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Proper Holomorphic Mappings: A Survey*

Franc Forstnerič

0. Introduction

A continuous mapping $f: X \to Y$ is called *proper* if $f^{-1}(K)$ is a compact subset of X whenever K is a compact subset of Y. If X and Y are complex spaces and $f: X \to Y$ is a proper holomorphic mapping, then $f^{-1}(y)$ is a compact subvariety of X for all points $y \in Y$. Consequently, if the space X is Stein, the preimage $f^{-1}(y)$ is finite for all $y \in Y$. A special class of proper holomorphic maps are the biholomorphic maps, i.e., the bijective holomorphic maps with a holomorphic inverse.

Proper holomorphic mappings between complex spaces were studied in the 1950s and early 1960s; (see Remmert-Stein [RS]). Perhaps the most important result from this time is the Grauert's theorem on the coherence of direct images of a coherent analytic sheaves [Nar2]. A special but important case of this result is the Proper Mapping Theorem of Remmert [GR]: *If $f: X \to Y$ is a proper holomorphic mapping of complex spaces, and if $A \subset X$ is a complex subvariety of X, then its image $f(A)$ is a complex subvariety of Y.* If both X and Y are Stein spaces and $A \subset X$ is an irreducible subvariety of dimension k, then $B = f(A)$ is an irreducible subvariety of Y of dimension k. Moreover, there exists a proper, nowhere dense subvari-

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ety $V \subset B$, such that $B \backslash V$ and $A \backslash f^{-1}(V)$ are complex manifolds and the restriction $f: A \backslash f^{-1}(V) \to B \backslash V$ is a finitely sheeted holomorphic covering projection.

In this survey we shall consider proper holomorphic mappings $f: D \to D'$ between bounded domains $D \subset\subset \mathbf{C}^n$ and $D' \subset\subset \mathbf{C}^N$. Such mappings are also known as 'finite mappings' since each preimage $f^{-1}(w)$ is a finite subset of D. A map $f: D \to D'$ is proper when for every sequence $\{z_j\} \subset D$ with $\lim_{j\to\infty} \text{dist}(z_j, bD) = 0$, we have $\lim_{j\to\infty} \text{dist}(f(z_j), bD') = 0$. (Note that $N \geq n$.) If f extends continuously to the closure of D, then the extended map takes the boundary bD into the boundary bD', and it satisfies the tangential Cauchy-Riemann equations on bD. Thus proper mappings $f: D \to D'$ lead naturally to the geometric theory of mappings from bD to bD'. A good reference for the structure of proper holomorphic mappings between domains in $\mathbf{C}^n$ is Chapter 15 in Rudin's book [Ru2].

In section 1 we survey the results on the regularity at the boundary of proper holomorphic mappings between smoothly bounded domains in $\mathbf{C}^n$. This has been a very active area of research ever since Fefferman proved in 1974 that biholomorphic mappings between smooth bounded strongly pseudoconvex domains in $\mathbf{C}^n$ extend smoothly to the boundary (Theorem 1.2 below). We first survey the results obtained by the method of the Bergman kernel and the related $\bar{\partial}$-Neumann problem. Most of these results have been covered in the survey by Bedford [Bed1]. There are some new results concerning the Condition $\mathbf{R}$, the most interesting ones due to Boas and Straube [BS1, BS2, BS3], as well as new regularity results for locally proper mappings, due to Bell and Catlin [BC2]. We also present an elementary approach to the regularity problem for mappings of strongly pseudoconvex domains, due to Pinčuk and Hasanov [PH] and Forstnerič [Fo6], that reduces the problem to the $\mathcal{C}^\infty$ version of the edge-of-the-wedge theorem. There is a new regularity result of Pinčuk and Tsyganov for continuous C-R mappings between strongly pseudoconvex hypersurfaces, and an optimal regularity result due to Hasanov.

In section 2 we consider the mappings between bounded domains in $\mathbf{C}^n$ with real-analytic boundaries; the main problem is to show that such mappings extend biholomorphically across the boundary. In one variable this is the classical Schwarz reflection principle. In several variables, this phenomenon was first discovered by Lewy [Lew] and Pinčuk [Pi2] in the case of strongly pseudoconvex boundaries. Very interesting and far reaching generalizations were obtained in recent years by several authors, and the research in this field is still very intensive. Most notably, the problem has been solved on pseudoconvex domains in $\mathbf{C}^n$ by Baouendi and Rothschild [BaR1] and Diederich and Fornæss [DF5].

In section 3 we collect results on mappings between some special classes of domains like the ball, the circular domains, the bounded symmetric domains, the Reinhardt domains, the ellipsoids, and the generalized ellipsoids. Besides the well-known results we present some of the recent developments. Perhaps the most interesting new result is the complete solution of the equivalence problem for Reinhardt domains by Shimizu [Shi].

In section 4 we present rather recent results on existence of proper holomorphic mappings of domains into special higher-dimensional domains like the ball and the polydisc. These new constructions of proper mappings, due mainly to Løw [Løw2, Løw3], Forstnerič [Fo1], and Stensønes [Ste], were motivated by the construction of inner functions in the early 1980s. The initial construction has been improved substantially by Stensønes [Ste].

Section 5 contains regularity results for mappings into higher-dimensional domains. Results in this field are still rather fragmentary in spite of some recent progress. Most of the results of sections 4 and 5 have been obtained since 1985. The reader should compare the surveys by Bedford [Bed1] from 1984 and by Cima and Suffridge [CS2] from 1987.

In section 6 we present results on the classification of proper holomorphic mappings between balls. Although there has been a lot of progress in this direction recently, mainly due to D'Angelo, we do not have a systematic theory yet. Because of its intrinsic beauty and its interesting connections with other areas of Mathematics, this problem would deserve more attention.

Since the literature on this subject is growing very rapidly, I had to omit certain topics in order not to make the paper excessively long. The selection necessarily reflects the personal choice of the author. Among the interesting topics that I have left out are: the branching behavior of a proper mapping at the boundary (see the survey by Bedford [Bed1] and the papers [Bed3], [BBe]); the proper holomorphic correspondences (see [BB2], [BB3], [Be4], [Pi6]); the holomorphic automorphism groups (see [GK], [Ro1], [Wong], [Fra]); the holomorphic continuation of mappings of real-analytic strongly pseudoconvex hypersurfaces (see [Pi3], [Vit]); the proper holomorphic mappings from the disc into higher dimensional domains (see [GS]), and others.

The survey is intended for the specialists in several complex variables. If the reader is looking for motivation, we would like to refer him to the recent survey of a more general nature by Steve Bell [Be7].

I have tried to be accurate with respect to credits. I do not strive for completeness; rather I tried to present the strongest results on each chosen topic. If I have overlooked, some important contributions, I wish to express my sincere apologies to their authors.

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1. Boundary Regularity

One of the central problems in the theory of proper holomorphic mappings is the question of regularity of mappings at the boundary:

Does every proper holomorphic mapping of bounded domains D, D' with smooth boundaries in $\mathbf{C}^n$ *extend smoothly to the boundary of D?*

It is a classical result due to Kellogg [Kel] that the answer is yes in dimension one. If D and D' are bounded by closed Jordan curves, then every biholomorphic mapping of D onto D' extends continuously to the closure of D according to Carathéodory [Ca].

The problem is much more interesting and difficult in dimensions greater than one. When we know that biholomorphic mappings between certain domains must extend smoothly to their boundaries, the question of biholomorphic equivalence is reduced to a geometric problem on the boundary. Using this approach, Chern and Moser [CMo] developed a theory of biholomorphic invariants of strongly pseudoconvex hypersurfaces. See also the papers of Burns et all [BSh], [BSW] on deformation of complex structures.

The regularity problem has been successfully solved on a wide class of pseudoconvex domains, as well as on some special cases of non-pseudoconvex domains. It is still open on general pseudoconvex domains and on most non-pseudoconvex domains.

The result is false for domains in non-Stein manifolds. Barrett [Ba2] found a family of bounded domains D_k ($k \in \mathbf{Z}_+$) contained in complex manifolds M_k of complex dimension two such that

(i) each D_k is a hyperbolic Stein manifold that has smooth real-analytic boundary in M_k,

(ii) each D_k is biholomorphic to D_1, but

(iii) the boundaries bD_{k_1} and bD_{k_2} are not isomorphic as C–R manifolds when $k_1 \neq k_2$.

This means that no biholomorphic mappings $f: D_{k_1} \to D_{k_2}$ extends smoothly to the boundary. The manifolds M_k are not Stein.

The most successful approach to the regularity problem is certainly the one via the Bergman kernel. This method has been developed mainly in the works of C. Fefferman [Fef], Webster [We1], Bell and Ligocka [BL], Bell [Be1, Be2, Be5]), Catlin [Cat1, Cat2], Bell and Catlin [BC1, BC2], and Diederich and Fornæss [DF2, DF4]. The Bergman kernel method gave a positive answer to our question on all pseudoconvex domains of finite type (see Theorem 1.4 below).

On strongly pseudoconvex domains there are at least three other approaches which avoid the Bergman kernel. One is due to Nirenberg, Webster, and Yang [NWY], the second one to Lempert [Le3, Le4], and the third one to Pinčuk and Hasanov [PH] and Forstnerič [Fo6].

We first mention a result on boundary continuity of proper mappings.

Theorem 1.1. *If D and D' are pseudoconvex domains with C^2 boundary in $\mathbf{C}^n$ and $f: D \to D'$ is a proper holomorphic mapping, there exists an $\epsilon > 0$ and constants $c_1 > 0, c_2 > 0$, such that*

$$c_1 \operatorname{dist}(z, bD)^{1/\epsilon} \leq \operatorname{dist}(f(z), bD') \leq c_2 \operatorname{dist}(z, bD)^{\epsilon}, \quad z \in D.$$

If, in addition, the infinitesimal Kobayashi metric on D' satisfies the esti-

mate

$$K_{D'}(z,X) \geq c|X|/\mathrm{dist}(z,bD')^{\delta}, \quad z \in D', X \in \mathbf{C}^n$$

for some $c > 0$ and $\delta > 0$, then f extends to a Hölder continuous map on $\overline{D}$.

This estimate on the Kobayashi metric holds in particular if D' is strongly pseudoconvex [Gra] (in this case f is Hölder continuous with the exponent $1/2$ on $\overline{D}$) or if D' is weakly pseudoconvex with real-analytic boundary [DF1, DF2]. The idea of the proof of Theorem 1.1 is to apply the Hopf lemma to the composition of the given mapping (or its inverse) with a bounded plurisubharmonic exhaustion function on D' (resp. D).

The history of Theorem 1.1 is as follows. Around 1970 the result was proved by Margulis [Mar] and Henkin [Hen] for biholomorphic mappings of strongly pseudoconvex domains, using the Carathéodory metric. Pinčuk [Pi1] generalized the method to proper holomorphic mappings. A similar result was proved independently by Vormoor [Vor]. The first inequality in Theorem 1.1 for weakly pseudoconvex domains was proved for the first time by Range [Ran], using the at that time new bounded plurisubharmonic exhaustion function of Diederich and Fornæss [DF8]. The Kobayashi metric was introduced into the picture in [DF2].

In 1974 C. Fefferman [Fef] made a remarkable discovery by proving:

Theorem 1.2. *Every biholomorphic mapping between bounded, strongly pseudoconvex domains with smooth boundaries in $\mathbf{C}^n$ extends to a smooth diffeomorphism of their closures.*

Fefferman's proof is based on a very careful study of geodesics of the Bergman metric which emanate from a point $z \in D$ close to the boundary in directions close to the normal direction towards the boundary. He showed that these geodesics give a smooth diffeomorphism of an open part of the unit sphere in the tangent space $T_z D$ onto an open part of the boundary bD. Since the Bergman metric is a biholomorphic invariant of the domain, the mapping carries geodesics in D to geodesics in D', and the regularity of the mapping on the boundary follows from its regularity in the domain.

Fefferman's theorem and its very difficult proof stimulated intensive work in this field, with attempts both to simplify the proof and extend the result to a wider class of domains. Another problem was that his proof did not apply immediately to locally biholomorphic or proper holomorphic mappings.

The first simplification was obtained by Webster [We1], who isolated a few crucial properties of the Bergman kernel that would imply extendability of the mapping.

The main progress in this direction was done by Bell and Ligocka [BL] and Bell [Be1] who discovered that the extendability of biholomorphic mappings follows from the global regularity of the Bergman projection on the two domains.

Let D be a bounded domain in $\mathbf{C}^n$. Denote by $L_2(D)$ the Hilbert space of functions on D that are square integrable with respect to the Lebesgue measure, and let $OL_2(D)$ be the subspace consisting of all the holomorphic functions in $L_2(D)$ (the Bergman space). Recall that the Bergman projector on D is the orthogonal projector $P\colon L_2(D) \to OL_2(D)$. More generally, for each $0 \le q \le n$, the Bergman projector P_q is the orthogonal projector from the Hilbert space of square integrable $(0,q)$-forms on D onto the subspace consisting of all $\overline{\partial}$-closed forms (so $P = P_0$).

Following Bell and Ligocka [BL] we say that D satisfies the Condition $\mathbf{R}$ if the projector P_0 is globally regular, in the sense that it maps the subspace $\mathcal{C}^\infty(\overline{D}) \subset L_2(D)$ of functions that are smooth up to the boundary of D to itself.

The following result was proved for biholomorphic maps by Bell [Be1] (see also Bell and Ligocka [BL]). It was extended to proper maps by Bell and Catlin [BC1] and, independently, by Diederich and Fornæss [DF4].

Theorem 1.3. *Let D and D' be bounded pseudoconvex domains in $\mathbf{C}^n$ with smooth boundaries and let D satisfy the Condition* $\mathbf{R}$. *Then every proper holomorphic map of D onto D' extends smoothly to the closure of D.*

An important independent step in the proof for mappings with non-trival branch locus is a division theorem for functions that are holomorphic on D and smooth up to the boundary; see [BC1] and [DF4]. This result holds also for relatively compact domains in Stein manifolds [BBC].

The pseudoconvexity hypothesis in the preceeding theorem may be dropped if both domains satisfy Condition $\mathbf{R}$ [BL].

An independent and rather difficult problem was to show that a large class of domains satisfies the Condition $\mathbf{R}$. The most successful approach so far has been the one via the regularity of the $\overline{\partial}$-Neumann problem. We shall recall the connection only very briefly since this may be found in several sources (see [Ko3], [Ko4], [Bed1]).

Let $D \subset \mathbf{C}^n$ be a bounded pseudoconvex domain with smooth boundary. When $1 \leq q \leq n$, the $\overline{\partial}$-Neumann operator N_q on D is the inverse of the complex Laplacian $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ on $(0,q)$-forms. For the general theory of the $\overline{\partial}$-Neumann operator see Folland and Kohn [FK] and the survey [Ko4].

The connection between the Bergman projection and the Neumann operator is given by the Kohn's formula

$$P_q = \mathrm{Id} - \overline{\partial}^* N_{q+1}\overline{\partial}, \qquad 0 \leq q \leq n-1.$$

Thus, if the Neumann operator N_1 is globally regular, i.e., it maps $C^\infty(\overline{D})$ to itself, then so is the Bergman projector $P = P_0$, whence Condition **R** holds. A more explicit connection between the regularity of the Bergman projection and the Neumann operator can be found in [BS2].

Kohn and Nirenberg proved in [KN2] that the Neumann operator N_q is globally regular when we have a *subelliptic estimate* at every boundary point of D. Kohn [Ko1,2] showed that these estimates hold under certain geometric hypotheses on the boundary; in particular, we have a subelliptic $1/2$ estimate on every strongly pseudoconvex domain. Similar results were obtained by Hörmander. Diederich and Fornæss [DF1] proved such an estimate on weakly pseudoconvex domains with real-analytic boundaries in $\mathbf{C}^n$. Catlin [Cat2] solved the problem completely on smoothly bounded pseudoconvex domains in $\mathbf{C}^n$. He shows that there is a subelliptic estimate at a point $z \in bD$ if and only if z is a *point of finite type* in bD in the sense of D'Angelo [DA1]. This condition means, roughly speaking, that there is an upper bound for the order of contact of bD at z with complex analytic curves passing through z.

As a consequence we see that every smoothly bounded pseudoconvex domain of finite type satisfies Condition **R**. Thus we have (see [BC1] and [DF4])

Theorem 1.4. *Let D and D' be bounded pseudoconvex domains with smooth boundaries in $\mathbf{C}^n$. If D is of finite type, then every proper holomorphic mapping of D onto D' extends smoothly to $\overline{D}$.*

Condition **R** may hold even if there are no subelliptic estimates. For instance, it holds on pseudoconvex domains that are weakly regular in the sense of Catlin [Cat1]. On these domains the $\bar{\partial}$-Neumann operator is globally regular, which implies Condition **R** . Every smooth pseudoconvex domain of finite type is weakly regular at each point, but the converse is not true. For instance, every smooth pseudoconvex domain that is strongly pseudoconvex except at a discrete set of points is weakly regular, so proper holomorphic mappings of such domains extend smoothly to the boundary.

