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Approximation of biholomorphic mappings by automorphisms of $\mathbf{C}^n$

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Introduction

In this paper we prove several results on approximation of biholomorphic mappings between domains in $\mathbf{C}^n$ ($n \geqq 2$) by holomorphic automorphisms of $\mathbf{C}^n$.

Recall that the group of holomorphic automorphisms of $\mathbf{C}^n$, denoted $\operatorname{Aut} \mathbf{C}^n$, consists of those holomorphic mappings $\Phi: \mathbf{C}^n \to \mathbf{C}^n$ that have a holomorphic inverse $\Phi^{-1}: \mathbf{C}^n \to \mathbf{C}^n$. With the topology of uniform convergence on compact sets, $\operatorname{Aut} \mathbf{C}^n$ is a topological group. While the group $\operatorname{Aut} \mathbf{C}^1$ of automorphisms of the complex plane consists only of the linear mappings $az + b$ ($a, b \in \mathbf{C}$, $a \neq 0$), the group $\operatorname{Aut} \mathbf{C}^n$ is very large and complicated when $n \geqq 2$. Let us choose a linear coordinate system on $\mathbf{C}^n$ and write the coordinates as $z = (z', w)$, where $z' = (z_1, \ldots, z_{n-1}) \in \mathbf{C}^{n-1}$ and $w = z_n \in \mathbf{C}$. Then $\operatorname{Aut} \mathbf{C}^n$ contains mappings of the form

$$z = (z', w) \mapsto (z', e^{h(z')} w + f(z')) ,$$

where f and h are holomorphic functions on $\mathbf{C}^{n-1}$. In [4] the mappings of this type are called *overshears*, and those with $h \equiv 0$ are called *shears* as in [17]. The Jacobian determinant of every shear is identically equal to one, i.e., the shears are volume preserving. We emphasize that we allow shears (and overshears) in any coordinate system on $\mathbf{C}^n$ that is obtained from the initial coordinates system by a linear transformation.

In [4] Andersén and Lempert proved that every biholomorphic mapping $\Phi: \Omega \to \mathbf{C}^n$ from a convex (or starshaped) domain $\Omega \subset \mathbf{C}^n$ ($n \geqq 2$) onto a Runge domain $\Phi(\Omega)$ can be approximated by finite compositions of overshears, uniformly on compact sets in Ω. If, in addition, Φ is volume preserving, then it can be approximated by finite compositions of shears. In particular, every automorphism of $\mathbf{C}^n$ can be approximated by finite compositions of overshears, uniformly on

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compact sets in $\mathbf{C}^n$, and the volume preserving automorphisms can be approximated by finite compositions of shears. On the other hand, the volume preserving automorphism

$$(z_1, z_2) \mapsto (z_1 e^{z_1 z_2}, z_2 e^{-z_1 z_2})$$

is not a finite composition of shears (Andersén [3]) nor overshears [4]. Thus, for $n \geqq 2$, the subgroup of $\mathrm{Aut}\,\mathbf{C}^n$ generated by overshears is dense in $\mathrm{Aut}\,\mathbf{C}^n$ but is not equal to $\mathrm{Aut}\,\mathbf{C}^n$. Similarly, the group generated by shears is dense in but not equal to the group $\mathrm{Aut}_1\,\mathbf{C}^n$ of volume preserving automorphisms of $\mathbf{C}^n$.

In Sect. 1 we state a more general theorem on approximation of parametrized families of biholomorphic maps by automorphisms (Theorem 1.1). From the converse part of Theorem 1.1 it is evident that our condition is close to being necessary and sufficient. This result was essentially proved by Andersén and Lempert in [4], but it was not stated in this generality. We do not want to claim any credit for this part. But in the sketch of proof, given for the sake of completeness, we change the presentation slightly by replacing the dynamics in $\mathrm{Aut}\,\mathbf{C}^n$ with dynamics in $\mathbf{C}^n$.

Our new results are explained in Sects. 2–6. In Sect. 2 we begin with a result on approximation by holomorphic automorphisms in a neighborhood of a polynomially convex set $K \subset \mathbf{C}^n$ (Theorem 2.1). We apply this to approximate independent motions of a collection of convex (or starshaped) bodies in $\mathbf{C}^n$ by a single automorphism of $\mathbf{C}^n$ (Theorem 2.2). The only obstruction to such approximation is the nontrivial polynomial hull of the union of the given bodies. For instance, a collection of three disjoint balls in $\mathbf{C}^n$ can be approximately moved to any position in $\mathbf{C}^n$ by an automorphism of $\mathbf{C}^n$ (Corollary 2.3). The same is true for three disjoint polydiscs in $\mathbf{C}^2$ with sides parallel to the axes. This answers a question raised by Rosay and Rudin in [17].

In Sect. 3 we consider the problem of approximately mapping a given embedded real-analytic submanifold $M_0 \subset \mathbf{C}^n$ onto another such submanifold $M_1 \subset \mathbf{C}^n$, diffeomorphic to M_0, by automorphisms of $\mathbf{C}^n$. Precisely speaking, two such real-analytic submanifolds are said to be $\mathbf{C}^n$-equivalent (Definition 2) if there exists a biholomorphic map $\Phi: \Omega \to \mathbf{C}^n$, defined in a neighborhood Ω of M_0 and taking M_0 onto M_1, such that Φ is the limit of a sequence of holomorphic automorphisms. We introduce a similar concept for real-analytic embeddings into $\mathbf{C}^n$. The notion of $\mathbf{C}^n$-equivalence seems a natural analogue of the concept of ambient equivalence in real differential topology. Even though the definition of $\mathbf{C}^n$-equivalence makes sense also for smooth submanifolds of $\mathbf{C}^n$, it does not seem to be natural there. We are aware that a similar approach is possible in the case of smooth manifolds, and this is explained (in the simple setting of smooth arcs in $\mathbf{C}^n$) in the forthcoming paper [16]. It seems likely that the same methods can be applied to more general smooth submanifolds. In this article we purposely deal only with real-analytic manifolds.

The main result of Sect. 3 (Theorem 3.1) is that, given a real-analytic submanifold $M_0 \subset \mathbf{C}^n$ that is totally real and polynomially convex in $\mathbf{C}^n$, M_0 is $\mathbf{C}^n$-equivalent to another real-analytic submanifold $M_1 \subset \mathbf{C}^n$ if and only if M_0 and M_1 are isotopic in $\mathbf{C}^n$ through a family of totally real, polynomially convex submanifolds $M_t \subset \mathbf{C}^n$ ($t \in [0, 1]$).

In Sects. 4 and 5 we apply Theorem 3.1 to obtain $\mathbf{C}^n$-equivalence for several classes of embedded real-analytic manifolds: totally real polynomially convex discs, arcs, embedded analytic discs, closed polynomially convex curves, and totally real, polynomially convex surfaces in $\mathbf{C}^n$ for $n \geqq 3$.

In Sect. 5 we also prove that the set of all smooth, totally real, polynomially convex embeddings of a compact, real, two-dimensional surface into $\mathbf{C}^n$ for $n \geqq 3$ is open and dense in the set of all embeddings, and every two such embeddings are isotopic through totally real, polynomially convex embeddings (Theorem 5.2). The same is true for closed curves in $\mathbf{C}^n$ for $n \geqq 2$. Similar result holds for embeddings of real k-dimensional manifolds into $\mathbf{C}^n$ provided that n is sufficiently large with respect to k, but we postpone this to a future publication in order not to make the paper too long.

In Sect. 6 we give a necessary and sufficient condition for the approximation of a given biholomorphic mapping $\Phi: \Omega \to \mathbf{C}^n$ by automorphisms of $\mathbf{C}^n$ in a neighborhood of a real-analytic, totally real, polynomially convex submanifold $M \subset \Omega$. The answer is in terms of certain holonomy group of homotopy classes of automorphisms of the complex normal bundle to M in $\mathbf{C}^n$. While the criterium seems difficult to check in general, we have a very explicit result for embedded real-analytic curves and surfaces $M \subset \mathbf{C}^n$ ($n \geqq 2$ resp. $n \geqq 3$): If M is totally real and polynomially convex, then a biholomorphic map Φ can be approximated by automorphisms in some neighborhood of M if and only if $\Phi(M)$ is also polynomially convex and the Jacobian determinant $J(\Phi): M \to \mathbf{C}\setminus\{0\}$ is homotopic to a constant in $\mathbf{C}\setminus\{0\}$ (Corollary 6.3).

In Sect. 7 we collect some examples and open problems.

1 The Andersén-Lempert theorem

In [4] the following theorem is essentialy proved, but not stated in this generality. By approximation on $\Omega \subset \mathbf{C}^n$ we shall always mean uniform approximation on compact subsets in Ω.

1.1. Theorem. *Let Ω be an open set in $\mathbf{C}^n$ ($n \geqq 2$). For every $t \in [0, 1]$, let Φ_t be a biholomorphic map from Ω into $\mathbf{C}^n$, of class $\mathscr{C}^2$ in $(t, z) \in [0, 1] \times \Omega$. Assume that each domain $\Omega_t = \Phi_t(\Omega)$ is Runge in $\mathbf{C}^n$.*

If Φ_0 can be approximated on Ω by holomorphic automorphisms of $\mathbf{C}^n$, then for every $t \in [0, 1]$ the map Φ_t can be approximated on Ω by holomorphic automorphisms of $\mathbf{C}^n$. Moreover, if every Φ_t is volume preserving (i.e., its Jacobian determinant equals one), and if Φ_0 can be approximated on Ω by volume preserving automorphisms of $\mathbf{C}^n$, then every Φ_t can be approximated on Ω by volume preserving automorphisms of $\mathbf{C}^n$.

Conversely, if Ω is a pseudoconvex Runge domain in $\mathbf{C}^n$ and $\Phi_1: \Omega \to \mathbf{C}^n$ is a biholomorphic map that can be approximated on Ω by automorphisms of $\mathbf{C}^n$, then for every compact set $K \subset \Omega$ there is an open set D, $K \subset D \subset \Omega$, and a family of biholomorphic maps $\Phi_t: D \to \mathbf{C}^n$, of class $\mathscr{C}^\infty$ in $(t, z) \in [0, 1] \times D$, such that Φ_0 is the identity map, Φ_1 is the given map, every Φ_t can be approximated on D by automorphisms of $\mathbf{C}^n$, and $\Phi_t(D)$ is Runge in $\mathbf{C}^n$ for each $t \in [0, 1]$.

A remark, at the end of this section, gives a more general statement, close to stating necessary and sufficient conditions.

Recall that a domain $\Omega \subset \mathbf{C}^n$ (not necessarily pseudoconvex) is said to be Runge in $\mathbf{C}^n$ if every holomorphic function on Ω can be approximated by entire functions, uniformly on compact sets in Ω. Every starshaped domain is Runge [10]. The envelope of holomorphy of a Runge domain is a pseudoconvex

Runge domain in $\mathbf{C}^n$, and therefore the pseudoconvexity hypothesis in the converse part of Theorem 1.1 is no loss of generality.

The original result of Andersén and Lempert [4] is that a biholomorphic mapping $\Phi: \Omega \to \Omega'$ from a starshaped domain $\Omega \subset \mathbf{C}^n$ can be approximated by automorphisms if and only if $\Omega' = \Phi(\Omega)$ is Runge. We may assume that $\Phi(0) = 0$. In this case the one-parameter family to which one applies Theorem 1.1 is $z \in \Omega \to \Phi(tz)/t$, $t \in [0, 1]$, that connects Φ to the linear map $D\Phi(0)$.

We obtain a slightly more general version of Theorem 1.1 if we allow the base domain Ω_t to vary smoothly with $t \in [0, 1]$. In that situation we obtain, under the same hypothesis as above, the approximation of the biholomorphic map $\Phi_t: \Omega_t \to \mathbf{C}^n$ by automorphisms of $\mathbf{C}^n$, uniformly on compact subsets of $\bigcap_{t \in [0,1]} \Omega_t$.

It will be useful to introduce the notion of isotopy of biholomorphic mappings:

Definition 1. Let Ω be a domain in $\mathbf{C}^n$. A family of biholomorphic mappings $\Phi_t: \Omega \to \mathbf{C}^n$, depending smoothly (of class at least $\mathscr{C}^2$) on $(t, z) \in [0, 1] \times \Omega$, is called an *isotopy of biholomorphic mappings* from Ω. The range $\Phi_t(\Omega) \subset \mathbf{C}^n$ depends on t.

Before proceeding to the proof of Theorem 1.1 we collect some simple but useful observations about biholomorphic mappings that can be approximated by global automorphisms.

Recall that a compact set $K \subset \mathbf{C}^n$ is polynomially convex if for every point $z_0 \in \mathbf{C}^n \backslash K$ there is a holomorphic polynomial $P(z)$ satisfying $|P(z_0)| > \sup_{z \in K} |P(z)|$. We denote by $\hat{K}$ the polynomially convex hull of K, that is, the smallest polynomially convex set in $\mathbf{C}^n$ containing K.

