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One Parameter Automorphism Groups on $\mathbf{C}^2$

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Given a polynomial holomorphic automorphism g of $\mathbf{C}^2$ which is not conjugate to an affine aperiodic map $(x,y) \mapsto (x+1, \beta y)(\beta \in \mathbf{C}\backslash\{0\})$, we find all real one parameter subgroups $\{\phi_t : t \in \mathbf{R}\}$ in the holomorphic automorphism group $\mathrm{Aut}\,\mathbf{C}^2$ whose time one map ϕ_1 equals g. For affine aperiodic g we find all such subgroups whose infinitesimal generator is polynomial. We also classify one parameter subgroups of the shear groups $\mathcal{S}(2)$ and $\mathcal{S}_1(2)$ on the plane $\mathbf{C}^2$.

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0. INTRODUCTION

Let $\mathrm{Aut}\,\mathbf{C}^2$ be the group of all holomorphic automorphisms of the complex plane $\mathbf{C}^2$ and $\mathrm{Aut}_1\mathbf{C}^2$ the group of automorphisms with Jacobian one. We denote by $\mathcal{G} \subset \mathrm{Aut}\,\mathbf{C}^2$ the subgroup of $\mathrm{Aut}\,\mathbf{C}^2$ consisting of all polynomial automorphisms of $\mathbf{C}^2$. The group $\mathrm{Aut}\,\mathbf{C}^2$ and its subgroups are topological groups in the topology of uniform convergence on compacts in $\mathbf{C}^2$.

A (real) one parameter subgroup of $\mathrm{Aut}\,\mathbf{C}^2$ is a family $\{\phi_t : t \in \mathbf{R}\} \subset \mathrm{Aut}\,\mathbf{C}^2$, depending continuously on $t \in \mathbf{R}$, satisfying $\phi_s \circ \phi_t = \phi_{s+t} (t,s \in \mathbf{R})$. Equivalently, $t \in \mathbf{R} \mapsto \phi_t \in \mathrm{Aut}\,\mathbf{C}^2$ is a continuous group homomorphism from $(\mathbf{R},+)$ to $\mathrm{Aut}\,\mathbf{C}^2$. Every such subgroup is smooth (and therefore real-analytic) in all variables, including t (see e.g. [1], p. 296). According to Corollary 2.2 in [13], every real one parameter subgroup of $\mathrm{Aut}\,\mathbf{C}^n$ extends to a complex one parameter subgroup $\{\phi_t : t \in \mathbf{C}\}$. Such subgroups will also be called *flows* since they are obtained by integrating complete holomorphic vector fields. Two flows ϕ_t, ψ_t are conjugate in $\mathrm{Aut}\,\mathbf{C}^2$ if there exists an $h \in \mathrm{Aut}\,\mathbf{C}^2$ such that $\phi_t = h^{-1} \circ \psi_t \circ h$ for all t. We refer the reader to [2] and [13] for further results.

In this paper we attempt to find all flows $\phi_t \in \mathrm{Aut}\,\mathbf{C}^2$ $(t \in \mathbf{R})$ whose time one map is polynomial: $\phi_1 \in \mathcal{G}$. The maps ϕ_t for non-integer values of t need not be polynomial. We succeed in all cases except when the time one map ϕ_1 is conjugate to an affine aperiodic map. In this case we identify all flows whose infinitesimal generator is a *polynomial* vector field on $\mathbf{C}^2$. Our results are summarized at the end of Section 1, and the precise statements and proofs are given in Sections 3–5.

All flows with polynomial time one map which we find in this paper are conjugate to flows that are entirely contained in the polynomial group $\mathcal{G}$.

Our work provides another proof of the classification theorem for polynomial flows $\{\phi_t : t \in \mathbf{R}\} \subset \mathcal{G}$ on the plane $\mathbf{C}^2$. These flows were classified in 1977 by

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Suzuki [22, 23]. In 1985 the result was rediscovered by Bass and Meisters [6] by using somewhat different methods. Both proofs rely on the observation that the polynomial degree of maps ϕ_t in a flow has an upper bound independent of t. The proof in [6] depends on results of Jung [17] and Van der Kulk [24] concerning the algebraic structure of the polynomial group $\mathcal{G}$ as an amalgamated free product of two special subgroups, the affine group $\mathcal{A}$ and the group $\mathcal{E}$ of *elementary automorphisms* (see Theorem 1 below).

In this paper we also classify one parameter subgroups (flows) in the shear group $\mathcal{S}(2)$ and in its subgroups consisting of $g \in \mathcal{S}(2)$ with constant Jacobian (resp. Jacobian one); see Theorem 7.1. Comparing the list of flows in $\mathcal{S}(2)$ with the list of polynomial flows from [6] we see that there is only one new type of flows in $\mathcal{S}(2)$.

In Section 6 we find all flows in $\mathrm{Aut}\,\mathbf{C}^2$ whose time one maps are shears (Theorem 6.1). In particular we show that most shears do not belong to any flow.

To find flows with polynomial time one map we use a recent result of G. Buzzard [9] (Theorem 2 below) to the effect that every polynomial automorphism of $\mathbf{C}^2$ which lies in a flow $\phi_t \in \mathrm{Aut}\,\mathbf{C}^2$ is conjugate in $\mathcal{G}$ to an elementary automorphism (see Section 1). This depends on a result of Friedland and Milnor [15] to the effect that every polynomial automorphism of $\mathbf{C}^2$ which is not conjugate to an elementary one has a discrete set of periodic points, and it has a periodic point which is not a fixed point.

We begin our analysis with an elementary automorphism E in normal form, provided by Theorem 3 below. Assuming that E is the time one map of a flow $\phi_t \in \mathrm{Aut}\,\mathbf{C}^2$ ($t \in \mathbf{R}$) we consider the *conjugation relation* between E and the infinitesimal generator V of ϕ_t : $DE \cdot V = V \circ E$ (Section 2). This gives two functional equations for the two component functions of the field V. Analysing these equations and integrating the system of ordinary differential equations for the flow we obtain the desired results. The same method is used to classify flows in the shear groups on $\mathbf{C}^2$.

