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Approximation by automorphisms on smooth submanifolds of $\mathbf{C}^n$

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Introduction

We denote by $\mathrm{Aut}\,\mathbf{C}^n$ the group of all holomorphic automorphisms of the complex Euclidean space $\mathbf{C}^n$. This is a very large and complicated group when $n \geqq 2$; for results in this direction see [5, 6, 11, 20]. In the present paper we further demonstrate this by showing that one can approximate smooth mappings $F : M \to \mathbf{C}^n$ on certain smooth submanifolds $M \subset \mathbf{C}^n$ by restrictions to M of holomorphic automorphisms of $\mathbf{C}^n$. We shall assume $n \geqq 2$ throughout the paper.

Recall that a real submanifold $M \subset \mathbf{C}^n$ is **totally real** if the tangent space T_zM at each point $z \in M$ contains no complex line. A compact set $K \subset \mathbf{C}^n$ is **polynomially convex** if for every $z \in \mathbf{C}^n \backslash K$ there exists a holomorphic polynomial P such that $|P(z)| > \sup_K |P|$. For $p \geqq 1$, a $\mathscr{C}^p$ **isotopy of embeddings** of M into $\mathbf{C}^n$ is a $\mathscr{C}^p$ map $F : [0,1] \times M \to \mathbf{C}^n$ such that for each fixed $t \in [0,1]$, $F_t = F(t, \cdot\,) : M \to \mathbf{C}^n$ is an embedding. The isotopy $\{F_t\}$ is totally real (resp. polynomially convex) if $F_t(M)$ is totally real (resp. polynomially convex) for every $t \in [0,1]$. In this paper all compact submanifolds $M \subset \mathbf{C}^n$ can be with or without boundary.

Main theorem. *If $M \subset \mathbf{C}^n$ $(n \geqq 2)$ is a compact, totally real, polynomially convex submanifold of class $\mathscr{C}^p$ $(2 \leqq p < \infty)$ and $F : M \to \mathbf{C}^n$ is a $\mathscr{C}^p$ mapping, then the following are equivalent:*
(i) *For each $\varepsilon > 0$ there exists a $\Phi \in \mathrm{Aut}\,\mathbf{C}^n$ such that $\|F - \Phi|_M\|_{\mathscr{C}^p(M)} < \varepsilon$.*
(ii) *For each $\varepsilon > 0$ there exists a totally real, polynomially convex isotopy $F_t : M \to \mathbf{C}^n$ $(t \in [0,1])$ of class $\mathscr{C}^p$ such that F_0 is the identity on M and $\|F_1 - F\|_{\mathscr{C}^p(M)} < \varepsilon$.*

Another main result of the paper, Theorem 4.1 and Corollary 4.2, concerns generic polynomial convexity of low dimensional submanifolds and their isotopies in $\mathbf{C}^n$. Together with the main theorem we obtain

Corollary 1. *Let $M \subset \mathbf{C}^n$ be a compact, totally real, polynomially convex submanifold of dimension at most $2n/3$ and of class $\mathscr{C}^p$, $2 \leqq p < \infty$. Then for every $\mathscr{C}^p$ mapping $F : M \to \mathbf{C}^n$ and for every $\varepsilon > 0$ there exists a holomorphic automorphism Φ of $\mathbf{C}^n$ such that $\|F - \Phi|_M\|_{\mathscr{C}^p(M)} < \varepsilon$. If M is real-analytic and if $F : M \to M' \subset \mathbf{C}^n$ is a real-analytic diffeomorphism onto another totally real, polynomially convex submanifold M' of $\mathbf{C}^n$, then F extends to a biholomorphic mapping in a neighborhood U of M which can be approximated, uniformly in U, by holomorphic automorphisms of $\mathbf{C}^n$.*

Proof of Corollary 1. Consider first the smooth case. Since M is compact and the dimension of the ambient space $\mathbf{C}^n$ exceeds $2 \dim M + 1$, F can be approximated in the $\mathscr{C}^p$ norm by an embedding $F' : M \to \mathbf{C}^n$ [15]. By Corollary 4.2 (Sect. 4) F' can be approximated arbitrarily close (in $\mathscr{C}^p(M)$) by the map F_1 at $t = 1$ in a totally real and polynomially convex isotopy $F_t : M \to \mathbf{C}^n$ which starts at $t = 0$ with the identity on M. Thus the condition (ii) in the main theorem is satisfied, and therefore F can be approximated by restrictions to M of holomorphic automorphisms of $\mathbf{C}^n$.

Suppose now that $F : M \to M'$ is a real-analytic diffeomorphism between totally real, polynomially convex submanifolds of $\mathbf{C}^n$ of dimension at most $2n/3$. By Corollary 4.2 (Sect. 4) there exists a totally real, polynomially convex isotopy $\{F_t\}$ connecting $F = F_1$ to the identity $F_0 = \mathrm{Id}_M$ on M. By approximation we can assume that the isotopy is real-analytic. The result now follows from Corollary 3.2 in Sect. 3 (or from Theorem 3.1 in [11]). This proves Corollary 1.

Recall that for $M \subset \mathbf{C}^n$ as in the main theorem, every $\mathscr{C}^p$ map $F : M \to \mathbf{C}^n$ can be approximated in the $\mathscr{C}^p$ norm on M by entire holomorphic maps. This follows from the approximation theorem of Range and Siu [18]: *If M is a totally real submanifold of class $\mathscr{C}^p$ in a complex manifold X, $1 \leqq p \leqq \infty$, then there exists a Stein open neighborhood U of M in X such that the set of restrictions to M of holomorphic functions in U is dense in the Fréchet space of all $\mathscr{C}^p$ functions on M.* Results in this direction were obtained also by Harvey and Wells [14], Hörmander and Wermer [16], Berndtsson [8], and others. Oka's theorem [17, p. 55] implies that, *if $X = \mathbf{C}^n$ and if M is also compact and polynomially convex, the set of restrictions to M of all holomorphic polynomials on $\mathbf{C}^n$ is dense in $\mathscr{C}^p(M)$.*

Sketch of proof of the main theorem. The proof relies to a large degree on methods developed in [5], [6], and [11]. The implication (i) $\Rightarrow$ (ii) follows from the fact that the group $\mathrm{Aut}\,\mathbf{C}^n$ is connected [11]. Given a $\Phi \in \mathrm{Aut}\,\mathbf{C}^n$ satisfying (i), we choose a family $\{\Phi_t\} \subset \mathrm{Aut}\,\mathbf{C}^n$, depending smoothly on the parameter $t \in [0, 1]$, such that Φ_0 is the identity and $\Phi_1 = \Phi$. The family of embeddings $F_t = \Phi_t|_M : M \to \mathbf{C}^n$ then satisfies (ii).

The main ingredient in the proof of (ii) $\Rightarrow$ (i) is Theorem 1.1 (Sect. 1) on approximation of parametrized families of biholomorphic mappings between

Runge domains in $\mathbf{C}^n$ by automorphisms of $\mathbf{C}^n$. This result is a stronger version of Theorem 1.1 in [11]. As we have already pointed out in [11], the result follows from methods developed by Andersén [5] and Andersén and Lempert [6].

The implication (ii) $\Rightarrow$ (i) is proved as follows. Given a totally real isotopy $F_t : M \to \mathbf{C}^n$, starting at $t = 0$ with the identity on M, we first construct an isotopy of biholomorphic mappings $\Psi_t : \Omega \to \mathbf{C}^n$ in a neighborhood Ω of M such that Ψ_0 is the identity and $\Psi_t|_M$ approximates F_t in $\mathscr{C}^p(M)$ (Theorem 3.1). This step only requires that the manifold $F_t(M)$ is totally real for each $t \in [0, 1]$. The proof depends on a jet approximation theorem, Theorem 2.1, which generalizes the Range-Siu theorem mentioned above. If we assume in addition that $F_t(M)$ is polynomially convex for every t, we can apply Theorem 1.1 to approximate Ψ by a map Φ such that $\Phi_t \in \operatorname{Aut}\mathbf{C}^n$ for each $t \in [0, 1]$ (Corollary 3.2 in Sect. 3). This proves the main theorem. Note that Corollary 3.2 gives a stronger result than is needed in the proof of the main theorem.

The analogue of our main theorem has been proved for real-analytic submanifolds $M \subset \mathbf{C}^n$ in our paper [11] with Rosay (Theorem 3.1). There we introduced a more restrictive notion of $\mathbf{C}^n$-equivalence as follows. Two embeddings $f_0, f_1 : M \to \mathbf{C}^n$ are $\mathbf{C}^n$-equivalent if there exists a biholomorphic mapping $\Phi : U \to U'$ in a neighborhood U of $f_0(M)$ satisfying $\Phi \circ f_0 = f_1$, and a sequence $\Phi^j \in \operatorname{Aut}\mathbf{C}^n$ such that $\lim_{j\to\infty} \Phi^j = \Phi$ uniformly on U. This notion of equivalence of subsets in $\mathbf{C}^n$ clearly implies equivalence of their polynomial hulls. According to Theorem 3.1 in [11], two real-analytic, totally real, polynomially convex embeddings $f_0, f_1 : M \to \mathbf{C}^n$ are $\mathbf{C}^n$-equivalent if and only if there exists a totally real, polynomially convex isotopy $f_t : M \to \mathbf{C}^n$ ($0 \leqq t \leqq 1$) connecting f_0 to f_1. This is precisely the condition (ii) in our main theorem. We offer a simpler proof of this result (in the real-analytic case) at the end of Sect. 1.