1.2. Proposition. *Suppose that a biholomorphic map $\Phi: \Omega \to \mathbf{C}^n$ from a domain $\Omega \subset \mathbf{C}^n$ onto $\Omega' = \Phi(\Omega) \subset \mathbf{C}^n$ can be approximated by automorphisms of $\mathbf{C}^n$. Then*
(a) *The inverse $\Phi^{-1}: \Omega' \to \Omega$ can be approximated by automorphisms of $\mathbf{C}^n$ on Ω'.*
(b) *If one of the compact sets $K \subset\subset \Omega$, $\Phi(K) \subset\subset \Omega'$ is polynomially convex then the other one is polynomially convex as well.*
(c) *If one of the domains Ω, Ω' is Runge in $\mathbf{C}^n$ then the other one is also.*
(d) *For each $\Psi \in \operatorname{Aut} \mathbf{C}^n$ there is an isotopy of automorphisms $\Psi_t \in \operatorname{Aut} \mathbf{C}^n$ ($t \in [0, 1]$) such that $\Psi_1 = \Psi$ and Ψ_0 is the identity.*

Proof. (a) If $\Phi = \lim_{j \to \infty} \Phi_j$ on Ω, $\Phi_j \in \operatorname{Aut} \mathbf{C}^n$, then $\Phi^{-1} = \lim_{j \to \infty} \Phi_j^{-1}$ on Ω'.
(b) Suppose $\Phi(K) = K' \subset\subset \Omega'$ is polynomially convex. Let $a \in \Omega \backslash K$. Choose a polynomial $P(w)$ satisfying $|P(\Phi(a))| > \sup_{K'} |P| = \sup_K |P \circ \Phi|$. If we approximate Φ sufficiently well by $\Psi \in \operatorname{Aut} \mathbf{C}^n$ on $K \cup \{a\}$, then $|P \circ \Psi(a)| > \sup_K |P \circ \Psi|$. Since $P \circ \Psi$ is an entire function, we conclude that $a \notin \hat{K}$. Thus $\hat{K} \cap \Omega = K$ which implies $\hat{K} = K$. If K is polynomially convex, we apply the same argument to Φ^{-1} to prove that K' is polynomially convex.
(c) Let Ω' be Runge in $\mathbf{C}^n$. Fix a compact set $K \subset\subset \Omega$ and a holomorphic function f on Ω. Choose an open set $R \subset\subset \Omega'$ containing $\Phi(K)$ and approximate the holomorphic function $g = f \circ \Phi^{-1}$ by a polynomial P, uniformly on $\bar{R}$. Also, approximate Φ by $\Psi \in \operatorname{Aut} \mathbf{C}^n$, uniformly on K, such that $\Psi(K) \subset R$. Then $P \circ \Psi$ is an entire function that approximates f on K. Thus Ω is Runge in $\mathbf{C}^n$. To get the converse we apply the same argument to Φ^{-1}.
(d) Let $\Psi \in \operatorname{Aut} \mathbf{C}^n$. The isotopy $\Psi + (t-1)\Psi(0)$ ($t \in [0, 1]$) connects Ψ to the automorphism $\Psi_0(z) = \Psi(z) - \Psi(0)$ satisfying $\Psi_0(0) = 0$. Next, the isotopy $\Theta_t(z) = \frac{1}{t} \Psi_0(tz)$ ($0 < t \leqq 1$) connects Ψ_0 to the linear automorphism $D\Psi_0(0) \in \mathbf{GL}(n, \mathbf{C})$, the derivative of Ψ_0 at the origin. Finally, every linear

automorphism can be connected by an isotopy to the identity since $\mathbf{GL}(n, \mathbf{C})$ is connected. If we combine these three isotopies into a single isotopy and reparametrize the interval [0, 1] suitably, we obtain a smooth isotopy $\Psi_t \in \operatorname{Aut} \mathbf{C}^n$ connecting Ψ to the identity.

Sketch of proof of Theorem 1.1 We first prove the converse part of Theorem 1.1. Suppose that $\Phi: \Omega \to \mathbf{C}^n$ can be approximated by automorphisms of $\mathbf{C}^n$ on Ω. Choose a compact set $K \subset \Omega$. We can find a Runge domain D satisfying $K \subset D \subset\subset \Omega$. If we approximate Φ sufficiently close by $\Psi \in \operatorname{Aut} \mathbf{C}^n$, uniformly on a neighborhood of $\bar{D}$, then the family $t\Phi + (1-t)\Psi: \Omega \to \mathbf{C}^n$ ($t \in [0, 1]$) is an isotopy of biholomorphic maps connecting $\Psi|_D$ to $\Phi|_D$. Further, the automorphism Ψ can be connected to the identity by an isotopy of automorphisms (Proposition 1.2(d)). Combining these two isotopies we obtain an isotopy $\Phi_t: D \to \mathbf{C}^n$ connecting $\Phi|_D$ to the identity. By construction, every Φ_t can be approximated by automorphisms on D. Since D is Runge in $\mathbf{C}^n$, it follows from Proposition 1.2(c) that every image domain $\Phi_t(D)$ is Runge in $\mathbf{C}^n$ as well. This establishes the converse part of Theorem 1.1.

As we said before, we will only sketch the proof of main part and refer to [3, 4]. However, we prefer to change slightly the presentation. Roughly speaking, in [3] and [4] the dynamics takes place in the group $\operatorname{Aut} \mathbf{C}^n$, while we prefer to 'see' the dynamics in $\mathbf{C}^n$.

Let X be a holomorphic vector field on $\mathbf{C}^n$, $X(z) = (X_1(z), \ldots, X_n(z))$, where $X_j(z)$ are entire holomorphic functions. The vector field X is said to be *complete* if and only if for every $z \in \mathbf{C}^n$, the ordinary differential equation

$$\frac{dR}{dt} = X(R(t)), \quad R(0) = z, \tag{$*$}$$

can be integrated for all times t from $-\infty$ to $+\infty$. If the field X is complete then for every $z \in \mathbf{C}^n$ and $t \in \mathbf{R}$ one sets $F_t(z) = R(t)$, where R solves ($*$) with the initial condition $R(0) = z$. Clearly F_t is an automorphism of $\mathbf{C}^n$ for every $t \in \mathbf{R}$. In fact, the family $\{F_t : t \in \mathbf{R}\}$ is a one-parameter subgroup of $\operatorname{Aut} \mathbf{C}^n$ since the field X is time independent.

Andersén [3] (in the special case of divergence zero) and Andersén-Lempert [4] proved the following fundamental lemma:

1.3. Lemma. *Every holomorphic vector field on $\mathbf{C}^n$ can be approximated, uniformly on compact sets, by finite sums of complete holomorphic vector fields. In fact, every polynomial vector field (resp. with divergence zero) is a finite sum of complete ones (resp. with divergence zero).*

Proof. We only give indications to help the reader to get the proof out of [3] and [4] (where complete vector fields are not explicitly mentioned).

If p is a polynomial in one variable and $a = (a_1, \ldots, a_{n-1}, 1)$, the vector field

$$X_{a,p}(z) = (p(a \cdot z), 0, \ldots, 0, -a_1 p(a \cdot z))$$

on $\mathbf{C}^n$ is complete. It gives rise to the flow (in each level set of $a \cdot z = \sum_{j=1}^n a_j z_j$)

$$z = (z_1, \ldots, z_n) \to z + tX_{a,p}(z) .$$

This corresponds to Lemma 5.5 in [3]. Then as in Lemma 5.6 in [3] this yields the desired result for vector fields $V(z) = (V_1(z), \ldots, V_n(z))$ with divergence zero ($\sum \partial V_j/\partial z_j = 0$). (Corresponding to Lemmas 5.3 and 5.4 in [3] notice that a vector field of the type $(0, \ldots, 0, \alpha(z_2, \ldots, z_{n-1}))$ corresponds to the flow $z \to (z_1, \ldots, z_{n-1}, z_n + t\alpha(z_2, \ldots, z_{n-1}))$. The case of general vector field is reduced to the case of divergence zero vector field as in [4, Proposition 3.8]. When reading Proposition 3.8 in [4] notice that, if p is a polynomial in one variable then the vector field $(0, \ldots, 0, p(z_1)z_n)$ has divergence $p(z_1)$ and is complete, it corresponds to the flow $(z_1, \ldots, z_n) \to (z_1, \ldots, z_{n-1}, e^{tp(z_1)}z_n)$.

By arguments standard in control theory one gets immediately

1.4. Lemma. *Let X be a holomorphic vector field (resp. divergence zero vector field) defined on all of $\mathbf{C}^n$. Let Ω be an open subset of $\mathbf{C}^n$ and let $t_0 > 0$. Assume that the differential equation $dR/dt = X(R(t))$ can be integrated for $0 \leqq t \leqq t_0$ with arbitrary initial condition $R(0) = z \in \Omega$. Set $F_t(z) = R(t)$ as above. Then F_t $(0 \leqq t \leqq t_0)$ is a biholomorphic map from Ω into $\mathbf{C}^n$ that can be approximated, uniformly on compact sets in Ω, by automorphisms of $\mathbf{C}^n$ (resp. by automorphisms with Jacobian one).*

Proof of Lemma 1.4 By Lemma 1.3 it suffices to consider the case when X is a finite sum of complete vector fields $X = X_1 + \cdots + X_k$. Then instead of integrating the vector field X from time 0 to time t one splits the time between the vector fields X_j. More precisely, choose a large integer N, flow along X_1 for time t/Nk, then flow along X_2 for time t/Nk, etc. After flowing along X_k in k-th step we return again to X_1, continue with X_2, etc., until we finally stop after Nk steps. This is a 'concatenation' of vector fields $X_1, \ldots, X_k$; clearly it is an automorphism of $\mathbf{C}^n$. As $N \to \infty$, the process converges to the flow $F_t(z)$ of X, uniformly on compact subsets of Ω.

We remark that Lemma 1.4 is a version of Proposition 2.1 in [6], with the additional uniformity with respect to initial data in compact sets. The details, to check uniformity, are left to the reader.

Theorem 1.1 is then proved in the following way. At each time $t_0 \in [0, 1]$ consider the vector field X_{t_0} on the domain $\Phi_{t_0}(\Omega) = \Omega_{t_0}$ obtained by differentiating Φ_t with respect to t at $t = t_0$:

$$X_{t_0}(w) = \frac{d}{dt} \Phi_t(\Phi_{t_0}^{-1}(w))|_{t=t_0}, \quad w \in \Omega_{t_0}.$$

Let $t_0 \in [0, 1]$. The map $F_{t_0} = \Phi_{t_0} \circ \Phi_0^{-1}$ is obtained by integrating the time dependent vector field X_t from time 0 to t_0. It can be approximated on compact sets in Ω by integrating the time independent vector field $X_{kt_0/N}$ from time kt_0/N to $(k+1)t_0/N$ $(0 \leqq k \leqq N-1, N$ large). Each of those vector fields can be approximated by holomorphic vector fields on $\mathbf{C}^n$ since Ω_t is Runge for every t. Finally one applies Lemma 1.4 and the proof is complete.

Remark. As seen from the proof, we can replace the hypothesis in Theorem 1.1 that each Ω_t is Runge by the hypothesis:

For each $t_0 \in [0, 1]$, the vector field

$$X_{t_0}(w) = \frac{d}{dt} \Phi_t(\Phi_{t_0}^{-1}(w))|_{t=t_0}, \quad w \in \Omega_{t_0},$$

can be approximated, uniformly on compact sets in Ω_{t_0}, by holomorphic vector fields defined on $\mathbf{C}^n$.

Also, even if we do not assume that Φ_0 is approximable by global biholomorphisms, we get that at any rate the composition $\Phi_t \circ \Phi_0^{-1}$ is approximable.

2 Approximation near polynomially convex sets

In this section we apply Theorem 1.1 to obtain results on approximation of biholomorphic maps in a neighborhood of polynomially convex sets. The important point is that every polynomially convex set in $\mathbf{C}^n$ has a basis of Stein neighborhoods that are Runge in $\mathbf{C}^n$ [14]. This observation allows us to prove

2.1. Theorem. *Let Ω be an open subset of $\mathbf{C}^n$ and let $\Phi_t: \Omega \to \mathbf{C}^n$, $t \in [0, 1]$, be an isotopy of biholomorphic maps such that Φ_0 is the identity map. Then for every compact polynomially convex subset $K \subset \Omega$ the following are equivalent:*
(a) *The set $K_t = \Phi_t(K)$ is polynomially convex for every $t \in [0, 1]$.*
(b) *There is a neighborhood U of K such that for every $t \in [0, 1]$, Φ_t can be approximated by automorphisms of $\mathbf{C}^n$, uniformly on U.*

The implication (b) $\Rightarrow$ (a) follows from Proposition 1.2(b) in Sect. 1. The main implication (a) $\Rightarrow$ (b) follows immediately from Theorem 1.1 and the following

2.2. Lemma. *Let Ω be an open subset of $\mathbf{C}^n$ and $\Phi_t: \Omega \to \mathbf{C}^n$ an isotopy of biholomorphic maps. If $K \subset \Omega$ is a compact polynomially convex set such that for each $t \in [0, 1]$ the set $K_t = \Phi_t(K)$ is polynomially convex, then there exists a basis of Stein neighborhoods U of K such that the domain $\Phi_t(U)$ is Runge in $\mathbf{C}^n$ for each $t \in [0, 1]$.*

Proof of the lemma. For every polynomially convex set $K \subset \mathbf{C}^n$ there exists a smooth plurisubharmonic exhaustion function $\rho \geqq 0$ on $\mathbf{C}^n$ that is strongly plurisubharmonic on $\mathbf{C}^n \setminus K$ and vanishes precisely on K. (This is a small extension of Theorem 2.6.11 in [14, p. 48].) Each sublevel set $U_\varepsilon = \{z \in \mathbf{C}^n : \rho(z) < \varepsilon\}$ is a Stein domain that is Runge in $\mathbf{C}^n$ (combine Theorem 2.7.3 and Lemma 4.3.1 in [14]).

Assuming that K_t is polynomially convex we claim that for every sufficiently small $\varepsilon > 0$ the domain $\Phi_t(U_\varepsilon)$ is Runge in $\mathbf{C}^n$.

The function $\rho_t = \rho \circ \Phi_t^{-1} \geqq 0$ is plurisubharmonic on $\Omega_t = \Phi_t(\Omega)$ and it vanishes precisely on K_t. Fix a $t \in [0, 1]$ and choose on open relatively compact neighborhood $V \subset\subset \Omega_t$ of K_t. Since K_t is polynomially convex, we can find a smooth plurisubharmonic exhaustion function $\tilde{\tau} \geqq 0$ on $\mathbf{C}^n$ that is positive and strongly plurisubharmonic on $\mathbf{C}^n \setminus V$ and vanishes on a smaller neighborhood $V_1 \subset\subset V$ of K_t (see [14, Theorem 2.6.1]). Let χ be a smooth cut-off function that equals one on V and is compactly supported in Ω_t. If we choose $\delta > 0$ sufficiently small then

$$\tau_t(z) = \tilde{\tau}(z) + \delta\chi(z)\rho_t(z)\,, \quad z \in \mathbf{C}^n\,,$$

is a strongly plurisubharmonic exhaustion function on $\mathbf{C}^n$ that vanishes precisely on K_t and equals $\delta\rho_t(z)$ for $z \in V_1$. Every sublevel set $\{z \in \mathbf{C}^n : \tau_t(z) < \varepsilon\}$ is Runge in $\mathbf{C}^n$.