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1. BACKGROUND AND RESULTS

In order to put our work in context we begin by recalling some known results concerning the structure of the polynomial automorphism group of $\mathbf{C}^2$. We denote the complex coordinates on $\mathbf{C}^2$ by $z = (x, y)$. Let $\mathcal{A}$ be the group of all affine automorphisms of $\mathbf{C}^2$:

$$A(x,y) = (\alpha x + \beta y + \xi, \gamma x + \delta y + \eta), \qquad \alpha\delta - \beta\gamma \neq 0.$$

Let $\mathcal{E}$ be the subgroup of $\mathcal{G}$ consisting of all automorphisms of the form

$$E(x,y) = (\alpha x + p(y), \beta y + \gamma),$$

where $\alpha,\beta \in \mathbf{C}^* = \mathbf{C}\backslash\{0\}$, $\gamma \in \mathbf{C}$, and p is a polynomial. Following Friedland and Milnor [15] we will call such automorphisms *elementary*. A theorem of Jung [17] asserts that the polynomial group $\mathcal{G}$ is generated by the subgroups $\mathcal{A}$ and $\mathcal{E}$ in the sense that every element of $\mathcal{G}$ is a finite composition of elements in $\mathcal{A}$ and $\mathcal{E}$. The following more precise result is due to Van der Kulk [24]; see also Rentschler [19] and Friedland and Milnor [15]. (It appears that Van der Kulk was not aware of the work of Jung [17].)

THEOREM 1 *The group $\mathcal{G}$ is an amalgamated free product of the subgroups $\mathcal{A}$ and $\mathcal{E}$.*

This means that every $g \in \mathcal{G}$ which does not belong to $\mathcal{A} \cap \mathcal{E}$ can be represented as a 'reduced word'

$$g = g_m \circ g_{m-1} \circ \cdots \circ g_1, \tag{1}$$

where every $g_j \in \mathcal{A} \cup \mathcal{E}\backslash(\mathcal{A} \cap \mathcal{E})$, and every two consecutive elements g_j, g_{j+1} belong to different groups $\mathcal{A}, \mathcal{E}$. Moreover, such a representation of g is unique up to compositions by elements in $\mathcal{A} \cap \mathcal{E}$ which should be thought of as units in this representation. In particular, no reduced word equals the identity. The existence of such a representation follows from Jung's theorem. The essential step in the proof of the uniqueness, given in [15] (see also [20]), is to show that the degree of the reduced word (1) equals the product of the degrees of the g_j's. Here, the degree of $g \in \mathcal{G}$ is the larger of the degrees of the two polynomial components of g.

From this representation of polynomial automorphisms it follows [15] that every $g \in \mathcal{G}$ is either conjugate in $\mathcal{G}$ to an element of $\mathcal{E}$ or else g is conjugate to a reduced word of even length $2r$ in which the end two elements belong to different groups $\mathcal{A}, \mathcal{E}$. Such a word is said to be *cyclically reduced*. The length of a cyclically reduced word is invariant under conjugation, and the degrees of the characters in the word are invariant up to cyclic permutations. The simplest maps of this type are $g = A \circ E$, with $A \in \mathcal{A}\backslash\mathcal{E}$ and $E = \mathcal{E}\backslash\mathcal{A}$. These include all complex Hénon mappings

$$h(x,y) = (y, p(y) - \delta x),$$

where $\delta \in \mathbf{C}^*$ and p is a polynomial. (It suffices to take $A(x,y) = (y,x)$ and $E(x,y) = (-\delta x + p(y), y)$). While maps in $\mathcal{E} \cup \mathcal{A}$ have rather simple dynamical properties, most Hénon maps (and other automorphisms represented by cyclically reduced words of even length) exhibit very interesting dynamics; we refer the reader to Hénon [16], Friedland and Milnor [15], Bedford and Smillie [7, 8], and Fornaess and Sibony [12].

The condition that a group $\mathcal{G}$ is an amalgamated free product $\mathcal{A} * \mathcal{E}$ implies that certain types of its subgroups $\mathcal{G}' \subset \mathcal{G}$ are conjugate to a subgroup of $\mathcal{A}$ or to a subgroup of $\mathcal{E}$. This is the case if $\mathcal{G}'$ has bounded length with respect to the amalgamated product decomposition of $\mathcal{G}$ (Serre ([21], Chap. I, No. 4.3, Théorèm 8). If $\mathcal{G}$ is a topological group and $\mathcal{B} = \mathcal{A} \cap \mathcal{E}$ is a closed subgroup, then the same holds for every connected abelian Lie group $\mathcal{G}' \subset \mathcal{G}$; in particular, every subgroup of $\mathcal{G}$ isomorphic to $\mathbf{R}$ or $\mathbf{C}$ is conjugate to a subgroup in $\mathcal{A}$ or in $\mathcal{E}$. This follows from a theorem of Moldavanski [18, 25]. Serre's result was used in the classification of polynomial flows by Bass and Meisters [6]. We shall use Moldavanski's result in Section 7 for the classification of one parameter subgroups of $\mathcal{S}(2)$.

The following result was communicated to us by G. Buzzard:

THEOREM 2 (Buzzard [9]) *Let $\phi_t \subset \mathrm{Aut}\,\mathbf{C}^2$ be a one parameter group of automorphisms of the plane such that $\phi_1 \in \mathcal{G}$. Then ϕ_1 is conjugate in $\mathcal{G}$ to an elementary map $E \in \mathcal{E}$.*

The proof uses the following result from [15]. If $g = g_{2r} \circ \cdots \circ g_1$ is a cyclically reduced word of even length $2r \geq 2$, then

(a) the number of fixed points of g is finite (bounded from above by the degree of g), and
(b) there exists a point $z_0 \in \mathbf{C}^2$ which is a periodic point of g but not a fixed point of g.

