z) \subset O^2(w, z) \subset \dots \subset O^s(w, z)$ of $s = m' - r_0 + 1$ complex subspaces of $\mathbf{C}^{m'}$ such that

$$\dim O^s(w, z) - \dim O^1(w, z) \leqq \varrho_1 - r_0 < m' - r_0 .$$

Hence there is a $j = j(\eta, \zeta)$ such that $O^j(\eta, \zeta) = O^{j+1}(\eta, \zeta)$. Since $\dim O^j(\eta, \zeta)$ is an upper semicontinuous function of (η, ζ), we can take j independent of (η, ζ) on an open subset of $Q(w, z)$.

Lemma 7.5 in [For] now implies that the image $g(Q(w, z))$ is contained in the affine subspace $g(\eta, \zeta) + O^s(\eta, \zeta) \subset \mathbf{C}^{m'}$ for some $(\eta, \zeta) \in Q(w, z)$. This, together with the fact $\dim O^s(\eta, \zeta) \equiv r_0$, shows that $O_s(\eta, \zeta)$ is independent of the choice of $(\eta, \zeta) \in Q(w, z)$, so the system (2.11) is also independent of (η, ζ).

Fix a point $p^2 \in \mathcal{W}^-$ close enough to p^1 so that the above conclusion holds for all (w, z) in a neighborhood $U_2 \subset \mathcal{W}^-$ of p^2. Locally near p^2 we can find an antiholomorphic mapping $(w, z) \to h_1(w, z) \in Q(w, z) \cap \mathcal{W}^+$ such that $h_1(p^2) = h(p^2)$. Now we solve the system (2.10) with h replaced by h_1. From what was said above we conclude that the solution set of the new system for (w, z) near p^2 is still given by (4.1). However, since the new system is (after conjugation) holomorphic in all variables, its solution is holomorphic, so the coefficients $c_{j,k}(w, z)$ in (4.1) are holomorphic functions near p^2. Since these functions are real-analytic on $\mathcal{W}^- \cap U$, they are holomorphic on all of $\mathcal{W}^- \cap U$.

We subsitute the solution (4.1) into the first Eq. (2.9) to obtain the corresponding solution w' (4.2). It remains to show that w' is also independent of the choice of the reference point $h(w, z)$. To see this we fix $(w, z) \in \mathcal{W}^-$ and let z' be any solution of (4.1). Consider the function

$$W'(\eta, \zeta) = f(\eta, \zeta) - 2i\langle Bg(\eta, \zeta), \bar{\zeta}\rangle$$

for $(\eta, \zeta) \in Q(w, z)$. Fix a point (η, ζ) and use the local coordinates $\Theta(x)$ (2.6) on $Q(w, z)$. Differentiating $W'(x)$ with respect to x at $x = 0$ we obtain the Eqs. (2.11) at the point $(\eta, \zeta; \bar{w}, \bar{z})$. Since z' is a solution of this system, the derivatives of W' all equal zero, whence $W'(\eta, \zeta)$ is constant on $Q(w, z)$. Thus the solution (4.2) is also independent of the choice of $h(w, z)$. If we choose h_1 as before to be antiholomorphic locally in $\mathcal{W}^-$, we conclude that the solution w' is holomorphic in all arguments. This proves Proposition 4.1.

Remark. Although we can choose h to be antiholomorphic *locally* on $\mathcal{W}^-$, there is no such global choice of h that would preserve M. Therefore we had to prove that different choices of h give the same solution set X_F.

4.2 Proposition. *The variety X_F extends to a complex analytic subvariety in $U_2 \times \mathbf{C}^{n'}$ for a suitably small open neighborhood $U_2 \subset U_1$ of 0 in $\mathbf{C}^n$. If M is the quadric* (0.3), *then X_F extends to a rational subvariety of $\mathbf{C}^n \times \mathbf{C}^{n'}$.*

Proof. We have seen that there is a neighborhood U_1 of 0 in $\mathbf{C}^n$ such that each coefficient $c_{j,k}$ and $d_{j,k}$ is a holomorphic function on the wedge $\mathscr{W}^- \cap U_1$ and extends continuously to the edge $M \cap U_1$. Its restriction to $M \cap U_1$ is a CR function on $M \cap U_1$.

Since M is minimal at 0, we can find a wedge $\mathscr{W}_1^+$ with edge M such that every CR function on $M \cap U_1$ extends holomorphically to $\mathscr{W}_1^+$ in a smaller neighborhood U_2 of 0. In particular, we can extend the coefficients $c_{j,k}, d_{j,k}$ to $\mathscr{W}_1^+$.