If $\varepsilon_0 > 0$ is sufficiently small (depending on t) then $\Phi_t(U_\varepsilon) \subset V_1$ and

$$\Phi_t(U_\varepsilon) = \{z \in V_1 : \rho_t(z) < \varepsilon\} = \{z \in \mathbf{C}^n : \tau_t(z) < \delta\varepsilon\}\,, \quad \varepsilon < \varepsilon_0\,,$$

whence $\Phi_t(U_\varepsilon)$ is Runge in $\mathbf{C}^n$. It follows from the construction that the same ε_0 is good for an open set of $t \in [0, 1]$. By compactness of $[0, 1]$ we can choose ε_0 to be independent of $t \in [0, 1]$, and the lemma is proved.

We now apply Theorem 2.1 to prove the following result that generalizes Theorem 8.1 of Rosay and Rudin [17]. We say that a set $K \subset \mathbf{C}^n$ is starshaped if there is a point $a \in K$ such that $a + t(z - a) \in K$ for every $z \in K$ and $t \in [0, 1]$.

2.3. Theorem. *Let $K_1, K_2, \ldots, K_s$ be pairwise disjoint, compact, starshaped subsets of $\mathbf{C}^n$ ($n \geqq 2$), and let $\Phi^j \in \operatorname{Aut} \mathbf{C}^n$ ($j = 1, \ldots, s$) be automorphisms of $\mathbf{C}^n$ such that the sets $K'_j = \Phi^j(K_j)$ are also pairwise disjoint. Assume that the unions $K = \bigcup_{j=1}^s K_j$ and $K' = \bigcup_{j=1}^s K'_j$ are polynomially convex. Then there exist neighborhoods U_j of K_j ($1 \leqq j \leqq s$) and a sequence of automorphisms $\Psi_l \in \operatorname{Aut} \mathbf{C}^n$, $l \in \mathbf{Z}_+$, such that*

$$\lim_{l \to \infty} \Psi_l(z) = \Phi^j(z)\,, \quad z \in U_j\,, \quad 1 \leqq j \leqq s\,,$$

the convergence being uniform on every U_j.

Note that we are approximating a set of *independent* motions of the sets K_j by a single automorphism of $\mathbf{C}^n$, provided that both unions are pairwise disjoint and polynomially convex. This answers a question raised by Rosay and Rudin in [17].

The union of two disjoint closed convex sets in $\mathbf{C}^n$ is clearly polynomially convex hence Theorem 2.3 applies. The union of three convex sets is in general not polynomially convex even in $\mathbf{C}^2$, see Kallin [9, 21] and Rosay [15]. However, the union of three disjoint closed balls in $\mathbf{C}^n$ is always polynomially convex (Kallin [9, 21]), and the same is true for three disjoint closed polydiscs in $\mathbf{C}^2$ with the sides parallel to the coordinate axes (Rosay [15]). We thus have

2.4. Corollary. *Let B_1, B_2, B_3 and B'_1, B'_2, B'_3 be two sets of pairwise disjoint closed balls in $\mathbf{C}^n$. Then there exists a sequence of automorphisms $\Psi_l \in \operatorname{Aut} \mathbf{C}^n$, converging uniformly on a neighborhood of each ball B_j to a linear map that takes B_j onto B'_j. The same is true for sets of three disjoint polydiscs in $\mathbf{C}^2$, with sides parallel to the coordinate axes.*

Proof of Theorem 2.3 In view of Theorem 2.1 it suffices to find for each $j \in \{1, 2, \ldots, s\}$ an isotopy of biholomorphic maps Φ^j_t, $t \in [0, 1]$, defined on a neighborhod U_j of K_j in $\mathbf{C}^n$, satisfying

(i) Φ^j_0 is the identity,
(ii) $\Phi^j_1 = \Phi^j$, and
(iii) the sets $K_{j,t} = \Phi^j_t(K_j)$ ($1 \leqq j \leqq s$) are pairwise disjoint and their union $\bigcup_{j=1}^s K_{j,t}$ is polynomially convex for each $t \in [0, 1]$.

Let U_j be a bounded neighborhood of K_j, starshaped with respect to a_j, such that $\bar{U}_j$ are pairwise disjoint for $j = 1, \ldots, s$, and also their images $\Phi^j(U_j) = U'_j$ are pairwise disjoint.

Every automorphism $\Phi^j \in \operatorname{Aut} \mathbf{C}^n$ can be smoothly deformed to the identity through an isotopy of automorphisms of $\mathbf{C}^n$ (Proposition 1.2). Choose such an isotopy Ψ^j_t, $t \in [0, 1]$, with $\Psi^j_0 = \mathrm{Id}$. Modifying each Ψ^j_t by a family of translations depending on t we may assume that the points $b_{j,t} = \Psi^j_t(a_j) \in \mathbf{C}^n$ for $j = 1, 2, \ldots, s$ are distinct for every $t \in [0, 1]$. If $\delta > 0$ is sufficiently small then the union of the closed balls $\bigcup_{j=1}^s \bar{\mathbf{B}}^n(b_{j,t}, \delta)$ is polynomially convex for every $t \in [0, 1]$. Fix such a δ.

Choose $\eta > 0$ such that Ψ_t^j maps the ball $\mathbf{B}^n(a_j, \eta)$ into the ball $\mathbf{B}^n(b_{j,t}, \delta)$ for every $t \in [0, 1]$. Let $R > 0$ be sufficiently large such that for every $1 \leqq j \leqq s$ the set U_j is contained in the ball $\mathbf{B}^n(a_j, R)$.

For each $t > 0$ and $j = 1, \ldots, s$ we set $\Theta_t^j(z) = a_j + t(z - a_j)$. The required isotopy Ψ_t^j is obtained by first squeezing U_j into $\mathbf{B}^n(a_j, \eta)$ using contractions Θ_t^j, then following the isotopy Ψ_t^j restricted to $\mathbf{B}^n(a_j, \eta)$, and finally expanding the image of U_j onto U_j'.

To give explicit formulas we determine $c > 0$ by $1 - c/3 = \eta/R$, hence $\Theta^j(U_j) \subset \mathbf{B}^n(a_j, \eta)$, and we define the isotopy Φ_t^j on U_j as follows:

$$\Phi_t^j = \begin{cases} \Theta_{1-ct}^j , & \text{if } 0 \leqq t \leqq 1/3; \\ \Psi_{3t-1}^j \circ \Theta_{\eta/R}^j , & \text{if } 1/3 < t \leqq 2/3; \\ \Phi^j \circ \Theta_{1+c(t-1)}^j , & \text{if } 2/3 < t \leqq 1 . \end{cases}$$

It follows from the construction that for each $t \in [0, 1]$ the set $\bigcup_{j=1}^s \Phi_t^j(K_j)$ is polynomially convex. For $0 \leqq t \leqq 1/3$ this holds since $\Phi_k^j(K_j) \subset K_j\,(1 \leqq j \leqq s)$, and similarly for $2/3 \leqq t \leqq 1$ we have $\Phi_t^j(K_j) \subset K_j'$. Here we are using the condition that the sets K_j are starshaped. On the middle interval $1/3 \leqq t \leqq 2/3$ we have $\Phi_t^j(K_j) \subset \bar{\mathbf{B}}^n(b_{j,t}, \delta)$, and the union of these balls is polynomially convex.

The constructed family of automorphisms Φ_t^j is only piecewise smooth in the t variable, but it can be made smooth in t by a reparametrization of the interval $[0, 1]$. We leave out the details. This completes the proof of Theorem 2.3.

3 Isotopies of submanifolds and $\mathbf{C}^n$-equivalence

We begin by the following definition:

Definition 2. (a) Two real-analytic submanifolds M_0 and M_1 in $\mathbf{C}^n$ are *$\mathbf{C}^n$-equivalent* if there exists a sequence of automorphisms $\Psi_j \in \operatorname{Aut} \mathbf{C}^n$ $(j \in \mathbf{Z}_+)$ that converges, uniformly on a neighborhood U of M_0, to a biholomorphic map $\Psi : U \to \Psi(U) \subset \mathbf{C}^n$ satisfying $\Psi(M_0) = M_1$.
(b) Two real-analytic embeddings $f_0, f_1 : M \to \mathbf{C}^n$ are *$\mathbf{C}^n$-equivalent* if there exists a sequence of automorphisms $\Psi_j \in \operatorname{Aut} \mathbf{C}^n$ $(j \in \mathbf{Z}_+)$ that converges, uniformly on a neighborhood U of $f_0(M)$, to a biholomorphic map $\Psi : U \to C^n$ satisfying $\Psi \circ f_0 = f_1$.

For a related notion for smooth manifolds see Rosay [16].

In order that M_0 and M_1 be $\mathbf{C}^n$-equivalence there must of course exist a real-analytic diffeomorphism $\phi : M_0 \to M_1$. The converse is not true in general. There exist local and global obstructions to extending ϕ to a biholomorphic map Φ in a neighborhood of M_0 in $\mathbf{C}^n$. Even if ϕ can be so extended, one may not be able to approximate Φ by automorphisms in any neighborhood of M_0. (See examples in Sect. 7.)

The problem is substantially simpler if we restrict our considerations to totally real submanifolds of $\mathbf{C}^n$. Recall that a submanifold $M_0 \subset \mathbf{C}^n$ is totally real if the real tangent space $\mathrm{T}_p M_0$ at each $p \in M_0$ contains no nontrivial complex subspace of $T_p\mathbf{C}^n$. Every continuous (or smooth) function on M_0 can be approximated (in appropriate norm) by functions holomorphic in a neighborhood of M_0. It is easily seen that a real-analytic diffeomorphism $\phi: M_0 \to M_1$ can be extended to a biholomorphic map Φ in some neighborhood of M_0 if and only if the complex normal bundles to M_0 and M_1 in $\mathbf{C}^n$ are isomorphic (see below for precise definitions). Thus the obstruction to the extension of ϕ in a neighborhood of M_0 is global along M_0.

In order to apply Theorem 1.1 the manifolds M_0 and M_1 must also have small Runge neighborhoods. When the manifolds are totally real this will hold if and only if they are polynomially convex in $\mathbf{C}^n$. If one of the manifolds, say $M_0 \subset \mathbf{C}^n$, is totally real and polynomially convex, and if $M_1 \subset \mathbf{C}^n$ is $\mathbf{C}^n$-equivalent to M_0, then M_1 is also totally real and polynomially convex (Proposition 1.2(b)).

Let M be a smooth manifold. Recall that an *isotopy* between smooth embeddings $f_0, f_1: M \to \mathbf{C}^n$ is a smooth mappings $\tau: [0,1] \times M \to \mathbf{C}^n$ such that for each $t \in [0,1]$ the map $\tau_t = \tau(t,\cdot): M \to \mathbf{C}^n$ is an embedding, $\tau_0 = f_0$, and $\tau_1 = f_1$. In this case the embeddings f_0 and f_1 are said to be *isotopic* in $\mathbf{C}^n$. The isotopy is real-analytic if M and the mapping τ are real-analytic.

Sometimes we are only interested in the submanifolds $M_t = \tau_t(M) \subset \mathbf{C}^n$ and not in the mapping τ. We say that the submanifolds $M_0, M_1 \subset \mathbf{C}^n$ are isotopic in $\mathbf{C}^n$ if there exists an isotopy $\tau: [0,1] \times M \to \mathbf{C}^n$ as above such that $\tau_0(M) = M_0$ and $\tau_1(M) = M_1$. With some obvious abuse of language we will say that M_0 and M_1 are isotopic in $\mathbf{C}^n$ through the family of submanifolds $M_t = \tau_t(M)$.

The main result of this section is

3.1. Theorem. *Let $M_0, M_1 \subset \mathbf{C}^n$ be compact real-analytic submanifolds (with or without boundary), and assume that M_0 is totally real and polynomially convex. Then M_0 and M_1 are $\mathbf{C}^n$-equivalent if and only if they are isotopic in $\mathbf{C}^n$ through a family M_t ($t \in [0,1]$) of totally real, polynomially convex submanifolds of $\mathbf{C}^n$.*

Similarly, if M is a compact real-analytic manifold and $f_0: M \to \mathbf{C}^n$ is a real-analytic embedding such that $f_0(M)$ is totally real and polynomially convex in $\mathbf{C}^n$, then f_0 is $\mathbf{C}^n$-equivalent to another real-analytic embedding $f_1: M \to \mathbf{C}^n$ if and only there is an isotopy of embeddings $\tau: [0,1] \times M \to \mathbf{C}^n$ such that $\tau_t = f_t$ for $t = 0, 1$, and $\tau_t(M) \subset \mathbf{C}^n$ is totally real and polynomially convex for every $t \in [0,1]$.

Recall that a compact polynomially convex subset K of $\mathbf{C}^n$ satisfies $\mathbf{H}^r(K, \mathbf{C}) = 0$ when $r \geqq n$ [134, p. 59]. Thus, if $M \subset \mathbf{C}^n$ is a closed, orientable, n-dimensional submanifold of $\mathbf{C}^n$, then M is not polynomially convex. Moreover, an n-dimensional totally real manifold $M \subset \mathbf{C}^n$ has vanishing Euler characteristic [22]. Thus the conditions in Theorem 3.1 require that $\dim_R M < n$, at least if we are considering closed orientable manifolds.

Theorem 3.1 can be compared with the standard result in real differential topology (cf. [13]) which asserts that, under certain obvious restrictions, every isotopy M_t of manifolds in $\mathbf{R}^n$ can be realized by a global *diffeotopy*, i.e., there exists an isotopy of global diffeomorphisms $\Psi_t: \mathbf{R}^n \to \mathbf{R}^n$, $t \in [0,1]$, satisfying $\Psi_t(M_0) = M_t$. Of course in the holomorphic case the best we can expect is to approximately realize the isotopy by automorphisms of $\mathbf{C}^n$ because of the identity principle. The proof of Theorem 3.1 is much more difficult than the proof of the corresponding result on extending isotopies to diffeotopies.

Proof of Theorem 3.1 It suffices to prove the second statement concerning the equivalence of embeddings, since the first part is a special case of this.