Recently Boas and Straube [BS2, BS3] found a new method to verify Condition **R** . Their main result is that the Bergman projector and the Neumann operator are globally regular (even exactly regular on Sobolev spaces) on smoothly bounded pseudoconvex domains in $\mathbf{C}^n$ that admit a defining function ρ whose complex Hessian is nonnegative at every boundary point $z \in bD$ in all directions. We shall say that such a function is plurisubharmonic at the boundary of D. (It does not need to be plurisubharmonic in any open set containing bD.) Note that this condition is somewhat stronger than that of pseudoconvexity where the Hessian at ρ must be nonnegative only in complex directions that are tangent to the boundary. Thus we have [BS3]

Theorem 1.5. *Let D and D' be bounded pseudoconvex domains in $\mathbf{C}^n$ with smooth boundaries, and let D admit a defining function that is plurisubharmonic at the boundary of D. Then every proper holomorphic map $f: D \to D'$ extends smoothly to the closure of D. This holds in particular when the domain D is (weakly) geometrically convex.*

It is not known whether Condition **R** holds on all pseudoconvex domains. Barrett has found a non-pseudoconvex domain with smooth real-analytic boundary in $\mathbf{C}^2$ on which Condition **R** fails [Ba1]. It is also not known if Condition **R** is necessary for the existence of a smooth extension to the boundary.

There is a local version of the regularity theorem, due to Bell [Be5] and Bell and Catlin [BC2]. This can be stated most naturally in terms of mappings of hypersurfaces that satisfy the tangential Cauchy-Riemann equations (in short, C-R mappings). If the mapping is merely continuous, one should understand that C-R condition in the distributional sense.

Theorem 1.6. *Suppose that $f: M_1 \to M_2$ is a continuous C-R mapping between smooth pseudoconvex hypersurfaces, and M_1 is of finite type. Let $z^0 \in M_1$ and $w^0 = f(z^0) \in M_2$. Suppose that at least one of the* *following hypotheses holds:*

(a) f is finite-to-one on M_1 near z^0;

(b) M_2 does not contain any one-dimensional analytic varieties, and the preimage $f^{-1}(w^0)$ is a compact subset of M_1.

Then f is of class C^∞ in a neighborhood of z^0. Furthermore, if f is a local C-R homeomorphism near z^0, then it is a local C-R diffeomorphism near z^0. In case (b) it follows that the preimage $f^{-1}(w^0)$ is finite and f is finite-to-one in its neighborhood.

Even though the Bergman kernel method has been very successful in the study of the boundary regularity problem, it involves the difficult analysis of the $\overline{\partial}$-Neumann operator and results from the theory of partial differential equations. For this reason the method cannot be considered elementary from the point of view of several complex variables.

There are at least three alternative approaches, due to Nirenberg, Webster, Yang [NWY], Lempert [Le3, Le4], and Pinčuk and Hasanov [PH] and Forstnerič [Fo6]. These methods only apply to mappings of strongly pseudoconvex domains. The advantage is that they rely on standard methods of several complex variables. The methods are entirely local and they apply also to the case when the boundaries have only a finite degree of smoothness. The method by Pinčuk and Hasanov is especially interesting and elementary.

The approach by Nirenberg, Webster, and Yang [NWY] is a natural generalization of the extension theorem of Lewy [Lew] and Pinčuk [Pi2], where the two boundaries are real-analytic, to the case of smooth boundaries. The crucial ingredient is the use of almost anti-holomorphic reflections across the boundary bD within complex lines that are transverse to the boundary. The extended mapping is the solution of a system of equations that are obtained by differentiating the initial equation and describing the condition $f(bD) \subset bD'$. The difficult part is to prove a transversality property for the mapping near the boundary (Condition A, p. 319 in [NWY]). The main result of the paper [NWY] is a different and more elementary proof of Fefferman's theorem (Theorem 1.2 above).

Lempert obtained a proof of Fefferman's theorem from his work [Le3] on the extremal discs for the Kobayashi metric on strongly convex domains in $\mathbf{C}^n$. Since the extremal discs are biholomorphically invariant, and since f is locally biholomorphic according to Pinčuk [Pi4], the smoothness of

f follows immediately from the smoothness of the extremal discs. One only needs to consider discs through points close to the boundary in the directions that are close to the complex tangent direction to the boundary at the nearest boundary point. Lempert constructed these extremal discs by the continuity method. One part involves solving a suitable Riemann-Hilbert boundary value problem for matrix-valued functions on the unit disc. The second part is to find apriori estimates for the extremal discs. Even though his construction is not easy, the method is very natural and self-contained. His foliation of strongly convex domains by the Kobayashi-extremal discs has proven very useful in many other problems.

Perhaps the most natural and simplest proofs of Fefferman's theorem were given by Pinčuk and Hasanov [PH] and by Forstnerič [Fo6]. Their approach is based on the reflection principle of Lewy [Lew] and Pinčuk [Pi2] (see section 2 below), and on the smooth version of the edge-of-the-wedge theorem from [PH]. The classical edge-of-the-wedge theorem was first used in this context by Webster [We3]. I wish to thank J. P. Rosay who called my attention to the work [PH].

The methods of Lempert [Le4], Pinčuk and Hasanov [PH], Forstnerič [Fo6], and Hurumov [Hur] give the following sharp regularity result for mappings of strongly pseudoconvex domains:

Theorem 1.7. *Let D and D' be bounded, strongly pseudoconvex domains in $\mathbf{C}^n$ with boundaries of class $C^m, m > 2$. Then every proper holomorphic mapping $f: D \to D'$ extends to a map of class $C^{m-1/2-0}(\overline{D})$.*

The fact that such a mapping is necessarily locally biholomorphic was proved by Pinčuk [Pi4].

Here, $m > 2$ may be any real number. If $m = k + \alpha$, with $0 < \alpha < 1$, then $C^m = C^{k,\alpha}$ is the usual Hölder class. Recall that $C^{m-0} = C^m$ if m is not an integer, and equals $\bigcup_{0<\alpha<1} C^{m-1,\alpha}$ if m is an integer.

This result was proved in [Le4], [PH], and [Fo6], with the weaker conclusion that $f \in C^{m-1-0}(\overline{D})$. The last step from C^{m-1-0} to $C^{m-1/2-0}$ was done recently by Hurumov [Hur], using some general regularity theory for elliptic equations. (Hurumov's result was announced by Pinčuk at the AMS Summer Research Institute 1989 in Santa Cruz.)

The following example, due to Hurumov, shows that the conclusion of Theorem 1.7 is sharp.

Example. Let $k > 3/2$ be a non-integer,

$$f(z_1, z_2) = (z_1 + z_2^k, z_2),$$
$$D = \{z \in \mathbf{C}^2 : x_2 + |z|^2 < 0\}.$$

Obviously f is a well-defined biholomorphic mapping from D onto some bounded domain $D' \subset \mathbf{C}^2$, $f \in C^k(\overline{D})$, and $f \notin C^l(\overline{D})$ for $l > k$. It is easy to check that D' is strongly pseudoconvex and that bD' is of class $C^{k+1/2}$.

Subsequently, Pinčuk and Tsyganov [PT] provided the same result for all local C-R mappings of strongly pseudoconvex hypersurfaces without assuming properness:

Theorem 1.8. *Every continuous C-R mapping $f: M \to M'$ of smooth strongly pseudoconvex hypersurfaces is smooth on M, with the loss of smoothness as in Theorem 1.7.*

From a previous result of Pinčuk [Pi4] it follows that such a map is either constant or a local diffeomorphism on every connected component of M.

Theorem 1.8, together with a theorem of Alexander (see Theorem 3.2 below), implies

Corollary 1.9. *Let $\Omega \subset b\mathbf{B}^n$ be an open connected subset of the unit sphere in $\mathbf{C}^n$, $n \geq 2$. If $f: \Omega \to b\mathbf{B}^n$ is a continuous C-R mapping with range in the sphere, then f extends to an automorphism of the ball $\mathbf{B}^n$.*

A similar result holds for mappings of certain quadric C-R manifolds of higher codimension in $\mathbf{C}^n$; see the papers [TH1, TH2], [Tum2], and [Fo8].

So far there is no comparable regularity result for continuous C-R mappings of weakly pseudoconvex hypersurfaces without further assumptions on the mapping; the reader should compare this with Theorem 1.6 above.

We shall briefly outline the proof of Fefferman's theorem given in [PH] and [Fo6]. For the sake of simplicity we shall assume that D and D' are strongly pseudoconvex domains with C^∞ boundaries, and $f: D \to D'$ is a biholomorphic mapping. Recall that f extends bicontinuously to the closures according to Theorem 1.1.

Following Webster [We3], we associate to f a holomorphic mapping

$$F(z, \Lambda) = (f(z), f'(z)\Lambda),$$

defined in the space of pairs (z, Λ) where z is a point in D and Λ is a complex $(n-1)$ plane in $\mathbf{C}^n$ passing through z. We may consider such $(n-1)$ planes as the points in the complex projective $(n-1)$-space $\mathbf{CP}^{n-1}$ and take as the ambient space the manifold $\mathbf{C}^n \times \mathbf{CP}^{n-1}$. Here, $f'(z)\Lambda$ is the image of Λ by the derivative of f at z.

The mapping F is holomorphic in the wedge domain $D \times \mathbf{CP}^{n-1}$ with the smooth, totally real edge $M = \{(z, T_z^C bD): z \in bD\}$. Here, $T_z^C bD$ is the maximal complex subspace of the real tangent space to bD at the point z. The total reality of M is equivalent to bD being Levi-nondegenerate, see Webster [We3].

The most difficult part of the proof is to show that there is a smaller wedge W^+ with the same edge M such that the restriction of F to W^+ extends continuously to the edge M and maps M to the totally real smooth manifold $M' = \{(z', T_z^C bD'): z' \in bD'\}$ associated to the second boundary bD'. Pinčuk and Hasanov proved this using the scaling method with non-homogeneous dilations and the theorem of Alexander [Al1] to the effect that all proper self-mappings of the ball are automorphisms. (See Theorem 3.1 below.) In the paper [Fo6] it was shown that the continuity of F up to the edge follows from the Julia-Carathéodory's theorem for maps between balls. The method developed in [Fo6] generalizes naturally to mappings of certain Cauchy-Riemann manifolds of higher codimension, and we have an analogue of Fefferman's theorem in this setting [Fo7].

To a smooth, generic, totally real submanifold $M \subset \mathbf{C}^{2n-1}$ one can associate a smooth reflection Φ defined in a neighborhood of M, which fixes M and is almost anti-holomorphic, in the sense that its holomorphic derivatives vanish to infinite order on M. Let Φ' be a similar reflection on M'. Denote by W^- the wedge $\Phi(W^+)$ with the same edge M. We extend the mapping F to W^- by setting $F(p) = \Phi' \circ F \circ \Phi(p)$ for $p \in W^-$. The extended mapping is almost anti-holomorphic on the double wedge $W^+ \cup W^-$ and continuous up to their common edge M. The C^∞ version of the edge-of-the-wedge theorem [PH] now implies that the restriction of F to M is smooth; hence, f is smooth on bD.

Pinčuk and Hasanov gave a direct proof of the last step using the Cauchy integrals. Alternatively, one can understand this step in terms of the C^∞ wave front set (see Hörmander [Hö1]). Composing F with a smooth diffeomorphism which maps the edge M to an open subset of $\mathbf{R}^{2n-1}$ and which is $\overline{\partial}$-flat on M (i.e., its $\overline{\partial}$ derivatives vanish to infinite order on M), we may deal with a wedge whose edge is an open subset of $\mathbf{R}^{2n-1}$. Passing to a smaller wedge, we may also assume that $W^+ = M + i\Gamma$ and $W^- = M - i\Gamma$, where Γ is an open convex cone in $\mathbf{R}^{2n-1}$. Let $\Gamma^0 \subset \mathbf{R}^{2n-1}$ be the closed dual cone (the polar of Γ). Since F is almost holomorphic in W^+, the C^∞ wave front set of $F|_M$ at a point $Z \in M$ is contained in Γ^0 (see [Hö1], p. 257). By the same argument for W^- the wave front set is contained in $-\Gamma^0$. Since the intersection $\Gamma^0 \cap -\Gamma^0$ is empty, the wave front set is empty at Z, i.e., F is smooth in a neighborhood of Z.

Several problems appear if one tries to use this method on domains that are not strongly pseudoconvex. First, the associated manifold M is no longer totally real. Furthermore, if f is branched, one cannot even define F as above. Also, the method of dilations does not apply directly since the limit domain would no longer be the unit ball. Even so, it would be nice to see this approach extended to a larger class of pseudoconvex domains.

We mention a theorem of a different nature. One may ask whether the continuity of a proper mapping at the boundary is the consequence of suitable *local* assumptions on the two boundaries. A result of this type is due to Forstnerič and Rosay [FR]; we shall only cite a special case.

Theorem 1.10. *Let $f: D \to D'$ be a proper holomorphic map of a domain $D \subset \mathbf{C}^n$ onto a bounded domain $D' \subset \mathbf{C}^n$. Assume that bD is of class C^2 and strongly pseudoconvex in a neighborhood of a point $z^0 \in bD$. If there is a sequence $\{z^j\} \subset D$ converging to z^0, such that the image sequence $\{f(z^j)\} \subset D'$ converges to a point $w^0 \in bD'$ and bD' is of class C^2 and strongly pseudoconvex in a neighborhood of w^0, then f extends to a Hölder continuous map with the exponent $1/2$ on a neighborhood of z^0 in $\overline{D}$.*

Note that there are no conditions on the two boundaries away from the points z^0, resp. w^0. One needs to assume much less on the two boundaries. What is important is to have good local holomorphic peaking functions on D' near w^0. The theorem was proved by a localization of the Kobayashi metric in a neighborhood of a certain pseudoconvex boundary point of a bounded domain in $\mathbf{C}^n$. Of course the theorem does not give anything new if the domains have globally smooth boundaries. However, a slightly stronger version of the theorem gives:

Corollary 1.11. *Let D be a domain in $\mathbf{C}^n$ with a plurisubharmonic defining function of class $C^{1+\epsilon}$ for some $\epsilon > 0$, and let D' be a bounded domain in $\mathbf{C}^n$ whose boundary is of class C^2 and strongly pseudoconvex outside a closed, totally disconnected subset $E \subset bD'$. Then every proper holomorphic mapping of D onto D' extends continuously to $\overline{D}$.*

If neither of the two domains $D, D' \subset \mathbf{C}^n$ has smooth boundary, there is no reason why a biholomorphic or proper map should extend continuously to the closure. Fridman [Fri] has shown that there is a bounded domain D in $\mathbf{C}^2$ with piecewise-smooth boundary and a biholomorphic mapping of D onto the unit polydisc in $\mathbf{C}^2$ that does not extend continuously to $\overline{D}$. Also, there is a domain D in $\mathbf{C}^2$ whose boundary is real-analytic and strongly pseudoconvex, except at one point $z \in bD$, and there is an automorphism of D that does not extend continuously to $D \cup \{z\}$ [FR]. Another example from [FR] shows that the condition in the Corollary is sharp, in the sense that the conclusion may be false if the exceptional set $E \subset bD'$ is a circle.

We mention a result of Lempert [Le5] in which the usual pseudoconvexity assumptions are replaced by a global geometric condition. Recall that a domain D is star-shaped with respect to a point $z^0 \in D$ if the line segment from any point $z \in D$ to z^0 is contained in D. The domain is strictly star-shaped if, in addition, the line segment from any boundary point $z \in bD$ to z^0 intersects bD at an angle that is uniformly bounded away from 0.

Theorem 1.12. *Let D and D' be bounded, strictly star-shaped domains with real-analytic boundaries in $\mathbf{C}^n$. Then every biholomorphic mapping of D onto D' extends to a Hölder continuous homeomorphism of $\overline{D}$ onto $\overline{D}'$.*

We close the section with some open problems.

1. As mentioned at the beginning of the section, it is not know whether mappings between bounded domains in $\mathbf{C}^n$ with smooth boundaries extend smoothly to the closure. A class of special domains on which the problem has not been solved are the Hartogs domains in $\mathbf{C}^n$ for $n > 2$, except of course the pseudoconvex domains of finite type. For domains in $\mathbf{C}^2$ see Theorem 3.12 below.
2. In the local extension theorem of Bell and Catlin (Theorem 1.5) one would like to remove the hypothesis that the fiber $f^{-1}(w_0)$ is com-

pact. More precisely, let $f: M_1 \to M_2$ be a continuous C-R mapping of smooth pseudoconvex hypersurfaces on finite type. Is f also smooth?