For smooth submanifolds the notion of $\mathbf{C}^n$-equivalence is obviously too strong since totally real submanifolds of the same dimension are not even locally equivalent in this sense. A possible approach for smooth submanifolds has been indicated (in the case of arcs) by Rosay [19]. In the present paper we consider only the approximation of maps $F : M \to \mathbf{C}^n$, without attempting to approximate at the same time the inverse F^{-1} by the inverses of automorphisms approximating F.

As an application we obtain results on 'filling' by totally real, polynomially convex manifolds with the given boundary. A smooth embedded m-disc in $\mathbf{C}^n$ is the image of an embedding of the closed ball $D^m \subset \mathbf{R}^m$ into $\mathbf{C}^n$. The following was proved in [11] for real-analytic submanifolds.

Corollary 2. (a) *Every simple, closed, polynomially convex $\mathscr{C}^2$ curve $\Gamma \subset \mathbf{C}^n$ ($n \geqq 2$) bounds an embedded, totally real, polynomially convex two-disc in $\mathbf{C}^n$.*

(b) *Every embedded two-sphere $S \subset \mathbf{C}^n$ ($n \geqq 3$) of class $\mathscr{C}^2$ which is totally real and polynomially convex bounds an embedded, totally real, polynomially convex three-disc in $\mathbf{C}^n$.*
(c) *Every closed, embedded, orientable, $\mathscr{C}^p$ surface $S \subset \mathbf{C}^n$ ($n \geqq 3, p \geqq 2$) which is totally real and polynomially convex bounds an embedded, totally real, polynomially convex three-manifold in $\mathbf{C}^n$.*

In each case we can take the manifold Σ bounded by the given curve or surface S to have the same smoothness as S.

Proof. (a) The curve $\Gamma_0 = \{(\zeta, \bar{\zeta}, 0, \ldots, 0) : |\zeta| = 1\} \subset \mathbf{C}^n$ bounds the totally real, polynomially convex disc $\Delta_0 = \{(\zeta, \bar{\zeta}, 0, \ldots, 0) : |\zeta| \leqq 1\}$. Choose a $\mathscr{C}^2$ diffeomorphism $F : \Gamma \to \Gamma_0$. By Corollary 1 we can approximate F in the $\mathscr{C}^2$ sense by automorphisms $\Phi \in \operatorname{Aut} \mathbf{C}^n$. The curve $\Phi(\Gamma) = \Gamma_1$ is then a small $\mathscr{C}^2$ perturbation of Γ_0. If $\|F - \Phi|_\Gamma\|_{\mathscr{C}^2(\Gamma)}$ is sufficiently small, the curve Γ_1 also bounds a totally real, polynomially convex disc Δ_1 which is obtained as a small $\mathscr{C}^2$ perturbation of Δ_0. (We used the well known fact that polynomial convexity is a stable property in the class of $\mathscr{C}^2$ totally real submanifolds of $\mathbf{C}^n$ [12].) Then $\Phi^{-1}(\Delta_1)$ is the required totally real, polynomially convex disc in $\mathbf{C}^n$ with boundary Γ.

In case (b) we apply Corollary 1 to a diffeomorphism $F : S \to S_0$ onto the unit sphere $S_0 \subset \mathbf{R}^3 \subset \mathbf{C}^n$. For part (c) observe that every closed orientable surface embeds into $\mathbf{R}^3 \subset \mathbf{C}^3$ and it bounds a domain in $\mathbf{R}^3$. This proves Corollary 2.

Recall that a closed curve $\Gamma \subset \mathbf{C}^n$ that is not polynomially convex bounds a one dimensional complex variety A and $\bar{A} = A \cup \Gamma$ is polynomially convex (Wermer [24], Stolzenberg [22], Alexander [2]). Thus we have an interesting alternative for $\mathscr{C}^2$ curves in $\mathbf{C}^n$: *either Γ bounds a complex variety or else it bounds a totally real, polynomially convex two-disc.*

The paper is organized as follows. In Sect. 1 we improve our result from [11] by including real parameters $x \in \mathbf{R}^k$. In Sect. 2 we prove an approximation theorem for jets of holomorphic functions on totally real submanifolds $M \subset \mathbf{C}^n$ (Theorem 2.1). The result follows essentially from the methods used in [18] and [7]. Since it had not been stated there explicitly and lacking a reference we include a proof. In Section 3 we prove Theorem 3.1 on approximation of parametrized families of diffeomorphisms of totally real manifolds by biholomorphic mappings defined in a neighborhood. Combining this with Theorem 1.1 gives Corollary 3.2 which yields the implication (ii) $\Rightarrow$ (i) in the main theorem. Section 4 contains results on generic polynomial convexity of low dimensional totally real submanifolds of $\mathbf{C}^n$, extending Theorem 5.2 in [11].

In the appendix we prove a result on decomposition of polynomial holomorphic vector fields on $\mathbf{C}^n$ into finite sums of complete fields of special types. This result, which is implicitly contained in the papers of Andersén [5] and Andersén and Lempert [6], is of fundamental importance in the approximation theorems in [5, 6, 11], and also in the present paper. Since the result has not been stated explicitly in [5] or [6] (or anywhere else in the literature), we state it here and give a short proof. We emphasize that we do not claim any originality on our part. (A less precise statement can be found in [11, Lemma 1.3].) For a similar result on decomposition of Hamiltonian vector fields on $\mathbf{C}^{2n}$ see [25].

1 An approximation theorem for biholomorphic mappings

The following result is a parametrized version of Theorem 1.1 in [11]. Recall that a domain $\Omega \subset \mathbf{C}^n$ is Runge if every holomorphic function on Ω can be approximated, uniformly on compacts in Ω, by holomorphic polynomials.

1.1 Theorem. *Let $\Omega \subset \mathbf{C}^n$ be a Runge domain, and let B be the closed unit ball in $\mathbf{R}^k$. Assume that $F : B \times \Omega \to \mathbf{C}^n$ is a mapping of class $\mathscr{C}^p$ ($0 \leqq p < \infty$) such that for each $x \in B$, $F_x = F(x, \cdot\,) : \Omega \to \mathbf{C}^n$ is a biholomorphic mapping onto a Runge domain $\Omega_x \subset \mathbf{C}^n$, and the map F_0 can be approximated by automorphisms of $\mathbf{C}^n$, uniformly on compact sets in Ω. Then for each compact set $K \subset \Omega$ and each $\varepsilon > 0$ there exists a smooth map $\Phi : B \times \mathbf{C}^n \to \mathbf{C}^n$ such that $\Phi_x = \Phi(x, \cdot\,)$ is a holomorphic automorphism of $\mathbf{C}^n$ for every $x \in B$ and $\|F - \Phi\|_{\mathscr{C}^p(B\times K)} < \varepsilon$. If in addition we have $F_0 \in \mathrm{Aut}\,\mathbf{C}^n$, then we can choose Φ such that $\Phi_0 = F_0$.*

Proof. Without loss of generality we may restrict ourselves to the case when Ω is a pseudoconvex Runge domain in $\mathbf{C}^n$. This is because the envelope of holomorphy $\tilde{\Omega}$ of every Runge domain $\Omega \subset \mathbf{C}^n$ is a (single sheeted) pseudoconvex Runge domain in $\mathbf{C}^n$, the maps $F_x : \Omega \to \mathbf{C}^n$ extend to $\tilde{\Omega}$, and approximation on Ω implies approximation on $\tilde{\Omega}$. By replacing F_x by $F_x \circ F_0^{-1}$ and Ω by $F_0(\Omega)$ we may also assume that F_0 is the identity on Ω.

Fix a compact $K \subset \Omega$ as in the theorem, and choose a smooth strongly pseudoconvex domain $D \Subset \Omega$ such that $K \subset D$ and $\bar{D}$ is polynomially convex in $\mathbf{C}^n$. Notice that Ω can be exhausted by such domains since it is pseudoconvex and Runge in $\mathbf{C}^n$. Since $F_x : \Omega \to \Omega_x$ is biholomorphic and Ω_x is Runge in $\mathbf{C}^n$, it follows that $F_x(D)$ is strongly pseudoconvex and $\overline{F_x(D)}$ is polynomially convex for every $x \in B$. Finally we choose a compact polynomially convex set $K_0 \subset \Omega$ such that $\bar{D} \subset \mathrm{Int}(K_0)$.