We first prove the 'only if' part of Theorem 3.1. Suppose that we have a neighborhood U of $M_0 = f_0(M)$ in $\mathbf{C}^n$ and a biholomorphic map $\Psi: U \to \mathbf{C}^n$ that is a limit of automorphisms on U and satisfies $\Psi \circ f_0 = f_1$. We may choose U to be Runge in $\mathbf{C}^n$ since M_0 is polynomially convex. The converse part of Theorem 1.1 gives a smaller Runge domain $\Omega \subset\subset U$ containing M_0 and an isotopy of biholomorphic maps $\Phi_t: \Omega \to \mathbf{C}^n$ ($t \in [0, 1]$) such that Φ_0 is the identity, $\Phi_1 = \Psi|_\Omega$, and each Φ_t can be approximated by automorphisms of $\mathbf{C}^n$. Then the family of embeddings $f_t = \Phi_t \circ f_0: M \to \mathbf{C}^n$ ($t \in [0, 1]$) is an isotopy of embeddings with the required properties. This proves the converse part of Theorem 3.1.

To prove the main implication in Theorem 3.1 let $\tau: [0, 1] \times M \to \mathbf{C}^n$ be a smooth isotopy as in the statement of the theorem. First we approximate τ by a real-analytic isotopy τ' satisfying $\tau'_t = \tau_t = f_t$ for $t = 0, 1$. To this end we choose a real-analytic function $\chi: [0, 1] \to [0, 1]$ such that $\chi(0) = 1$, $\chi(1) = 0$, and $\chi(t)$ is very close to 0 for $\delta \leqq t \leqq 1$ for some small $\delta > 0$. If $\tilde{\tau}: [0, 1] \times M \to \mathbf{R}$ is any real-analytic map approximating τ, then the mapping

$$\tau'(t, x) = \chi(t) f_0(x) + \chi(1 - t) f_1(x) + (1 - \chi(t) - \chi(1 - t))\, \tilde{\tau}(t, x)$$

approximates τ and satisfies the boundary conditions. (Alternatively, one may appeal to Cartan's Theorem A for Stein manifolds [14, p. 190].) If the approximation of τ by τ' is sufficiently close in $\mathscr{C}^2$ norm then $\tau'_t(M)$ is still totally real and polynomially convex for every $t \in [0, 1]$. Thus we may assume from the outset that the isotopy τ is real-analytic. The proof now follows by combining Lemma 2.2 with the following

3.2. Proposition. *Let M be a compact real-analytic manifold and let $\tau: [0, 1] \times M \to \mathbf{C}^n$ be a real-analytic isotopy of embeddings such that for every $t \in [0, 1]$, the submanifold $M_t = \tau_t(M)$ is totally real and polynomially convex. Then there exists an open set $U \subset \mathbf{C}^n$ containing M_0 and an isotopy of biholomorphic mappings $\Phi_t: U \to \mathbf{C}^n$ such that Φ_0 is the identity map and $\Phi_t \circ \tau_0 = \tau_t$ for all $t \in [0, 1]$.*

Applying Theorem 1.1 (in a smaller neighborhood of $f_0(M)$) we conclude that Φ_1 is the limit of automorphisms near $f_0(M)$, hence the embeddings f_0 and f_1 are $\mathbf{C}^n$-equivalent. This proves Theorem 3.1, provided that Proposition 3.2 holds.

Proof of Proposition 3.2 Let $\tau: [0, 1] \times M \to \mathbf{C}^n$ be the given real-analytic isotopy. The map $\phi_t = \tau_t \circ \tau_0^{-1}: M_0 \to M_t$ is a real-analytic diffeomorphism depending analytically on $t \in [0, 1]$. Our goal is to extend ϕ_t to an isotopy Φ_t of biholomorphic mappings, defined on a tubular neighborhood U of M_0 in $\mathbf{C}^n$, satisfying $\Phi_t|_{M_0} = \phi_t$, with Φ_0 the identity map.

The result is immediate if $m = \dim_{\mathbf{R}} M$ equals n; in this case Φ_t is obtained by simply complexifying ϕ_t. Since ϕ_t is real-analytic in t, the neighborhood of M_0 in which the complexification Φ_t is defined can be chosen independently of t.

From now on we shall assume that $m < n$ and set $d = n - m$. Fix a $t \in [0, 1]$ and a point $z \in M_t$. The tangent space $\mathrm{T}_z\mathbf{C}^n$ can be decomposed as a direct sum

$$\mathrm{T}_z\mathbf{C}^n = \mathrm{T}_z M_t \oplus J(\mathrm{T}_z M_t) \oplus (N_t)_z\,, \tag{1}$$

where J is the almost complex structure operator on $\mathrm{T}_z\mathbf{C}^n$ and $(N_t)_z$ is the (hermitian) orthogonal complement of $\mathrm{T}^{\mathbf{C}}_z M_t = \mathrm{T}_z M_t \oplus J(\mathrm{T}_z M_t)$ in $\mathrm{T}_z\mathbf{C}^n$. Since M_t is totally real in $\mathbf{C}^n$, $(N_t)_z$ is a complex subspace of dimension $d = n - m$.

Denote by $\pi_t : N_t \to M_t$ the complex vector subbundle of $T\mathbf{C}^n|_{M_t}$ or rank $d = n - m$ with fibers $(N_t)_z$. In general this bundle may be nontrivial which prevents us from simply reducing the proof to the case when $\dim M_t = n$. Let $\mathcal{N} \to [0, 1] \times M$ be the complex vector bundle of rank d whose fiber at (t, x) equals $(N_t)_{\tau(t,x)}$. Thus, the part $\mathcal{N}_t$ of $\mathcal{N}$ lying over $\{t\} \times M$ is just the pull-back of $N_t \to M_t$ by the map τ_t.

Every vector bundle $\pi : \mathcal{N} \to [0, 1] \times M$ over the product manifold $[0, 1] \times M$ is isomorphic (over $[0, 1] \times M$) to the product bundle $[0, 1] \times \mathcal{N}_0$, where $\mathcal{N}_0 = \pi^{-1}(\{0\} \times M)$. For real vector bundles a good reference is [13, p. 90]; the proof can can easily be adapted to complex vector bundles, and is totally elementary.

Since the complex vector bundle $\mathcal{N}$ is real-analytic in the base variable, there exists (by approximation) a real-analytic vector bundle isomorphism $\Psi : [0, 1] \times \mathcal{N}_0 \to \mathcal{N}$ over $[0, 1] \times M$ that is complex linear on every fiber, and $\Psi(0, \cdot)$ is the identity on $\mathcal{N}_0$. Then $F_t = \Psi(t, \cdot) : \mathcal{N}_0 \to \mathcal{N}_t$, $t \in [0, 1]$, is a family of complex vector bundle isomorphisms, depending analytically on t, such that F_0 is the identity on $\mathcal{N}_0$.

We now embed a neighborhood V_t of the zero section in $\mathcal{N}_t$ into $\mathbf{C}^n$ by the mapping $S_t : \mathcal{N}_t \to \mathbf{C}^n$, $S_t(x, v) = \tau_t(x) + v$. The zero section is mapped diffeomorphically onto $M_t = \tau_t(M)$, and S_t is non-singular at every point of the zero section. Hence S_t is an embedding near the zero section. The image $\Sigma_t = S_t(V_t)$ is a generic, real-analytic, Cauchy-Riemann submanifold of $\mathbf{C}^n$, of CR dimension d, containing M_t. (In fact, Σ_t is foliated by open sets of complex d-dimensional planes, the images of the fibers of $\mathcal{N}_t$.) Since everything is analytic in the t variable, we can choose the neighborhoods V_t such that $F_t(V_0) = V_t$ for every $t \in [0, 1]$. The mapping

$$\Theta_t = S_t \circ F_t \circ S_0^{-1} : \Sigma_0 \to \Sigma_t$$

is a real-analytic CR diffeomorphism of Σ_0 onto Σ_t, depending analytically on $t \in [0, 1]$. By construction, Θ_t extends the diffeomorphism $\phi_t : M_0 \to M_t$, and Θ_0 is the identity.

It remains to extend the CR map Θ_t from Σ_0 to a holomorphic map $\Phi_t : U \to U_t$ in a neighborhood U of M_0, mapping U onto a neighborhood U_t of M_t. By analyticity of Θ_t in the t variable we can choose U independently of $t \in [0, 1]$. Φ_0 is the identity since it extends the identity on Σ_0. Shrinking U if necessary we can insure that every Φ_t is biholomorphic on U. Since every M_t is polynomially convex, we may shrink U further to insure that for every $t \in [0, 1]$ the set $U_t = \Phi_t(U)$ is Runge in $\mathbf{C}^n$ (Lemma 2.2). This completes the proof of Proposition 3.2.

Remark. Of course it is possible to construct the isomorphisms F_t and Θ_t directly, without appealing to splitting of the bundle $\mathcal{N} \to [0, 1] \times M$. For instance, for small values of $t > 0$ we can take Θ_t to be the orthogonal projection of $(N_0)_z$ ($z \in M_0$) onto $(N_t)_{\phi_t(z)}$ in the direction of the tangent space $T_z^{\mathbf{C}} M_0$. This works for $0 \leqq t \leqq t_1$ for some $t_1 > 0$ that only depends on the data. For $t > t_1$ but close to t_1 we can project in a similar way the fibers of N_{t_1} onto the corresponding fibers of N_t, and then compose the new projection with the one from the first step to get the map Θ_t. The required isomorphism Θ_t can thus be constructed in a finite number of steps for every $t \in [0, 1]$. Finally we may approximate this family Θ_t with another family that depends analytically on t.

Another, more canonical approach to construction of Θ_t is by choosing a connection and performing the parallel transport.

4 $\mathbf{C}^n$-equivalence of discs and curves

In this section we apply Theorem 3.1 to several classes of embedded real-analytic sub-manifolds of $\mathbf{C}^n$ ($n \geqq 2$): totally real polynomially convex discs, arcs, embedded analytic discs, and closed polynomially convex curves. In the next section we deal with real-analytic surfaces in $\mathbf{C}^n$ for $n \geqq 3$. In the first three cases the $\mathbf{C}^n$-equivalence already follows from the result of Andersén and Lempert [4] on approximation of biholomorphic mappings from convex domains onto Runge domains.

4.1. Corollary. (*All manifolds are assumed to be embedded real-analytic in* $\mathbf{C}^n$.)
(a) *Any two totally real, polynomially convex, k-dimensional discs in* $\mathbf{C}^n$ ($k \leqq n$) *are* $\mathbf{C}^n$*-equivalent* (*in the sense of Definition* 2).
(b) *Any two arcs in* $\mathbf{C}^n$ *are* $\mathbf{C}^n$*-equivalent.*
(c) *Any two embedded analytic discs in* $\mathbf{C}^n$ *are* $\mathbf{C}^n$*-equivalent.*
Moreover, if $M \subset \mathbf{C}^n$ *is as in* (b) *or* (c), *then every biholomorphic map* $\Phi: D \to \mathbf{C}^n$ *defined in a neighborhood D of M in* $\mathbf{C}^n$ *can be approximated, uniformly on a smaller neighborhood of M, by automorphisms of* $\mathbf{C}^n$. *If M is a k-disc as in* (a), *then* Φ *can be so approximated near M if and only if the image* $\Phi(M)$ *is also polynomially convex in* $\mathbf{C}^n$.

Here, a k-disc in $\mathbf{C}^n$ is the image of an embedding of the standard closed ball D^k in $\mathbf{R}^k$. Similarly, an analytic disc in $\mathbf{C}^n$ is the image of a holomorphic embedding of the closed unit disc $\bar{\Delta} \subset \mathbf{C}$ into $\mathbf{C}^n$ (the embedding must be analytic in some neighborhood of $\bar{\Delta}$). Every analytic disc in $\mathbf{C}^n$ is polynomially convex according to Wermer [23]. Similarly, every smooth arc in $\mathbf{C}^n$ is polynomially convex, so (b) follows from (a).

Corollary 4.1 means, intuitively speaking, that 'there is only one arc (analytic disc, totally real polynomially convex disc of certain dimension, etc.) in $\mathbf{C}^n$'.

Remark. There is an interesting algebraic result, due to Abhyankar and Moh [1], to the effect that for every proper polynomial embedding $f: \mathbf{C} \to \mathbf{C}^2$ the image $f(\mathbf{C})$ can be mapped onto $\mathbf{C} \times \{0\}$ by a polynomial automorphism of $\mathbf{C}^2$. It would be of interest to know whether the analogous result holds for proper holomorphic embedding of $\mathbf{C}$ in $\mathbf{C}^2$, i.e., can we always straighten the image by a holomorphic automorphism of $\mathbf{C}^2$?

Proof of Corollary 4.1 (a) Let E be an arbitrary embedded, real-analytic, totally real, polynomially convex k-disc in $\mathbf{C}^n$. Denote by E^k the standard linear k-disc in $\mathbf{R}^k \times \{0\} \subset \mathbf{C}^n$, and choose a real-analytic diffeomorphisms $\psi: E^k \to E$. We can extend ψ to a biholomorphic map Ψ from a small convex neighborhood U of E^k in $\mathbf{C}^n$ onto a neighborhood $V = \Psi(U)$ of E. If U is chosen sufficiently small then V is Runge according to Lemma 2.2.

Now we may apply the result of Andersén and Lempert [4] (or Theorem 1.1) to conclude that Ψ is the limit of a sequence of automorphisms of $\mathbf{C}^n$. This means that E and E^k are $\mathbf{C}^n$-equivalent.

Fix this Ψ. Then every other biholomorpohic map Φ defined near E can be written as a composition $\Phi = \Phi' \circ \Psi^{-1}$, where Φ' is a biholomorphic map defined in a neighborhood of E^k and $\Phi'(E^k) = \Phi(E)$. Thus, if $\Phi(E)$ is polynomially convex, then Φ' and hence Φ can be approximated by automorphisms of $\mathbf{C}^n$. This proves part (a).

Part (b) is a special case of (a). Part (c) is proved in the same way as (a) except that one uses the standard analytic disc $A_0 = \bar{\Delta} \times \{0\}^{n-1} \subset \mathbf{C}^n$ instead of E^k.

Remark. There exist embedded totally real 2-discs in $\mathbf{C}^2$ that are not polynomially convex (Wermer). A simple example of this kind is the disc

$$M = \{(z, w) : |z| \leqq 1, w = \bar{z}(1 - |z|^2) e^{i|z|^2}\} .$$

Its boundary is the circle $\{(z, 0) : |z| = 1\}$ bounding the analytic disc $\{(z, 0) : |z| \leqq 1\}$ which is therefore contained in the polynomial hull of M. It follows that no biholomorphic map taking M onto the disc $D^2 \subset \mathbf{R}^2 \subset \mathbf{C}^2$ can be approximated by automorphisms of $\mathbf{C}^2$.