3. Is it possible to extend the elementary approach by Pinčuk and Hasanov to a wider class of domains?
4. Study the regularity of C-R mappings between smooth generic C-R submanifolds in $\mathbf{C}^n$ of codimension $d > 1$. It is well-known that the one-sided holomorphic extension of C-R functions is replaced by holomorphic extension into wedges (see section 2 below). What is the correct analogue of a proper mapping in this setting? Under what geometric conditions is every local C-R homeomorphism of such C-R manifolds smooth? For partial results in this direction see [Fo7].
5. Bedford and Bell proved in [BB2] that every proper holomorphic self-mapping of a bounded pseudoconvex domain $D \subset \mathbf{C}^n$ with real-analytic boundary is biholomorphic. Is the same true when bD is smooth?

2. Analyticity of C-R Mappings

In this section we present results on the following extension problems:

Local extension problem. Let M, $M' \subset \mathbf{C}^n$ be (germs of) smooth real-analytic hypersurfaces at the origin. Locally near the origin M splits the space in two open half-spaces M^- and M^+. Suppose that a continuous mapping $f: M^- \cup M \to \mathbf{C}^n$ is holomorphic on M^- and maps M into M'. Under what conditions does f extend holomorphically to an open neigborhood of the origin in $\mathbf{C}^n$?

Global extension problem. Let D and D' be relatively compact domains with smooth real-analytic boundaries in $\mathbf{C}^n$ (or in n-dimensional Stein manifolds). Does every proper holomorphic mapping $f: D \to D'$ extend holomorphically to a neighborhood of $\overline{D}$?

In one variable the map in the local problem extends according to the Schwarz reflection principle. In the global problem one first applies the classical theorem of Carathéodory to obtain a continuous extension of f to $\overline{D}$.

From now on we shall always assume that $n \geq 2$. The first extension result in several variables was obtained in 1974 by Lewy [Lew] and Pinčuk [Pi2] for strongly pseudoconvex hypersurfaces (see Theorem 2.1

below). Even though the global extension problem is still open on non-pseudoconvex domains in $\mathbf{C}^n$, it has been solved for the pseudoconvex domains by Baouendi, Bell, and Rothschild [BaBR, BaR1, BaR2] and, independently, by Diederich and Fornæss [DF5]. (See Theorem 2.2 below.) It was also solved on domains in C^2 with smooth algebraic boundaries [DF6]. Results in this direction have been obtained in recent years by several authors, see [We2, We3, We5, BB1, Be2, Der, Han, DW, BaJT].

As we have mentioned in section 1, Barrett [Ba2] constructed examples of biholomorphic mappings between domains in non-Stein manifolds that do not extend even continuously to the boundary. The common feature of such examples is that the boundaries of the domains contain complex analytic varieties. This is not possible for bounded domains with real-analytic boundaries in $\mathbf{C}^n$ [DF1].

The formulation of the problem is intentionally a bit vague. Typically one would like to prove the existence of a holomorphic extension under few or no extra conditions on the map f. Of course, one has to put some conditions on M and M'. For instance, if M and M' are Levi-flat, f need not extend. The simplest example of this type is $M = M' = \{(z_1, z_2) \in \mathbf{C}^2 : \Re z_1 = 0\}, f(z_1, z_2) = \big(z_1, z_2 + g(z_1)\big)$, where $g(\zeta)$ is a holomorphic function on $\{\Re\zeta < 0\}$, smooth on $\{\Re\zeta \leq 0\}$, that does not extend holomorphically across $\{\Re\zeta = 0\}$.

The conditions on the two hypersurfaces can sometimes be traded with the conditions on f. In most results one assumes that f is not merely continuous, but also smooth. Often the smoothness of f follows from one of the regularity theorems in section one.

The set M^- plays an auxiliary role. The hypothesis that f is holomorphic on M^- guarantees that the restriction $f|_M$ is a C-R map. It follows by a usual complexification argument that f will extend holomorphically to a neighborhood of the origin if and only if $f|_M$ is real-analytic at $0 \in M$. Thus our problem is a special case of the following

Regularity problem. Let $f: M \to M'$ be a C-R mapping of smooth real-analytic hypersurfaces at the origin in $\mathbf{C}^n$. Under what conditions is f real-analytic at the origin?

If M does not contain a complex hypersurface passing through the origin, then every C-R function f on M extends holomorphically to one side of M, say M^-, according to Trepreau [Tre]. Hence, the two problems are equivalent in this case.

The second formulation of the problem can be generalized immediately to C-R mappings of real-analytic C-R manifolds of higher codimension in $\mathbf{C}^n$. The one-sided holomorphic extension of f is replaced in this case by holomorphic extension to wedges with edge M. We shall mention some results of this type below.

If the Levi form of M at 0 has at least one positive and one negative eigenvalue, then every C-R function (and even every C-R distribution) on M extends to a holomorphic function in a neighborhood of the origin in $\mathbf{C}^n$ [BaCT]. Thus the problem is only interesting when the eigenvalues of the Levi form at 0 have the same sign.

Our subject started around 1974 with the following 'reflection principle' due to Lewy [Lew] and, independently, Pinčuk [Pi2]:

Theorem 2.1. *Let M and M' be real-analytic hypersurfaces at the origin in $\mathbf{C}^n$. Assume that $f\colon M^- \cup M \to \mathbf{C}^n$ is a map of class C^1 that is holomorphic in M^- and $f(M) \subset M'$. If the derivative $f'(0)$ is non-degenerate and M' is strongly pseudoconvex at 0, then f extends holomorphically to a neighborhood of the origin in $\mathbf{C}^n$. If M is also strongly pseudoconvex at 0, then $f'(0)$ is necessarily non-degenerate.*

We recall briefly the idea of the proof. The tangential C-R operators on the hypersurface M play a very important role. These are complex vector fields of type $(0,1)$ that are tangent to M. Let r be a real-analytic real-valued defining function of M near the origin, i.e.,

$$M = \{z \in \mathbf{C}^n : r(z, \bar{z}) = 0\}, \tag{2.1}$$

where $r(0) = 0$, $dr \neq 0$. We may assume that $\partial r/\partial \bar{z}_n(0) \neq 0$. Then we may take as the basic tangential C-R operators on M

$$L_j = \frac{\partial}{\partial \bar{z}_j} - \frac{\partial r/\partial \bar{z}_j}{\partial r/\partial \bar{z}_n}\frac{\partial}{\partial \bar{z}_n}, \qquad 1 \leq j \leq n-1.$$

Suppose that r' is a real-analytic defining function of the second hypersurface M'. Let $0 \in M$ and $f(0) = 0 \in M'$. Applying the L_j's to the identity

$$r'\big(f(z), \overline{f(z)}\big) = 0, \qquad z \in M, \tag{2.2}$$

we obtain $n-1$ additional identities on M:

$$\sum_{j=1}^{n} \frac{\partial r'}{\partial \overline{z}'_j}\big(f(z), \overline{(f(z))}\big) L_s \overline{f_j(z)} = 0, \qquad z \in M, 1 \leq s \leq n-1. \tag{2.3}$$

We consider (2.2)-(2.3) as a system of n holomorphic equations in the unknowns $(f_1, \ldots, f_n)$. A simple calculation shows that its complex Jacobian with respect to f is nonvanishing at the origin, provided that the Levi form of r' is nondegenerate on $T_0^C M'$ and $f'(0)$ is injective on $T_0^C M$. Solving the system, we obtain a new set of identities

$$f = Q(\overline{f}, L_1\overline{f}, \ldots, L_{n-1}\overline{f}), \tag{2.4}$$

where Q is a holomorphic function in all arguments.

Recall that f is holomorphic in M^-. Fix a complex line $\Gamma \subset \mathbf{C}^n$ transverse to M. Its intersection with M is a real-analytic curve γ that splits Γ locally in two half-lines $\gamma^+ = \Gamma \cap M^+$ and $\gamma^- = \Gamma \cap M^-$. Since $f|_\gamma$ extends holomorphically to γ^-, its conjugate $\overline{f}|_\gamma$ extends holomorphically to γ^+. By extending the real-analytic coefficients of L_j from γ to holomorphic function on Γ we can also extend the functions $L_j\overline{f}|_\gamma$ to γ^+. Thus the right-hand side of (2.4) defines a holomorphic extension of f to γ^+. This can be done uniformly on a family of parallel lines, so we obtain a holomorphic extension of f across the origin. This proves Theorem 2.1.

If M' is not strongly pseudoconvex at the origin, or if the derivative $f'(0)$ is degenerate, the implicit function theorem does not apply. A natural idea then is to differentiate the equations (2.3) further with respect to the operators L_j, obtaining more and more equations. This idea was pursued by several authors, see the papers [Der, Han], and the papers cited below. Of course, the map f must be smooth to begin with.

The result has been generalized to mappings between much wider classes of hypersurfaces, and also to mappings between C-R manifolds of higher codimension in $\mathbf{C}^n$. The following outstanding global result follows from the independent works of Baouendi, Bell, and Rothschild [BaBR], [BaR1], Diederich and Fornæss [DF5], and from Theorem 1.4 above:

Theorem 2.2. *Let D and D' be bounded pseudoconvex domains with smooth real-analytic boundaries in $\mathbf{C}^n$. Then every proper holomorphic mapping of D onto D' extends to a proper holomorphic mapping from a neighborhood of $\overline{D}$ onto a neighborhood of $\overline{D}'$.*

The analytic part of this theorem is a consequence of more general *local* extension results, obtained in the papers [DF5], [BaR1], [BaBR], [BaR2], [BaBR4], that we shall now describe. Because of the larger number of results following the pioneering work of Lewy and Pinčuk we shall concentrate on the most important contributions. We try to be as accurate as possible with respect to credits and priorities, which seems a difficult task in the present case.

Essentially there have been two main lines of development; one by Webster [We2, We3], Diederich and Webster [DW], and Diederich and Fornæss [DF5]; the other one by Baouendi, Jacobowitz, and Treves [BaJT], Baouendi, Bell and Rothschild [BaBR], and Baouendi and Rothschild [BaR1, BaR2, BaR4]. We should also mention a paper by Bedford and Pinčuk [BP2].

We shall first deal with mappings of hypersurfaces, deferring the mappings between C-R manifolds of higher codimension to the end of this section.

In the paper [We3], Webster showed how Theorem 2.1 can be proved using the classical edge of the wedge theorem, at least when the hypersurfaces are strongly pseudoconvex. An important contribution of Webster in [We2] and [We3] is the introduction of the so called 'Segre varieties' to the mapping problem; see (2.5) below. Implicitly these varieties already appeared in the works of Lewy and Pinčuk , but Webster made this much more explicit.

An important step in extending the result to more general hypersurfaces was the paper [DW] by Diederich and Webster in 1980. They proved that a biholomorphic mapping $f\colon D \to D'$ between pseudoconvex domains with real-analytic boundaries, such that f and f^{-1} are sufficiently smooth up to the boundary, extends holomorphically to every point $z \in bD \backslash E$ outside a real-analytic subset $E \subset bD$ of real codimension at least two in bD. The technique developed in this paper was later improved by Diederich and Fornæss [DF5] in 1987, where they proved a rather general local extension result (see Theorem 2.3 below) that implies Theorem 2.2.

In the paper [DW], the authors recognized the importance of the condition of *essential finiteness*, see definition below. This terminology is due to Baouendi, Jacobowitz, and Treves [BaJT], who introduced the condition in a slightly different but equivalent form in 1985.

The paper by Baouendi, Jacobowitz, and Treves was an important contribution in the development of this theory. They showed in particular that, if the hypersurface M is essentially finite and M' is of finite type at the origin, then every smooth C-R diffeomorphism $f: M \to M'$, $f(0) = 0$, is real-analytic near 0. The point is that they proved extendability of the mapping at *every point* where the two requirements are met. The authors replaced the implicit function theorem by a more delicate method of systematic elimination of variables in a set of analytic equations. They also obtained results on extendability in the case of C-R manifolds of higher codimension. Their approach was developed further in the papers [BaBR] and [BaR1, BaR2, BaR4]. We shall return to the paper [BaJT] at the end of this section; see Theorem 2.8 below.

We shall now explain the main idea of the papers by Diederich and Webster [DW] and Diederich and Fornæss [DF5]. Suppose that, near the origin in $\mathbf{C}^n$, the hypersurface M is defined by (2.1). If we set $w = \overline{z}$ and vary z and w independently, we obtain a holomorphic function $r(z, w)$ of $2n$ variables. The set

$$M^0 = \{(z, w) \in \mathbf{C}^{2n}: r(z, w) = 0\}$$

is a local complex hypersurface in $\mathbf{C}^{2n}$, called the *polar* of M. It is the complexification of the totally real submanifold $\{(z, \overline{z}): z \in M\} \subset \mathbf{C}^{2n}$.

For each fixed $w \in \mathbf{C}^n$ we define the *Segre variety*

$$Q_w = \{z \in \mathbf{C}^n: r(z, \overline{w}) = 0\}. \tag{2.5}$$

This is a nonsingular local complex hypersurface near the origin in $\mathbf{C}^n$, the intersection of the polar M^0 with a hyperplane $\overline{w} = constant$. These hypersurfaces were apparently first introduced by Segre [Seg] in a different context. Their importance in the mapping problem comes from the following simple observation, due to Webster [We2]. Let $M' = \{z \in \mathbf{C}^n: r'(z, \overline{z}) = 0\}$ be another real-analytic hypersurface, and let $f: U \to U'$ be a holomorphic mapping between small neighborhoods of 0 in $\mathbf{C}^n$, satisfying $f(M) \subset M'$ and $f(0) = 0$. This means that $r'(f(z), \overline{f(z)}) = 0$ for $z \in M$. Hence we have an identity

$$r'(f(z), \overline{f(z)}) = p(z, \overline{z}) r(z, \overline{z})$$

for $z \in \mathbf{C}^n$ near 0, with $p(z, \overline{z})$ a real-analytic function. Setting $\overline{z} = \overline{w}$ and varying z and w independently we have an identity

$$r'(f(z), \overline{f(w)}) = p(z, \overline{w}) r(z, \overline{w})$$

for z and w in a suitably smaller neighborhood of the origin in $\mathbf{C}^n$. If we fix w and let $z \in Q_w$, both sides of the identity are zero, so we conclude that $f(z) \in Q'_{f(w)}$. Here, $Q'_{w'}$ is the Segre variety associated to M'. This shows that

$$f(Q_w) \text{ is contained in } Q'_{f(w)}$$

for all $w \in \mathbf{C}^n$ near 0. The same is true if $M' \subset \mathbf{C}^N$ with $N > n$; this was exploited in [Fo4]. If f is locally biholomorphic, we may apply the same argument to f^{-1}, thus showing that the family of varieties $\{Q_w\}$ is invariantly attached to M.

Let A_w denote the fiber of the mapping $w \to Q_w$:

$$A_w = \{z \in \mathbf{C}^n : Q_z = Q_w\}. \tag{2.6}$$

This is also a local complex variety in $\mathbf{C}^n$. We list some elementary but important properties of these sets (see [DW] or [DF5]):

(a) $z \in Q_w$ if and only if $w \in Q_z$,
(b) $z \in Q_z$ if and only if $z \in M$,
(c) $z \in A_z$,
(d) if $z \in M$, then A_z is a complex subvariety of M, and
(e) $A_w = \cap\{Q_z : z \in Q_w\}$.

Suppose now that f is holomorphic only in one side M^- of the hypersurface and smooth on $M^- \cup M$. If $w \in M^+$ is close to 0, then $Q_w \cap M^-$ is a nonempty complex hypersurface, so we can try to find a point $w' \in \mathbf{C}^n$ such that

$$f(Q_w) \subset Q'_{w'}. \tag{2.7}$$

If such a w' exists and is unique, it is natural to set $f(w) = w'$ and hope that this would give a desired holomorphic extension of f.

There are several problems. To show that there exists at least one point w' satisfying (2.7) we must use the condition that f is smooth on $M^- \cup M$ and maps M to M'. The condition (2.7) makes no sense for points $w \in M$ since Q_w may then be contained in $M^+ \cup \{w\}$. However, in actual proof, (2.7) is replaced by a differential condition involving the Taylor coefficients of the Segre varieties and the mapping which makes sense also for $w \in M$.

A more serious problem is related to the fact that the mapping $w \in Q_w$ need not be one-to-one. It turns out that it is sufficient to require that this mapping be finite-to-one. The importance of this condition was first recognized by Diederich and Webster [DW]. In a slightly different but equivalent form it was introduced and given a name by Baouendi, Jacobowitz, and Treves [BaJT]:

Definition. *A smooth real-analytic hypersurface $M \subset \mathbf{C}^n$ is said to be essentially finite at $0 \in M$ if the subvariety*

$$A_0 = \{z \in \mathbf{C}^n : Q_z = Q_0\} = \cup\{Q_z : z \in Q_0\}$$

of M is equal to $\{0\}$ (in the sense of germs at the origin).

Before proceeding, we collect some observations about this property (see [DW] and [BaJT]).