In the first step we approximate F by a polynomial map $\Theta : \mathbf{R}^k \times \mathbf{C}^n \to \mathbf{C}^n$ in variables (x,z) such that $\Theta(0,z) = z$ ($z \in \mathbf{C}^n$), and

$$\|F - \Theta\|_{\mathscr{C}^p(B\times K_0)} < \varepsilon/3 \,.$$

Such approximations exists according to the following lemma which we state in a more general form for a future reference.

1.2 Lemma. *Let $f(x,z)$ be a function of class $\mathscr{C}^p$ ($0 \leqq p < \infty$) on a domain of the form $\tilde{\Omega} = \bigcup_{x\in B}\{x\} \times \Omega_x \subset \mathbf{R}^k \times \mathbf{C}^n$, where each Ω_x is a Runge domain in $\mathbf{C}^n$ and B is a closed ball in $\mathbf{R}^k$. If $f(x, \cdot\,)$ is holomorphic on Ω_x for each fixed $x \in B$, then for every $\varepsilon > 0$ and for every compact set $E \subset \tilde{\Omega}$ there is a polynomial $g(x,z)$ in variables $(x,z) \in \mathbf{R}^k \times \mathbf{C}^n$, satisfying $\|f - g\|_{\mathscr{C}^p(E)} < \varepsilon$.*

If $z \mapsto f(x^0,z)$ is a polynomial in z for some fixed $x^0 \in B$, we can choose g as above such that $f(x^0, \cdot) = g(x^0, \cdot)$.

Proof. For a fixed $x^0 \in B$ we develop f in Taylor series in the x-direction at x^0:

$$f(x,z) = \sum_{|\alpha| \leqq p} \frac{1}{\alpha!} D_x^\alpha f(x^0,z)(x-x^0)^\alpha + R_p(x,x^0,z),$$

where $|R_p(x,x^0,z)| = o(|x-x^0|^p)$, uniformly with respect to z in a compact subset of Ω_{x^0}. We approximate the coefficients $D_x^\alpha f(x^0,z)$ on Ω_{x^0}, uniformly in a neighborhood of $E \cap (\{x_0\} \times \Omega_{x^0})$, by holomorphic polynomials $g_\alpha(z)$ ($|\alpha| \leqq p$). The polynomial

$$\sum_{|\alpha| \leqq p} \frac{1}{\alpha!} g_\alpha(z)(x-x^0)^\alpha$$

in the variables (x,z) approximates f in the $\mathscr{C}^p$ norm on E for x close to x^0. We can patch these approximations, which are local in x but global in z, into a global approximation g on E by using a suitable approximate partition of unity on B, consisting of polynomials in x. The last statement in Lemma 1.2 is clear from the construction. This proves Lemma 1.2.

We continue with the proof of Theorem 1.1. The Cauchy estimates in the z-variable imply that on the smaller set $\bar{D} \subset K_0$ the map $\Theta_x = \Theta(x, \cdot)$ approximates F_x in any $\mathscr{C}^s$ norm, uniformly with respect to $x \in B$. If we choose $s = 2$ and if $\varepsilon > 0$ is sufficiently small, then Θ_x is a biholomorphic map in a neighborhood of $\bar{D}$, and it maps D onto a strongly pseudoconvex domain $D_x = \Theta_x(D) \subset \mathbf{C}^n$ for every $x \in B$. Moreover, since D_x is a small $\mathscr{C}^2$ deformation of the strongly pseudoconvex domain $F_x(D) \subset \Omega_x$ with polynomially convex closure, $\bar{D}_x$ is also polynomially convex in $\mathbf{C}^n$ for every $x \in B$, provided that ε is sufficiently small. (Recall from [12] that the polynomial convexity and the Runge property are $\mathscr{C}^2$-stable properties in the class of strongly pseudoconvex domains in $\mathbf{C}^n$.)

We consider Θ as the time one map of the flow $t \in [0,1] \to \Theta(tx,z)$, starting with the map $(x,z) \to z$ at $t = 0$. Its infinitesimal generator is the time dependent vector field on $\mathbf{C}^n$,

$$\begin{aligned} V(t,x,z) &= \frac{d}{ds}\Theta\big(sx, \Theta_{tx}^{-1}(z)\big)|_{s=t} \\ &= \sum_{j=1}^{n} x_j(\partial\Theta/\partial x_j)\big(tx, \Theta_{tx}^{-1}(z)\big), \quad z \in \bar{D}_{tx}, \end{aligned}$$

whose coefficients are real-analytic functions on the set $\tilde{D} = \bigcup_{(t,x)\in[0,1]\times B} (t,x) \times \bar{D}_{tx}$ which are holomorphic on $z \in \bar{D}_{tx}$ for each fixed (t,x).

It will be convenient to extend the parameter x to nearby complex values. There exist an open neighborhood $B_0 \subset \mathbf{C}^k$ of $B \times \{i0\} \subset \mathbf{C}^k$ and a number $T > 1$ (close to 1) such that V and its flow are defined for $t \in [0,T]$ and $x \in B_0$.

Using Lemma 1.2 we can approximate V in the $\mathscr{C}^p$ norm on $\tilde{D}$ by a polynomial vector field

$$W(t,x,z) = \sum t^j x^\alpha W^l(z), \quad j,l \in \mathbf{Z}_+, \ \alpha \in \mathbf{Z}_+^k ,$$

where each $W^l(z)$ is a holomorphic polynomial vector field on $\mathbf{C}^n$ and the sum is finite. Let $\Xi(t,x,z)$ be the flow of W with respect to the time variable t, subject to the initial condition $\Xi(0,x,z) = z$. Write $\Xi_t = \Xi(t, \cdot, \cdot)$. If the approximation of V by W is sufficiently close, there is an open neighborhood B_1 of $B \times \{i0\} \subset \mathbf{C}^k$ such that $\Xi(t,x,z)$ is defined for all $t \in [0,1]$ and all $(x,z) \in B_1 \times \tilde{D}$, and we have

$$\|\Xi_1 - \Theta\|_{\mathscr{C}^p(B\times\tilde{D})} < \varepsilon/3 .$$

The lemma in the Appendix implies that each polynomial vector field $W^l(z)$ on $\mathbf{C}^n$ is a finite sum of complete polynomial vector fields whose flows are one parameter subgroups of $\operatorname{Aut}\mathbf{C}^n$. By changing the enumeration we may suppose that each field W^l above is of this type, with the corresponding flow $t \in \mathbf{R} \mapsto \Xi_t^l \in \operatorname{Aut}\mathbf{C}^n$.

To complete the proof of Theorem 1.1 it remains to approximate the flow Ξ of W by suitable compositions of flows Ξ^l as in [11]. Here is the precise result we need (for time independent fields). An excellent reference is Abraham and Marsden [1, p.78].

1.3 Proposition. *Let V^j $(1 \leqq j \leqq m)$ be $\mathscr{C}^1$ vector fields on a manifold M, with flows F_t^j. Let F_t be the flow of $V = \sum_{j=1}^m V^j$. Then*

$$F_t(x) = \lim_{N\to\infty} \left(F_{t/N}^1 \circ \cdots \circ F_{t/N}^m\right)^N(x), \quad x \in M .$$

(Each side is defined if and only if the other is.) The convergence is uniform on any compact set $K \subset M$ such that $F_t(x)$ is defined for all $x \in K$.

To prove Theorem 1.1 we choose a large integer L and subdivide the time interval $[0,1]$ into L subintervals $[m/L,(m+1)/L]$. For $t \in [m/L,(m+1)/L]$ we replace $W(t,x,z)$ by the time independent field on $\mathbf{C}^n$

$$W(m/L,x,z) = \sum_{j,\alpha,l} (m/L)^j x^\alpha W^l(z) .$$

The flow of $(m/L)^j x^\alpha W^l(z)$ for a fixed $x \in B_1$ consists of automorphisms of $\mathbf{C}^n$ (generalized shears). Applying Proposition 1.3 on each segment $[m/L,(m+1)/L]$ and letting $L \to \infty$ we obtain a uniform approximation of the time one map Ξ_1 by holomorphic maps $\Phi : B_2 \times \mathbf{C}^n \to \mathbf{C}^n$, where $B_2 \subset B_1$ is a smaller neighborhood of B in $\mathbf{C}^k$, such that $\Phi_x \in \operatorname{Aut}\mathbf{C}^n$ for each $x \in B_2$. If the approximation is sufficiently close on $B_2 \times \tilde{D}$, the Cauchy estimates imply

$$\|\Phi - \Xi_1\|_{\mathscr{C}^p(B\times K)} < \varepsilon/3 .$$

Thus $\|F - \Phi\|_{\mathscr{C}^p(B\times K)} < \varepsilon$ as required. Theorem 1.1 is proved.