Next we consider closed real-analytic curves $\Gamma \subset \mathbf{C}^n$. According to Wermer [23, 24, p. 77] (see also Stolzenberg [20]) such a curve is polynomially convex if and only if it does not bound a one dimensional complex variety. The same is true for simple closed rectifiable curves [2], but we shall not need this fact. The set of polynomially convex curves is open and dense in the $\mathscr{C}^\infty$ topology on the space of all curves in $\mathbf{C}^n$.

4.2. Theorem. *Let T be the circle. If $f_0, f_1 : T \to \mathbf{C}^n$ ($n \geqq 2$) are real-analytic embeddings such that both curves $f_0(T)$ and $f_1(T)$ are polynomially convex, then f_0 and f_1 are $\mathbf{C}^n$-equivalent. Moreover, given an embedded, real-analytic, polynomially convex curve $\Gamma \subset \mathbf{C}^n$ and a biholomorphic map $\Phi : D \to \mathbf{C}^n$ defined on a neighborhood of Γ, Φ can be approximated by automorphisms of $\mathbf{C}^n$ in some neighborhood of Γ if and only if $\Phi(\Gamma)$ is also polynomially convex and the winding number of the Jacobian $J(\Phi) = \det D\Phi$ along Γ equals zero.*

In particular, every two simple closed real-analytic curves in $\mathbf{C}^n$ ($n \geqq 2$) that are polynomially convex are $\mathbf{C}^n$-equivalent. Unlike the previous results, Theorem 4.2 does not follow immediately from the Andersén-Lempert theorem since closed curves do not have small starshaped neighborhoods. One must use Theorem 1.1 instead.

The first assertion of Theorem 4.2 follows from Theorem 3.1 and Lemma 4.4 below. The second assertion will be proved in section 6 below (after Theorem 6.2).

Theorem 4.2 implies the following result that may be of independent interest:

4.3. Corollary. *Every simple, closed, real-analytic curve $\Gamma \subset \mathbf{C}^n$ ($n \geqq 2$) that is polynomially convex bounds a smooth, totally real, polynomially convex disc.*

The result is most interesting in $\mathbf{C}^2$ since in $\mathbf{C}^n$ for $n \geqq 3$ a generic embedded disc is totally real and polynomially convex (see Sect. 5). Most likely the same result holds also for smooth polynomially convex curves.

Proof of Corollary 4.3 The curve

$$\Gamma_0 = \{(e^{i\theta}, e^{-i\theta}, 0, \ldots, 0) \in \mathbf{C}^n : \theta \in \mathbf{R}\}$$

bounds the totally real, polynomially convex disc

$$M_0 = \{(\zeta, \bar{\zeta}, 0, \ldots, 0) \in \mathbf{C}^n : |\zeta| \leqq 1\} .$$

Every curve that is sufficiently close in the $\mathscr{C}^\infty$ sense to Γ_0 also bounds a totally real, polynomially convex disc that can be obtained by a small $\mathscr{C}^\infty$ perturbation of

M_0. According to Theorem 4.2 there exist automorphisms $\Psi \in \mathrm{Aut}\,\mathbf{C}^n$ such that the curve $\Gamma' = \Psi(\Gamma)$ is arbitrary close to Γ_0 in the $\mathscr{C}^\infty$ sense. If M' is a totally real, polynomially convex disc bounded by Γ' then $M = \Psi^{-1}(M')$ is a similar disc bounded by Γ. This proves Corollary 4.3.

In this remainder of this section we shall prove the following result:

4.4. Lemma. *Let T be the circle. For every pair of smooth embeddings $f_0, f_1 : T \to \mathbf{C}^n$ $(n \geqq 2)$ such that $f_0(T)$ and $f_1(T)$ are polynomially convex there is an isotopy $\tau: [0, 1] \times T \to \mathbf{C}^n$ satisfying $\tau_t = f_t$ for $t = 0, 1$, and such that $\tau_t(T)$ is polynomially convex for every $t \in [0, 1]$.*

Proof. Here we give a proof using Wermer's (or Stolzenberg's) theorem on hulls of curves. A more elementary proof is possible along the lines of the proof of Theorem 5.2 in Sect. 5.

If a smooth closed curve $\gamma \subset \mathbf{C}^n$ fails to be polynomially convex, then by Stolzenberg's theorem [20] it bounds a pure one-dimensional complex variety V, in the sense of Stokes' theorem. If $\omega(z) = \alpha_1(z)\,dz_1 + \cdots + \alpha_n(z)\,dz_n$ is a holomorphic $(1, 0)$ form on $\mathbf{C}^n$, its differential $d\omega$ is a $(2, 0)$ form which therefore vanishes on V, hence $\int_\gamma \omega = \int_V d\omega = 0$ for every such ω. Thus, in order for γ to be polynomially convex, it suffices that $\int_\gamma \omega \neq 0$ for at least one such form ω.

Conversely, if γ is polynomially convex, we can approximate every continuous function on γ by a holomorphic polynomial according to Oka-Weyl's theorem [14, p. 91], hence we can construct holomorphic $(1, 0)$ forms ω on $\mathbf{C}^n$ satisfying $\int_\gamma \omega \neq 0$.

Let $T = \{e^{i\theta} \in \mathbf{C}: \theta \in \mathbf{R}\}$. Every two embeddings $f_0, f_1 : T \to \mathbf{R}^k$ for $k \geqq 4$ are isotopic. (In general, every two embeddings of k-dimensional manifolds into $\mathbf{R}^{2k+2}$ are isotopic. This is an immediate consequence of Whitney's one-one immersion theorem [12, p. 61] which is itself a consequence of Thom's jet transversality theorem [12, p. 54].) Every isotopy τ: $[0, 1] \times T \to \mathbf{C}^n$ is of the form

$$\tau(t, \theta) = \sum_{j \in \mathbf{Z}} c_j(t)\, e^{ij\theta}\,, \quad t \in [0, 1]\,, \ \theta \in \mathbf{R}\,,$$

where c_j: $[0, 1] \to \mathbf{C}^n$ are smooth functions.

First we approximate f_0 resp. f_1 by embeddings f_0' resp. f_1' given by finite trigonometric polynomials of the form $\sum_{j=-N}^{N} c_j e^{ij\theta}$. If the approximations are close enough, we can isotop f_0 into f_0' by taking $f_0^t = (1 - t) f_0 + t f_0'$ $(t \in [0, 1])$, and every intermediate curve is still polynomially convex. We do the same thing for f_1. This shows that we may replace f_0 by f_0' and f_1 by f_1'.

Now we choose an isotopy τ: $[0, 1] \times T \to \mathbf{C}^n$ from f_0 to f_1. By approximation we may assume that each τ_t is a trigonometric polynomial, of degree N that is independent of t. We denote by $\mathscr{C}_N$ the space of trigonometric polynomials of degree at most N, with values in $\mathbf{C}^n$. The spaces $\mathscr{C}_N$ is isomorphic to $\mathbf{C}^{2N+1} \times \mathbf{C}^n$, with the coefficients $c_j = (c_{j,1}, \ldots, c_{j,n})$ as complex coordinates.

Since $\Gamma_0 = f_0(T)$ is polynomially convex, there is a holomorphic $(1, 0)$-form ω with polynomial coefficients on $\mathbf{C}^n$, satisfying $\int_T f_0^* \omega \neq 0$. The function G: $\mathscr{C}_N \to \mathbf{C}$ defined by $G(f) = \int_T f^* \omega$ is a holomorphic polynomial in the coefficients $c_{j,k}$ of $f \in \mathscr{C}_N$. Thus the zero set $Z_G = \{f \in \mathscr{C}_N\colon G(f) = 0\}$ is a complex hypersurface in $\mathscr{C}_N$.

By a standard result on analytic varieties [7] we can approximate the smooth path $t \to \tau_t \in \mathscr{C}_N$ by another smooth path τ'_t connecting f_0 to f_1 such that $G(\tau'_t) \neq 0$ for $0 \leqq t < 1$. If the approximation is sufficiently close then $\Gamma'_t = \tau'_t(T)$ is an embedded polynomially convex curve for every $t \in [0, 1]$ as required. Lemma 4.4 is proved.

5 Polynomial convexity and $\mathbf{C}^n$-equivalence of surfaces

In this section we prove the following result on $\mathbf{C}^n$-equivalence of surfaces for $n \geqq 3$. Here, a surface is a smooth, compact, connected manifold of real dimension two, with or without boundary.

5.1. Theorem. *Let M be a compact real-analytic surface and let f_0, f_1: $M \to \mathbf{C}^n$ ($n \geqq 3$) be real-analytic embeddings. If the surfaces $M_0 = f_0(M)$ and $M_1 = f_1(M)$ are totally real and polynomially convex in $\mathbf{C}^n$ then f_0 and f_1 are $\mathbf{C}^n$-equivalent.*

Recall that a compact surface without boundary embedded into $\mathbf{C}^2$ is never polynomially convex [14, p. 59], [25]. On the other hand, we will prove that surfaces in $\mathbf{C}^n$ for $n \geqq 3$ are generically polynomially convex. Theorem 5.1 follows immediately from Theorem 3.1 and from the part (b) of the following

5.2. Theorem. *If M is a smooth compact surface and $n \geqq 3$ then:*
(a) *The set of totally real embeddings f: $M \to \mathbf{C}^n$ whose image $f(M)$ is polynomially convex is open and everywhere dense in the set of all embeddings of M into $\mathbf{C}^n$ (in the Whitney $\mathscr{C}^\infty$ topology).*
(b) *If f_0, f_1: $M \to \mathbf{C}^n$ are totally real embeddings whose images are polynomially convex then there is an isotopy of embeddings τ: $[0, 1] \times M \to \mathbf{C}^n$ connecting f_0 to f_1 such that for every $t \in [0, 1]$, the embedding $\tau_t = \tau(t, \cdot)$: $M \to \mathbf{C}^n$ is totally real and $\tau_t(M)$ is polynomially convex.*

In fact we will prove that every isotopy of embeddings τ: $[0, 1] \times M \to \mathbf{C}^n$ can be approximated (in the $\mathscr{C}^\infty$ topology on the space of mappings) by isotopies τ' such that $\tau'_t(M)$ is totally real and polynomially convex for every $t \in [0, 1]$. If τ_0 and τ_1 are already totally real and polynomially convex then τ' can be chosen to agree with τ at $t = 0$ and $t = 1$.

First we shall prove the following slightly more general result on density of totally real embeddings and isotopies.

5.3. Lemma. *Let M be a compact smooth manifold of dimension k.*
(a) *If $k \leqq (2n + 1)/3$ then the set of smooth totally real embeddings $M \to \mathbf{C}^n$ is open and everywhere dense in the set of all smooth embeddings of M into $\mathbf{C}^n$.*
(b) *If $k \leqq 2n/3$ then every pair of totally real embeddings of M into $\mathbf{C}^n$ can be joined by an isotopy of totally real embeddings of M into $\mathbf{C}^n$.*

In the proof of this and other results in this section we shall frequently appeal to the following 'jet transversality theorem' of Thom [12, p. 54]:

Theorem (Thom Transversality Theorem). *Let X and Y be smooth manifolds and W a closed submanifold of the bundle $J^m(X, Y)$ of m-jets of smooth mappings $X \to Y$. Then the set of all maps f: $X \to Y$ for which the m-jet map $j^m f$: $X \to J^m(X, Y)$ is transverse to W is open and everywhere dense in the space of all smooth mappings $X \to Y$ (in the $\mathscr{C}^\infty$ topology).*

Recall that, if $\dim X + \dim W < \dim J^m(X, Y)$, then the transversality of the jet-map $j^m f$ to W means that the image of $j^m f$ misses W.

For immersions (and embeddings) $f: X \to \mathbf{R}^N$ there is another version of the transversality theorem for one-jets as follows. Let $k = \dim X$, and let $\mathscr{G}(k, N)$ denote the Grassman manifold consisting of all k-dimensional subspaces of $\mathbf{R}^N$. For each immersion $f: X \to \mathbf{R}^N$ we consider the Gauss map $\tilde{f}: X \to \mathscr{G}(k, N)$ that sends $x \in X$ to the tangent plane $f_*(T_x X)$ to f at $f(x)$. Given a closed submanifold $W \subset \mathscr{G}(k, N)$, there is an open and dense set of immersions for which $\tilde{f}$ is transverse to W. This result is an immediate consequence of the general transversality theorem for one-jets.

Proof of Lemma 5.3. Let $\mathscr{G} = \mathscr{G}(k, 2n)$ be the Grassman manifold of real k-dimensional subspaces (k-planes) of $\mathbf{C}^n = \mathbf{R}^{2n}$, and let $\mathscr{H} \subset \mathscr{G}(k, 2n)$ be the subset consisting of k-planes that contain a nontrivial complex subspace (a complex line). Clearly $\mathscr{H}$ is a real-analytic subset of $\mathscr{G}$. Recall that $\dim_{\mathbf{R}} \mathscr{G}(k, 2n) = k(2n - k)$.

Next we calculate $\dim \mathscr{H}$. Clearly $\mathscr{H}$ is empty if $k \leqq 1$, so we may assume $k \geqq 2$. Every k-plane $\Lambda \in \mathscr{H}$ is of the form $\Lambda = \Lambda_0 \oplus \Lambda_1$, where Λ_0 is a complex line in $\mathbf{C}^n$ and Λ_1 is a real $(k - 2)$-dimensional subspace of $\Lambda_0^{\perp}$ (= the orthogonal complement of Λ_0 in $\mathbf{C}^n$). This shows that

$$\dim_{\mathbf{R}} \mathscr{H} = \dim_{\mathbf{R}} \mathbf{CP}^{n-1} + \dim_{\mathbf{R}} \mathscr{G}(k - 2, 2n - 2) = (2n - 2) + (k - 2)(2n - k).$$

A simple calculation shows that the condition $\dim M + \dim \mathscr{H} < \dim \mathscr{G}$ is satisfied if and only if $k = \dim M \leqq (2n + 1)/3$. If this holds then the transversality theorem implies that for a generic embedding $f: M \to \mathbf{C}^n$, the tangent plane $f_*(T_p M)$ never belongs to $\mathscr{H}$ for any $p \in M$, hence f is totally real. (Here, as always, 'generic' means that the set of such embeddings is open and dense in the set of all embeddings.)