Granted this, Theorem 2 is proved as follows (Buzzard). Assume that $\phi_1 \in \mathcal{G}$ is conjugate to a reduced word g of even length $2r \geq 2 : \phi_1 = h^{-1} \circ g \circ h$. Replacing ϕ_t by the conjugate subgroup $h \circ \phi_t \circ h^{-1}$ we may as well assume $\phi_1 = g$. Let $z_0 \in \mathbf{C}^2$ be a periodic point of g which is not a fixed point of g. Then $g^n(z_0) = z_0$ for some $n \geq 2$. Using the group property of ϕ_t and the assumption $g = \phi_1$ (hence $g^n = \phi_n$) we have

$$g^n(\phi_t(z_0)) = \phi_n(\phi_t(z_0)) = \phi_t(\phi_n(z_0)) = \phi_t(g^n(z_0)) = \phi_t(z_0).$$

This means that the entire orbit $\{\phi_t(z_0) : t \in \mathbf{R}\}$ (which is a nontrivial closed curve since $\phi_1(z_0) \neq z_0$) belongs to the fixed point set of g^n. Since g^n is a cyclically reduced word of length $2rn \geq 2$, this contradicts the property (a) above. Therefore ϕ_1 must be conjugate to an elementary automorphism. ∎

In the case when $\phi_t \in \mathcal{G}$ is a polynomial flow the result can be seen as follows. Suppose that $\phi_1 = E$ is a reduced word of even length $2r \geq 2$. Choose an integer $n > r$ and consider the map $g = \phi_{1/n} \in \mathcal{G}$. If g were conjugate to an element in $\mathcal{E}$, then so would be $g^n = E$ which is not the case. Thus g is conjugate to a cyclically reduced word $\tilde{g}$ of even length $2m \geq 2$. Then $E = g^n$ is conjugate to $\tilde{g}^n$ which is a cyclically reduced word of length $2mn \geq 2n > 2r$. This is a contradiction since the length of a cyclically reduced word is invariant under conjugation in $\mathcal{G}$.

The last argument shows more: *If $g \in \mathcal{G}$ is a cyclically reduced word of even length $2r$, and if $g = h^n$ for some $h \in \mathcal{G}$ and $n > 1$ (i.e., h is an nth root of g in the sense of composition), then n divides r and h is conjugate in $\mathcal{G}$ to a reduced word of length $2r/n$.* In particular, a nonlinear Hénon map (or any element of the form $A \circ E$ with $A \in \mathcal{A} \backslash \mathcal{E}$ and $E \in \mathcal{E} \backslash \mathcal{A}$) does not have a root of any order in $\mathcal{G}$.

In our analysis of flows we will refer to the following classification of elementary maps, due to Friedland and Milnor [15].

THEOREM 3 *Every element in $\mathcal{E}$ is conjugate in $\mathcal{E}$ to one of the following maps:*

(a) $(x,y) \mapsto (\alpha x, \beta y)$,
(b) $(x,y) \mapsto (x+1, \beta y)$ *or* $(x,y) \mapsto (\alpha x, y+1)$,
(c) $(x,y) \mapsto (\beta^d(x + y^d), \beta y)$,
(d) $(x,y) \mapsto (\beta^d(x + y^d q(y^r)), \beta y)$.

In each case we have $\alpha, \beta \in \mathbf{C}^*$. In case (c) *we have $d \geq 1$. In case* (d) *we have $d \geq 0$, β is a primitive rth root of 1 for some $r \geq 1$, and q is a nonconstant polynomial satisfying $q(0) = 1$.*

A special but important class of automorphisms are *shears*

$$E(x,y) = (x + p(y), y), \tag{2}$$

where p is an entire function. If p is constant, say $p = 1$, this is a map of type (b) with $\beta = 1$. If p is a nonconstant polynomial, and if we normalize the coefficient with the lowest power of y in p to 1, then (2) is a map of type either (c) or (d), again with $\beta = 1$.

We now describe the results of this paper; the precise statements and proofs are given in Sections 3–6. Let E be one of the normal forms in Theorem 3 and let $\phi_t \in \mathrm{Aut}\,\mathbf{C}^2$ ($t \in \mathbf{R}$) be a real one parameter group (flow) such that $\phi_1 = E$. We prove the following:

1. If E is an aperiodic diagonal linear map of type (a) with no nontrivial resonances ($\alpha^p\beta^q \neq 1$ when $p, q \geq -1$, p and q not both 0), then the flow ϕ_t is diagonal linear (Theorem 3.1):

$$\phi_t(x,y) = (e^{\mu t}x, e^{\lambda t}y), \qquad e^\mu = \alpha, \quad e^\lambda = \beta.$$

When there are nontrivial resonances between α and β, there exist nonlinear flows ϕ_t satisfying $\phi_1 = E$, but each of them is conjugate to a linear flow.

2. If E is of type (c) and β is not a root of one, then ϕ_t is of the form

$$\phi_t(x,y) = (e^{d\lambda t}(x + ty^d), e^{\lambda t}y), \qquad e^\lambda = \beta. \tag{3}$$

(See Theorem 4.2.) If β is a root of one, there exists an infinite dimensional family of flows with $\phi_1 = E$ (also non-polynomial ones), but each of them is conjugate in $\mathrm{Aut}\,\mathbf{C}^2$ to a flow of the form (3).

3. If p is an entire function on $\mathbf{C}$ with at least two different zeros, we show (Theorem 4.1) that the only flow $\phi_t \in \mathrm{Aut}\,\mathbf{C}^2$ whose time one map is the shear (2) is

$$\phi_t(x,y) = (x + tp(y), y). \tag{4}$$

It follows that *maps* $E \in \mathcal{E}$ *of type* (d) *with* $\beta \neq 1$ *do not belong to any flow* (part (a) of Theorem 4.2). This was proved independently by Buzzard [9].