1. If M does not contain any positive dimensional complex subvariety, then it is essentially finite. (This is trivial since V is a complex subvariety of M.)
2. From 1. and a result of Diederich and Fornæss [DF1] it follows that every compact real-analytic hypersurface in $\mathbf{C}^n$ is essentially finite at each point.
3. If M contains a germ of a complex hypersurface through the point 0, then M is not essentially finite.
4. If a real-analytic hypersurface M is essentially finite, then it is of finite type in the sense of Bloom and Graham [BlG]. (This is in fact a restatement of 3.)
5. The real hypersurface $M \subset \mathbf{C}^3$ defined by

$$\Re w = |z_1|^2 + |z_1 z_2|^2 + |z_2|^2(z_1^4 + \bar{z}_1^4)$$

is pseudoconvex in a neighborhood of the origin, and it is essentially finite even though it contains the complex line $z_1 = w = 0$.

The assumption that M is essentially finite at 0 implies that $\dim A_z = 0$ for all z close to the origin. Hence, for each w, there will be at most finitely many points w' satisfying (2.7). It turns out that the set

$$X = \{(w, w') : w \in M^+, f(Q_w \cap M^-) \subset Q'_{w'}\},$$

localized suitably near the origin, is an n-dimensional complex subvariety in $\mathbf{C}^{2n}$ that is a branched covering over M^+.

If M is essentially finite at 0, then it can not contain a germ of a complex hypersurface at 0 according to the property 3 above. Hence we may assume, according to a result of Baouendi and Treves [BaT] and Trepreau [Tre], that all holomorphic functions on M^+ extend holomorphically to a full neighborhood of the origin (otherwise M^- has this property and we can simply extend f across 0). By extending the canonical defining functions of the analytic cover $X \to M^+$ we can extend X to a complex n-dimensional subvariety in a full neighborhood of the origin in $\mathbf{C}^{2n}$. This extension contains the graph

$$\Gamma_f = \{(z, f(z)) : z \in M^- \cup M\}$$

of f. Hence each component of f satisfies an identity $P_j(f_j; z) = 0$, where $P_j(t; z)$ is a Weierstrass polynomial in t. Since f_j is C^∞ on M, it follows from the Artin-Rees lemma that $f|_M$ is real-analytic at 0.

This outline can be used to prove the following local extension result, due to Diederich and Fornæss [DF5] and, independently and simultaneously, to Baouendi and Rothschild [BaR1] (see also [BaBR] for the case $n = 2$). Let $z = (z_1, \ldots, z_n)$ be the coordinates in $\mathbf{C}^n$.

Theorem 2.3. *Let $M, M' \subset \mathbf{C}^n$ be smooth real-analytic hypersurfaces at the origin, with tangent space $\Re z_n = 0$ at 0. Assume that M is essentially finite at the origin. If $f = (f_1, \ldots, f_n)$ is a smooth mapping on $M^- \cup M$ that is holomorphic in M^- and satisfies the conditions $f(M) \subset M'$, $f(0) = 0$, and $\partial f_n(0)/\partial z_n \neq 0$, then f extends holomorphically to a neighborhood of 0 in $\mathbf{C}^n$.*

Actually, both papers [BaR1] and [DF5] give a similar result under somewhat weaker hypotheses, but they do get more technical. We refer the interested reader to the original papers. In the case when $f: M \to M'$ is a smooth local C-R diffeomorphism at 0, Theorem 2.3 was also proved by Bedford and Pinčuk [BP2].

The transversality condition $\partial f_n(0)/\partial z_n$ is satisfied when M and M' are pseudoconvex and f maps the pseudoconvex side of M to the pseudoconvex side of M' [Fn2]. Since every compact real-analytic hypersurface in $\mathbf{C}^n$ is essentially finite, Theorems 1.4 and 2.3 imply Theorem 2.2 above.

In the subsequent paper [BaR4], Baouendi and Rothschild have obtained a sharper local result on holomorphic extendability (see Theorem 6 in [BaR4]). With the appropriate choice of local holomorphic coordinates it suffices to require that M is essentially finite and the transverse component f_n of the given smooth C-R mapping $f: M \to M'$ is not flat at 0.

The conclusion then is that $f|_M$ is real-analytic at 0. They also proved extendability under the asssumption that M' is essentially finite at 0 and the mapping f is either of *finite multiplicity* or else it is *not totally degenerate* at 0; we refer the reader to [BaR4] for the definitions and precise formulation. The first of these two cases falls under the scope of Theorem 2.3. Namely, if M' is essentially finite, then it is of finite type in the sense of Bloom and Graham. If, in addition, f is of finite multiplicity at 0, it follows that the transverse component f_n satisfies $\partial f_n(0)/\partial z_n \neq 0$ [BaR4, Theorem 1], and also that M is essentially finite [BaR1]. Thus Theorem 2.3 applies.

In the case when both M and M' are Levi nondegenerate, it suffices to use the first order information carried by Q_w, i.e., the tangent space $T_z Q_w$ at a point $z \in Q_w$. The mapping $(z, T_z Q_w) \to (w, T_w Q_z)$ is an anti-holomorphic reflection that fixes the totally real manifold $\{(z, T_z M) : z \in M\}$. Using these reflections one can reduce the extension problem to the edge-of-the-wedge theorem. (See Webster [We3].) We explained this method in connection with the result of Pinčuk and Hasanov (Theorem 1.7 above).

The results that we have mentioned so far are not sharp, in the sense that they only give sufficient conditions for extendability that are, generally speaking, not necessary. Baouendi and Rothschild [BaR2] have obtained a particularly beautiful sharp result in $\mathbf{C}^2$ that we shall now describe.

Let M be a smooth real-analytic hypersurface at the origin in $\mathbf{C}^2$. If $f: M^- \to \mathbf{C}^2$ is a holomorphic mapping that extends smoothly to $M^- \cup M$, we shall say, following [BaR2], that f is *not totally degenerate* at 0 if its Jacobian determinant $\det(\partial f_j/\partial z_k)$ is not flat at 0, i.e., its Taylor series at 0 does not vanish identically. We say that a real-analytic hypersurface M has the *reflection property* at 0 if any holomorphic mapping defined on one side of M as above and not totally degenerate at 0, mapping M into another real-analytic hypersurface $M' \subset \mathbf{C}^2$, extends holomorphically to a full neighborhood of 0 in $\mathbf{C}^2$. The main result of [BaR2] is

Theorem 2.4. *A real-analytic hypersurface $M \subset \mathbf{C}^2$ has the reflection property at 0 if and only if M is not locally biholomorphically equivalent to the hypersurface $\{\Im z_2 = 0\}$.*

We have seen above that the hypersurface $\{\Im z_2 = 0\}$ does not have the reflection property. Also, if M is any hypersurface in $\mathbf{C}^2$ on which there is a smooth C-R function g which extends holomorphically only to M^-, then the mapping $f = (g, 0)$ is a totally degenerate mapping from M to $M' = \{\Im z_2 = 0\}$ which does not extend holomorphically in any neighborhood of 0. (This example is taken from [BaR2]). Hence Theorem 2.4 is optimal.

Recall that a bounded domain $D \subset \mathbf{C}^n$ is said to be *algebraic* if there exists a real polynomial $r(z, \bar{z})$ on $\mathbf{C}^n$ such that D is a connected component of the set $\{z \in \mathbf{C}^n : r(z, \bar{z}) < 0\}$ and $dr(z) \neq 0$ for $z \in bD$. Using the Q_w varieties, Webster proved [We2] that biholomorphic mappings between strongly pseudoconvex algebraic domains in $\mathbf{C}^n$ are *algebraic*, i.e., the graph of the mapping is contained in an n-dimensional algebraic subvariety X of $\mathbf{C}^{2n}$. Recently, Diederich and Fornæss [DF6] proved the following results concerning mappings of algebraic domains.

Theorem 2.5. *Let $D, D' \subset\subset \mathbf{C}^2$ be algebraic domains and $f: D \to D'$ a biholomorphic mapping. Then f extends holomorphically to a neighborhood of $\overline{D}$.*

Theorem 2.6. *Every proper holomorphic mapping $f: D \to D'$ between algebraic domains $D, D' \subset\subset \mathbf{C}^n$ extends continuously to $\overline{D}$.*

There is an older result, due to Diederich and Fornæss [DF3], for mappings of non-pseudoconvex domains in $\mathbf{C}^2$.

Theorem 2.7. *Let $D_1, D_2 \subset \mathbf{C}^2$ be bounded domains with smooth real-analytic boundaries. Assume that the set of strongly pseudoconvex boundary points of D_j is separated from the set of strongly pseudoconcave boundary points by a real-analytic, totally real submanifold $M_j \subset bD_j$ for $j = 1, 2$. Then every biholomorphic mapping $f: D_1 \to D_2$ extends continuously to $\overline{D}_1$.*

It follows that f extends holomorphically to a neighborhood of $\overline{D}_1$. Since f clearly maps M_1 into M_2, the extendability of f at points of M_1 follows from the edge-of-the-wedge theorem.

We now return to the paper [BaJT] by Baouendi, Jacobowitz, and Treves, where the authors considered the analyticity of smooth C-R diffeomoprhisms between generic real-analytic C-R manifolds in $\mathbf{C}^n$. Such a manifold M is given locally by a set of real equations

$$r_1(z,\overline{z}) = 0, \ldots r_d(z,\overline{z}) = 0, \tag{2.8}$$

where each r_j is real-analytic in a neighborhood of the origin, and $\partial r_1 \wedge \ldots \wedge \partial r_d \neq 0$. Everything should be understood in the sense of germs at the origin. Write $n = m + d$; then we have $\dim M = 2m + d$ and CR $\dim M = m$. As before we define the polar of M by

$$M^0 = \{(z,w) \in \mathbf{C}^{2n} : r_j(z,w) = 0, 1 \leq j \leq d\},$$

and for each fixed $w \in \mathbf{C}^n$ we define a local complex submanifold of $\mathbf{C}^n$ of complex dimension $n - d$ by

$$Q_w = \{(z,w) \in \mathbf{C}^n : r_j(z,\overline{w}) = 0, 1 \leq j \leq d\}.$$

Just as before we pose the following

Definition. (See[BaJT].) *A generic real-analytic C-R manifold $M \subset \mathbf{C}^n$ is said to be essentially finite at $0 \in M$ if the subvariety $V = \cap\{Q_z : z \in Q_0\}$ of M is equal to $\{0\}$ (in the sense of germs at the origin).*

Let $\Lambda \subset \mathbf{R}^d$ be a non-empty open convex cone with vertex 0 in $\mathbf{R}^d$. Suppose that M is given by (2.8). We define a wedge $\mathcal{W}(\Lambda) \subset \mathbf{C}^n$ with edge M by

$$\Im w - \phi(z,\overline{z},\Re w) \in \Lambda.$$

As usual we think of $\mathcal{W}(\Lambda)$ as a germ of a domain at the origin.

The main result of the paper [BaJT] is:

Theorem 2.8. *Suppose that M and M' are real-analytic generic C-R submanifolds at the origin in $\mathbf{C}^n$ of C-R dimension $m > 0$, satisfying the following two conditions:*

(a) There is a nonempty open convex cone $\Gamma \subset \mathbf{R}^d$ such that every C-R function on M extends (in the sense of distributions) to a holomorphic function in a wedge $\mathcal{W}(\Lambda)$ with edge M;

(b) M' is essentially finite at 0.

Then every C^∞-smooth C-R diffeomorphism $f: M \to M'$ with $f(0) = 0$ is real-analytic at 0 (and therefore it extends holomorphically to a neighborhood of 0).

There are several known results on holomorphic extension of C-R functions to wedges; see the papers [BaCT, BaR3]. We mention an interesting and strong result of Tumanov [Tum1].

Definition. *A generic C-R manifold $M \subset \mathbf{C}^n$ is called minimal at the point $z \in M$ if there exists no C-R manifold $N \subset M$, pasing through z, of smaller dimension than M, but of the same C-R dimension.*

Tumanov proved that at every point $z \in M$ at which M is minimal, all C-R functions on M can be extended to some wedge with edge on M. If M is a hypersurface in $\mathbf{C}^n$, then it is minimal when it does not contain a complex subvariety of maximal dimension $n - 1$. In this case the result of Tumanov is just the one-sided extension theorem of Baouendi and Treves [BaT] and Trépreau [Tre]. The result of Tumanov, together with Theorem 2.8, implies

Corollary 2.9. *Let M and M' be generic real-analytic C-R manifolds at the origin in $\mathbf{C}^n$. If M is minimal at 0 and M' is essentially finite at 0, then every smooth C-R diffeomorphism $f: M \to M'$ with $f(0) = 0$ is real-analytic at 0.*

On Levi-nondegenerate C-R manifolds the result of Theorem 2.8 holds if f is merely of class C^1, see [BaJT]. In this case the same result was proved earlier by Webster [We6]. On hypersurfaces, Theorem 2.3 above is stronger than Theorem 2.8 where the mapping is assumed to be non-branched. An alternative proof of Theorem 2.8 for hypersurfaces was given by Bedford and Pinčuk [BP2].

We conclude the section by mentioning some open problems.

1. One of the assumptions in Theorem 2.4 is that the mapping f is not totally degenerate. Suppose that the real-analytic hypersurfaces M, $M' \subset \mathbf{C}^n$ are not biholomorphically equivalent to $\Im z_n = 0$. If $f: M \to M'$ is a smooth C-R mapping that is totally degenerate at 0, does it follow that f is constant?
2. Find necessary and sufficient conditions for extendability of holomorphic mappings in dimension $n > 2$ (see Theorem 2.4 for $n = 2$).
3. In a typical extension result one first uses one set of ideas to prove that the mapping is smooth on the boundary and another set of ideas to prove that it extends across the boundary. It would be more sat-

isfactory to be able to go from the small initial amount of regularity directly to holomorphic extension. On strongly pseudoconvex hypersurfaces the methods of Lempert [Le4], Pinčuk and Hasanov [PH] and Forstnerič [Fo6] are of this kind. Also, Bell proved [Be2] that the holomorphic extendability of a mapping $f: D \to D'$ between pseudoconvex real-analytic domains in $\mathbf{C}^n$ follows from the global analytic hypoellipticity of the $\overline{\partial}$-Neumann problem for D, which is known to hold on strongly pseudoconvex domains. It seems that there is so far no successful approach of this kind for arbitrary pseudoconvex real-analytic domains in $\mathbf{C}^n$.

4. Suppose that $f: M \to M'$ is a C-R homeomorphism of generic real-analytic C-R manifolds in $\mathbf{C}^n$, $f(0) = 0$. If both M and M' are minimal at 0 (in the sense of Tumanov [Tu]), is f real-analytic at 0? This problem seems to be open even on Levi non-degenerate C-R manifolds (see [Fo7]).

3. Mappings of Special Domains

One of the most special domains in $\mathbf{C}^n$ is certainly the unit ball:

$$\mathbf{B}^n = \{z \in \mathbf{C}^n : |z|^2 = \sum_{j=1}^{n} |z_j|^2 < 1\}.$$

Recall that the automorphism group of $\mathbf{B}^n$ is generated by the unitary group $\mathrm{U}(n)$, together with the involutions

$$\phi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle},$$

where $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$, $P_a z = \frac{\langle z,a\rangle}{\langle a,a\rangle} a$, $Q_a z = z - P_a z$, and $s_a = (1 - |a|^2)^{1/2}$. See Rudin [Ru2].

It follows that the group $\mathrm{Aut}\, \mathbf{B}^n$ acts transitively on the ball. Conversely, when $n > 1$, $\mathbf{B}^n$ is the only bounded, strongly pseudoconvex domain in $\mathbf{C}^n$ with a transitive group of automorphisms, according to Wong [Wong] and Rosay [Ro1]. See also [Ru2, p. 327].

We now consider the proper holomorphic maps of $\mathbf{B}^n$ onto itself. When $n = 1$, such maps are precisely the finite Blaschke products [Ru1, p.164]:

$$F(z) = \prod_{j=1}^{m} e^{i\theta_j} \frac{z - a_j}{1 - \overline{a}_j z}, \qquad \theta_j \in \mathbf{R}, a_j \in \mathbf{B}^1, m \in \mathbf{Z}_+.$$

Thus the space of proper holomorphic maps $\mathbf{B}^1 \to \mathbf{B}^1$ is infinite dimensional, and every such map is rational.

The situation is quite different for $n > 1$. It is much more difficult for a domain $D \subset \mathbf{C}^n$ with $n > 1$ to admit proper self-mappings that are not automorphisms. The first evidence of this was the following theorem, due to Alexander [Al1]:

Theorem 3.1. *If $n > 1$ and $f: \mathbf{B}^n \to \mathbf{B}^n$ is a proper holomorphic mapping, then f is an automorphism of $\mathbf{B}^n$.*

An elementary proof of this theorem can be found in Rudin's book [Ru2, p.316]. The theorem can be reduced to the following local result (see [Ru2, p. 311]):

Theorem 3.2. *Let $n > 1$. Suppose that Ω_j is an open subset of $\mathbf{B}^n$ $(j = 1, 2)$ whose boundary $b\Omega_j$ contains an open subset Γ_j of $b\mathbf{B}^n$, and suppose that f is a biholomorphic map of Ω_1 onto Ω_2.*

If there is a sequence $\{a_k\}$ in Ω_1, converging to a point $a \in \Gamma_1$ which is not a limit point of $\mathbf{B}^n \cap b\Omega_1$, such that the sequence $b_k = f(a_k)$ converges to a point $b \in \Gamma_2$ which is not a limit point of $\mathbf{B}^n \cap b\Omega_2$, then f extends to an automorphism of $\mathbf{B}^n$.