Proof of the main theorem for real-analytic manifolds. Suppose that the manifold $M \subset \mathbf{C}^n$ is real-analytic. By approximation we can assume that the isotopy $F_t : M \to M_t \subset \mathbf{C}^n$ satisfying (ii) is real-analytic as well. Let V be the infinitesimal generator of $\{F_t\}$:

$$V(t,w) = \frac{d}{ds}F_s(z)|_{s=t}, \quad w = F_t(z),\ z \in M,\ 0 \leqq t \leqq 1 .$$

Then $F_t(z)$ is the solution of the ordinary differential equation

$$\frac{d}{dt}F_t(z) = V(t,F_t(z)), \quad F_0(z) = z \in M .$$

Set $\tilde{M} = \bigcup_{0 \leqq t \leqq 1}\{t\} \times F_t(M) \subset \mathbf{C}^{n+1}$. Since $F_t(M)$ is totally real and polynomially convex for each $t \in [0,1]$, $\tilde{M}$ is a totally real and polynomially convex submanifold of $\mathbf{C}^{n+1}$. The vector field V is real-analytic on $\tilde{M}$, and hence it extends to a holomorphic vector field (with values in $\mathbf{C}^n$) in a neighborhood of $\tilde{M}$ in $\mathbf{C}^{n+1}$. The flow $\mathscr{F}_t$ of this extended field V is a family of biholomorphic mappings, defined in a neighborhood Ω of M and depending analytically on t, whose restriction to M equals F_t.

By polynomial convexity of $\tilde{M}$ we can approximate V, uniformly in a fixed neighborhood $\tilde{U}$ of $\tilde{M}$ in $\mathbf{C}^{n+1}$, by a holomorphic polynomial field $W(t,z)$ (with values in $\mathbf{C}^n$). If the approximation is sufficiently close, the flow of W is defined for all z in a neighborhood $U \subset \Omega$ of M and for all $t \in [0,1]$, and it approximates $\mathscr{F}_t$ uniformly on U for each fixed t.

We conclude the proof exactly as above, by decomposing W into a finite sum of complete fields W^l (Lemma in the Appendix) and applying Proposition 1.3. This gives uniform approximation of $\mathscr{F}_t$ by automorphisms of $\mathbf{C}^n$ in a smaller neighborhood U_1 of M, and hence the $\mathscr{C}^p$ approximation on M for any p (by Cauchy estimates).

2 A jet approximation theorem

Let $z = (z_1,\dots,z_n)$, $z_j = x_j + iy_j$, be the complex coordinates on $\mathbf{C}^n$, and let $(x_1,\dots,x_n,y_1,\dots,y_n)$ be the corresponding real coordinates. If $U \subset \mathbf{C}^n$ is an open subset, $f \in \mathscr{C}^p(U)$, and $\alpha \in \mathbf{Z}_+^{2n}$ is a multiindex of total weight $|\alpha| = \sum_{j=1}^{2n}\alpha_j \leqq p$, we denote

$$D^\alpha f(z) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1}\dots\partial y_n^{\alpha_{2n}}}(z), \quad z \in U .$$

Similarly we set for $\alpha \in \mathbf{Z}_+^n$, $|\alpha| \leqq p$,

$$\partial_\alpha f = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1}\dots\partial z_n^{\alpha_n}}, \quad \bar{\partial}_\alpha f = \frac{\partial^{|\alpha|} f}{\partial \bar{z}_1^{\alpha_1}\dots\partial \bar{z}_n^{\alpha_n}} .$$

We shall also write $\bar{\partial}_j f = \partial f/\partial \bar{z}_j$, $1 \leqq j \leqq n$.

For a closed set $K \subset \mathbf{C}^n$ and $p \in \mathbf{Z}_+$ we denote by $\mathscr{E}_p(K)$ the space of jets of $\mathscr{C}^p$ functions on K. Each such jet is determined by a $\mathscr{C}^p$ function f, defined in an open neighborhood of K in $\mathbf{C}^n$ (depending on f); two functions f and g determine the same p-jet on K if $f - g$ has vanishing derivatives up to order p at all points of K. Clearly the addition and multiplication of jets are well defined operations, hence $\mathscr{E}_p(K)$ is an algebra. If K is compact, we endow $\mathscr{E}_p(K)$ with the norm

$$\|f\|_{p,K} = \sup\{|D^\alpha f(z)| : z \in K,\ 0 \leqq |\alpha| \leqq p\} .$$

A p-jet $f \in \mathscr{E}_p(M)$ is said to be **holomorphic** if

$$D^\beta(\bar{\partial}_j f)(z) = 0, \quad z \in K,\ 1 \leqq j \leqq n,\ \beta \in \mathbf{Z}_+^{2n},\ |\beta| \leqq p - 1 .$$

We denote by $\mathscr{Z}_p(K)$ the vector space of all holomorphic p-jets on K. Notice that the product of two holomorphic p-jets on K is again a holomorphic p-jet on K.

If $M \subset \mathbf{C}^n$ is a totally real submanifold of class $\mathscr{C}^p$ and $K \subset M$ is a compact subset in M, then every function $f \in \mathscr{C}^p(M)$ can be extended to a $\mathscr{C}^p$ function in a neighborhood of K such that the extension determines a holomorphic p-jet on K [9, p. 147]. Moreover, if M is generic (of real dimension n) and K is the closure of an open subset of M, then every $f \in \mathscr{C}^p(M)$ determines a unique holomorphic p-jet on K, since all derivatives of order $\leqq p$ at points $z \in K$ are determined by the tangential derivatives of f on M via the Cauchy-Riemann equations. If, on the other hand, the real dimension of M is smaller than n, then a holomorphic p-jet which extends $f \in \mathscr{C}^p(M)$ is not uniquely determined.

2.1 Theorem (Jet approximation theorem). *Let $M \subset \mathbf{C}^n$ be a totally real submanifold of class $\mathscr{C}^p$, $p \geqq 1$. There exist Stein open neighborhoods D of M in $\mathbf{C}^n$ with the following property. Given a holomorphic p-jet $f \in \mathscr{Z}_p(M)$, a compact set $K \subset M$, and an $\varepsilon > 0$, there exists a holomorphic function F in D such that $\|F - f\|_{p,K} < \varepsilon$.*

In the generic case when $\dim_{\mathbf{R}} M = n$, Theorem 2.1 follows immediately from the approximation theorem of Range and Siu [18] since the approximation in $\mathscr{C}^p(M)$ implies p-jet approximation. Notice, however, that we can not reduce the result to the generic case since our manifold M need not be contained in any generic totally real submanifold of $\mathbf{C}^n$.

The result of Theorem 2.1 holds also if we replace $\mathbf{C}^n$ by an arbitrary complex manifold X, provided that we define the topology on the spaces of jets with respect to a fixed Riemann metric on X.

Proof of Theorem 2.1. We may assume that M has real dimension $m < n$, as the result follows from [18] for $m = n$. For each $z \in M$ we set $\mathrm{T}_z^{\mathbf{C}} M = \mathrm{T}_z M \oplus i\,\mathrm{T}_z M$, and we let $\mathrm{T}^{\mathbf{C}} M \to M$ be the complex vector bundle of rank m with fibers $\mathrm{T}_z^{\mathbf{C}} M$. There exists a complex vector bundle $N \to M$ over M, of rank $d = n - m > 0$ and of class $\mathscr{C}^p$, such that $\mathrm{T}^{\mathbf{C}} M \oplus N = \mathrm{T}\mathbf{C}^n|_M$. If M is

$\mathscr{C}^\infty$, we can take N to be the complex normal bundle to M whose fiber N_z at $z \in M$ is the orthogonal complement of $T_z^{\mathbf{C}} M$ in $T_z\mathbf{C}^n$. In general this bundle is only of class $\mathscr{C}^{p-1}$, but we can approximate it by a $\mathscr{C}^p$ bundle.

We identify N_z with the corresponding d-dimensional affine complex plane passing through z. If U_0 is sufficiently small tubular neighborhood of M in $\mathbf{C}^n$, then $\Sigma = \left(\bigcup_{z\in M} N_z\right) \cap U_0$ is a $\mathscr{C}^p$ submanifold in U_0 containing M. Moreover, Σ is foliated by pieces of d-dimensional complex planes $\Sigma_z = N_z \cap U_0$, and hence it is a Levi flat C-R manifold of C-R dimension $d = n - m$.

Let $\pi : \Sigma \to M$ be the projection with fibers $\pi^{-1}(z) = \Sigma_z$, $z \in M$. We cover M by open sets $U_j \subset U_0$ in $\mathbf{C}^n$ such that $\Sigma_z \subset U_j$ whenever $z \in U_j \cap M$, and such that there exists a closed totally real submanifold $M_j \subset U_j$ of dimension n and of class $\mathscr{C}^p$, containing $M \cap U_j$. To obtain M_j we let U_j be small enough such that the bundle N is trivial over $M \cap U_j$, and we let M_j correspond to a totally real subbundle of rank d in $N|_{M\cap U_j}$; that is, M_j is foliated by pieces of totally real planes of dimension d in the fibers Σ_z.