4. If E is an affine map of type (b), we find all flows ϕ_t with $\phi_1 = E$ whose infinitesimal generator is a *polynomial* vector field on $\mathbf{C}^2$. (If the flow itself is polynomial, its infinitesimal generator is also polynomial, but the converse is not true.) When β is not a root of 1, every such flow with a polynomial generator is of the form

$$\phi_t(x,y) = (x + t, e^{\lambda t}y), \qquad e^\lambda = \beta.$$

(See Theorem 5.1.) If β is a root of one, there exist other flows with $\phi_1 = E$, but they are all conjugate to flows of this form. There exist many other flows through E whose infinitesimal generator is not polynomial, but we don't know how to get all of them.

5. A *generalized shear* on $\mathbf{C}^2$ is an automorphism of the form

$$f(x,y) = (e^{g(y)}x + h(y), y),$$

where g and h are entire functions on $\mathbf{C}$. These and the shears (2) play a very important role in the approximation theorems; see [4, 5, 14, 20]. Of course f is not polynomial unless g is constant and h is a polynomial. If g is nonconstant, we show that f is the time one map of a flow $\phi_t \in \operatorname{Aut}\mathbf{C}^2$ if and only if

$$h(y) = \frac{e^{g(y)} - 1}{g(y) + 2\pi i k} b(y)$$

for some entire function b and some $k \in \mathbf{Z}$ (Theorem 6.1). In particular, h must vanish on the set $\{y \in \mathbf{C} : g(y) = 2\pi i m,\ m \in \mathbf{Z},\ m \neq -k\}$ for some $k \in \mathbf{Z}$. Hence *most generalized shears do not lie in any flow.*

At the end of Section 5 we give an example of a polynomial (quadratic) flow ϕ_t whose time one map ϕ_1 is affine but not elementary ($\phi_1 \in \mathcal{A}\backslash\mathcal{E}$), and such that $\phi_t \notin \mathcal{A} \cup \mathcal{E}$ for any time $0 < t < 1$. This example invalidates the proof of Theorem 2.1 in [10, p. 348]. More precisely, the example shows that the part of the proof on p. 351 in [10] is incorrect. The mistake has been corrected in the forthcoming paper [11] by the same author.

2. SPECIAL SYSTEMS OF ORDINARY HOLOMORPHIC DIFFERENTIAL EQUATIONS

In this section we outline our method and we recall how to solve certain systems of ordinary differential equations in the plane.

Let $\phi_t \in \operatorname{Aut}\mathbf{C}^2$ ($t \in \mathbf{R}$) be a one parameter automorphism group of the plane $\mathbf{C}^2$. Its infinitesimal generator is the holomorphic vector field $V = (F, G)$ on $\mathbf{C}^2$, defined by

$$V(x,y) = \frac{d}{dt}\phi_t(x,y)\bigg|_{t=0}.$$

Conversely, the flow $\phi_t(x,y) = (X, Y)$ is the solution at time t of the system of ordinary differential equations

$$\dot{X} = F(X,Y), \qquad \dot{Y} = G(X,Y), \qquad X(0) = x, \qquad Y(0) = y. \tag{5}$$

We refer the reader to Abraham and Marsden [2] for the general properties of solutions of such systems.

By differentiating the identity $\phi_t \circ \phi_s(x,y) = \phi_{t+s}(x,y)$ on s at time $s = 0$ we get

$$D\phi_t(x,y) \cdot V(x,y) = V(\phi_t(x,y)), \qquad t \in \mathbf{R}.$$

Here $D\phi_t$ denotes the derivative with respect to the space variables, while the dot always denotes the time derivative. At time $t = 1$ this gives the following condition relating the vector field V and the time one map $\phi_1 = E$:

$$DE(x,y) \cdot V(x,y) = V(E(x,y)). \tag{6}$$

Our classification of flows with $\phi_1 = E \in \mathcal{E}$ is based on the analysis of this condition, called the *commutation relation*, when $E \in \mathcal{E}$ is one of the normal forms from

Theorem 3 (Section 1). The condition is a set of two functional equations for the component functions F and G of the vector field V.

In most cases to be considered the condition (6) implies that the second component G is independent of $x : G = G(y)$. Granted this one can explicitly integrate the system of equations (5) as follows. The second equation in (5) defines a complete flow in each complex line $\{x = \text{const}\}$. Therefore G is linear in y, $G(y) = \lambda y + c$. If $\lambda \neq 0$, we can write $G(y) = \lambda(y - y_0)(y_0 = -c/\lambda)$, and the solution equals

$$Y(t,y) = e^{\lambda t}(y - y_0) + y_0. \tag{7}$$

If $G = c = \text{const.}$, we have

$$Y(t,y) = y + ct. \tag{8}$$

We insert this solution into the first equation for $\dot{X}$ in (5):

$$\dot{X} = F(X, Y(t,y)), \qquad X(0) = x.$$

By hypothesis this equation is integrable for all $t \in \mathbf{R}$ and all initial conditions $X(0) = x \in \mathbf{C}$, $y \in \mathbf{C}$. We denote the solution by $X(t,x,y)$. For each fixed $t \in \mathbf{R}$ and $y \in \mathbf{C}$ the map $x \in \mathbf{C} \to X(t,x,y) \in \mathbf{C}$ is injective holomorphic and therefore linear. It follows that $F(x,y)$ is linear in x,

$$F(x,y) = a(y)x + b(y)$$

for some entire functions a, b on $\mathbf{C}$. We thus have an inhomogeneous, time dependent linear equation for X:

$$\dot{X} = a(Y(t,y))X + b(Y(t,y)), \qquad X(0) = x.$$

If we set

$$A(t,y) = \int_0^t a(Y(\tau,y))\,d\tau,$$

$$B(t,y) = \int_0^t e^{-A(\tau,y)} b(Y(\tau,y))\,d\tau,$$

where $Y(\tau,y)$ is given either by (7) or (8), then the solution is

$$X(t,x,y) = e^{A(t,y)}(x + B(t,y)).$$

The flow $\phi_t(x,y) = (X(t,x,y), Y(t,y))$ maps every line $y = \text{const.}$ to another such line. The flow is polynomial if and only if $a(y) = \mu = \text{const.}$ and b is a polynomial. It is worthwhile writing down this special case explicitly because it will arise in several cases treated below. Assume that the second component of the flow is given by (7). Let $b(y) = \sum_{k \geq 0} b_k (y - y_0)^k$. Then $A(t,y) = \mu t$ and

$$X(t,x,y) = e^{\mu t}\left(x + \sum_{k \geq 0} b_k (y - y_0)^k c_k(t)\right),$$

where

$$c_k(t) = \int_0^t e^{(-\mu+k\lambda)\tau}\, d\tau = \begin{cases} (e^{(k\lambda-\mu)t} - 1)/(k\lambda-\mu); & \text{if } \; k\lambda-\mu \neq 0, \\ t; & \text{if } \; k\lambda-\mu = 0. \end{cases}$$

3. FLOWS WHOSE TIME ONE MAP IS DIAGONAL LINEAR

The main result of this section, Theorem 3.1, gives a complete classification of flows $\phi_t \in \mathrm{Aut}\,\mathbf{C}^2$ ($t \in \mathbf{R}$) whose time one map is diagonal linear and aperiodic (of type (a) in Theorem 3). The nondiagonal linear maps (Jordan blocks) are of type (c) with $d = 1$, and this case is covered by Theorem 4.2 below. For the periodic case see Propositions 3.2.

In the classification theorem we will need the following notion of *resonance*.

DEFINITION *The numbers $\alpha,\beta \in \mathbf{C}^*$ have a resonance of type (p,q) if $\alpha^p\beta^q = 1$, where p and q are integers. The resonance is nontrivial if at least one of the numbers p,q is nonzero.*

Clearly the set of all resonances between α and β is an additive subgroup of $\mathbf{Z}^2$. A resonance of type $(0,q)$ means that β is a qth root of 1, and a resonance of type $(-1,q)$ mean that $\alpha = \beta^q$. If none of the numbers α,β is a root of 1, then any nontrivial resonance is an integer multiple of a base resonance.

THEOREM 3.1 *Let $\phi_t \in \mathrm{Aut}\,\mathbf{C}^2$ be a flow whose time one map $\phi_1 = E$ is diagonal linear and aperiodic:*

$$E(x,y) = (\alpha x, \beta y), \qquad \alpha,\beta \in \mathbf{C}^*, \quad E^r \neq 1 \quad (r \geq 1).$$

(a) *If α and β have no nontrivial resonances (p,q) with $p,q \geq -1$, then the flow ϕ_t is diagonal linear:*

$$\phi_t(x,y) = (e^{\mu t}x, e^{\lambda t}y), \qquad e^\mu = \alpha, \quad e^\lambda = \beta. \tag{9}$$

If $\alpha = \beta$ (a resonance of type $(-1,1)$) then ϕ_t is linearly conjugate (but not necessarily equal) to a flow (9).

(b) *If α is not a root of 1 but β is a primitive qth root of 1 for some $q > 1$ (a resonance of type $(0,q)$), then*

$$\phi_t(x,y) = \left(x \exp\left(\mu t + \sum_{n\geq 1} a_n (y-y_0)^{nq} c_{nq}(t) \right), e^{\lambda t}(y-y_0) + y_0 \right), \tag{10}$$

where $e^\mu = \alpha$, $e^\lambda = \beta$, $a_n \in \mathbf{C}$, $y_0 = 0$ unless $\beta = 1$, and $c_k(t) = (e^{\lambda k t} - 1)/\lambda k$. Every one of these flows is conjugate to the flow (9).

(c) *If α,β have a base resonance (p,q) for some $p,q \geq 1$, then*

$$\phi_t(x,y) = \left(x \exp\left(\int_0^t f(e^{c\tau}x^p y^q)\, d\tau \right), y \exp\left(\int_0^t g(e^{c\tau}x^p y^q)\, d\tau \right) \right), \tag{11}$$

where $c = 2\pi i n$ for some $n \in \mathbf{Z}$ and f, g are entire functions on $\mathbf{C}$ satisfying $pf + qg = c$, $e^{f(0)} = \alpha$, $e^{g(0)} = \beta$. If $c = 0$ then f and g must be constant. The flow (11) *is conjugate to the linear flow* (9) *with $\mu = f(0)$ and $\lambda = g(0)$.*

(d) *If $\alpha = \beta^q$ for some $q \geq 2$ (a resonance of type $(-1, q)$) then ϕ_t is either of the form* (9) *or of the form*

$$\phi_t(x,y) = (e^{\mu t}(x + by^q c(t)), e^{\lambda t} y), \tag{12}$$

where $e^{\mu} = \alpha$, $e^{\lambda} = \beta$, $q\lambda - \mu = 2\pi i n \neq 0$, $b \in \mathbf{C}$, and $c(t) = (e^{2\pi i n t} - 1)/2\pi i n$. The flow (12) *is conjugate to the linear flow* (9).

Remarks 1. In parts (b) and (d) the analogous results hold if we switch the roles of α and β. The flow (10) with $y_0 = 0$ is a special case of (11) with $p = 0$ and $g = c/q = \lambda$ constant.

2. If $\alpha = \beta^d$ (case (d)) and β is an rth root of 1 for some $r \geq 1$, then the map E (which is now periodic with period r) is the time one map of many other flows. Some of them are given by (18) in Sect. 4 below (with $e^{\lambda} = \beta$ and $y_0 = 0$).