The best result in this direction is due to Pinčuk and Tsyganov [PT], see Corollary 1.9 above.

It is worth mentioning the history of this problem. In 1907 Poincaré [Po] proved the following result for $n = 2$: Let U be an open ball centered at a point $a \in b\mathbf{B}^n$ and let $f: U \to V \subset \mathbf{C}^n$ be a biholomorphic map that takes $b\mathbf{B}^n \cap U$ to $b\mathbf{B}^n \cap V$. Then f extends to an automorphism of $\mathbf{B}^n$. This was proved for arbitrary $n > 1$ in 1962 by Tanaka [Ta] who was apparently not aware of Poincaré's work. The same result was rediscovered by Pelles [Pe] and Alexander [Al1].

Subsequently Alexander [Al2] proved Theorem 3.1 using the fact that the map f extends smoothly to $\overline{\mathbf{B}}^n$ according to a theorem of Fefferman (Theorem 1.2). Pinčuk reduced the smoothness requirement C^1 [Pi2]. The proof of Theorem 3.2 given in Rudin's book [Ru2] avoids Fefferman's theorem and is rather elementary. Finally Pinčuk and Tsyganov proved in [PT] that every nonconstant local continuous C-R mapping of $b\mathbf{B}^n$ to itself extends to an automorphism of $\mathbf{B}^n$ (Corollary 1.9).

Alexander's theorem has been extended to several classes of domains. Diederich and Fornæss proved in [DF7] that every proper holomorphic mapping from a smoothly bounded, strongly pseudoconvex domain in $\mathbf{C}^n$ onto another smoothly bounded domain in $\mathbf{C}^n$ is locally biholomorphic, hence a holomorphic covering projection. If the target is simply connected, the map is biholomorphic. Thus, if $D \subset\subset \mathbf{C}^n$ is strongly pseudoconvex and simply connected, then every proper self-map of D is an automorphism. Also, every proper holomorphic self-mapping of a bounded pseudoconvex domain in $\mathbf{C}^n$ with smooth real-analytic boundary is an automorphism according to Bedford and Bell [BB2]. It is an open problem whether the same holds for smooth bounded domains of finite type in $\mathbf{C}^n$. In this direction, Pan [Pan] proved that every proper holomorphic self-map of a pseudoconvex Reinhardt domain $D \subset \mathbf{C}^n (n > 1)$, such that the Levi determinant of D does not vansh identically on bD, is an automorphism of D. He proved the same result for certain special pseudoconvex domains that are not of finite type.

In a different direction, Henkin and Novikov [HN] and Henkin and Tumanov [TH1, TH2, Tum2] proved the same result for all classical Cartan domains, as well as for a wide class of Siegel domains of the second kind. We shall state only a special case of these results:

Theorem 3.3. *Let $D \subset \mathbf{C}^n (n > 1)$ be an irreducible bounded symmetric domain (in the sense of E. Cartan; see Piatetsky-Shapiro* [Pia]*). Then every proper holomorphic map $f: D \to D$ is an automorphism of D.*

Next we consider mappings between *bounded circular domains* D, $D' \in \mathbf{C}^n$. Recall that a domain D is circular if it is invariant under the rotation $T_\theta z = e^{i\theta} z$ for all $\theta \in \mathbf{R}$. The following classical result is due to H. Cartan [Car1]:

Theorem 3.4. *If $f: D \to D'$ is a biholomorphic mapping of bounded circular domains in $\mathbf{C}^n$ containing the origin, and if $f(0) = 0$, then f is a linear mapping.*

This result has been generalized to proper holomorphic mappings by Bell [Be6].

Theorem 3.5. *Suppose that D and D' are bounded circular domains in $\mathbf{C}^n$ which contain the origin. If $f: D \to D'$ is a proper holomorphic* *mapping such that $f(0) = 0$, then f must be algebraic. If we further assume that $f^{-1}(0) = \{0\}$, then f is a polynomial mapping.*

Recall that a mapping is algebraic if its graph $\Gamma = \{(z, w): w = f(z)\}$ is an algebraic variety, i.e., it is defined by polynomial equations. Bell first proved this result in the papers [Be3] and [Be4] using the Bergman kernel function. In [Be4] he also proved that every proper holomorphic correspondence f of D onto D' satisfying $f^{-1}(0) = \{0\}$ is algebraic. The proof given in [Be6] is elementary; it does not involve the behavior of f near the boundary or the properties of the Bergman kernel.

Next we consider the mappings between bounded *Reinhardt domains*. For each $a \in \mathbf{C}^n$ we denote by $T_a: \mathbf{C}^n \to \mathbf{C}^n$ the linear map $T_a z = (a_1 z_1, \ldots, a_n z_n)$. A domain $D \subset \mathbf{C}^n$ is called a Reinhardt domain if $T_a(D) = D$ for each $a = (a_1, \ldots, a_n)$ with $|a_j| = 1$ for $1 \leq j \leq n$.

Recently Shimura [Shi] solved the equivalence problem for arbitrary bounded Reinhardt domain. A similar result was obtained independently by Barrett [Ba3] under an additional but rather weak hypothesis on the two domains.

Theorem 3.6. ([Shi], Theorem 1, page 131.) *If two bounded Reinhardt domains D, D' in $\mathbf{C}^n$ are biholomorphically equivalent, then there exists a biholomorphic mapping $f: D \to D'$ of the form*

$$f_i(z) = b_i z_1^{a_{1i}} \ldots z_n^{a_{ni}}, \quad 1 \leq i \leq n, \tag{3.1}$$

where $b_i \in \mathbf{C}\backslash\{0\}$ for $i = 1, \ldots, n$ and $A = (a_{ij}) \in \mathrm{GL}(n, \mathbf{Z})$. Moreover, if we assume in addition that D and D' contain the origin, then there exists a biholomorphic map $D \to D'$ of the form $z_i \to r_i z_{\sigma(i)}$ where $r_i > 0$, $1 \leq i \leq n$, and σ is a permutation of the indices.

The last result when both domains contain the origin has been proved before by Sunada [Sun]. For $n = 2$ the result goes back to Thullen [Thu].

While the approach by Barrett is analytic (he studied the Bergman kernel function), the approach by Shimizu is algebraic. Denote by $T(D)$ the subgroup of the holomorphic automorphism group $\mathrm{Aut}(D)$ consisting of all maps $T_a, a = (a_1, \ldots, a_n), |a_j| = 1$ for $1 \leq j \leq n$. The most important step in the proof of Shimizu is to show that $T(D)$ is a maximal torus in the connected component $G(D)$ of $\mathrm{Aut}(D)$ containing the identity map. If $f: D \to D'$ is a biholomorphic map, then the set $\widetilde{T}(D') = \{f \circ T_a \circ f^{-1}: T_a \in T(D)\}$ is another maximal torus in $G(D')$. By the conjugacy theorem of Hochschild, the maximal tori $\tilde{T}(D')$ and $T(D')$ in the connected Lie group $G(D')$ are conjugate, i.e., there is an automorphism $g \in G(D')$ such that $gT(D')g^{-1} = \tilde{T}(D')$. Hence we have $g(fT(D)f^{-1})g^{-1} = (gf)T(D)(gf)^{-1} = T(D')$. This implies that the map $gf: D \to D'$ takes the tori $\{T_a z: |a_1| = \cdots |a_n| = 1$ in D into tori in D'. Shimizu proved that such a map has to be algebraic of the form given in Theorem 3.6 above.

Shimizu obtained further results on the automorphism group of bounded Reinhardt domains $D \subset (\mathbf{C}^*)^n$. The logarithmic image of such a domain is the set

$$\log(D) = \{(-\log|z_1|, \ldots - \log|z_n|): z \in D\} \subset \mathbf{R}^n.$$

The second main result of [Shi] (page 136, Theorem 2) is

Theorem 3.7. *If D is a Reinhardt domain in $(\mathbf{C}^*)^n$ whose logarithmic image has the convex hull containing no complete straight lines, then every automorphism in the connected component of the identity is of the form $(z_1, \ldots, z_n) \to (a_1 z_1, \ldots, a_n z_n)$ where $|a_j| = 1$ for $1 \leq j \leq n$. It follows that every automorphism of D is of the form (3.1), with $|b_i| = 1$.*

A description of the automorphism group of certain Kobayashi hyperbolic Reinhardt domains has been obtained independently by Kruzhilin [Kru]. For instance, if such a domain does not intersect the coordinate hyperplanes, then all its automorphisms have the form (3.1), with $|b_i| = 1$. (See [Kru], Theorem 2 and its Corollary.)

The paper by Shimizu [Shi] containing several further results on the automorphism group of two-dimensional Reinhardt domains.

The classification of *proper* holomorphic mappings between Reinhardt domains will necessarily be more complicated. For instance, we can map an annulus in $\mathbf{C}$ properly onto the disc.

Barrett proved in [Ba3] the following extension result for proper holomorphic mappings of Reinhardt domains.

Theorem 3.8. *Let $f: D \to D'$ be a proper holomorphic mapping of bounded Reinhardt domains in $\mathbf{C}^n$. Suppose that there is an integer k, $0 \leq k \leq n$, such that $D \cap \{z_j = 0\} \neq \emptyset$ for $j = 1, \ldots, k$, and $\overline{D} \cap \{z_j = 0\} = \emptyset$ for $j = k+1, \ldots, n$. Then f extends holomorphically to a neighborhood of $\overline{D}$.*

Denote the integer k in the formulation of the last theorem by $k(D)$. Clearly, $k(D) = 0$ means that $\overline{D}$ is contained in $(\mathbf{C}^*)^n$. For such domains we have the following rigidity result due to Bedford [Bed2]:

Theorem 3.9. *Every bounded Reinhardt domain $D \subset (\mathbf{C}^*)^n$ is rigid in the sense of Cartan: If $f\colon D \to D$ is a holomorphic mapping such that the induced map $f_*\colon H_1(D, \mathbf{R}) \to H_1(D, \mathbf{R})$ on the first homology group of D is non-singular, then f is an automorphism of D.*

Bedford also introduced a metric on $H_1(D, \mathbf{R})$ such that every holomorphic mapping $f\colon D \to D'$ of bounded Reinhardt domains $D, D' \subset (\mathbf{C}^*)^n$ for which f_* is an isometry of the form $f(z) = (c_1 z^{\alpha_1}, \ldots, c_n z^{\alpha_n})$.

Landucci [Lan] and Dini and Primicerio [DP] have obtained results on mappings between Reinhardt domains of the form

$$\Sigma_n(\alpha) = \{(z_1, \ldots, z_n) \in \mathbf{C}^n\colon |z_1|^{2\alpha_1} + |z_2|^{2\alpha_2} + \ldots + |z_n|^{2\alpha_n} < 1\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n_+$. If all α_i equal 1, we have the unit ball. Otherwise $\Sigma_n(\alpha)$ is a weakly pseudoconvex Reinhardt domain that can be mapped properly onto the ball $\mathbf{B}^n$ by the mapping $(z_1, \ldots, z_n) \to (z_1^{\alpha_1}, \ldots, z_n^{\alpha_n})$. These domains are usually called *generalized ellipsoids*. The following theorem was proved by Landucci [Lan] in the case when $\alpha_i \in \mathbf{Z}_+$, and was generalized to $\alpha_i \in \mathbf{R}_+$ by Dini and Primicerio [DP3]:

Theorem 3.10. *There exists a proper holomorphic mapping $f\colon \Sigma_n(\alpha) \to \Sigma_n(\beta)$ if and only if after a permutation of indices $\alpha_i/\beta_i = h_i \in \mathbf{N}$.*

Moreover, if the α's and β's are integers, then every such mapping is equivalent up to reordering of indices and an automorphism of $\Sigma_n(\beta)$, to the map

$$(z_1, \ldots, z_n) \to (z_1^{h_1}, \ldots, z_n^{h_n}). \tag{3.2}$$

Dini and Primicerio [DP4] have extended this result to proper holomorphic mappings $f\colon R \to D$, where R is a Reinhardt domain and D is a smoothly bounded strongly pseudoconvex domain in $\mathbf{C}^n$, $n > 1$. The conclusion then is that, up to an automorphism of D, f is equivalent to the mapping (3.2).

In [DP3] and [DP4] there are also results on factorization of proper mappings by groups of automorphisms, similar to that of Rudin [Ru3] for mappings from the ball.

The last theorem is partly contained in the more general result of Bedford [Bed3, Theorem 3].

In the papers [DP1] and [DP2] Dini and Primicerio characterized the Reinhardt domains $D \subset \mathbf{C}^n$ that admit a proper holomorphic mapping onto a generalized ellipsoid.

Theorem 3.11. *Let D be a Reinhardt domain in $\mathbf{C}^n$ containing the origin. If there exists a proper holomorphic mapping $f\colon D \to \Sigma_n(\alpha)$, then there also exists a proper polynomial mapping of D onto $\Sigma_n(\alpha)$. If D is a Reinhardt domain in $\mathbf{C}^n$ and $f\colon D \to \Sigma_n(\alpha)$ is a proper polynomial mapping, then there are $a \in (\mathbf{C}^*)^n$ and $\beta \in \mathbf{N}^n$ such that $D = T_a(\Sigma_n(\beta))$.*

Here, $T_a(z_1, \ldots, z_n) - (\alpha_1 z_1, \ldots, a_n z_n)$. From Theorem 3.10 it then follows that $\beta_i/\alpha_i \in \mathbf{N}$.

Another special class of domains that have been classified up to biholomorphic mappings are the real ellipsoids; see the paper by Webster [We2]. In the same paper Webster found sufficient conditions on two algebraic real hypersurfaces in $\mathbf{C}^n$ that force every biholomorphic mapping between the two hypersurfaces to be birational.

In [La2] Landucci treated the proper holomorphic mappings in certain class of bounded pseudoconvex Reinhardt domains with center 0 that lies in the boundary of these domains, and the boundaries are not smoth at 0. The global condition R is not satisfied, but the mappings are nevertheless nice away from 0.

Finally we mention a result of Boas and Straube [BS1] on mappings of complete Hartogs domains in $\mathbf{C}^2$.

Let $t_\alpha(z', z_n) = (z', \alpha z_n)$ for $\alpha \in \mathbf{C}$. Recall that a domain $D \subset \mathbf{C}^n$ is called a *Hartogs domain* with respect to the variable z_n if $T_\alpha(D) \subset D$ for all $|\alpha| = 1$. A domain is *complete Hartogs* if $T_\alpha(D) \subset D$ for all $|\alpha| \leq 1$, i.e., it is of the form $D = \{|z_n| < h(z_1, \ldots, z_{n-1})\}$.

Boas and Straube proved in [BS1] that on smooth, bounded, complete Hartogs domains in $\mathbf{C}^2$, the Bergman projection exactly preserves the differentiability of the functions as measured by Sobolev norms. This implies

Theorem 3.12. *Every biholomorphic mapping between smooth bounded complete (not necessarily) pseudoconvex) Hartogs domains in C^2 extends smoothly to the boundaries. If the boundaries are real-analytic,* *then the mapping extends holomorphically to a neighborhood of the closure.*

The correspondent problem seems open in dimensions $n > 2$.

4. Existence of Proper Holomorphic Mappings into Balls

It is well-known that each pseudoconvex domain $D \subset \mathbf{C}^n$ and, more generally, each Stein manifold D admits proper holomorphic mappings $f: D \to \mathbf{C}^{n+1}$ and proper holomorphic embeddings $f: D \to \mathbf{C}^{2n+1}$ (see [Hö2]).

The existence of such mappings into bounded domains in $\mathbf{C}^N$ is a much more delicate problem. An obvious necessary condition is that D admits plenty of bounded holomorphic functions. However, the unit polydisc $\Delta^n \subset \mathbf{C}^n$ for $n > 1$ can not be mapped properly into any ball $\mathbf{B}^N \subset \mathbf{C}^N$ or, more generally, into any bounded domain $\Omega \subset \mathbf{C}^N$ on which every boundary point $p \in b\Omega$ is a local peak point [Ru2, p. 306]. There is even a quantitative explanation for the non-existence of proper mappings $f: \Delta^n \to \mathbf{B}^N$; see [Ru2, p. 308], [Al3], and [Le2].

Suppose now that $D \subset \mathbf{C}^n$ is a bounded, strongly pseudoconvex domain with $\mathcal{C}^k$ boundary for some $k \geq 2$. When $n = 1$, Stout proved in [Sto] that there is a proper holomophic embedding $f: D \to \mathbf{B}^3$, smooth of class $\mathcal{C}^{k-0}$ on $\overline{D}$, such that the components of f have constant modulus on bD. There also exist proper holomorphic mappings $f: D \to \mathbf{B}^2$ with these properties.