According to the local approximation theorem of Range and Siu [18, Theorem 2.11] every point $a \in M \cap U_j$ has a pair of neighborhoods $W \Subset V \Subset U_j$ in $\mathbf{C}^n$ such that for every $\mathscr{C}^p$ function f on $\mathbf{C}^n$, with $\operatorname{supp} f \cap \Sigma \subset W$, there is a sequence of holomorphic polynomials F_k on $\mathbf{C}^n$ such that $F_k|_{V\cap M_j} \to f|_{V\cap M_j}$ in $\mathscr{C}^p(V \cap M_j)$ as $k \to \infty$, and $F_k \to 0$ uniformly in an open neighborhood $E \subset \mathbf{C}^n$ of any compact set $L \subset (V\backslash \bar{W}) \cap M_j$.

The proof of this local approximation result in [18] (see also [8]) uses kernel methods. The same local result can be obtained in a simpler way by convolution with the Gauss kernel as in Baouendi and Treves [7]. We recall the idea. Choose local coordinates such that $a = 0$ and $T_0 M_j = \mathbf{R}^n \oplus \{i0\}$. We assume that M_j is small so that $T_z M_j$ is sufficiently close to $T_0 M_j$ for all $z \in M_j$. Let $f \in \mathscr{C}^p(\mathbf{C}^n)$ have compact support contained in an open set $W \subset \mathbf{C}^n$ such that $W \cap M_j \Subset M_j$. Set

$$h_\tau(z) = \left(\frac{\tau}{\pi}\right)^{n/2} \int_{M_j} f(\zeta) e^{-\tau[z-\zeta]^2}\, d\zeta_1 \wedge \ldots \wedge \zeta_n$$

(the convolution of f with the complex Gauss kernel), where $[z-\zeta]^2 = \sum_{j=1}^n (z_j - \zeta_j)^2$. As $\tau \to +\infty$, this family of entire functions converges to f in $\mathscr{C}^p(M_j)$ (see [7]). Moreover, at $\tau \to +\infty$, the kernel function converges to zero exponentially in the cone $\Gamma_\zeta = \{z \in \mathbf{C}^n : \Re\, [z-\zeta]^2 > 0\}$ with vertex ζ and axis $\zeta + \mathbf{R}^n$. If $\zeta \in M_j$ and M_j is sufficiently small, we have $M_j \subset \{\zeta\} \cup \Gamma_\zeta$. The intersection $\Gamma_W = \bigcap_{\zeta \in W\cap M_j} \Gamma_\zeta$ contains in its interior a compact set $L \subset (V\backslash \bar{W}) \cap M_j$ which disconnects M_j. Therefore $\operatorname{Int} \Gamma_W$ also contains an open neighborhood $E \subset \mathbf{C}^n$ of L, and the sequence h_τ converges to 0 in E as $\tau \to \infty$ for every such f. This gives the local approximation result.

In order to make the approximating function F_k well defined in a neighborhood of M and small outside the given neighborhood V of a, we solve a Cousin problem as in [18, Sect. 3]. We can choose a pair of open sets D_1, $D_2 \subset \mathbf{C}^n$ such that

(i) $D = D_1 \cup D_2 \subset U_0$ is a Stein domain containing M,
(ii) $M \cap W \subset D_2 \subset V$, $M \setminus V \subset D_1 \subset U_0 \setminus W$, and
(iii) $D_1 \cap D_2 \subset E$, where E is chosen as above.

Since $D = D_1 \cup D_2$ is Stein, the Cousin problem $F_k^1 - F_k^2 = F_k$ on $D_1 \cap D_2$ has solutions $F_k^j \in \mathcal{O}(D_j)$ $(j = 1,2)$. Moreover, as $F_k \to 0$ on $D_1 \cap D_2 \subset E$ when $k \to \infty$, the open mapping theorem for Fréchet spaces implies that there are solutions F_k^1, F_k^2 converging to zero on compact sets in D_1 resp. D_2 as $k \to \infty$. Setting

$$f_k = \begin{cases} F_k^1 & \text{on } D_1 , \\ F_k + F_k^2 & \text{on } D_2 , \end{cases}$$

we get a sequence of holomorphic functions in D which converges to f in $\mathscr{C}^p(M_j \cap D_2)$ and to zero in D_1.

Let $K \subset M$ be a compact subset as in the theorem, and let $\{U_j\}$ be the open sets chosen at the beginning of the proof. We cover K by finitely many sets $W_j \Subset V_j$ as above such that $V_j \subset U_i$ for some $i = i(j)$. For each W_j we also have a Stein neighborhood D_j of M as above. Since there are only finitely many W_j's, we may take $D_j = D$ independent of j (by taking their intersection).

Choose a $\mathscr{C}^p$ partition of unity χ_j on a neighborhood of K in M, with $\operatorname{supp} \chi_j \subset W_j \cap M$. Then $\psi_j = \chi_j \circ \pi : \Sigma \to \mathbf{R}$ is a C-R function on Σ (since it is constant on the fibers Σ_z), supported in $U_{i(j)} \cap \Sigma$, and $\{\psi_j\}$ is a $\mathscr{C}^p$ partition of unity on a neighborhood of K in Σ. We extend each ψ_j to a $\mathscr{C}^p$ function on $\mathbf{C}^n$ such that $\bar{\partial}\psi_j$ vanishes to order $p-1$ on Σ [9, p. 147]. The extended function ψ_j then determines a holomorphic p-jet on M. Finally we choose a smooth function χ on $\mathbf{C}^n$ which equals one in a neighborhood $D' \subset D$ of K and has compact support in. D. We choose the support of χ sufficiently close to M so that $\operatorname{supp}(\chi\psi_j) \subset W_j$.

Let f be a $\mathscr{C}^p$ function on $\mathbf{C}^n$ which determines a holomorphic p-jet on M. We have $f = \sum_j f\chi\psi_j$ on $D' \cap \Sigma$. Since the support of $f\chi\psi_j$ is contained in W_j, there exists a holomorphic function f_j in D which approximates $f\chi\psi_j$ as close as we want in $\mathscr{C}^p(M_{i(j)} \cap V_j \cap D)$ and is as small as we want on $D \setminus V_j$. Since the p-jet of $f\chi\psi_j$ at M is holomorphic and since the manifold $M_{i(j)}$ is generic, it follows that $\|f_j - f\chi\psi_j\|_{p,K}$ is small, i.e., the p-jet of f_j approximates the p-jet of $f\chi\psi_j$ on K. The sum $F = \sum_j f_j$ is then a holomorphic function in D whose p-jet on K approximates the p-jet of f on K. This completes the proof of Theorem 2.1.

3 Approximation of diffeomorphisms on totally real manifolds

In this section we prove the following approximation result.

3.1 Theorem. *Let $M \subset \mathbf{C}^n$ be a compact, totally real submanifold in $\mathbf{C}^n$ of class $\mathscr{C}^p$ $(2 \leqq p < \infty)$, with or without boundary, and let B be the closed unit ball in $\mathbf{R}^k$. Assume that $F : B \times M \to \mathbf{C}^n$ is a $\mathscr{C}^p$ map such that for* *each fixed* $x \in B$, $F_x = F(x, \cdot\,) : M \to \mathbf{C}^n$ *is a* $\mathscr{C}^p$*-diffeomorphism of* M *onto another totally real submanifold* $M_x \subset \mathbf{C}^n$, *and* F_0 *is the identity on* $M = M_0$. *Then for each* $\varepsilon > 0$ *there exists an open set* $\Omega \subset \mathbf{C}^n$ *containing* M *and a real-analytic map* $\Phi : B \times \Omega \to \mathbf{C}^n$ *such that* $\Phi_x = \Phi(x, \cdot\,) : \Omega \to \Phi_x(\Omega)$ *is biholomorphic for each* $x \in B$, Φ_0 *is the identity, and* $\|F - \Phi\|_{\mathscr{C}^p(B\times M)} < \varepsilon$. *If* M *and* F *are real-analytic, there is a* Φ *as above such that* $\Phi|_{B\times M} = F$.

3.2 Corollary. *Let* $F : B \times M \to \mathbf{C}^n$ *be as in Theorem 3.1, and assume in addition that the manifold* $M_x = F_x(M) \subset \mathbf{C}^n$ *is polynomially convex for each* $x \in B$. *Then for each* $\varepsilon > 0$ *there exists a real-analytic map* $\Phi : B \times \mathbf{C}^n \to \mathbf{C}^n$ *such that* $\|F - \Phi\|_{\mathscr{C}^p(B\times M)} < \varepsilon$, $\Phi_x \in \operatorname{Aut}\mathbf{C}^n$ *for each* $x \in B$, *and* Φ_0 *is the identity.*

Proof of Corollary 3.2. Choose a map $\Phi : B \times \Omega \to \mathbf{C}^n$ as in Theorem 3.1. Since $F_x(M)$ is polynomially convex for each $x \in B$, the same is true for $\Phi_x(M)$, provided that ε is sufficiently small. By Lemma 2.2 in [11] there is a smaller neighborhood $U \subset \Omega$ of M in $\mathbf{C}^n$ such that $\Phi_x(U)$ is Runge in $\mathbf{C}^n$ for every $x \in B$. Now apply Theorem 1.1 to Φ. This proves Corollary 3.2, provided that Theorem 3.1 holds.