3. The only polynomial flows of types (10) and (11) are the linear ones. The flows (11) with $c = 0$ and f, g nonconstant are of interest even though they are not a part of our classification. They can be written in the form

$$\phi_t(x,y) = (xe^{tqh(u)}, ye^{-tph(u)}), \qquad u = x^p y^q,$$

where h is an entire function on $\mathbf{C}$. These flows remain in the level sets of $x^p y^q$, and hence they are *proper* in terminology of M. Suzuki [22]; for a classification of such flows see Theorem 4 in [22]. The time one map of these flows is linear only if h is constant. When $p = q = 1$ ($\alpha\beta = 1$) we get the flows

$$\phi(x,y) = (xe^{th(xy)}, ye^{-th(xy)})$$

which are volume preserving (symplectic). We refer the reader to Section 6 in [13] for the classification of the complex volume preserving flows $\phi_t \in \operatorname{Aut}\mathbf{C}^2$ ($t \in \mathbf{C}$).

Proof of Theorem 3.1 The commutation relations (6) for the infinitesimal generator $V = (F, G)$ are

$$\alpha F(x,y) = F(\alpha x, \beta y), \qquad \beta G(x,y) = G(\alpha x, \beta y). \tag{13}$$

We expand F and G in power series in x and y and compare the coefficients of terms $x^p y^q$. If this term appears in F (with a nonvanishing coefficient), we get $\alpha^{p-1}\beta^q = 1$, i.e., a resonance of type $(p-1, q)$. Similarly, if this term appears in G, we get a resonance of type $(p, q-1)$.

Case (a) If α, β have no nontrivial resonances (p, q) with $p, q \geq -1$, it follows from (13) that $F = \mu x$ and $G = \lambda y$ for some $\mu, \lambda \in \mathbf{C}$. Hence the flow ϕ_t is of the form (9). If $\alpha = \beta \neq 1$, (13) only shows that F and G are linear in x and y, and hence the flow is linear: $\phi_t(x,y) = \exp(At)(x,y)^t$ for some 2×2 matrix A. Since $\exp(A)$ is the scalar matrix representing E, A is linearly conjugate to a diagonal matrix with eigenvalues μ, λ satisfying $e^\mu = e^\lambda = \alpha$, but μ and λ need not be equal. Therefore the flow ϕ_t is conjugate (but not necessarily equal) to a flow (9). In fact, since E is self-conjugate by any linear map, any flow ϕ_t which is linearly conjugate to a flow (9) has the same time one map E. However, the matrix representing ϕ_t for non-integer values of t need not be diagonal, and we get non-diagonal linear flows with $\phi_1 = E$.

CASE (b) Consider first the case when $\beta = 1$. The commutation relations (13) imply that G is independent of x and F contains only terms which are linear in x. Thus $G = G(y)$ and $F(x,y) = a(y)x$ for some entire function a. By completeness of the flow we get $G(y) = \lambda(y - y_0)$ (the case $G(y) = c \neq 0$ is impossible since it would integrate to the flow $Y(t,y) = y + ct$ whose time one map is $y + c \neq y$). Integrating the equations (5) as in Section 2 we get the flow (10) with $\mu = a(y_0)$ and $q = 1$.

The case when β is a primitive qth root of 1 for some $q > 1$ can be reduced to the case $\beta = 1$ by replacing E with its iterate $E^q(x,y) = (\alpha^q x, y)$. We first find all flows ϕ_t satisfying $\phi_q = E^q$. These are given by (10) with $e^{q\lambda} = 1$, $e^{q\mu} = \alpha^q$, and with nq replaced by n in the summation. In this family we then identify the flows which also satisfy $\phi_1 = E$. This requires $e^\mu = \alpha$, $e^\lambda = \beta$, $y_0 = 0$, and $c_n(1) = 0$ whenever the term y^n appears in the sum (with a nonvanishing coefficient). The last condition is satisfied when n is a multiple of q, and we get the flow (10). The automorphism

$$h(x,y) = \left(x \exp\left(-\sum_{n\geq 1} a_n (y - y_0)^{nq}/nq\lambda \right), y - y_0 \right)$$

conjugates this flow to (9).

CASE (c) We now have a resonance $\alpha^p \beta^q = 1$ for some $p, q \geq 1$. Since E is not periodic, none of the numbers α, β is a root of 1, and therefore every resonance is a multiple of a base resonance. Thus we may assume that (p,q) is a base resonance. The commutation relations (13) imply that F and G have the following special form:

$$F(x,y) = xf(x^p y^q), \qquad G(x,y) = yg(x^p y^q)$$

for some entire functions f and g. The resonance condition implies that E and it iterates remain in the level sets of the function $u = x^p y^q$.

We now calculate the derivative of u along the flow ϕ_t. We use the lower case letters for the initial position and the upper case letters for the value along the flow. We have

$$\begin{aligned} \dot{U} &= pX^{p-1}Y^q \dot{X} + qX^p Y^{q-1} \dot{Y} \\ &= pX^p Y^q f(U) + qX^p Y^q g(U) \\ &= U(pf(U) + qg(U)). \end{aligned}$$

By hypothesis this is a complete flow in the U-plane, and therefore $pf(U) + qg(U) = c = \text{const}$. By integration we get

$$U(t,x,y) = e^{ct}x^p y^q, \qquad e^c = 1.$$

The last condition $e^c = 1$ follows from $U \circ E = U$. We now insert this solution into the equations for the flow $\phi_t = (X,Y)$:

$$\dot{X} = Xf(e^{ct}x^p y^q), \qquad \dot{Y} = Yg(e^{ct}x^p y^q), \qquad X(0) = x, \quad Y(0) = y.$$

Integrating this system of homogeneous equations we get the flow (11).

Let h and k be the entire functions on $\mathbf{C}$ satisfying $h(0) = k(0) = 0$ and

$$h'(\zeta) = (f(\zeta) - f(0))/c\zeta, \qquad k'(\zeta) = (g(\zeta) - g(0))/c\zeta.$$

Then the map

$$\Phi(x,y) = (xe^{h(x^p y^q)}, ye^{k(x^p y^q)})$$

is an automorphism of $\mathbf{C}^2$ such that $\Phi^{-1} \circ \phi_t \circ \Phi$ is the linear flow (9), with $\mu = f(0)$ and $\lambda = g(0)$.