From now on we shall assume that $n > 1$. By a well-known theorem of Fornæss [Fn1] and Henkin [HČ] there is a proper holomorphic embedding $f: D \to \Omega$ into a strongly convex domain $\Omega \subset \mathbf{C}^N$ with $\mathcal{C}^k$ boundary, such that f is holomorphic in a neighborhood of $\overline{D}$ and is transverse to $b\Omega$. The integer N is large and depends on D.

A question arises whether a similar result could hold if we insist that Ω be some special domain, say the unit ball $\mathbf{B}^N \subset \mathbf{C}^N$ for some N. If we also insist that f is holomorphic on $\overline{D}$, then the boundary of D must be real-analytic since it is defined by the analytic equation $\sum |f_j(z)|^2 = 1$. If bD is smooth of class $\mathcal{C}^k$, it would be natural to require that f be of class $\mathcal{C}^s(\overline{D})$ for some $s \leq k$.

In this and the next section we shall describe the present state of the knowledge on the following questions:

(1) Which domains $D \subset \mathbf{C}^n$ admit proper holomorphic mappings (embeddings) $f: D \to \mathbf{B}^N$? How large must N be in terms of D and n?
(2) How smooth can such mappings be on $\overline{D}$?
(3) Do there exist proper holomorphic mappings $f: D \to \mathbf{B}^N$ with 'wild' boundary behavior?
(4) Suppose that bD is real-analytic and f is somewhat smooth on $\overline{D}$. Does this imply that f extends holomorphically across bD?

The first positive evidence for the existence of proper mappings of strongly pseudoconvex domains into balls was given by Lempert [Le1] in 1981:

Theorem 4.1. *If $D \subset \mathbf{C}^n$ is a bounded, strongly pseudoconvex domain with C^2 boundary, there exists a sequence of functions $\{f_j\}$, each of them holomorphic and nonconstant in a neighborhood of $\overline{D}$, such that the series $\sum_{j=1}^{\infty} |f_j(z)|^2$ converges for all $z \in \overline{D}$ and equals 1 for $z \in bD$.*

Thus, $f = (f_1, f_2, \ldots): D \to \ell_2(\mathbf{C})$ is a holomorphic mapping of D into the complex Hilbert space $\ell_2(\mathbf{C})$ of all square-convergent complex sequences, such that bD is mapped to the unit sphere.

In his recent paper [Le6] Lempert proved that for strongly pseudoconvex domains $D \subset\subset \mathbf{C}^n$ with real-analytic boundary we can find a sequence $\{f_j\}$ as above such that the series $\sum\limits_{j=1}^{\infty} |f_j(z)|^2$ is locally uniformly convergent on an open neighborhood of $\overline{D}$. In other words, we can find C-R embeddings of every compact strongly pseudoconvex real-analytic hypersurface $M \subset \mathbf{C}^n$ into the unit sphere of the Hilbert space $\ell_2(\mathbf{C})$.

Around 1984 it was shown by Faran (unpublished) and, independently, by Forstnerič [Fo1] that the analogue of the convex embedding theorem of Fornæss and Henkin, with Ω being the unit ball of any finite dimensional complex space, fails:

Theorem 4.2. *For each $n > 1$ there exist bounded strongly pseudoconvex domains D in $\mathbf{C}^n$ with smooth real-analytic boundaries such that no proper holomorphic mapping $f: D \to \mathbf{B}^N$ for any N extends to a C^∞ map on $\overline{D}$.*

The set of domains satisfying the conclusion of this theorem is of the second category in the natural topology on the space of domains. There exist formal obstructions for the existence of local holomorphic embeddings of real-analytic hypersurfaces into spheres of any dimension. The main idea of the proof is similar to the heuristic argument of Poincaré [Po] for the existence of local biholomorphic invariants of hypersurfaces in $\mathbf{C}^n$ for $n > 1$.

These methods did not shed any light on question (1). The breakthrough came in 1984 when Løw proved that the unit ball can be embedded as a closed complex subvariety of a high-dimensional polydisc [Løw2]. In fact, his method applies to every bounded strongly pseudoconvex domain instead of ball. Løw's technique was inspired by the construction of inner functions developed by Hakim and Sibony [HS] and Løw [Løw1]. A modification of his method also gives embeddings of bounded strongly pseudoconvex domains $D \subset \mathbf{C}^n$ into balls (Forstnerič [Fo1] and Løw [Løw3]). Similar results have been obtained independently by Aleksandrov [Ale]. To summarize, we have

Theorem 4.3. *Let D be a bounded, strongly pseudoconvex domain with C^2 boundary in $\mathbf{C}^n$. There are integers N_1 and N_2 such that*

(a) For each $N \geq N_1$ there exist proper holomorphic embeddings $f: D \to \mathbf{B}^N$. Some of these embeddings extend continuously to $\overline{D}$, but there also exist embeddings that are not continuous at bD.

(b) For each $N \geq N_2$ there exist proper holomorphic embeddings $f: D \to \Delta^N$.

In fact, if N is sufficiently large, there are plenty of proper maps $f: D \to \mathbf{B}^N$, in the sense that every holomorphic mapping $g: D \to \mathbf{C}^N$ with $\sup_{z\in D}|g(z)| < 1$ can be approximated, uniformly on compacts in D, by proper holomorphic maps $f: D \to \mathbf{B}^N$ [Løw3].

The proof of Theorem 4.3 is based on a constructive procedure found by Løw [Løw2]. It also resembles the construction of inner functions on strongly pseudoconvex domains due to Hakim and Sibony [HS] and Løw [Løw1].

We consider the case when the target is $\mathbf{B}^N$. Starting with a holomorphic map $f = f^0: D \to \mathbf{B}^N$, smooth on $\overline{D}$, one constructs a sequence of maps $f^k: D \to \mathbf{B}^N$, converging uniformly on $\overline{D}$, such that $\lim_{k\to\infty} |f^k(z)| = 1$ for $z \in bD$. The limit map $f = \lim_{k\to\infty} f^k$ is continuous on $\overline{D}$, holomorphic on D, and it maps D properly to $\mathbf{B}^N$. At each step of the procedure we add to f^k a correction term g^k to obtain the next map $f^{k+1} = f^k + g^k$. Each correction term is a combination of holomorphic peak functions that are smooth on $\overline{D}$, chosen in such a way that the infimum $\inf_{z\in bD}|f^k(z)|$

increases sufficiently fast towards one, while at the same time the sequence of corrections $g^k(z)$ is uniformly convergent on $\overline{D}$. The main idea is to choose the correction term $g^k(z)$ to be orthogonal to the vector $f^k(z)$ for each point z in a discrete subset of the sphere $b\mathbf{B}^n$. Moreover, for nearby points of the given discrete set the vectors g^k at these points are approximately orthogonal to each other. In this way we assure that the norm of the mapping increases in a controlled way at each step.

A similar construction gives proper maps into polydiscs; this time we have to push the maximal value of the components $\max_{1\le j\le n}(\sup_{z\in D} |f_j(z)|)$ towards 1.

Using this technique Løw proved the following, somewhat more general result [Løw3]:

Theorem 4.4. *Let $D\subset \mathbf{C}^n$ be a bounded, strongly pseudoconvex domain with C^2 boundary. Then for all sufficiently large m the following holds. If ϕ is a continuous positive function on bD, $f\colon bD \to \mathbf{C}^m$ is a continuous mapping satisfying $|f(z)| < \phi(z)$ for each $z \in bD$, K is a compact subset of D and $\epsilon > 0$, there exists a continuous mapping $g\colon \overline{D} \to \mathbf{C}^m$, holomorphic in D, such that*

$$|f(z) + g(z)| = \phi(z) \quad \textit{for} \quad z \in bD; \quad |g(z)| < \epsilon \quad \textit{for} \quad z \in K.$$

The mapping g can be chosen to vanish to any prescribed order at an interior point of D.

Using the above theorem we can construct proper holomorphic embeddings into balls that do not extend continuously to the boundary. We choose a holomorphic map $h\colon \mathbf{B}^n \to \mathbf{C}^p$ which does not extend continuously to $\overline{\mathbf{B}}^n$, but whose norm $|h(z)|$ extends continously to $\overline{\mathbf{B}}^n$ and satisfies $|h(z)| < 1$ for all $z \in \overline{\mathbf{B}}^n$. If we apply Theorem 4.4 with $f = 0$ and $\phi(z) = (1 - |h(z)|^2)^{1/2}$, we find a map $g\colon \overline{\mathbf{B}}^n \to \mathbf{C}^m$ such that $F = (h, g)\colon \mathbf{B}^n \to \mathbf{B}^{p+m}$ is proper holomorphic. Clearly F does not extend continously to $\overline{\mathbf{B}}^n$. If h is a one-to-one holomorphic immersion, then F is an embedding. Thus Theorem 4.4 implies the following corollary which was proved independently by Forstnerič [Fo1].

Corollary 4.5. *If N is sufficiently large (depending on n), then there exist proper holomorphic embeddings $F\colon \mathbf{B}^n \to \mathbf{B}^N$ which do not extend continuously to $\overline{\mathbf{B}}^n$.*

This is in sharp contrast with the results on smooth extension of proper holomorphic mappings between domains in $\mathbf{C}^n$.

The technique of Løw was used by Globevnik [Glo] to prove the following interpolation result for proper holomorphic mappings.

Theorem 4.6. *Let $n \geq 1$. There is an integer $N(n)$ such that for every $N \geq N(n)$ the following holds: Let $K \subset b\mathbf{B}^n$ be an interpolation set for the algebra $A(\mathbf{B}^n)$. Then every continuous mapping $f: K \to b\mathbf{B}^N$ has a continuous extension $\tilde{f}: \overline{\mathbf{B}}^n \to \overline{\mathbf{B}}^N$ which is holomorphic on $\mathbf{B}^n$ and satisfies $\tilde{f}(b\mathbf{B}^n) \subset b\mathbf{B}^N$.*

We can choose the interpolation set $K \subset b\mathbf{B}^n$ such that there exists a continuous surjection $g: K \to b\mathbf{B}^N$. Extending g as above one obtains the following result [Glo]:

Corollary 4.7. *Let $n \geq 1$. For every $N \geq N(n)$ there is a continuous map $g: \overline{\mathbf{B}}^n \to \overline{\mathbf{B}}^N$ that is holomorphic in $\mathbf{B}^n$ and satisfies $g(b\mathbf{B}^n) = b\mathbf{B}^N$.*

The corollary shows that proper holomorphic maps into higher dimensional domains may have a very large boundary cluster set, even when they extend continuously to the boundary. In particular, *there exist non-rational proper holomorphic maps between balls of different dimensions.*

In the above results, the dimension N is large compared to n, and it depends on D. New ideas were introduced to this problem by B. Stensønes [Ste]. By a very careful refinement of Løw's technique, she proved

Theorem 4.8. *Let D be a bounded, strongly pseudoconvex domain with C^∞ boundary in $\mathbf{C}^n$. Then there exist proper holomorphic mappings $f: D \to \Delta^{n+1}$ into the $(n+1)$- dimensional polydisc.*

The key improvement due to Stensønes is the use of very carefully selected holomorphic peaking functions in D that are smooth on $\overline{D}$. The analogous result for ball as the target domain was proved by Dor [Dor1] and, independently, by Hakim [Ha]:

Theorem 4.9. *Let D be as in Theorem 4.8. Then there exist proper holomorphic mappings $f: D \to \mathbf{B}^{n+1}$ that extend continuously to $\overline{D}$. Also, for each n there exist proper holomorphic mappings from $\mathbf{B}^n$ to $\mathbf{B}^{n+1}$ that are not rational.*

We shall see in section 5 below that rational proper maps of $\mathbf{B}^n$ to $\mathbf{B}^N$ map affine complex hyperplanes in $\mathbf{C}^n$ to affine hyperplanes in $\mathbf{C}^N$. Dor and Hakim constructed proper mappings $f: \mathbf{B}^n \to \mathbf{B}^{n+1}$ that do not have this property. According to Theorem 5.1 such maps cannot be of class C^2 on any open part of the boundary of $\mathbf{B}^n$.

In his recent paper [Dor2], Dor proved the following extension result for proper mappings between balls. We consider $\mathbf{B}^n$ as a subset of $\mathbf{B}^{n+1}$ via the embedding $z \to (z, 0)$.

Theorem 4.10. *Every proper holomorphic mapping $f: \mathbf{B}^{n-1} \to \mathbf{B}^N$, $1 < n < N$, can be extended to a proper holomorphic mapping $F: \mathbf{B}^n \to \mathbf{B}^N$.*

It seems likely that these methods can be applied to a more general class of domains. All that one needs are good holomorphic peaking functions for points in bD. An evidence for this is the recent result of Noel and Stensønes [NSt]:

Theorem 4.11. *For every bounded pseudoconvex domain $D \subset \mathbf{C}^2$ with real-analytic boundary there exists a proper holomorphic mapping from D into the unit polydisc in $\mathbf{C}^3$, and a uniformly continuous proper holomorphic map from D into the unit ball of $\mathbf{C}^3$.*

In the other direction, Sibony [Sib] constructed the following counterexample:

Theorem 4.12. *There is a smoothly bounded pseudoconvex domain $D \subset \mathbf{C}^2$ that does not admit a proper holomorphic mapping into any convex domain $\Omega \subset \mathbf{C}^N$.*

The domain D is not of finite type; in fact, it contains an open set foliated by dense complex analytic curves.

The following two questions are of interest:

(1) Does every pseudoconvex domain with real-analytic boundary in $\mathbf{C}^n$ admit a proper holomorphic map into some ball?

(2) Does every strictly linearly convex domain $D \subset \mathbf{C}^n$ (i.e., $x, y \in \overline{D}$, $x \neq y$, and $t \in (0,1)$ imply that $tx + (1-t)y \in D$) admit such maps?

The known results give rather satisfactory positive answers to questions 1 and 3 stated at the beginning, and a partial negative answer to question 2.

However, no progress whatsoever has been made toward the construction of mappings $f: D \to \mathbf{B}^N$ that would be somewhat smooth on the boundary. It seems plausible that such maps should exist if N is large and that their smoothness should increase with N. The present methods do not yield anything better than $f \in \mathbf{C}(\overline{D})$. To be very specific, we pose the following

Open Problem. *Let $n > 1$. Does there exist a proper holomorphic mapping $f: \mathbf{B}^n \to \mathbf{B}^N$ for some N such that f is of class C^1 on $\overline{B}^n$, but f is not holomorphic in a neighborhood of $\mathbf{B}^n$ (and hence not a rational mapping in view of Theorem 5.1)?*

5. Regularity of Mappings into Higher Dimensional Domains

Recall that, by a theorem of Alexander [Al1] (Theorem 3.1 above), every proper holomorphic self-mapping of the unit ball $\mathbf{B}^n (n > 1)$ is an automorphism of $\mathbf{B}^n$ and, therefore, a rational map (linear fractional). We mentioned in section 4 (Theorem 4.9) that for $N > n$ there exist non-rational proper holomorphic mappings $f: \mathbf{B}^n \to \mathbf{B}^N$ that are continuous on $\overline{\mathbf{B}}^n$. However, if f is sufficiently smooth on $\overline{\mathbf{B}}^n$, then f must be rational. More precisely:

Theorem 5.1. Forstnerič [Fo4]) *Let U be an open ball centered at a point $z \in b\mathbf{B}^n$. If $N > n > 1$ and $f: \mathbf{B}^n \cap U \to \mathbf{C}^N$ is a mapping of class C^{N-n+1} that is holomorphic in $\mathbf{B}^n \cap U$ and satisfies $f(b\mathbf{B}^n \cap U) \subset b\mathbf{B}^N$, then f is rational, $f = (p_1, \ldots, p_N)/q$, where the p_j's and q are holomorphic polynomials of degree at most $N^2(N - n + 1)$.*

Cima and Suffridge proved [CS4] that such a mapping has no singularities on the closed ball $\overline{\mathbf{B}}^n$, and thus it extends to a proper holomorphic map of $\mathbf{B}^n$ to $\mathbf{B}^N$ that is holomorphic in a neighborhood of $\overline{\mathbf{B}}^n$. The same was proved before by Pinčuk , but not published.

Corollary. *If $N > n > 1$ and $f: \mathbf{B}^n \to \mathbf{B}^N$ is a proper holomorphic map that extends to a map of class C^{N-n+1} on $\overline{\mathbf{B}}^n$, then f is rational, of degree at most $N^2(N - n + 1)$.*

Most likely the given bound on the degree of f is not optimal. The existence of the bound implies that the space of all rational proper mappings

$f: \mathbf{B}^n \to \mathbf{B}^N$ is finite dimensional when $n > 1$. Theorem 5.1 may be important in the classification of proper maps between balls; see section 6 below.