We would like to conclude that these functions extend holomorphically to a neighborhood of 0 in $\mathbf{C}^n$. This is so if we can show that they are holomorphic on a double wedge with edge M, since we can then use the edge-of-the-wedge theorem as in Sect. 3 above. The trouble is that $\mathscr{W}_1^+ \cup \mathscr{W}^-$ need not be a double wedge, i.e., the convex hull of the union of the corresponding cones in $\mathbf{R}^d$ need not be all of $\mathbf{R}^d$.

To avoid this difficulty, we associate to $\mathscr{W}_1^+$ the opposite wedge

$$\mathscr{W}_1^- = \{(w,z) \in U_2 : h(w,z) \in \mathscr{W}_1^+\}.$$

Since the original mapping F is CR on a set containing $M \cap U_1$, we may also assume that F is holomorphic on $\mathscr{W}_1^+$. We repeat the construction of X_F with $\mathscr{W}^+$ resp. $\mathscr{W}^-$ replaced by the wedges $\mathscr{W}_1^+$ resp. $\mathscr{W}_1^-$. For points $(w,z) \in M \cap U_1$ the solution (4.1)–(4.2) of the system (2.9)–(2.10) does not depend on the choice of the wedges. The upshot is that we obtain a holomorphic extension of the functions $c_{j,k}, d_{j,k}$ from $M \cap U_2$ to the double wedge $\mathscr{W}_1^+ \cup \mathscr{W}_1^-$, so the edge-of-the-wedge theorem applies. Thus X_F extends as a complex analytic variety to $U_2 \times \mathbf{C}^{n'}$ for a suitably small neighborhood U_2 of 0 in $\mathbf{C}^n$.

If M is the quadric (0.3) whose cone Γ_A has nonempty interior, then Cramer's formula shows that each coefficient in the solution (4.1)–(4.2) is of the form (3.1). Since these coefficients are holomorphic near the origin, Proposition 3.2 implies that they extend to rational functions on $\mathbf{C}^n$, so X_F extends to a rational subvariety of $\mathbf{C}^n \times \mathbf{C}^{n'}$. Proposition 4.2 is proved.

We can now prove that $F|_{M \cap U_2}$ is real-analytic, so it extends holomorphically to a neighborhood of $M \cap U_2$. The proof is very similar to the one in [For, pp. 56–57].

First we observe that for each point $p \in M \cap U_1$ we have

$$F(p) \in X(p) \subset Q'(F(p)).$$

Here, $X(p) \subset \mathbf{C}^{n'}$ is the fiber of X_F over p, an affine subspace of $\mathbf{C}^{n'}$. The first inclusion is the content of Proposition 2.3, and the second is a consequence of the Eq. (2.9).

Recall that M' is strongly pseudoconvex, so we may assume, after a linear change of w coordinates, that B_1 is strongly positive definite. Let $\phi(w',z') = \Im w'_1 - \langle B_1 z', \bar{z}' \rangle$. A calculation shows that for each point $(w^0, z^0) \in M'$, the restriction of ϕ to $Q'(w^0, z^0)$ equals $-\langle B_1(z'-z^0), \bar{z}'-\bar{z}^0 \rangle$, so it has a unique nondegenerate critical point at $z' = z^0$.

The same is then true for the restriction of ϕ to the affine subspace $X(p)$ of $Q'(F(p))$ for $p \in M \cap U_2$. Recall that the equations of X_p are given by (4.1)–(4.2). If we insert these equations into ϕ, we obtain a function $\varrho'(z'_1, \ldots, z'_\delta)$ with a unique nondegenerate critical point at $z'_j = F_{d'+j}(w,z)$. This point is the unique solution of the system of linear equations

$$\partial \varrho'/\partial \bar{z}'_j = 0, \qquad 1 \leqq j \leqq \delta$$

in the variables $(z'_1, \ldots, z'_{d'})$. By Cramer's formula the solution is a rational expression in the coefficients of this system, which are themselves polynomial expressions of the functions $c_{j,k}, d_{j,k}$ from (4.1)–(4.2). Therefore the functions $F_{d'+j}(w,z)$, $1 \leqq j \leqq \delta$, are real-analytic functions of $(w,z) \in M \cap U_2$. The same is then true for the remaining components of F since these may be computed from the system (4.1)–(4.2). Hence F extends holomorphically to a neighborhood of $M \cap U_2$.

When M is a quadric, we already know that the coefficients in (4.1)–(4.2) are rational functions of (w,z), so it follows that $F|_{M \cap U_2}$ coincides with a rational function of the variables $(w, z; \bar{w}, \bar{z})$. Since F is holomorphic on the wedge $\mathscr{W}^0$, it follows that F is complex rational. We shall omit the proof of this last step since it is very similar to the proof of Proposition 7.7 in [For].

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