Proof of Theorem 3.1. Set $M_x = F_x(M) \subset \mathbf{C}^n$. Let $N^x \to M_x$ be a complex normal bundle to M_x as defined in the proof of Theorem 2.1 (Sect. 2), depending smoothly of class $\mathscr{C}^p$ on both variables (x,z) $(x \in B, z \in M_x)$.

3.3 Lemma. *There exists a family of complex bundle isomorphisms* $\mathscr{F}^x : N^0 \to N^x$, *of class* $\mathscr{C}^p$ *in* $(x,\xi) \in B \times N^0$, *such that the following diagram commutes*:

$$\begin{array}{ccc} N^0 & \xrightarrow{\mathscr{F}^x} & N^x \\ \downarrow & & \downarrow \\ M_0 & \xrightarrow{F_x} & M_x\,. \end{array}$$

Proof. This follows from the well known fact that a vector bundle $N = \bigcup_{x\in B}\{x\} \times N^x$ over a product manifold $B \times M$ is isomorphic to the product bundle $B \times N^0$ (see [15, p. 90]). An elementary proof was indicated (in the case when $B = [0,1]$) in [11] (remark at the end of Sect. 3). We omit the details.

We continue with the proof of Theorem 3.1. For each point $z \in M_x$ we identify the fiber N^x_z with the corresponding affine complex subspace of $\mathbf{C}^n$. The map $(z,\xi) \in N^x \mapsto z + \xi$ embeds a small neighborhood of the zero section in N^x onto a generic Levi flat C-R manifold $\Sigma^x \subset \mathbf{C}^n$ containing M_x. Let $f^x : \Sigma^0 \to \Sigma^x$ be the map which corresponds to $\mathscr{F}^x : N^0 \to N^x$. Since $\mathscr{F}^x$ is complex linear on the fibers of N^0, f^x is a C-R map for every $x \in B$. We consider B as a subset in $\mathbf{R}^k \subset \mathbf{C}^k$. Let $f(x,z) = f^x(z)$ be the corresponding C-R map of class $\mathscr{C}^p$ on the C-R submanifold $B \times \Sigma_0 \subset \mathbf{C}^{k+n}$. By construction f extends the map $F : B \times M \to \mathbf{C}^n$.

When M and F are real-analytic, the above construction gives a real-analytic C-R mapping f on the generic, real-analytic C-R manifold $B \times \Sigma_0$. It follows that f extends to a holomorphic map Φ in a neighborhood of $B \times \Sigma_0$ in $\mathbf{C}^{k+n}$ [9, p. 141].

In the smooth case we can extend f to a $\mathscr{C}^p$ map on $\mathbf{C}^{k+n}$ such that $\bar{\partial} f$ vanishes to order $(p-1)$ on $B \times \Sigma_0$ [9, p. 147]. The extension thus determines a holomorphic p-jet on the totally real submanifold $B \times M \subset \mathbf{C}^{k+n}$ with values in $\mathbf{C}^n$ (Sect. 2). According to Theorem 2.1 in Sect. 2 we can approximate the p-jet of f arbitrarily close in the p-jet norm on $B \times M$ by the p-jet of a holomorphic mapping $\Phi : U \times \Omega \to \mathbf{C}^n$, defined on an open neighborhood of $B \times M$ in $\mathbf{C}^{k+n}$.

By construction, the partial one-jet of $f_x = f(x, \cdot\,)$ with respect to the z-variable is nondegenerate at every point $z \in M = M_0$. Hence the derivative $D_z\Phi(x, \cdot\,)$ is also nondegenerate for every $z \in M$, provided that the one-jet of Φ is sufficiently close to the one-jet of f on $B \times M$. Moreover, as f_x is diffeomorphic on M for every $x \in B$, the same is true for $\Phi_x = \Phi(x, \cdot\,)$. It follows that the neighborhood Ω of M can be chosen sufficiently small such that $\Phi_x : \Omega \to \mathbf{C}^n$ is biholomorphic onto its image for every $x \in B$. Finally, since Φ_0 approximates the identity f^0 on Σ^0, we may replace Φ_x for small $x \in B$ by a suitable convex combination of Φ_x with the identity map and thus arrange that Φ_0 is the identity. Theorem 3.1 is proved.

Similarly one can prove the following result.

3.4 Proposition. *Let $F : M \to M'$ be a $\mathscr{C}^p$ diffeomorphism between compact, totally real submanifolds $M, M' \subset \mathbf{C}^n$. Then the following are equivalent:*
(i) *There is a sequence of neighborhoods U_j of M and biholomorphic maps $\Phi_j : U_j \to \mathbf{C}^n$ such that* $\lim_{j\to\infty} \|\Phi_j|_M - F\|_{\mathscr{C}^p(M)} = 0$.
(ii) *There exists a complex bundle isomorphism $\mathscr{F} : N \to N'$ of the complex normal bundles $\pi : N \to M$, $\pi' : N' \to M'$, such that $\pi' \circ \mathscr{F} = F \circ \pi$ on N.*

Proof. (i) $\Rightarrow$ (ii) If Φ is a biholomorphic map on a neighborhood of M, taking M onto $M_1 = \Phi(M)$, it induces an isomorphism $\tilde{\Phi} : N \to N^1$ of the complex normal bundles $N \to M$ resp. $N^1 \to M^1$ by taking $\tilde{\Phi} = \tau \circ D\Phi$, where $\tau : \mathrm{T}\mathbf{C}^n|_{M_1} \to N^1$ is the linear projection with kernel $\mathrm{T}^C M_1$. Moreover, if $\|\Phi - F\|_{\mathscr{C}^p(M)}$ is small, the map $\phi = \Phi \circ F^{-1} : M' \to M_1$ is a diffeomorphism which is close to the identity on M'. Thus ϕ can be covered by a vector bundle isomorphism $\Theta : N' \to N^1$ of the corresponding complex normal bundles; it suffices to take Θ_z to be the linear projection of the fiber N'_z to $N^1_{\phi(z)}$ in the direction of $\mathrm{T}^C_z M'$. Then $\mathscr{F} = \Theta^{-1} \circ \tilde{\Phi} : N \to N'$ satisfies (ii).

(ii) $\Rightarrow$ (i) The isomorphism $\mathscr{F} : N \to N'$, which we may assume to be of class $\mathscr{C}^p$ by approximation, gives rise to a nondegenerate holomorphic p-jet f along M as in the proof of Theorem 3.1 above. According to Theorem 2.1 we can approximate the p-jet f on M by the p-jet of a holomorphic mapping Φ, defined on a neighborhood U of M. If the approximation is close enough in the sense of jets on M, the map Φ in biholomorphic in a smaller neighborhood of M, and (i) holds. This proves Proposition 3.4.

4 Generic polynomial convexity of low dimensional submanifolds in $\mathbf{C}^n$

All manifolds in this section are compact, with or without boundary. If $M \subset \mathbf{C}^n$ is a totally real submanifold, we denote by NM its complex normal bundle as in the proof of Theorem 2.1; i.e., $NM \to M$ is a complex vector bundle of rank $n - \dim M$ whose direct sum with the complexified tangent bundle to M is $T\mathbf{C}^n|_M$.

The following result was proved in [11, Theorem 5.2] for curves and surfaces.

4.1 Theorem. (a) *Let $M \subset \mathbf{C}^n$ be a totally real $\mathscr{C}^p$ submanifold ($p \geqq 2$) with* $\dim_R M < n$. *If the complex normal bundle NM admits a section over M with at most finitely many zeros, then there exist arbitrary small $\mathscr{C}^p$ deformations $\tilde{M}$ of M in $\mathbf{C}^n$ which are polynomially convex (and totally real).*

(b) *Let $F_t : M \to \mathbf{C}^n$ ($0 \leqq t \leqq 1$) be a $\mathscr{C}^p$ isotopy ($p \geqq 2$) such that* $\dim M < n$ *and $F_t(M) = M_t$ is totally real for every $t \in [0, 1]$. If the complex normal bundle to M_0 admits a section with at most finitely many zeros, then there exist arbitrarily small $\mathscr{C}^p$ deformations $\{\tilde{F}_t\}$ of $\{F_t\}$ such that the manifold $\tilde{F}_t(M)$ is polynomially convex for every $t \in [0, 1]$. Moreover, if M_0 and M_1 are already polynomially convex, we may choose $\{\tilde{F}_t\}$ such that $F_t = \tilde{F}_t$ for $t = 0$ and $t = 1$.*

Remark 1. Most likely the additional condition on the complex normal bundle to M is not necessary, but at this time we do not know how to prove the result without it. Notice that for totally real isotopies as in the theorem the complex normal bundles to M_t are isomorphic. Therefore the existence of such a section for one value of t implies the existence for all values of t. The condition is always satisfied when $\dim M \leqq 2n/3$ since in this case a generic section of NM_0 (which is a vector bundle of real rank $2n - 2\dim M \geqq \dim M$) has at most finitely many zeros.