CASE (d) We now have $\alpha = \beta^q$ $(q \geq 2)$. Since E is aperiodic, α and β are not roots of 1, and thus $(-1,q)$ is the only resonance between α and β with both numbers ≥ -1. The relation (13) implies

$$G(y) = \lambda y, \qquad F(x,y) = \mu x + by^q, \qquad \lambda, \mu, b \in \mathbf{C}.$$

Integrating the system (5) we obtain the flow (12) when $q\lambda - \mu \neq 0$, and the flow

$$\phi_t(x,y) = (e^{q\lambda t}(x + tby^q), e^{\lambda t}y)$$

when $q\lambda = \mu$. In the last case the time one map ϕ_1 is not linear unless $b = 0$ and the flow is of the form (9).

If $q\lambda - \mu \neq 0$, we set $d = b/(q\lambda - \mu)$ and $\Phi(x,y) = (x + dy^q, y)$. Then $\Phi \in \operatorname{Aut}\mathbf{C}^2$ and $\Phi^{-1} \circ \phi_t \circ \Phi$ is the linear flow (9). This completes the proof of Theorem 3.1. ■

In Theorem 3.1 we have excluded the case when E is periodic since our methods do not apply in this case. However, a result of Suzuki [22] gives:

PROPOSITION 3.2 *If $E \in \operatorname{Aut}\mathbf{C}^2$ is a periodic map and $\phi_t \in \operatorname{Aut}\mathbf{C}^2$ $(t \in \mathbf{R})$ is a flow satisfying $\phi_1 = E$, then ϕ_t is conjugate in* $\operatorname{Aut}\mathbf{C}^2$ *to a linear flow* (9).

Proof By Corollary 2.2 in [13] every flow $\{\phi_t : t \in \mathbf{R}\} \subset \operatorname{Aut}\mathbf{C}^2$ extends to all a complex flow $\{\phi_t : t \in \mathbf{C}\} \subset \operatorname{Aut}\mathbf{C}^2$. Since E is periodic, ϕ_t is also periodic, say $\phi_k = 1$. Hence $\phi_t = \psi_{\exp(2\pi i t/k)}$, where ψ_s $(s \in \mathbf{C}^*)$ is a holomorphic action of the group $\mathbf{C}^*$ on $\mathbf{C}^2$. According to Theorem 5 in [22] every such action is conjugate in $\operatorname{Aut}\mathbf{C}^2$ to an action

$$\psi_s(x,y) = (s^m x, s^n y), \qquad s \in \mathbf{C}^*, \quad m,n \in \mathbf{Z}.$$

This implies Proposition 3.2. ■

4. FLOWS WHOSE TIME ONE MAP IS OF TYPE (c) OR (d)

In this section we find all flows $\phi_t \in \mathrm{Aut}\,\mathbf{C}^2$ $(t \in \mathbf{R})$ such that $\phi_1 = E$ is of type (c) or (d) in Theorem 3. We begin with the simplest case when E is a shear.

THEOREM 4.1 *Let g be an entire function on* $\mathbf{C}$ *and*

$$E(x,y) = (x + g(y), y), \qquad (x,y) \in \mathbf{C}^2.$$

Suppose that $\phi_t \in \mathrm{Aut}\,\mathbf{C}^2$ $(t \in \mathbf{R})$ is a flow satisfying $\phi_1 = E$.

(a) *If g has at least two distinct zeros, then ϕ_t is the flow*

$$\phi_t(x,y) = (x + tg(y), y). \tag{14}$$

(b) *If $g(y) = c(y - y_0)^d$ for some $d \geq 1$ and $c \in \mathbf{C}^*$, then ϕ_t is either the flow* (14) *or else it is conjugate in* $\mathrm{Aut}\,\mathbf{C}^2$ *to a flow*

$$\psi_t(x,y) = (e^{d\lambda t}(x + tc(y - y_0)^d), e^{\lambda t}(y - y_0) + y_0), \qquad e^\lambda = 1. \tag{15}$$

Proof Let $V = (F, G)$ be the infinitesimal generator of ϕ_t. The commutation relation (6) gives the following equations:

$$\begin{aligned} F(x,y) + g'(y)G(x,y) &= F(x + g(y), y) \\ G(x,y) &= G(x + g(y), y). \end{aligned} \tag{16}$$

Iterating the second equation we get

$$G(x,y) = G(x + kg(y), y), \qquad k \in \mathbf{Z}.$$

Assume that g has a zero, say $g(y_0) = 0$. Fix an $\epsilon \in \mathbf{C}^*$ and choose a sequence $y_k \in \mathbf{C}$ such that $kg(y_k) = \epsilon$ and $\lim_{k\to\infty} y_k = y_0$. This gives

$$G(x, y_k) = G(x + y_k g(y_k), y_k) = G(x + \epsilon, y_k).$$

Since the holomorphic functions $G(x,\cdot)$ and $G(x + \epsilon, \cdot)$ coincide along the sequence y_k with an accumulation point in $\mathbf{C}$, they coincide identically by the uniqueness principle. Hence $G(x,y) = G(x + \epsilon, y)$. Since this holds for every $\epsilon \in \mathbf{C}^*$, it follows that G is independent of $x : G = G(y)$. Hence our system of differential equations for the flow is of the special type treated in Section 2:

$$G(y) = \lambda y + c, \qquad F(x,y) = a(y)x + b(y).$$

We cannot have $\lambda = 0$ and $c \neq 0$ since in this case the second component of the flow would be $Y(t,y) = y + tc$, and this does not equal y (the second component of E) at time $t = 1$. Thus we have

$$G(y) = \lambda(y - y_0), \qquad Y(t,y) = e^{\lambda t}(y - y_0) + y_0.$$

The condition $Y(1,y) = y$ implies $e^\lambda = 1$. The first equation in (16) now gives

$$a(y) = \lambda(y - y_0)\frac{g'(y)}{g(y)}. \tag{17}$$

If $\lambda = 0$, we have $a = 0$ and the first equation for the flow is $\dot{X} = b(y)$. The flow equals

$$\phi_t(x,y) = (x + tb(y), y).$$

Comparing with E at time 1 we see that $b = g$, and we get the flow (14).