Similarly, if $k \leqq 2n/3$, we have $\dim \mathscr{G} - \dim \mathscr{H} \geqq k + 2$, hence the transversality theorem implies that for a generic isotopy $\tau: [0, 1] \times M \to \mathbf{C}^n$, the Gauss map $(t, p) \in [0, 1] \times M \to (\tau_t)_*(T_p M) \in \mathscr{G}$ misses $\mathscr{H}$. This means that τ_t is totally real for every $t \in [0, 1]$. Lemma 5.3 is proved.

Proof of Theorem 5.2. First we shall prove part (a). Lemma 5.3 implies that every embedding of M into $\mathbf{C}^n (n \geqq 3)$ can be approximated by a totally real embedding. We identify M with its totally real image in $\mathbf{C}^n$.

We will now show that M can be perturbed as little as we want in $\mathbf{C}^n$ so as to become polynomially convex. The required perturbation will be done in several steps, using the following well known facts:

(i) The total reality is a stable property under small $\mathscr{C}^1$ perturbations.
(ii) Polynomial convexity is a stable property under small $\mathscr{C}^2$ perturbations of totally real manifolds [11].
(iii) If a compact set $K \subset \mathbf{C}^n$ projects under the first coordinate projection $\pi: \mathbf{C}^n \to \mathbf{C}$ onto a smooth arc $\gamma \subset \mathbf{C}$, and if the fiber $M_z = \pi^{-1}(z) \cap M$ is polynomially convex for every $z \in \gamma$, then M is polynomially convex.

Let $\pi: \mathbf{C}^n \to \mathbf{C}$ be the coordinate projection $\pi(z_1, z') = z_1 = u + iv$. A standard application of Thom transversality theorem shows that after a small $\mathscr{C}^\infty$ perturbation of $M \subset \mathbf{C}^n$ the function $\rho = \Re\pi|_M$ is a Morse function on M.

Let $\rho(M) = [a, b] \subset \mathbf{R}$. Every level set $M(u) = \rho^{-1}(u)$ $(a \leqq u \leqq b)$ is either a union of finitely many smooth curves (this happens for all but finitely many values of u), or a finite union of curves such that one of them has a transverse self-intersection, or a point. This structure of the fibers $M(u)$ enables us to perturb M slightly in the direction $v = \mathfrak{J} z_1$ in such a way that the function $\mathfrak{J}\pi$ is nonconstant on every closed curve contained in any of the fibers $M(u)$. In fact, we can choose the perturbation such that all fibers of $\pi|_M$ are finite. It follows that every fiber of $\pi|_M$ is polynomially convex whence the same is true for every fiber $M(u) = \bigcup_{v \in \mathbf{R}} \pi^{-1}(u + iv) \cap M$, $u \in [a, b]$.

Since M is totally real, it is locally holomorphically convex. Thus we can choose for each u an open neighborhood W_u of $M(u)$ in $\mathbf{C}^n$ such that $\bar{W}_u$ is polynomially convex and $M \cap W_u$ is holomorphically convex in W_u. It follows that for every sufficiently small interval $J \subset [a, b]$ the slice $M(J) = \rho^{-1}(J) = \bigcup_{u \in J} M(u)$ is polynomially convex in $\mathbf{C}^n$. By compactness we can subdivide $[a, b]$ into a finite number of closed intervals $J_k = [u_{k-1}, u_k]$, where $a = u_0 < u_1 < u_2 < \cdots < u_m = b$, such that every slice $M(J_k) \cup M(J_{k+1}) = M(J_k \cup J_{k+1})$ $(1 \leqq k < m)$ is polynomially convex.

Now we perform another perturbation within each slice $M(J_k)$ with the purpose of splitting the polynomial hull of $M = \bigcup_{k=1}^{m} M(J_k)$ to the union of hulls of individual slices, thus making M polynomially convex.

Fix a k, $1 \leqq k \leqq m$, and write $J = J_k$. Choose a closed interval J' contained in the interior of J such that every $u \in J'$ is a regular value of ρ, and choose a pair of disjoint closed intervals I^0, I^1 contained in the interior of J'. By a small perturbation of $M(J)$ we may assume that the slice $M(J')$ is smooth real-analytic. The complexification of $M(J')$ is then a local complex manifold Σ of complex dimension 2. Let $N \to M(J')$ be the complex normal bundle to $M(J')$ in $\mathbf{C}^n$, i.e., every fiber N_z for $z \in M(J')$ is the complex plane of dimension $n - 2 \geqq 1$ that is orthogonal to the tangent space $T_z M$.

By our choice of J' each fiber $M(u)$ for $u \in J'$ is a finite collection of smooth curves, and the fibers depend smoothly on $u \in J'$. Hence the manifold $M(J')$ is homotopic to a finite family of disjoint closed curves, and therefore the complex bundle N is trivial over $M(J')$. This implies that we can pick a holomorphic function g on a neighborhood of $M(J')$ that vanishes identically on $M(J')$ (and therefore on Σ), but its gradient dg is nonvanishing at every point of $M(J')$.

By polynomial convexity of $M(J')$ we can approximate g uniformly on a neighborhood of $M(J')$ by holomorphic polynomials $P(z)$. Then the derivative of P also approximates the derivative of g near $M(J')$, hence the level sets of P near $M(J')$ are smooth complex hypersurfaces that are close to the level sets of g in the $\mathscr{C}^\infty$ topology.

We now deform the interior of $M(J')$ by pushing the slice $M(I^0)$ into the level set $\{P = 0\}$, and similarly we push $M(I^1)$ into the level set $\{P = \varepsilon\}$ for a small $\varepsilon \neq 0$. The deformation can be made arbitrary small in the $\mathscr{C}^\infty$ sense, provided that the approximation of g by P is close enough and ε is sufficiently small. Thus the slices $M(J_{k-1} \cup J_k)$ and $M(J_k \cup J_{k+1})$ remain totally real and polynomially convex after the perturbation. We leave out the obvious but tedious details.

We perform the perturbation described above for every $k = 1, 2, \ldots, m$. We claim that the new manifold M is then polynomially convex. To see this, fix a point $z_0 \in \hat{M}$ and let ν be the Jensen representing measure for z_0 supported on M [21, p. 108]. Fix a k, $1 \leqq k \leqq m$, and let P be the polynomial constructed above for the slice $M(J_k)$. If ν has any mass on the set $M(I^0)$ where $P = 0$ then the Jensen's inequality $\log|P(z_0)| \leqq \int_M \log|P| dv$ implies $P(z_0) = 0$. Similarly, if v has any mass on $M(I^1)$ where $P = \varepsilon$ then Jensen's inequality, applied to $\log|P(z) - \varepsilon|$, shows that $P(z_0) = \varepsilon$.

Thus, v has no mass on at least one of the slices $M(I^0)$, $M(I^1)$. Since the support of the projection $\tilde{v} = \rho_*(v)$ of v onto the real axis always has connected support, it follows that v is supported on only one of the slices $M(J_{k-1} \cup J_k)$. But every such slice is polynomially convex by construction, hence v must be the point mass at $z_0 \in M$. This proves that the constructed surface M is polynomially convex as claimed.

We proved part (a) of Theorem 5.2. To prove part (b) we suppose that $f_t(M) = M_t$ for $t = 0, 1$ are totally real embedded polynomially convex surfaces in $\mathbf{C}^n$ $(n \geqq 3)$. According to Lemma 5.3(b) there exists a totally real isotopy $\tau: [0, 1] \times M \to \mathbf{C}^n$ connecting $f_0 = \tau_0$ to $f_1 = \tau_1$. We shall perturb this isotopy so as to make every manifold $M_t = \tau_t(M) \subset \mathbf{C}^n$ polynomially convex but keeping it fixed at $t = 0$ and $t = 1$. Since M_t is already totally real and polynomially convex for t sufficiently close to 0 or 1, it suffices to consider perturbations without fixed ends since we can then patch the new isotopy with the old one near the endpoints $t = 0, 1$.

5.4. Lemma. *Let M be a smooth manifold of dimension k, $1 \leqq k < n$, and let τ: $[0, 1] \times M \to \mathbf{R}^n$ be a smooth isotopy of embeddings. Suppose that $l_1, \ldots, l_s$ are linear functionals on $\mathbf{R}^n$ such that* $\dim \bigcap_{j=1}^{s} \ker l_j < k$. *Then we can approximate the isotopy τ in the $\mathscr{C}^\infty$ sense by an isotopy $\tilde{\tau}$ such that for every $t \in [0, 1]$ at least one of the functions l_j is a Morse functions when restricted to the manifold $\tilde{\tau}_t(M)$.*

Proof. Let $J^2 = J^2([0, 1] \times M, \mathbf{R}^n)$ be the 2-jet bundle of smooth mappings of $[0, 1] \times M$ into $\mathbf{R}^n$. We denote the local coordinates on M by x, and we let $(j_x^2 \tau)(t, x)$ be the x-part of the full 2-jet $(j^2 \tau)(t, x)$ of τ at (t, x). For each linear function l:$\mathbf{R}^n \to \mathbf{R}$ the composition $l \circ (j_x^2 \tau)(t, x)$ is then the 2-jet of $l \circ \tau_t$: $M \to \mathbf{R}$ at $x \in M$.

For a linear functional l on $\mathbf{R}^n$ we denote by $\Sigma_l \subset J^2$ the subset consisting of all 2-jets $(j^2 \tau)(t, x) \in J^2$ for which the jet $l \circ (j_x^2 \tau)(t, x)$ fails to be the jet of a Morse function at $x \in M$. Clearly the definition is independent of the choice of local coordinates x on M. A jet as above is not Morse if and only if the one-jet $(j_x^1 \tau)(t, x)$ belongs to the kernel of l (this gives $k = \dim M$ independent conditions), and the second order jet $l \circ (j_x^2 \tau)(t, x)$ is degenerate (this gives another condition-the vanishing of the real Hessian determinant). Thus Σ_l is a real-analytic subset of J^2 of codimension

$$\mathrm{Codim}_{\mathbf{R}}(\Sigma_l, J^2) = \dim M + 1 .$$

The Thom transversality theorem implies that for a generic smooth map τ: $[0, 1] \times M \to \mathbf{R}^n$ its 2-jet map $j^2 \tau$: $[0, 1] \times M \to J^2$ avoids the singular part of Σ_l and intersects its regular part transversely at a finite set of points $E_l \subset [0, 1] \times M$. The same is true for a finite collection of the manifolds Σ_l.

If the functionals $l_1, \ldots, l_s$ are chosen as in the lemma, it follows that no point (t, x) is mapped by $j^2 \tau$ to all $\Sigma_j = \Sigma_{l_j}$ simultaneously, i.e., $\bigcap_{j=1}^{s} E_j = \varnothing$. After a small generic perturbation of τ supported in pairwise disjoint neighborhoods of the points in $\bigcup_{j=1}^{s} E_j$ we can arrange that the projections of the sets $E_1, \ldots, E_s$ onto $[0, 1]$ are pairwise disjoint. This means that for each $t \in [0, 1]$ at least one of the functions $l_j \circ \tau_t$ is Morse on M_t, and Lemma 5.4 is proved.

We apply Lemma 5.4 to τ: $[0,1]\times M\to\mathbf{C}^n$, with the l_j's being the real and imaginary parts of the n coordinate projections π_j: $\mathbf{C}^n\to\mathbf{C}$. We replace τ by τ' as in Lemma 5.4. The lemma allows us to partition the interval $[0,1]$ into a finite number of subintervals I_j such that for each j one of the projections $\Re\pi_k$, $\Im\pi_k$ (k depending on j) is a Morse function on every $M_t=\tau_t(M)$, $t\in I_j$.

It suffices to describe the required perturbation of τ over each interval I_j. Even though the perturbation of τ for $t\in I_j$ will affect τ_t also for nearby values of t, we shall keep all perturbations sufficiently small in the $\mathscr{C}^2$ norm, thereby not destroying any of the relevant properties of τ over the adjacent intervals I_{j-1} and I_{j+1}.

Thus we may assume from now on that for some coordinate projection π: $\mathbf{C}^n\to\mathbf{C}$, the real part $\Re\pi$ is a Morse function on M_t for every $t\in[0,1]$. The proof now proceeds just as in the case (a) by noting that for each $t_0\in[0,1]$ all the required perturbations of M_t can be done uniformly for t sufficiently close to t_0. More precisely, we can partition the interval $[0,1]$ into a finite number of subintervals $0=t_0<t_1<t_2<\cdots<t_p=1$ such that for $t_{j-1}\leqq t_j$, the perturbations of M_t described in part (a) can be performed on a set of intervals $I^0_{k,j}$, $I^1_{k,j}\subset J'_{k,j}\subset J_{k,j}(1\leqq k\leqq m_j)$ that are chosen independently of $t\in[t_{j-1},t_j]$. (We are using the same notation as in the proof of part (a), except that we added indices in the obvious way.)

Recall that the larger intervals $J_{k,j}\subset\mathbf{R}$ $(1\leqq k\leqq m_j)$ are chosen such that for each $t\in[t_{j-1},t_j]$, the union of every two consecutive slices $M_t(J_{k,j})$, $M_t(J_{k+1,j})$ is polynomially convex. Moreover, we choose the smaller intervals $J'_{k,j}\subset\subset J_{k,j}$ $(1\leqq k\leqq m_j)$ to be pairwise disjoint from the corresponding intervals $J'_{k,i}$ $(1\leqq k\leqq m_i)$ whenever $i\neq j$.

We start with the interval $[0,t_1]$ and perform the perturbations described in the proof of part (a) that make every manifold M_t for $0\leqq t\leqq t_1$ polynomially convex. The required perturbations are supported on the slices $M_t(J'_{k,1})$ $(1\leqq k\leqq m_1)$. This changes the manifold M_{t_1} on the same slices, but we can smoothen out the perturbation over the intervals $J'_{k,1}(1\leqq k\leqq m_1)$ and over the values of t that are slightly larger than t_1. If the perturbations are sufficiently small, this will not destroy the polynomial convexity of the slices $M_t(J_{k,2})$.