Remark 2. Theorem 4.1 fails entirely for closed submanifolds $M \subset \mathbf{C}^n$ of $\dim M \geqq n$, as the polynomial hull $\hat{M}$ of such a manifold has topological dimension at least $\dim M + 1$ [3, 4, 13]. The result is false even for n-dimensional totally real discs in $\mathbf{C}^n$. In fact, a small modification of the lagrangian disc in $\mathbf{C}^2$ with nontrivial polynomial hull, constructed by Duval [10], gives a lagrangian disc $D \subset \mathbf{C}^2$ with the property that the polynomial hull of every small deformation of D contains an open subset of $\mathbf{C}^2$. (This was communicated to me by Duval.) This example is particularly striking since lagrangian discs (and discs which are sufficiently close to being lagrangian) do not bound any analytic varieties.

4.2 Corollary. *Let $2 \leqq p < \infty$. For every compact $\mathscr{C}^p$ submanifold $M \subset \mathbf{C}^n$ with* $\dim M \leqq 2n/3$ *there exist arbitrarily small $\mathscr{C}^p$ deformations $\tilde{M} \subset \mathbf{C}^n$ which are totally real and polynomially convex. Moreover, if $F : M \to F(M) \subset \mathbf{C}^n$ is a $\mathscr{C}^p$ diffeomorphism such that both M and $F(M)$ are totally real and polynomially convex, there exists a totally real, polynomially convex $\mathscr{C}^p$ isotopy $F_t : M \to \mathbf{C}^n$ ($t \in [0, 1]$) such that $F_0 = \mathrm{Id}_M$ and $F_1 = F$.*

Proof of Corollary 4.2. According to Lemma 5.3 in [11] every submanifold $M \subset \mathbf{C}^n$ with $\dim M \leqq 2n/3$ can be deformed to a totally real submanifold by an arbitrary small smooth deformation. Moreover, if $F : M \to F(M)$ is a diffeomorphism such that both M and $F(M)$ are totally real submanifolds of $\mathbf{C}^n$ of dimension $\leqq 2n/3$, there exists a totally real isotopy $F_t : M \to \mathbf{C}^n$ such that $F_0 = \mathrm{Id}_M$ and $F_1 = F$. The condition on the normal bundle required in Theorem 4.1 is satisfied since $\dim M \leqq 2n/3$. Therefore Theorem 4.1 implies that we can deform the isotopy $\{F_t\}$ to an isotopy which is totally real and polynomially convex. This proves Corollary 4.2.

Proof of Theorem 4.1. We shall first describe the basic step in the proof, the *separation of hulls procedure*, abbreviated SOH. The main idea was developed in [11] in the case of curves and surfaces in $\mathbf{C}^n$ for $n \geqq 2$ resp. $n \geqq 3$.

It suffices to consider submanifolds of class $\mathscr{C}^\infty$. Let $M \subset \mathbf{C}^n$ be a smooth, compact, totally real submanifold of dimension $m < n$, with or without boundary. Let $\Pi \subset \mathbf{C}^n$ be a real hyperplane which intersects both M and its boundary bM transversely. Denote by M^+ resp. M^- the intersections of M with the closed half spaces of $\mathbf{C}^n$ defined by Π.

4.3 Lemma (Separation of hulls–SOH). *Suppose that M^+ and M^- are polynomially convex, and the complex normal bundle $NM \to M$ admits a nonzero section over $M \cap \Pi$. Then for every open neighborhood U of $M \cap \Pi$ there exist arbitrarily small smooth deformations $\tilde{M}$ of M such that $\tilde{M}$ is polynomially convex and $\tilde{M} \backslash U = M \backslash U$.*

Proof of Lemma 4.3. We may assume that $\Pi = \{z \in \mathbf{C}^n : \Re\, \pi(z) = 0\}$ for some complex linear projection $\pi : \mathbf{C}^n \to \mathbf{C}$. For each set $I \subset \mathbf{R}$ we define $M(I) = \{z \in M : \Re\, \pi(z) \in I\}$. Thus $M^+ = M([0,\infty))$ and $M^- = M((-\infty,0])$. Fix a neighborhood U of $M \cap \Pi$ in $\mathbf{C}^n$.

For sufficiently small $\varepsilon > 0$ the sets $M_1 = M((-\infty,\varepsilon])$ and $M_2 = M([-\varepsilon,\infty))$ are still polynomially convex, and their intersection $M_0 = M_1 \cap M_2 = M([-\varepsilon,\varepsilon])$ is contained in U. By decreasing ε if necessary we may also assume that every $t \in [-\varepsilon,\varepsilon]$ is a regular value of $\Re\, \pi|_M$, and there is a nonzero section v of NM over M_0. Fix such an ε.

After a small deformation of M, supported in U, we may assume that M and the transverse vector field v are real-analytic on M_0, and all other properties are preserved. We claim that there exists a holomorphic function g in a neighborhood $V \subset U$ of M_0 such that $g = 0$ on M_0 and $dg \neq 0$ on M_0. To construct g we first observe that the field v splits the bundle N into a direct sum $N = N^0 \oplus \mathbf{C}v$ of real-analytic bundles over M_0. Let $\Sigma \subset V$ be the germ of real-analytic submanifold near M_0 (and containing M_0), consisting of points of the form $z(q,w,s) = q + w + sv(q)$ for $q \in M_0$, $w \in N^0_q$, and $s \in \mathbf{R}$. Here, N^0_q is the fiber of N^0 at q. Clearly Σ is a C-R manifold of C-R dimension $m = n - \dim M - 1$, foliated by complex subspaces N^0_q of dimension m. The function on Σ defined by $g(z(q,w,s)) = s$ is real-analytic and C-R, and therefore it extends to a holomorphic function g in a neighborhood of M_0 [9, p. 141].

We now choose two disjoint closed intervals $J_0, J_1 \subset (-\varepsilon, \varepsilon)$. Since M_0 is polynomially convex, we can approximate g in some fixed neighborhood of M_0 arbitrarily close by holomorphic polynomials P. Then dP approximates dg near M_0, so $dP \neq 0$ in a fixed neighborhood of M_0. Thus the level sets $\{P = c\}$ near M_0 are smooth complex hypersurfaces which approximate the level sets of g. If the approximation is close enough, we can deform M as little as necessary so that we push the sector $M(J_0)$ into the level set $\{P = 0\}$, and at the same time we push the sector $M(J_1)$ into another level set $\{P = c\}$ for some arbitrary small $c \neq 0$. The deformation can be made within any given neighborhood of $M(J_0) \cup M(J_1)$, and it will not affect the polynomial convexity of M_1 and M_2, provided that it is small enough.

We claim that the deformed manifold, still denoted by M, is polynomially convex. To prove this, choose any point $z_0 \in \hat{M}$ and let ν be a Jensen representing measure for z_0, supported on M [23, p. 108]. If ν has any support on the set $M(J_0)$, then

$$\log |P(z_0)| \leqq \int_M \log |P| d\nu = \int_{M(J_0)} \log |P| d\nu + \int_{M \setminus M(J_0)} \log |P| d\nu = -\infty$$

since $P = 0$ on $M(J_0)$; hence $P(z_0) = 0$. Analogous argument applied to $|P - c|$ shows that, if ν has any support on $M(J_1)$, then $P(z_0) = c$. This means that the support of ν does not meet at least one of the two sets $M(J_0)$, $M(J_1)$. Hence the projection $\pi(\operatorname{supp} \nu) \subset \mathbf{C}$ does not meet at least one of the two strips $J_0 \times i\mathbf{R}$, $J_1 \times i\mathbf{R}$. The Runge approximation theorem implies that $\pi(\operatorname{supp} \nu)$ can not be disconnected by such a strip. It follows that ν is either supported entirely by M_1 or entirely by M_2. Since these two sets are polynomially convex, we have $z_0 \in M_1 \cup M_2 = M$. Thus $\hat{M} = M$ as claimed, and Lemma 4.3 is proved.

We now turn to the proof of Theorem 4.1. We first consider case (a). The idea is to apply Lemma 4.3 finitely many times, starting with sufficiently small open pieces of M which are polynomially convex (since M is totally real). The pieces can be obtained by slicing M in $2n$ real coordinate directions $\xi_1, \ldots, \xi_{2n}$ on $\mathbf{C}^n$ by suitably chosen family of parallel hyperplanes $\xi_j = c_{j,l}$ ($j = 1, \ldots, 2n$; $l \in \mathbf{Z}$) which intersect the manifold M and its boundary bM transversely. Since the transverse field v has at most finitely many zeros on M, we can choose the cuts so that none of the zeros of v belong to any of these hyperplanes. We denote the resulting pieces of M by M_α, where $\alpha = (\alpha_1, \ldots, \alpha_{2n}) \in \mathbf{Z}^{2n}$ are multiindices. The component α_j denotes the position of the piece in the direction ξ_j. Of course $M_\alpha = \emptyset$ for large $|\alpha|$.