Special cases of Theorem 5.1 have been proved by several authors: Webster [We4] for $N = n + 1 \geq 4$ and $f \in \mathcal{C}^3(\overline{\mathbf{B}}^n)$; Faran [Fa1] for $N = n + 1 = 3$ and $f \in \mathcal{C}^3(\overline{\mathbf{B}}^n)$; and Cima and Suffridge [CS1] for $N = n + 1$ and $f \in \mathcal{C}^2(\overline{\mathbf{B}}^n)$. The proof of Theorem 5.1 was inspired by the methods of the paper by Cima and Suffridge [CS1].

There is an analogous result, due to Forstnerič [Fo4], for proper holomorphic mappings of more general domains of different dimensions.

Theorem 5.2. *Let $D \subset \mathbf{C}^n$ and $D' \subset \mathbf{C}^N (N > n > 1)$ be pseudoconvex domains, bounded in part by strongly pseudoconvex, real-analytic hypersurfaces $M \subset \mathbf{C}^n$ (resp. $M' \subset \mathbf{C}^N$). If $f: D \cup M \to \mathbf{C}^N$ is a $\mathcal{C}^\infty$ mapping that is holomorphic in D and maps M to M', then f extends holomorphically to a neighborhood of every point in an open, everywhere dense subset of M. In the case when the target D' is the unit $\mathbf{B}^N$, the same conclusion holds under the weaker assumption that f is of class $\mathcal{C}^{N-n+1}$ on $D \cup M$.*

In the case $N = n + 1, D'$ is the ball $\mathbf{B}^N$, and $f \in \mathcal{C}^3(\overline{D})$, the result of Theorem 5.2 was proved by Webster [We4]. A theorem of this type in codimension one for arbitrary strongly pseudoconvex domains with real-analytic boundaries was also announced in [CKS], but the proof given there does not appear to be entirely correct.

According to Pinčuk (personal communication), f extends holomorphically across each point of the hypersurface M, provided that the target D' is the ball $\mathbf{B}^N$. A result of this type was announced in [Pi5], but a proof has not been published.

Theorem 5.2 was proved in [Fo4] using the Q_w varieties associated to the hypersurfaces M and M' (see (2.6) in Section 2 above). The method is similar to the one developed by Webster, Diederich, and Fornæss [DW, DF5]. In the paper [Fo4] the author associated to the mapping f in an invariant way an upper semicontinuous, integer valued function $\nu: M \to \mathbf{Z}_+$, called the *deficiency* of f, that measures the rate of degeneracy of f at the given point $z \in M$. If f is holomorphic in a neighborhood of z in $\mathbf{C}^n$, then

$\nu(z)$ is the dimension of the analytic variety

$$S_z = \{w' \in \mathbf{C}^N : f(Q_z) \subset Q'_{w'}\}$$

at the point $f(z)$. Thus $\nu(z) = 0$ means that $f(z)$ is an isolated point of S_z, which says that the restriction of f to Q_z locally determines $Q'_{f(z)}$ among all varieties $Q'_{w'}$ for w' close to $f(z)$. If this happens, we say that f is non-degenerate at z. If f is only smooth on bD, the definition of ν is more complicated. Instead of simply restricting f to Q_z one has to consider the Taylor development of f along Q_z. (See section 4 in [Fo4].) The main result of [Fo4] is the following pointwise extension theorem:

Theorem 5.3. (See [Fo4], Theorem 6.1.) *Assume that the hypotheses of Theorem 5.2 hold. If the deficiency function $\nu: M \to \mathbf{Z}_+$ is constant in a neighborhood of the point $z \in M$, then f extends holomorphically to a neighborhood of z in $\mathbf{C}^n$. In particular, f extends to a neighborhood of each non-degenerate point $z \in M$.*

Since ν is upper semicontinuous, it is locally constant on an open dense subset of bD, so Theorem 5.2 follows from Theorem 5.3.

Similar results on holomorphic extensions were obtained by Faran [Fa3]. In the setting of Theorem 5.2, with f merely of class C^2 on $D \cup M$, he proved that f extends at $z \in M$ provided that it satisfies certain non-degeneracy condition involving first and second derivatives in the complex tangent directions to M at z. This restricts the range of possible codimensions. The main result of [Fa3] is

Theorem 5.4. *Let U_1 be an open set in $\mathbf{C}^n, U_2$ an open set in $\mathbf{C}^N$. Let $\Omega_j \subset U_j$ be proper open subsets so that each $M_j = b\Omega_j \cap U_j$ is a real-analytic, strongly pseudoconvex hypersurface. Let $f: U_1 \to U_2$ be a mapping of class C^2 that is holomorphic on Ω_1 and maps M_1 into M_2. If the second fundamental form of f is non-degenerate at a point $z \in M_1$, then f extends holomorphically to an open neighborhood of z in $\mathbf{C}^n$.*

The appropriate definition of the second fundamental form can be found on page 8 of [Fa3]. As a corollary, Faran obtained the result of Theorem 5.2 in the case when $N = n + 1$ and f is merely of class C^3 on $D \cup M$.

If f is holomorphic in a neighborhood of $z \in M_1$, and if the second fundamental form of f at z is non-degenerate, then the deficiency $\nu(z)$ equals zero, so the condition of Theorem 5.3 is fulfilled. The improvement in Theorem 5.4 is that f is only assumed to be of class $\mathcal{C}^2$.

In the setting of Theorem 5.2 it is not known whether f extends holomorphically to a neighborhood of each boundary point of D. The proof given in [Fo4] does not apply at a point where the deficiency is not locally constant. The result of Theorem 5.2 is proved in [Fo4] under slightly weaker conditions on M and M'.

In the remainder of this section we shall describe the idea of the proof of Theorem 5.1. (We refer the reader to Section 7 in [Fo1] for the details.)

Let $\langle z, w\rangle = \sum_{j=1}^{n} z_j w_j$ for $z, w \in \mathbf{C}^n$. The conditions on f imply that there is a function $p \in \mathcal{C}^{s-1}(\overline{\mathbf{B}}^n \cap U)$, where $s = N - n + 1$, such that

$$\langle f(z), \overline{f(z)}\rangle - 1 = p(z)\big(\langle z, \overline{z}\rangle - 1\big), \qquad z \in \overline{\mathbf{B}}^n \cap U.$$

Fix a point $z^0 \in b\mathbf{B}^n \cap U$. If f is holomorphic in a full neighborhood of z^0, then p is real-analytic near z^0, so that we can polarize the above identity to obtain

$$\langle f(z), \overline{f}(\overline{w})\rangle - 1 = p(z, \overline{w})\big(\langle z, \overline{w}\rangle - 1\big), \qquad z, w \in \mathbf{C}^n \text{ near } z^0.$$

For $w \in \mathbf{C}^n \backslash \{0\}$ we denote by Q_w the affine hyperplane

$$Q_w = \{z \in \mathbf{C}^n : \langle z, \overline{w}\rangle = 1\} = w/|w|^2 + w^{\perp}.$$

The above polarized identity implies that for each w near z^0, f maps the affine hyperplane $Q_w \subset \mathbf{C}^n$ into the analogous affine hyperplane $Q'_{f(w)} \subset \mathbf{C}^N$ associated to $b\mathbf{B}^n$. This is the fundamental property of proper mappings between balls that are holomorphic across the boundary.

If f is merely of class $\mathcal{C}^s$ on $\overline{\mathbf{B}}^n \cap U$ one can prove by a standard complexification arguement that the Taylor polynomial f^s of f at a boundary point $z \in b\mathbf{B}^n$ maps $Q_z = z^{\perp}$ to $Q'_{f(z)} = f(z)^{\perp}$. Let $\Lambda_z \subset Q'_{f(z)}$ be the smallest affine subspace in $\mathbf{C}^N$ containing the image of $f^s(Q_z)$. In the non-degenerate case we have $\Lambda_z = Q'_{f(z)}$ for some $z \in b\mathbf{B}^n$, so $Q'_{f(z)}$ is the only affine hyperplane in $\mathbf{C}^N$ containing $f^s(Q_z)$. The same is then true for points $w \in U \backslash \overline{\mathbf{B}}^n$ close to z: there is a unique $w' \in \mathbf{C}^N$ such that the affine hyperplane $Q'_{w'}$ contains $f(Q_w \cap \mathbf{B}^n)$. One can then show that the induced mapping $w \to w'$ is a holomorphic extension of f to a neighborhood of z

in $\mathbf{C}^n$. By studying the form of the extension one can also prove that f is a rational mapping. This part is due to Cima and Suffridge [CS1].

In the degenerate case the dimension of Λ_z is smaller than $N-1$ for each $z \in b\mathbf{B}^n \cap U$. Fix a point for which this dimension is maximal, say, k. If U is a sufficiently small neighborhood of z in $\mathbf{C}^n$, then one can show that the set $X \subset (U \backslash \overline{\mathbf{B}}^n) \times \mathbf{C}^N$, defined by

$$X = \{(w, w'): f(Q_w \cap \mathbf{B}^n \cap U) \subset Q'_{w'}\},$$

is a complex analytic variety whose fibers X_w are affine subspaces of $\mathbf{C}^N$ of dimension $m = N - k - 1$. Moreover, by analyzing the form of the defining equations, one can prove that X extends to a rational variety in $\mathbf{C}^n \times \mathbf{C}^N$ whose intersection with the product of the two spheres $(U \cap b\mathbf{B}^n) \times b\mathbf{B}^N$ equals the graph $\{(z, f(z)): z \in U \cap b\mathbf{B}^n\}$ of the mapping f on bD. It follows that f itself extends to a rational mapping.

Examples show that the rational variety X that contains the graph of f may in fact be larger than the graph. It would be interesting to see whether the variety X has any relevance in the classification problem for proper mappings between balls (see section 6).

It seems that the regularity of mappings into higher dimensional hypersurfaces is still very poorly understood.

6. Classification of Mappings between Balls

One would like to know what are all the proper holomorphic mappings $f: \mathbf{B}^n \to \mathbf{B}^N$ for a given pair $N \geq n \geq 1$. If $N = n = 1$, these are the finite Blaschke products, and if $N = n > 1$, these are the automorphisms of $\mathbf{B}^n$ according to the theorem of Alexander (Theorem 3.1). If $N > n$, such maps are abundant according to the results in section 4, and we cannot expect to have any reasonable classification unless we pose some additional conditions on the map.

Further results are known if the map f is smooth of class $C^s(\overline{\mathbf{B}}^n)$ on the closed ball for a sufficiently large s, depending on n and N. Since Theorem 5.1, with $s = N - n + 1$, covers all known regularity results for mappings of balls, we shall assume from now on that the mapping f is *rational*. Recall that f is then holomorphic in a neighborhood of $\overline{\mathbf{B}}^n$ according to Cima and Suffridge [CS4].

Two proper mappings $f, g: \mathbf{B}^n \to \mathbf{B}^N$ are said to be *equivalent* if there exist automorphisms ϕ, ψ of the respective balls such that $g = \psi \circ f \circ \phi$.

It seems that the first result in the case of positive codimension was obtained by Webster [We4]: If $n \geq 3$, then every proper holomorphic map $f: \mathbf{B}^n \to \mathbf{B}^{n+1}$ that is of class $\mathcal{C}^3$ on $\overline{\mathbf{B}}^n$ is equivalent to the linear embedding $(z_1, \ldots, z_n) \to (z_1, \ldots, z_n, 0)$. An alternative proof was given by Cima and Suffridge [CS1] who reduced the smoothness assumption to $f \in \mathcal{C}^2(\overline{\mathbf{B}}^n)$. This was extended by Faran [Fa2] to the case $N \leq 2n-2$. Thus we have

Theorem 6.1. *If $f: \mathbf{B}^n \to \mathbf{B}^N$ is a rational proper mapping and $N \leq 2n-2$, then f is equivalent to the linear embedding $z \to (z, 0)$.*

For $N = 2n-1$ there is a proper polynomial mapping $E_n: \mathbf{B}^n \to \mathbf{B}^{2n-1}$,

$$E_n(z_1, \ldots, z_n) = (z_1, \ldots, z_{n-1}, z_1 z_n, z_2 z_n, \ldots, z_n z_n),$$

that is not equivalent to a linear mapping.

There are precisely four equivalence classes of mappings $\mathbf{B}^2 \to \mathbf{B}^3$. Their classification is due to Faran [Fa1] and, independently, to Cima and Suffridge [CS3]:

Theorem 6.2. *Every rational proper mapping $f: \mathbf{B}^2 \to \mathbf{B}^3$ is equivalent to one the mappings:*

$$\begin{aligned}
&(z, w) \to (z, w, 0), \\
&(z, w) \to (z^2, \sqrt{2} zw, w^2), \\
&(z, w) \to (z^3, \sqrt{3} zw, w^3), \\
&(z, w) \to (z^2, zw, w).
\end{aligned}$$

Further results on classification were obtained by D'Angelo [DA2, DA3, DA4]. For instance, he found that there are precisely 15 non-equivalent proper mappings $\mathbf{B}^2 \to \mathbf{B}^4$ whose components are monomials (see [DA2]). Besides the four mappings to the three-ball listed above there are nine new discrete examples and two one-parameter families.

D'Angelo's approach to the classification problem relies on two key observations. One is that the classification of proper monomial mappings of balls can be reduced to an algebraic problem. If we set $|z_j|^2 = t_j$, then the mapping $f(z) = (c_\alpha z^\alpha), z \in \mathbf{C}^n$, with N components is proper from $\mathbf{B}^n$ to $\mathbf{B}^N$ if and only if $|f(z)|^2 = \sum_\alpha |c_\alpha|^2 t^\alpha = 1$ when $|z|^2 = t_1 + t_2 + \ldots t_n = 1$. Conversely, each polynomial in the variables $t = (t_1, \ldots, t_n)$ with nonnegative coefficients that has constant value 1 on the hyperplane

$\sum_{j=1}^{n} t_j = 1$ gives rise to a proper monomial mappings from $\mathbf{B}^n$ to some ball.

The second observation is that the study of proper polynomial mappings of balls can be reduced to monomial mappings. Suppose that $f = (f_1, \ldots, f_N): \mathbf{C}^n \to \mathbf{C}^N$ is a polynomial mapping. Write each component as a sum of monomials: $f_j = \sum_\alpha c_{j,\alpha} z^\alpha$, and let $h = (c_{j,\alpha} z^\alpha)$ be the mapping from $\mathbf{C}^n$ to $\mathbf{C}^M$ whose components are all the monomials $c_{j,\alpha} z^\alpha$ that appear in f. The order of the components is irrelevant. We then have:

Proposition 6.3. *Let $f: \mathbf{C}^n \to \mathbf{C}^N$ be a polynomial mapping, and let $h: \mathbf{C}^n \to \mathbf{C}^M$ be the associated monomial mapping as above. If f is a proper mapping from $\mathbf{B}^n$ to $\mathbf{B}^N$, then h is a proper mapping from $\mathbf{B}^n$ to $\mathbf{B}^M$.*

This simple but very useful fact was noticed by Forstnerič [Fo5, p. 67, Lemma 4.1] and, independently, by D'Angelo [DA2]. This result can be formulated by saying that each polynomial proper mapping $f: \mathbf{B}^n \to \mathbf{B}^N$ is a composition $L \circ h$ of a proper monomial mapping $h: \mathbf{B}^n \to \mathbf{B}^M$ and a linear mapping $L: \mathbf{C}^M \to \mathbf{C}^N$.

Using simple algebraic methods, D'Angelo obtained in [DA3] an interesting decomposition theorem for proper polynomial mappings of balls. We define the *extend* operation E that acts on proper mappings from $\mathbf{B}^n$ to other balls. The operation E associates to each proper mapping $f = (f_1, \ldots, f_N): \mathbf{B}^n \to \mathbf{B}^N$ the proper mapping $Ef: \mathbf{B}^n \to \mathbf{B}^{N+n-1}$ with the components $Ef = (f_1, \ldots, f_{n-1}, z_1 f_N, \ldots z_n f_N)$. For example, if $n = 2$, $E(id)(z, w) = (z, zw, w^2)$. This operation is the analogue of multiplication by z in one variable. We also have the inverse E^{-1}: If $f = Eg$ for some proper mapping $g: \mathbf{B}^n \to \mathbf{B}^M$, then $E^{-1} f = g$. Using the observation concerning the monomial maps and Proposition 6.3, D'Angelo proved in [DA3] the following:

Theorem 6.4. *Each polynomial proper mapping $f: \mathbf{B}^n \to \mathbf{B}^N$ can be obtained by a successive application of a finite number of operations A_j on proper mappings from $\mathbf{B}^n$ to balls, starting with the identity map on $\mathbf{B}^n$. Each operation A_j is either the extend E, its inverse E^{-1}, or the composition with a linear mapping.*

This result can be thought of as the several variables analogue of de-composing a proper map $f\colon \Delta \to \Delta$ to a finite Blaschke product. Another result of D'Angelo [DA2] that is useful in the classification problem is:

Theorem 6.5. *Suppose that f and g are polynomial proper maps between balls that map the origin to the origin. If f and g are equivalent, then they are unitarily equivalent, i.e., there are unitary maps U_1, U_2 of the respective spaces such that $g = U_2 \circ f \circ U_1$.*

Corollary. *There exist one-parameter families of inequivalent proper maps from $\mathbf{B}^n$ to $\mathbf{B}^{2n}$. An example when $n = 2$ is the family $\{f_t\}$ is defined by*

$$f_t(z, w) = (z, \cos(t)\, w, \sin(t)\, zw, \sin(t)\, w^2).$$

For each $t, s \in [0, \pi/2]$, f_t is equivalent to f_s only when $t = s$.