Now we go to the next interval $t\in[t_1,t_2]$, and we perturb the manifolds M_t on the slices $M_t(J'_{k,2})$ $(1\leqq k\leqq m_2)$. Since these new intervals are disjoint from the ones used in step one, the previous perturbations do not affect the new perturbations at all. Again we smoothen out the new perturbation near both endpoints $t=t_1$, $t=t_2$ and over the intervals $J'_{k,2}$. Even though we changed again the manifolds M_t for certain $t\in[0,t_1]$, we may suppose that the new perturbations are sufficiently small that they do not destroy the polynomial convexity of those manifolds M_t.

Continuing this way we complete the required perturbation of the isotopy $M_t(t\in[0,1])$ to an isotopy with polynomially convex sets M_t in a finite number of steps. This completes the proof of Theorem 5.2.

6 Obstructions to approximation near submanifolds

In this section we give a necessary and sufficient condition for the existence of approximation of a given biholomorphic mapping Φ_1: $\Omega\to\mathbf{C}^n$ by automorphisms of $\mathbf{C}^n$ in a neighborhood of a real-analytic,. totally real, polynomially convex submanifold $M\subset\Omega$.

From Theorem 3.1 we know that, in order for the approximation to exist near M, there must exist an isotopy of embeddings $\tau\colon [0, 1] \times M \to \mathbf{C}^n$ satisfying
(a) τ_0 is the identity on M,
(b) $\tau_1 = \Phi_1|_M$, and
(c) for every $t \in [0, 1]$, the submanifold $M_t = \tau_t(M) \subset \mathbf{C}^n$ is totally real and polynomially convex.

The converse of this is not always true. Suppose that an isotopy satisfying (a)–(c) exists. Applying Theorem 3.1 we get a biholomorphic map Ψ near M satisfying $\Psi|_M = \Phi_1|_M$, such that Ψ is the limit of a sequence of automorphisms in some neighborhood of M. By construction the composition $\Phi = \Psi^{-1} \circ \Phi_1$ is the identity on M, and Φ_1 can be approximated by automorphisms if and only if Φ can be so approximated. We have thus reduced the approximation problem to biholomorphic mappings that fix the manifold M pointwise.

If M is a generic submanifold of $\mathbf{C}^n$, it follows that Φ is the identity, hence $\Phi_1 = \Psi$ is the limit of automorphisms. Thus, in the case of a generic manifold M, Φ_1 is a limit of automorphisms near M if and only if the manifolds M and $M_1 = \Phi_1(M)$ are $\mathbf{C}^n$-equivalent.

The more interesting case is when M is not generic. Simple examples show that in general Φ is then not approximable by automorphisms (see Sect. 7).

Let $N \to M$ be the complex normal bundle to M in $\mathbf{C}^n$ (1), of rank $d = n - \dim M$. Denote by $\pi\colon T\mathbf{C}^n|_M \to N$ the projection onto the normal bundle in the direction of the complex tangent bundle $T^{\mathbf{C}}M$. The derivative $D\Phi$ defines a complex automorphism of the normal bundle $N \to M$ by

$$\Phi_*(z)\,v = \pi(D\Phi(z)\,v)\,, \qquad z \in M\,,\ v \in N_z\,. \tag{2}$$

6.1. Proposition. *Suppose that $M \subset \mathbf{C}^n$ is a compact, real-analytic submanifold that is totally real and polynomially convex. Let Φ be a biholomorphic mapping in a neighborhood of M such that $\Phi(z) = z$ for all $z \in M$. If the automorphism Φ_* of the normal bundle $N \to M$ defined by* (2) *is homotopic to the identity in the group* Aut N, *then Φ is the limit of a sequence of automorphisms of $\mathbf{C}^n$ in some neighborhood of M.*

Remark. By the techniques explained in the proof of Proposition 3.2 above we can construct for every real-analytic automorphisms ϕ of the complex normal bundle $N \to M$ a biholomorphic map Φ near M that fixes M pointwise and satisfies $D\Phi|_N = \phi$. It suffices to embed N locally near its zero section into $\mathbf{C}^n$ and to extend ϕ to a holomorphic map near M.

Proof of Proposition 6.1. We consider N locally near its zero section as an embedded real-analytic CR submanifold of $\mathbf{C}^n$. The condition means that there is a family of automorphisms $\Theta_t \in \operatorname{Aut} N (t \in [0, 1/3])$ such that Θ_0 is the identity and $\Theta_{1/3} = \Phi_*$. We may choose Θ_t to depend analytically on all variables, including t. Furthermore, between time $t = 1/3$ and $t = 2/3$ we can join Φ_* to $D\Phi|_N$ through complex automorphisms Θ_t of the bundle $T\mathbf{C}^n|_M$ that restrict to the identity on $T^{\mathbf{C}}M$. Finally, from time $t = 2/3$ to $t = 1$ we join $D\Phi|_N$ to Φ by simply taking convex combinations of the two.

As in the proof of Proposition 3.2 we extend each Θ_t $(0 \leqq t \leqq 1)$ to a biholomorphic map $F_t\colon U \to \mathbf{C}^n$, defined in a neighborhood U of M, beginning with the identity F_0 and ending with $F_1 = \Phi$. Every map F_t fixes M pointwise. Since M is assumed to be polynomially convex, Theorem 2.1 implies that Φ is the limit of a sequence of automorphisms on a smaller neighborhood of M. This proves Proposition 6.1.

The condition in Proposition 6.1 is sufficient but not necessary for the approximation. We can formulate the necessary condition as follows. Let $T = \mathbf{R}/2\pi\mathbf{Z}$ be the circle. Consider isotopies of embeddings $\tau\colon T\times M \to \mathbf{C}^n$ of M into $\mathbf{C}^n$, parametrized by $t \in T$, satisfying
(i) $\tau_0 = \tau_1$ is the inclusion $M \to \mathbf{C}^n$, and
(ii) for every $t \in T$ the submanifold $\tau_t(M) \subset \mathbf{C}^n$ is totally real and polynomially convex.

Let $\mathcal{N} \to T\times M$ be the complex vector bundle of rank $d = n - \dim M$ whose fiber over (t, x) is the complex normal space to $\mathrm{T}_{\tau(t,x)} M$ as in (1). In other words, the part $\mathcal{N}_t$ lying over $\{t\}\times M$ is just the pull-back of the complex normal bundle $N_t \to M_t$ by τ_t. If we fix a connection on $\mathcal{N}$, then the parallel transport along the loops $T\times\{x\}$ yields an automorphism $\tau_* \in \operatorname{Aut}\mathcal{N}_0$ of the bundle $\mathcal{N}_0 \to \{0\}\times M$ (the monodromy automorphism). Even though τ_* depends on the choice of the connection, it is easily seen that its homotopy class $[\tau_*]$ only depends on τ.

We may identify $\mathcal{N}_0$ with the normal bundle $N \to M$ and think of τ_* as an element of Aut N. We denote by Π the group of all homotopy classes of automorphisms of $N \to M$. The group operation is composition. Let $\Pi_0 \subset \Pi$ be the subgroup consisting of all homotopy classes of the form $[\tau_*]$, where $\tau\colon T\times M \to \mathbf{C}^n$ is any isotopy satisfying the properties (i) and (ii) above; we shall call Π_0 the *monodromy group* corresponding to the isotopies satisfying (i), (ii). We can now apply Theorem 3.1 in the same way as in the proof of Proposition 6.1 to obtain

6.2. Theorem. (*Notation as above.*) *Let $M \subset \mathbf{C}^n$ be a compact, real-analytic submanifold that is totally real and polynomially convex. Let Φ be a biholomorphic map defined near M such that $\Phi(z) = z$ for all $z \in M$. Then the following are equivalent:*
(a) *Φ is a limit of a sequence of automorphisms in a neighborhood of M.*
(*b*) *The homotopy class $[\Phi_*] \in \Pi$ of the automorphism defined by* (2) *belongs to the monodromy group Π_0.*

As was noted above this also gives a necessary and sufficient condition for approximation of biholomorphic mappings Φ_1 near M that do not fix M.

The homotopy classification of automorphisms of a vector bundle $N \to M$ is equivalent to the homotopy classification of sections of an associated bundle $G \to M$ whose fiber at $z \in M$ is the set of all complex linear isomorphisms of the fiber N_z. Thus G is a principal $\mathbf{GL}(d, \mathbf{C})$ bundle over M. Even though the classification of sections of $G \to M$ is a rather complicated matter in general (we refer the reader to the Part III of Steenrod's book [19]), it is much simpler in the case when the bundle $N \to M$ is trivial. If we choose an isomorphism of N with $M \times \mathbf{C}^d$, we obtain a reference frame $X_1, X_2, \ldots, X_d$ on N, i.e., global sections of N over M that form a basis of every fiber N_x, $x \in M$. In this basis, every automorphism of N gives a mapping of the base space M into the group $\mathbf{GL}(d, \mathbf{C})$ of the fiber, and vice versa. Moreover, two automorphisms of N are homotopic if and only if the corresponding maps $M \to \mathbf{GL}(d, \mathbf{C})$ are homotopic. Thus, the monodromy group Π_0 is a subgroup of the group

$$\Pi = [M, \mathbf{GL}(d, \mathbf{C})] = [M, \mathbf{U}(d)]\,,$$

where $\mathbf{U}(d)$ is the unitary group.

We shall now consider in detail the case of closed curves in $\mathbf{C}^n$ for $n \geqq 2$ and two dimensional surfaces in $\mathbf{C}^n$ for $n \geqq 3$. For the relevant results of homotopy theory we refer the reader to Steenrod [19, pp. 131–132].

Case 1: M is a simple closed curve in $\mathbf{C}^n$. In this case the complex normal bundle $N \to M$ is trivial (every orientable bundle over the circle is trivial). The group of homotopy classes of automorphisms of $N \to M$ is

$$\Pi = [M, \mathbf{U}(d)] = \pi_1(\mathbf{U}(d)) = \mathbf{Z}$$

for every d. Moreover, it is easily seen that the class of Φ_* (2) is just the winding number of the Jacobian determinant of Φ along M. Since M is homotopic to a point in $\mathbf{C}^n$, this winding number vanishes for automorphisms of $\mathbf{C}^n$, hence the monodromy group Π_0 is trivial. Conversely, every Φ satisfying $[\Phi_*] = 0$ can be approximated by automorphisms according to Proposition 6.1.

If Φ_1 is a biholomorphic map near M that maps M onto another polynomially convex curve $M_1 = \Phi_1(M)$, then by the first part of Theorem 4.2 there exists a biholomorphic map Ψ that is the limit of automorphisms near M and satisfies $\Psi|_M = \Phi_1|_M$. Since the winding number of the Jacobian determinant of Ψ along M equals zero, the maps Φ_1 and $\Phi = \Psi^{-1} \circ \Phi_1$ give the same winding number. Thus, if the winding number of $J(\Phi_1)$ equals zero and $\Phi_1(M)$ is polynomially convex in $\mathbf{C}^n$, it follows that Φ and therefore Φ_1 are approximable by automorphisms near M. This proves the second part of Theorem 4.2.

Case 2: M is a two-dimensional surface in $\mathbf{C}^n$ for $n \geqq 3$. Of course we assume that M satisfies all other properties mentioned at the beginning. For every such $M \subset \mathbf{C}^n$ the complex normal bundle $N \to M$ is trivial which can be seen as follows. First, every two totally real embeddings of M into $\mathbf{C}^n (n \geqq 3)$ are totally real isotopic by Lemma 5.3, hence the bundle N is the same for every such embedding. If M is orientable, we can embed it as a hypersurface into a totally real plane $\mathbf{R}^3 \subset \mathbf{C}^3$. The normal bundle of M in $\mathbf{C}^3$ is then generated by the normal vector field to M in $\mathbf{R}^3$. In general, we can embed M into $\mathbf{C}^2$ with only isolated complex tangents. Let $h: \mathbf{C}^2 \to \mathbf{C}$ be any smooth function whose differential dh is not complex linear on $\mathrm{T}_p M$ at any complex tangent $p \in M$. Then the graph $M' = \{(z, h(z)) \colon z \in M\} \subset \mathbf{C}^3$ is a totally real embedding of M into $\mathbf{C}^3$ whose complex normal bundle is generated by the projection of the constant vector field $X = (0, 0, 1)$ onto N in the direction of $\mathrm{T}^{\mathbf{C}} M$. Thus the bundle $N \to M$ is trivial as claimed.

The group Π equals $[M, \mathbf{U}(d)]$. If $d = 1$, $\mathbf{U}(1) = S^1$ is the circle, and $\Pi = [M, S^1]$. For $d = 2$ the group $\mathbf{U}(2)$ is homeomorphic to the product $S^1 \times S^3$, and

$$\Pi = [M, \mathbf{U}(2)] = [M, S^1] \times [M, S^3] = [M, S^1]$$

since $[M, S^3] = 0$ for every two dimensional surface M. In general, under the natural embedding $\mathbf{U}(d-1) \subset \mathbf{U}(d)$, the quotient $\mathbf{U}(d)/\mathbf{U}(d-1)$ is homeomorphic to the sphere S^{2d-1}, hence every map of a two dimensional manifold M to $\mathbf{U}(d)$ is homotopic to a map into $\mathbf{U}(1) = S^1$. Thus, $\Pi = [M, \mathbf{U}(d)] = [M, S^1]$ for every two dimensional surface M and for every d. The homotopy class $[\Phi_*] \in [M, S^1]$ (2) is evidently just the homotopy class of the Jacobian determinant

$$J(\Phi) = \det \mathrm{D}\Phi \colon M \to \mathbf{C} \backslash \{0\} . \qquad (3)$$

Since this class vanishes when Φ is approximable by automorphisms, the monodromy group Π_0 is trivial. Together with Theorem 5.1 this implies

6.3. Corollary. *Let $M \subset \mathbf{C}^n (n \geqq 3)$ be a compact, embedded, real-analytic surface that is totally real and polynomially convex. A biholomorphic mapping $\Phi\colon U \to \mathbf{C}^n$ defined in a neighborhood of M can be approximated by automorphisms of $\mathbf{C}^n$ near M if and only if the image $\Phi(M)$ is polynomially convex and the Jacobian* (3) *is homotopic to a constant. In particular, if M is a two dimensional sphere, then Φ can be approximated by* Aut $\mathbf{C}^n$ *near M if and only if $\Phi(M)$ is polynomially convex.*

7 Examples and open problems

We have seen in Sect. 6 above that the only topological obstruction to approximating a biholomorphic map by automorphisms near curves resp. surfaces $M \subset \mathbf{C}^n$ ($n \geqq 2$ resp. $n \geqq 3$) lies in the group $[M, S^1]$. This group vanishes when M is simply connected. Of course, if we consider manifolds M of higher dimension, we will have many more homotopy obstructions.