Every two pieces M_α, M_β for which $|\alpha - \beta| = 1$ satisfy the conditions of Lemma 4.3 (or else they do not meet at all). Hence we can make the union of these two pieces polynomially convex by a small smooth deformation, supported in a prescribed neighborhood of the common border of the two pieces. Of course this will affect the other neighboring pieces, but since our deformation can be made arbitrarily small, we may assume that the pieces of M and their unions that were already polynomially convex stay polynomially convex (and totally real).

In order to be able to apply Lemma 4.3 at every stage of our deformation, we must perform the procedure in certain order. We fix $\alpha' = (\alpha_2, \ldots, \alpha_{2n}) \in \mathbf{Z}^{2n-1}$ and perform the SOH procedure finitely many times in the first coordinate direction ξ_1, once for each pair of adjacent pieces $M_{(\alpha_1,\alpha')}$ and $M_{(\alpha_1+1,\alpha')}$. If two such pieces do not intersect or one of them is empty, we do not have to do anything in that step. To simplify the notation we denote the perturbed manifold by the same letter. This process gives a new perturbed manifold for which the union of pieces $M_{(\alpha_1,\alpha')}$ over all values of $\alpha_1 \in \mathbf{Z}$ is polynomially convex. We denote this union by $M_{\alpha'}$, with $\alpha' \in \mathbf{Z}^{2n-1}$.

We perform this procedure for every value of $\alpha' \in \mathbf{Z}^{2n-1}$ for which there are any nonempty pieces $M_{(\alpha_1,\alpha')}$. After that we fix a multiindex $\alpha'' = (\alpha_3, \ldots, \alpha_{2n}) \in \mathbf{Z}^{2n-2}$ and perform the SOH procedure with the polynomially convex pieces $M_{(\alpha_2,\alpha'')}$, $\alpha_2 \in \mathbf{Z}$, obtained in the previous step. This produces larger polynomially convex pieces $M_{\alpha''}$ of M for all $\alpha'' \in \mathbf{Z}^{2n-2}$.

Repeating the procedure in all coordinate directions we obtain in finitely many steps a perturbation of M which is polynomially convex. This completes the proof of Theorem 4.1 in case (a). The same proof is easily adapted to totally real isotopies in part (b) (see [11] for $\dim M = 2$). We leave out the details.

Remark. Observe that in the SOH procedure we can avoid any finite set of points of M or, more generally, any closed subset of zero length. Hence Theorem 4.1 also holds for manifolds M which have a finite number of points $p \in M$ at which the tangent space $\mathrm{T}_p M$ contains a complex line (complex tangents), provided that M is locally polynomially convex near such points. The problem becomes more serious if the set of complex tangents of M contains curves. It would be interesting to know whether the result holds even for such manifolds, as long as they are locally polynomially convex and of dimension less than n.

Appendix: Decomposition of polynomial vector fields on $\mathbf{C}^n$

A vector field X on $\mathbf{C}^n$ is said to be **complete** (in real time) if the differential equation $\dot{z} = X(z)$, $z(0) = z^0$, can be integrated for all $t \in \mathbf{R}$, starting at any point $z^0 \in \mathbf{C}^n$. The following result is essentially due to Andersén [5] and Andersén and Lempert [6].

Lemma. *Every holomorphic polynomial vector field X on $\mathbf{C}^n$ is a finite sum of complete vector fields of the following two types:*

$$V(z) = f(Az)v, \qquad W(z) = f(Az)\langle z, v\rangle v\,, \tag{$*$}$$

where $v \in \mathbf{C}^n$, $|v| = 1$, A is a complex linear form on $\mathbf{C}^n$ satisfying $Av = 0$, f is a polynomial in one variable, and $\langle z, v\rangle = \sum_{j=1}^n z_j \bar{v}_j$. If $\operatorname{div} X = \sum_{j=1}^n \partial X_j/\partial z_j = 0$, then X is a finite sum of vector fields of the first type.

Proof. A vector field $V(z) = f(Az)v$ of the first type in $(*)$ is complete, with the flow

$$F_t(z) = z + tf(Az)v, \quad z \in \mathbf{C}^n, \ t \in \mathbf{C}.$$

Each F_t is a polynomial automorphism of $\mathbf{C}^n$ with Jacobian one. Automorphisms of this type are called *shears* [20]. The vector field $W(z) = f(Az)\langle z, v\rangle v$ of the second type is also complete, with the flow

$$G_t(z) = z + \left(e^{tf(Az)} - 1\right)\langle z, v\rangle v, \quad t \in \mathbf{C}.$$

Automorphisms of this type are called **generalized shears**, or **overshears** [6].

We claim that $\operatorname{div} V = 0$ in the first case and $\operatorname{div} W(z) = f(Az)$ in the second case. This can be verified directly, but the following proof is more elegant. Choose an $A \in SU(n, \mathbf{C})$ satisfying $Av = e_n$ (the base vector in the coordinate direction z_n). Then, writing $z = (z', z_n)$, we have $A \cdot V(z) = \tilde{V}(Az)$ and $A \cdot W(z) = \tilde{W}(Az)$, where

$$\tilde{V}(z) = g(z')e_n, \quad \tilde{W}(z) = g(z')z_n e_n,$$

and the function $g(z') = f(AA^{-1}z)$ is a polynomial independent of z_n (since $AA^{-1}e_n = Av = 0$). Clearly $\operatorname{div}\tilde{V} = 0$ and $\operatorname{div}\tilde{W}(z) = g(z')$. Since conjugation by $A \in SU(n, \mathbf{C})$ preserves divergence, we get $\operatorname{div} V = 0$ and $\operatorname{div} W(z) = g(Az) = f(Az)$ as claimed. Notice that the conjugate flows are of the form

$$A \circ F_t \circ A^{-1}(z) = (z', z_n + tg(z')),$$
$$A \circ G_t \circ A^{-1}(z) = (z', e^{tg(z')}z_n).$$

Suppose now that $X = (X_1, \ldots, X_n)$ is a polynomial vector field on $\mathbf{C}^n$. Consider first the case $\operatorname{div} X = 0$. Since divergence lowers the order of terms in the Taylor expansion of X by one, it suffices to prove the decomposition result in the case when each component of X is a homogeneous polynomial of degree m. Choose multiplicatively independent numbers $a_1, \ldots, a_{n-1}$, and set $a_n = 1$. For instance, one may take $a_1, \ldots, a_{n-1}$ to be the first $n-1$ primes. Write $a^k \cdot z = \sum_{j=1}^n (a_j)^k z_j$. A simple argument [5, p. 231] shows that there exists a number $M \in \mathbf{Z}_+$ such that every homogeneous polynomial $p(z)$ of degree m on $\mathbf{C}^n$ can be written in the form

$$p(z) = \sum_{k=0}^{M} c_k (a^k \cdot z)^m$$

for some coefficients $c_0, c_1, \ldots, c_M \in \mathbf{C}$. Applying this to the component X_j of X for $j \in \{1, \ldots, n-1\}$ we get

$$X_j(z) = \sum_{k=0}^{M} c_{j,k} (a^k \cdot z)^m.$$

Set

$$V^{j,k}(z) = c_{j,k}(a^k \cdot z)(e_j - a_j^k e_n),$$

where $e_1, \ldots, e_n$ denotes the standard basis of $\mathbf{C}^n$. Clearly each vector field $V^{j,k}$ is of the first type in $(*)$. Let

$$V = \sum_{j,k} V^{j,k} = (V_1, \ldots, V_n),$$

the sum running over $1 \leqq j \leqq n-1$ and $0 \leqq k \leqq M$. Then $V_i = X_i$ for $1 \leqq i \leqq n-1$, hence $X = V + (X_n - V_n)e_n$. Since $\operatorname{div} X = 0 = \operatorname{div} V$ we get

$$0 = \operatorname{div}(X - V) = \partial(X_n - V_n)/\partial z_n .$$

Thus the function $(X_n - V_n)$ is independent of z_n, and hence $(X_n - V_n)e_n$ is again a vector field of the first type in $(*)$. This completes the proof for divergence zero vector fields.

In the general case when $\operatorname{div} X \neq 0$ we decompose the divergence into a finite sum

$$\operatorname{div} X(z) = \sum q_j(\lambda_j z),$$

where the q_j's are polynomials in one variable and the λ_j's are linear forms on $\mathbf{C}^n$ [6, Proposition 3.7]. For each j we choose a vector $v_j \in \mathbf{C}^n$ satisfying $|v_j| = 1$ and $\lambda_j v_j = 0$, and set

$$V_j(z) = q_j(\lambda_j z)\langle z, v_j\rangle v_j, \quad 1 \leqq j \leqq m .$$

Each V_j is a vector field of the second type in $(*)$, and $\operatorname{div} V_j(z) = q_j(\lambda_j z)$. Writing $X = \sum_{j=1}^m V_j + \tilde{X}$ we have $\operatorname{div} \tilde{X} = 0$. By the first part of the proof $\tilde{X}$ is a finite sum of vector fields of the first type in $(*)$. This proves the lemma.

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