As the codimension $N - n$ increases further, one obtains multiparameter families of proper monomial maps. As we remarked above, the problem of finding all monomial proper maps from $\mathbf{B}^n$ to $\mathbf{B}^N$ amounts to finding all polynomials in n real variables $x = (x_1, \ldots, x_n)$ with at most N terms, where the coefficients are positive, and the polynomial has a fixed value on the hyperplane $\sum x_j = 1$. We illustrate this procedure in a simple case. Suppose that we wish to find all quadratic monomial maps f from $\mathbf{B}^2$ with $f(0,0) = 0$. There are 5 nontrivial monomials of degree at most 2 in 2 variables. The corresponding real polynomial can be written $p(x,y) = ax + bx^2 + cy + dy^2 + exy$, where the coefficients are non-negative. Setting $p(x, 1-x) = 1$, one obtains a system of linear equations of rank 3. Solving this system, one sees that p can be written as

$$p(x,y) = ax + (1-a)x^2 + cy + (1-c)y^2 + (2-a-c)xy.$$

Put $a = A^2$ and $c = C^2$. Assume A and C are non-negative. We see that the general monomial map of degree at most 2 from the two-ball can be written as a member of the 2-parameter family

$$f(z,w) = (Az, \sqrt{1, A^2}\, z^2, Cw, \sqrt{1 - C^2}\, w^2, \sqrt{2 - A^2 - C^2}\, zw), \quad A, B \geq 0 \tag{6.1}$$

Of course one can apply diagonal linear isometries to this family, but such maps will be equivalent to f. One uses Theorem 6.5 to show that the maps in this family are inequivalent, except for the possibility of interchanging the roles of z and w.

To get higher dimensional families, one must consider monomials of higher degree. To list these requires that the map must have sufficiently high target dimension. We describe one more example. If f is a monomial map from $\mathbf{B}^2$ of degree at most 8 and $f(0) = 0$, there are 44 undetermined coefficients. The condition that f be proper is that $p(x, 1-x) = 1$. This amounts to 9 independent linear equations in these 44 coefficients. Thus every such map f is equivalent to one of those maps into the ball $\mathbf{B}^{44}$, and one must specify 35 parameters to determine the map up to equivalence. If one restricts the target dimension, one obtains additional linear equations on the coefficients. For example, in (6.1), if we wish f to map to the 4-ball, we must fix one parameter to be either 0 or 1. If we wish f to map to the 3-ball, we must specify both parameters.

We repeat that, according to Proposition 6.3, every polynomial map is simply the composition of a linear map and a monomial map, so that the general classification of proper polynomial maps reduces to linear algebra. However, the problem is not a trivial one even after this reduction.

One can use similar ideas to show that, for every n, there are polynomial proper maps from $\mathbf{B}^n$ that are not equivalent to monomial ones. From each $\mathbf{B}^n$, there are rational proper maps that are not equivalent to any polynomial map. These results are not surprising. In dimension 1 it is easy to see that a Blaschke product with with three distinct factors is not equivalent to a polynomial. Every Blaschke product with two factors is equivalent to the map $z \to z^2$.

It is perhaps worthwhile to note that the theory of proper maps between balls can be cast into a general framework that does not treat the one dimensional case differently. The analogue of finite Blaschke product in several variables is the composition product as described in Theorem 6.4 above. The main difference from the case $n = 1$ is that multiplication increases the target dimension when $n > 1$, so that one needs larger dimensional balls to see non-trivial polynomial or rational maps.

D'Angelo has further results on the classification problem in the recent article [DA4].

In the remainder of this section we shall explain the connection between proper holomorphic maps from $\mathbf{B}^n$ and the finite subgroups of the unitary group $U(n)$. The reference for this part are the papers [Car2], [Ru3] and

[Fo3].

If $\Gamma \subset U(n)$ is a finite unitary group, then quotient $\mathbf{C}^n/\Gamma$ can be realized as a normal algebraic subvariety V in some $\mathbf{C}^N$ according to a theorem of Cartan [Car2]. In order to do this we choose a finite number of homogeneous, Γ-invariant holomorphic polynomials $q_1, \ldots, q_N$ which generate the algebra of all Γ-invariant polynomials (this is possible by the Hilbert basis theorem). Recall that a function f is called Γ-invariant if $f(z) = f(\gamma(z))$ for all $\gamma \in \Gamma$ and all points z in $\mathbf{B}^n$. The induced map $Q = (q_1 \ldots, q_N): \mathbf{C}^n \to \mathbf{C}^N$ is proper and induces a homeomorphism of the quotient $\mathbf{C}^n/\Gamma$ onto the image $V = Q(\mathbf{C}^n) \subset \mathbf{C}^N$. The restriction of Q to the ball $\mathbf{B}^n$ maps the ball properly onto a domain Ω in V.

Recall that a *reflection* in $\mathrm{U}(n)$ is an element of finite order that fixes a complex hyperplane, i.e., $n-1$ of its eigenvalues equal one, and the remaining eigenvalue is a root of unity. A finite subgroup $\Gamma \subset \mathrm{U}(n)$ is called a *reflection group* if it is generated by reflections. It is known that the algebra of Γ-invariant polynomials always requires at least n generators, and it is generated by n elements if and only if Γ is generated by reflections. Thus, the quotient $\mathbf{B}^n/\Gamma$ is non-singular at the origin if and only if Γ is a *finite reflection group*. In this case $\mathbf{C}^n/\Gamma = \mathbf{C}^n$, and the ball $\mathbf{B}^n$ is mapped by Q properly onto a bounded domain $\Omega \subset \mathbf{C}^n$. (See [Fe3] and the references therein.)

Rudin [Ru3] proved a converse of this for proper holomorphic mappings from $\mathbf{B}^n$ to domains in $\mathbf{C}^n$:

Theorem 6.6. *If $f: \mathbf{B}^n \to \Omega$ is a proper holomorphic map onto a domain $\Omega \subset \mathbf{C}^n, n \geq 2$, then there are a finite reflection group $\Gamma \subset \mathrm{U}(n)$ and an automorphism ϕ of $\mathbf{B}^n$ such that $f = \eta \circ Q \circ \phi$, where $Q: \mathbf{B}^n \to \mathbf{B}^n/\Gamma$ is the quotient projection, and $\eta: \mathbf{B}^n/\Gamma \to$ is a biholomorphic map.*

Rudin proved the theorem under the additional hypothesis that f is of class $\mathcal{C}^1$ on the closed ball $\overline{\mathbf{B}}^n$. A result of Bedford and Bell [BB3, BB4] on proper holomorphic correspondences of strongly pseudoconvex domains shows that this smoothness hypothesis is not required. Bedford and Bell proved a similar result for mappings from smoothly bounded pseudoconvex domains in $\mathbf{C}^n$ to normal complex analytic spaces of dimension n. (See also [BaB].)

Since there exists a classification of all finite reflection groups, we have the corresponding classification of proper holomorphic mappings from $\mathbf{B}^n$

to domains in $\mathbf{C}^n$ (see [Ru3]). If $U \subset \Gamma$ is a reflection that fixes the complex hyperplane $\Sigma \subset \mathbf{C}^n$, then every map f that is invariant under Γ must branch at all points of Σ. It follows that the image $\Omega = f(\mathbf{B}^n)$ does not have smooth boundary.

On the other end of the scale among the subgroups of $\mathrm{U}(n)$ lie the *fixed point free* groups. These are the finite subgroups of $\mathrm{U}(n)$ with the property that 1 is not an eigenvalue of any $\gamma \in \Gamma\backslash\{1\}$. Equivalently, γ has no fixed points except the origin. The action of Γ on $\mathbf{C}^n\backslash\{0\}$ is without fixed points, so the origin is the only singularity of the quotient variety $\mathbf{C}^n/\Gamma$. The quotient of the sphere $b\mathbf{B}^n/\Gamma$ is a real-analytic manifold, called the *spherical space form.* The quotient projection can again be realized by a polynomial mapping $Q\colon \mathbf{C}^n \to \mathbf{C}^N$, where N is the minimal number of generators for the algebra of Γ-invariant polynomials on $\mathbf{C}^n$. We can find a smoothly bounded, strongly pseudoconvex domain $\Omega \subset \mathbf{C}^N$ such that $Q\colon \mathbf{B}^n \to \Omega$ is a proper mapping.

Again we have a converse of this, due to Forstnerič [Fo3]:

Theorem 6.7. *Let $f\colon \mathbf{B}^n \to \Omega, n \geq 2$, be a proper holomorphic mapping into a relatively compact, strongly pseudoconvex domain Ω in a complex manifold. If f extends to a C^1 map on $\overline{\mathbf{B}}^n$, then there exists a finite fixed point free unitary group $\Gamma \subset \mathrm{U}(n)$ and an automorphism ϕ of $\mathbf{B}^n$ such that $f = \eta \circ Q \circ \phi$, where $Q\colon \mathbf{B}^n \to \mathbf{B}^n/\Gamma$ is a quotient projection, and $\eta\colon \mathbf{B}^n/\Gamma \to f(\mathbf{B}^n)$ is the normalization of the subvariety $f(\mathbf{B}^n)$ of Ω.*

The set $f(\mathbf{B}^n)$ is a subvariety of Ω according to the theorem of Remmert. For the notion of normalization see [Nar]. The proof of this result is considerably simpler that the proof of Theorem 6.6. It is easy to understand why Γ must be fixed point free. Since f is assumed to be of class C^1 on the boundary, it does not branch there, hence the branch locus is compactly contained in $\mathbf{B}^n$ and thus finite. On the other hand, $f \circ \phi^{-1}$ is branched along the fixed point set of every $\gamma \in \Gamma\backslash\{1\}$, which is a linear subspace of $\mathbf{C}^n$. It follows that this subspace is the trivial one, so γ acts without fixed points on $\mathbf{C}^n\backslash\{0\}$.

One difference between Theorems 6.6 and 6.7 is that in the first one, the map $f \circ \phi^{-1}$ is precisely Γ-invariant, in the sense that the quotient space is biholomorphic to the image $f(\mathbf{B}^n)$. In Theorem 6.7 this is no longer true. An example is the map $(z, w) \to (z^2, zw, w)$ that is one-one except on the disc $w = 0$ where it is two-to-one. It seems that most proper maps of balls are not invariant under any non-trivial group.

There is a classification of finite fixed point-free unitary groups that was carried out in order to solve the Clifford-Klein *spherical space form problem*. Spherical space forms are the complete, connected Riemannian manifolds of constant positive curvature. Every such is the quotient of a sphere $S_r = \{x \in \mathbf{R}^n : |x| = r\}$ by a finite fixed point free orthogonal group $\Gamma \subset O(n)$. The classification of real fixed point-free groups can be reduced to the complex ones, so it suffices to treat the unitary case $\Gamma \subset \mathrm{U}(n)$. A beautiful and detailed exposition can be found in Chapters 5-7 of Wolf's book [Wolf]. For a brief introduction see also the paper [Fo3].

As we remarked above, every fixed point free group $\Gamma \subset \mathrm{U}(n)$ induces a proper, Γ-invariant holomorphic map f from $\mathbf{B}^n$ to some strongly pseudoconvex domain $\Omega \subset \mathbf{C}^N$. The problem becomes interesting if we require the target domain Ω to be the unit ball $\mathbf{B}^N$ for some N. If the map is sufficiently smooth on $\overline{\mathbf{B}}^n$ and therefore rational, we obtain severe restrictions on the group Γ. In order to explain the main result of [Fo3], we have to recall that the basic structure of fixed point free groups $\Gamma \subset \mathrm{U}(n)$.

All Sylow p-subgroups of Γ for odd p are cyclic. The Sylow 2-subgroups are either cyclic (groups of Type A) or generalized quaternioic (groups of type B). The fixed point-free groups of Type A have two generators; they are classified in [Wolf, p. 168]. The simplest such groups are the cyclic groups generated by ϵI, where ϵ is a root of unity.

Groups of type B are subdivided into five subtypes; they are classified in Section 7.2 of [Wolf]. The finite subgroups of SU(2) are all fixed point free, and they play a special role in the classification. Besides the cyclic groups (of type A), SU(2) contains binary dihedral and binary polyhedral groups which arise as the preimages of the rotation groups of regular Platonic solids in $\mathbf{R}^3$. These groups are of type B. Each of them has a basis of three invariant polynomials q_1, q_2, q_3, satisfying one relation. These invariants can already be found in Klein's book [Kl, pp. 50-63]. We do not know of any systematic treatment of the invariant theory of other fixed point free groups.

Recall that a unitary representation of an abstract group Γ is a homomorphism $\pi: \Gamma \to \mathrm{U}(n)$. The integer n is the degree of π. The representation is fixed point free if 1 is not an eigenvalue of $\pi(\gamma)$ for any $\gamma \neq 1$. The representation is irreducible if it is not the direct sum of two other representations. (See Chapter 4 in [Wolf] for a concise account of the representation theory of finite groups.)

Theorem 6.8. *Let $\pi: \Gamma \to \mathrm{U}(n), n \geq 2$, be a fixed point-free representation. If there exists $\pi(\Gamma)$-invariant rational proper mapping $f: \mathbf{B}^n \to \mathbf{B}^N$ for some N, then the group Γ is of type* **A**, *i.e., all of its Sylow subgroups are cyclic, and the irreducible fixed point free representations of Γ are of odd degree. If $n = 2^k$ for some k, then Γ is cyclic.*

It is not known which fixed point-free groups $\Gamma \subset \mathbf{C}^n$ of type **A** actually induce a Γ-invariant proper map from $\mathbf{B}^n$ to some ball. For $n = 2$ it seems that there are only two known examples:

(1) Γ is the cyclic group in $\mathrm{U}(2)$, generated by the matrix $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$, where $\epsilon = e^{2\pi i/k}$. The corresponding invariant mapping $f: \mathbf{B}^2 \to \mathbf{B}^{n+1}$ has components $\sqrt{\binom{k}{j}} z^j w^{n-j}, 0 \leq j \leq k$, so it is homogeneous of order k. An example of this kind exists for every n.

(2) Γ is the cyclic group generated by $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^2 \end{pmatrix}$, where $\epsilon = e^{2\pi i/k}$ and $k = 2r + 1$ for some $r \in \mathbf{Z}_+$. A basis of invariants is

$$z^{2r+1}, z^{2r-1}w, \; z^{2r-3}w^2, \ldots, w^{2r+1}.$$

D'Angelo proved in [DA3] that there is, up to equivalence, exactly one proper mapping $f: \mathbf{B}^2 \to \mathbf{B}^{r+2}$ whose components are constant multiples of these monomials. He obtained an explicit expression for the coefficients and listed the first few of these maps [DA3, p. 214].

Another way of asking the question about the existence of proper maps between balls that are invariant under a group is in terms of C-R embeddings. If $\Gamma \subset \mathrm{U}(n)$ is a finite fixed point free group, the quotient $S(\Gamma) = b\mathbf{B}^n/\Gamma$ is a C-R manifold that is locally equivalent to the sphere at each point. The question is for which Γ does $S(\Gamma)$ admit a global C-R immersion or embedding into a sphere $b\mathbf{B}^n$ for a sufficiently large N? Clearly such an immersion extends to a proper holomorphic map of $\mathbf{B}^n$ into $\mathbf{B}^N$, and vice versa, the restriction of a Γ-invariant proper mapping $f: \mathbf{B}^n \to \mathbf{B}^N$, smooth on $\overline{\mathbf{B}}^n$, to the sphere $b\mathbf{B}^n$ induces a C-R immersion of $S(\Gamma)$ to $b\mathbf{B}^n$. Since f induces a mapping of tangent bundles that respects the C-R structure, it may be possible to obtain obstructions to such C-R immersions by the methods of algebraic topology (Chern classes, etc.). Perhaps there is an alternative proof of Theorem 6.8, using topological methods.

Although there has been a substantial progress toward the classification of maps between balls in recent years, there are still a lot of open problems. The composition theorem of D'Angelo (Theorem 6.4) allows us in principle to list all proper polynomial maps of certain degree between a given pair of balls, although the corresponding algebraic problem is not always easy to solve. The classification problem for rational maps between balls seems to be much more difficult.

In our opinion, the classification problem for mappings between balls deserves more attention since it combines in a nice way the methods of complex analysis with those from algebra, the representation theory of fixed point free groups, the theory of invariants, differential geometry, and possibly other areas of Mathematics.

Institut of Mathematics,
University of Ljubljana,
Ljubljana, Yugoslavia

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