An obvious obstruction to approximation of a biholomorphic mapping $\Phi\colon \Omega \to \mathbf{C}^n$ by automorphisms of $\mathbf{C}^n$ in a neighborhood of a submanifold $M \subset \Omega$ is the (nontrivial) homotopy class of the derivative $D\Phi\colon M \to \mathbf{GL}(n, \mathbf{C})$, since this mapping is homotopic to a constant whenever Φ can be so approximated. We will give two examples of this type. In Example 7.1 the obstruction to approximation lies in the fundamental group $\pi_1(M)$, and in Example 7.2 it lies in the group $\pi_3(M)$.

7.1. Example. The curve

$$M = \{(e^{i\theta}, e^{-i\theta})\colon \theta \in \mathbf{R}\} \subset \mathbf{C}^2$$

is polynomially convex. The mapping

$$\Phi(z, w) = (z, w + (1 - zw)(1/z - 1))$$

is biholomorphic near M and fixes M pointwise. Its Jacobian equals $J(\Phi)(z, w) = z$, with winding number along M equal to one. Hence Φ can not be approximated by automorphisms of $\mathbf{C}^2$ in any neighborhood of M.

7.2. Example. We construct a real-analytic, polynomially convex, three dimensional sphere $M \subset \mathbf{C}^5$ and a biholomorphic map Φ near M that fixes M pointwise but is not a limit of automorphisms in any neighborhood of M. This is a slight modification of Example 5.4 in [4].

Let M be the three-sphere S^3, embedded as the unit sphere in the totally real 4-plane $\mathbf{R}^4 \times \{0\} \subset \mathbf{C}^5$. Clearly M is polynomially convex. We choose orthonormal vector fields X_1, X_2, X_3, X_4 along M such that the first three are tangent to M while X_4 is the outward unit normal field to M in $\mathbf{R}^4 \times \{0\}$. The complex normal bundle $N \to M$ is a trivial bundle of rank two, generated by the fields X_4 and $X_5 = E_5 = (0, 0, 0, 0, 1)$. For each $z \in M$ let $A(z) \in \mathbf{U}(5)$ be the map that sends the standard basis vectors $E_1, \dots, E_5$ of $\mathbf{C}^5$ to the basis $X_1, \dots, X_5$ at z.

The group $\mathbf{U}(2)$ is the product $\mathbf{U}(2) = \mathbf{U}(1) \times \mathbf{SU}(2) = S^1 \times S^3$. Choose a real-analytic mapping $F\colon M \to \mathbf{SU}(2)$ that generates the third fundamental group $\pi_3(\mathbf{U}(2)) = \pi_3(S^3) = \mathbf{Z}$. The same map then generates $\pi_3(\mathbf{U}(d))$ for every $d \geqq 2$

under the standard embedding $\mathbf{U}(2) \subset \mathbf{U}(d)$ [19, p.132]. Write F in matrix notation $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c, d: $M \to \mathbf{C}$ are real-analytic functions. Let Φ be a biholomorphic mapping in a neighborhood of M that fixes M pointwise and satisfies

$$\mathrm{D}\Phi(z) X_j(z) = X_j(z), \qquad 1 \leqq j \leqq 3,$$

$$\mathrm{D}\Phi(z) X_4(z) = a(z) X_4(z) + c(z) X_5(z),$$

$$\mathrm{D}\Phi(z) X_5(z) = b(z) X_4(z) + d(z) X_5(z),$$

for all $z \in M$. Clearly the map $\mathrm{D}\Phi$: $M \to \mathbf{U}(5)$ is conjugate to $\mathrm{Id} \oplus F$ by A, that is,

$$A^{-1}(z) \circ \mathrm{D}\Phi(z) \circ A(z) = 1 \oplus F(z), \qquad z \in M.$$

It follows that the derivative map $z \in M \to \mathrm{D}\Phi(z) \in \mathbf{U}(5)$ represents a nontrivial element of $\pi_3(\mathbf{U}(5))$. Therefore Φ can not be approximated by automorphisms of $\mathbf{C}^5$ near M.

Remark. Note that, even if we extend Φ to $\mathbf{C}^5 \times \mathbf{C}^m$ as the identity in the extra variables, we still have a non-approximable map, no matter how large m is. Of course it is possible to add a linear 'twist' in the space $\mathbf{C}^m$ for $m \geqq 2$ that 'undoes' the twist of Φ on the normal bundle N such that the new map is approximable by automorphisms.

7.3. Example. The following is an example of a four dimensional real submanifold $\Sigma \subset \mathbf{C}^5$ with a single complex tangent at the origin that can not be made totally real by a small $\mathscr{C}^1$ perturbation near the origin:

$$\Sigma = \{(z_1, z_2, |z_1|^2, |z_2|^2, \bar{z}_1 - \bar{z}_2)\colon z_1, z_2 \in \mathbf{C}\} \subset \mathbf{C}^5.$$

This example is related to Lemma 5.3(a). It shows that, when attempting to generalize our results on surfaces in $\mathbf{C}^3$ (Sect. 5) to higher dimensional manifolds, one must in general have sufficiently high codimension as well.

To prove our claim, notice that every submanifold $\Sigma' \subset \mathbf{C}^5$ that is close enough to Σ in the $\mathscr{C}^1$ topology will be a graph over $\mathbf{C}^2$:

$$\Sigma' = \{(z_1, z_2, |z_1|^2 + \phi, |z_2|^2 + \psi, \bar{z}_1 - \bar{z}_2 + \lambda)\colon z_1, z_2 \in \mathbf{C}\},$$

where ϕ, ψ, λ are functions of (z_1, z_2) of small $\mathscr{C}^1$ norm.

For $\alpha \in \mathbf{C}$ we let $L_\alpha = \partial/\partial \bar{z}_1 + \alpha \partial/\partial \bar{z}_2$. One obtains a complex tangency of Σ' at the point $(z_1, z_2, \ldots) \in \Sigma'$ in the direction of the complex line $\zeta \to (\zeta, \bar{\alpha}\zeta, \ldots)$ by looking for (z_1, z_2) close to 0 and $\alpha \in \mathbf{C}$ such that

$$L_\alpha(|z_1|^2 + \phi) = L_\alpha(|z_2|^2 + \psi) = L_\alpha(\bar{z}_1 - \bar{z}_2 + \lambda) = 0.$$

This gives three equations with three unknowns z_1, z_2, α:

$$z_1 + \frac{\partial \phi}{\partial \bar{z}_1} + \alpha \frac{\partial \phi}{\partial \bar{z}_2} = 0,$$

$$\alpha z_2 + \frac{\partial \psi}{\partial \bar{z}_1} + \alpha \frac{\partial \psi}{\partial \bar{z}_2} = 0,$$

$$1 - \alpha + \frac{\partial \lambda}{\partial \bar{z}_1} + \alpha \frac{\partial \lambda}{\partial \bar{z}_2} = 0.$$

It has to be thought of as a system of six real equations in six real unknowns. When $\phi = \psi = \lambda = 0$ this system reduces to

$$z_1 = 0\,, \qquad \alpha z_2 = 0\,, \qquad 1 - \alpha = 0$$

that has a unique solution (0, 0, 1) and is of maximal rank at (0, 0, 1). Thus every small perturbation of this system of maximal rank will have a solution close to (0, 0, 1).

Remark. The four dimensional manifold

$$\Sigma_1 = \{(z_1, z_2, |z_1|^2, |z_2|^2, 0)\} \subset \mathbf{C}^5$$

with complex tangencies along both z_1 and z_2 axes can easily be deformed into a totally real manifold. Indeed, consider the manifolds $\{(z_1, z_2, |z_1|^2, |z_2|^2 + \varepsilon\bar{z}_1, \varepsilon\bar{z}_2)\}$ for small $\varepsilon \neq 0$.

7.4. Problem. Let $\Omega \subset \mathbf{C}^n$ be a domain and Φ: $\Omega \to \mathbf{C}^n$ a biholomorphic map. Suppose that the derivative map $D\Phi$: $D \to \mathbf{GL}(n, \mathbf{C})$ is homotopic to a constant through a family of holomorphic maps into $\mathbf{GL}(n, \mathbf{C})$. Does it follow that Φ is the limit of automorphisms? The condition that $D\Phi$ be homotopic to a constant on every compact subset of Ω is necessary for the approximation according to the converse part of Theorem 1.1.

7.5. Problem. Let $\Omega \subset \mathbf{C}^n$ be a pseudoconvex Runge domain with smooth boundary such that $\bar{\Omega}$ is diffeomorphic to the ball. If Φ: $\Omega \to \mathbf{C}^n$ is a biholomorphic map whose image $\Phi(\Omega)$ is Runge in $\mathbf{C}^n$, is Φ the limit of automorphisms?

7.6. Problem. Let Ω be as in the previous problem. Is the set of all biholomorphic mappings from Ω to domains in $\mathbf{C}^n$ connected?

7.7. Problem. Does there exist a Fatou-Bieberbach domain $D \subset \mathbf{C}^n$ that is not Runge? It seems that in all known constructions of such domains (see for instance [5], [8], and [17]) one obtains a biholomorphic mapping Φ: $\mathbf{C}^n \to D$ (Fatou-Bieberbach map) as a limit of a sequence of automorphisms, hence D is Runge. Note that, if Φ: $\mathbf{C}^n \to D$ is a biholomorphic map whose image D is Runge in $\mathbf{C}^n$, then Φ a limit of automorpohisms of $\mathbf{C}^n$ according to Theorem 1.1.

7.8. Problem. If f: $\mathbf{C} \to \mathbf{C}^2$ is a proper holomorpohic embedding, can $f(\mathbf{C})$ be straightened (i.e., mapped onto a line) by an automorphism of $\mathbf{C}^2$? The answer is positive for polynomial embeddings into $\mathbf{C}^2$ [1], and is negative for holomorphic embeddings $f : \mathbf{C} \to \mathbf{C}^3$ [18].

References

1. Abhyankar, S.S., Moh, T.T.: Embeddings of the line in the plane. J. Reine Angew. Math. **276**, 148–166 (1975)
2. Alexander, H.: The polynomial hull of a rectifiable curve in $\mathbf{C}^n$. Am. J. Math. **110**, 629–640 (1988)
3. Andersén, E.: Volume preserving automorphisms of $\mathbf{C}^n$. Complex Variables **14**, 223–235 (1990)
4. Andersén, E., Lempert, L.: On the group of holomorphic automorphisms of $\mathbf{C}^n$. Invent. Math. **110**, 371–388 (1992)
5. Bochner, S., Martin, W.T.: Several Complex Variables. Princeton: Princeton University Press 1948

6. Bony, J.-M.: Principle du maximum, inegalité de Harnack ... Ann. Inst. Fourier **19**, 277–304 (1969)
7. Chirka, E.M.: Complex Analytic Sets. (Math. Appl., vol. 46) Dordrecht: Kluwer 1989; Original in Russian, Moskva: Nauka 1985
8. Dixon, P.G., Esterle, J.: Michael's problem and the Poincaré-Fatou-Bieberbach phenomenon. Bull. Amer. Math. Soc., New. Ser. **15**, 127–187 (1986)
9. Kallin, E.: Polynomial convexity: the three spheres problem. In: Aeppli, A. et al. (eds.) Proc. Conf. Complex Analysis, Minneapolis 1964, pp. 301–304. Berlin Heidelberg New York: Springer 1965
10. El Kasimi, A.: Approximation polynômiale dans les domaines étoilés de $\mathbf{C}^n$. Complex Variables **10**, 179–182 (1988)
11. Forstneric, F.: Stability of polynomial convexity of totally real sets. Proc. Am. Math. Soc. **96**, 489–494 (1986)
12. Golubitsky, M., Guillemin, V.: Stable Mappings and Their Singularities. (Grad. Texts Math., vol. 14) Berlin Heidelberg New York: Springer 1973
13. Hirsch, M.: Differential Topology. (Grad. Texts Math., vol. 33) Berlin Heidelberg New York: Springer 1976
14. Hörmander, L.: An Introduction to Complex Analysis in Several Variables, 3rd ed. Amsterdam: North Holland 1990
15. Rosay, J.-P.: The polynomial hull of non-connected tube domains, and an example of E. Kallin. Bull. Lond. Math. Soc. **21**, 73–78 (1989)
16. Rosay, J.-P.: Straightening of arcs. (Preprint 1992)
17. Rosay, J.-P., Rudin, W.: Holomorphic maps from $\mathbf{C}^n$ to $\mathbf{C}^n$. Trans. Am. Math. Soc. **310**, 47–86 (1988)
18. Rosay, J.-P., Rudin, W.: Holomorphic embeddings of $\mathbf{C}$ in $\mathbf{C}^n$. In: Proc. Inst. Mittag-Leffler. Djursholm, 1987–88. Princeton: Princeton University Press 1993
19. Steenrod, N.: The Topology of Fibre Bundles. Princeton: Princeton University Press 1951
20. Stolzenberg, G.: Polynomially and rationally convex sets. Acta Math. **109**, 259–289 (1963)
21. Stout, E.L.: The Theory of Uniform Algebras. Tarrytown on Hudson: Bogden and Quigley 1971
22. Wells, R.O.: Compact real submanifolds of a complex manifold with nondegenerate holomorphic tangent bundles. Math. Ann. **179**, 123–129 (1969)
23. Wermer, J.: The hull of a curve in $\mathbf{C}^n$. Ann. Math. **68**, 550–561 (1958)
24. Wermer, J.: Banach Algebras and Several Complex Variables. Berlin Heidelberg New York: Springer 1976
25. Forstneric, F.: Complements of Runge domains and polynomial hulls. Michigan Math. J. (to appear)