Suppose now $\lambda \neq 0$. If g has a zero at $y_1 \in \mathbf{C}$, then g'/g has a pole at y_1. Since a is an entire function, the equation (17) has no solutions unless $y_1 = y_0$. Thus, if g has at least two different zeros, (17) has no solutions unless $\lambda = 0$ and the flow is given by (14).

If g has only one zero y_0, we can write $g(y) = (y - y_0)^d e^{h(y)}$ for some $d \geq 0$ and some entire function h. Then the function a defined by (17) is entire, and we can proceed by integrating the system of equations for ϕ_t as in Section 2. Of course we must compare the time one map ϕ_1 with E, and this gives further conditions on the functions h and b. We will carry out this process explicitly only in the case when g is a polynomial, $g(y) = c(y - y_0)^d$. The equation (17) then gives $a = d\lambda = \text{const}$. For every entire function $b(y) = \sum_k b_k(y - y_0)^k$ we get a solution $\phi_t = (X, Y)$ given by

$$\begin{aligned} X(t,x,y) &= e^{d\lambda t}\left(x + tc(y - y_0)^d + \sum_{k\neq d} b_k(y - y_0)^k c_k(t)\right), \\ Y(t,x,y) &= e^{\lambda t}(y - y_0) + y_0, \end{aligned} \tag{18}$$

where

$$e^{\lambda} = 1, \qquad c_k(t) = \frac{e^{(k-d)\lambda t} - 1}{\lambda(k-d)} \qquad (k \neq d).$$

Notice that the power series defining X converges for each $t \in \mathbf{C}$ and for all $(x,y) \in \mathbf{C}^2$, and hence the above is a complex one parameter group of automorphisms of $\mathbf{C}^2$. Each of these groups satisfies $\phi_1 = E$ since the terms with $k \neq d$ disappear at the integer values of t. It is verified easily that $\phi_t = h^{-1} \circ \psi_t \circ h$, where $h \in \operatorname{Aut}\mathbf{C}^2$ is the automorphism

$$h(x,y) = \left(x - \sum_{k\neq d} \frac{b_k}{\lambda(k-d)} y^k, y\right)$$

and ψ_t is the flow (15). This completes the proof of Theorem 4.1. ■

The next theorem classifies flows whose time one map if of type (c) or (d) in Theorem 3 (Section 1).

THEOREM 4.2

(a) *If E is an elementary map of type* (d) *and if $\beta \neq 1$, then E does not belong to any flow $\phi_t \in \operatorname{Aut}\mathbf{C}^2$. If $\beta = 1$, the only flow satisfying $\phi_1 = E$ is*

$$\phi_t(x,y) = (x + ty^d q(y^r), y). \tag{19}$$

(b) *If E is of type* (c) *in Theorem* 3 *and β is not a root of one, then every flow ϕ_t satisfying $\phi_1 = E$ is of the form*

$$\phi_t(x,y) = (e^{d\lambda t}(x + ty^d), e^{\lambda t}y), \qquad e^{\lambda} = \beta. \tag{20}$$

(c) *If E is of type* (c) *and $\beta \neq 1$ is a primitive rth root of* 1 *for some $r > 1$, then all flows satisfying $\phi_1 = E$ are described by* (21) *below, and each of them is conjugate in* Aut $\mathbf{C}^2$ *to one of the flows* (20). *If $\beta = 1$, the flows satisfying $\phi_1 = E$ are given by* (14) *or* (15) *in Theorem* 4.1 *above.*

G. Buzzard [9] proved by a different method that maps $E \in \mathcal{E}$ of type (d) do not belong to any flow unless $\beta = 1$.

Proof Suppose first that E is of type (d) with $\beta \neq 1$:

$$E(x,y) = (\beta^d(x + y^d q(y^r)), \beta y), \qquad \beta^r = 1, \quad d \geq 0,$$

where q is a nonconstant polynomial with $q(0) = 1$. Then E^r equals

$$E^r(x,y) = (x + ry^d q(y^r), y).$$

Since q has a zero $c \neq 0$, $q(y^r)$ has zeros at all roots of $y^r = c$. Since $r > 1$, part (a) in Theorem 4.1 shows that the only flow satisfying $\phi_r = E^r$ is (19). This flow satisfies $\phi_1 = E$ only if $\beta = 1$.This proves part (a).

Consider now the case when E is of type (c) in Theorem 3:

$$E(x,y) = (\beta^d(x + y^d), \beta y), \qquad d \geq 1.$$

If $\beta^r = 1$ for some $r > 0$, then

$$E^r(x,y) = (x + ry^d, y).$$

We choose r so that β is a primitive rth root of one. In order to use Theorem 4.1 we rescale the time and write $\phi_t = \psi_{rt}$. Then the flow ψ_t satisfies $\psi_1 = E^r$ and hence is of the form (18) with $e^\lambda = 1$. Going back to ϕ_t we must replace λ by λ/r; hence ϕ_t is of the form (18) with $e^\lambda = \beta$ and $c = 1$. The condition $\phi_1 = E$ holds if and only if $y_0 = 0$ and $b_k c_k(1) = 0$ for all $k \neq d$. Since $e^\lambda = \beta$ is a primitive rth root of 1, we have $c_k(1) = 0$ if and only if $k - d$ is a multiple of r. This shows that the only nonzero coefficients b_k in (18) are those for which $k - d = mr$ for some integer m (and of course $k \geq 0$). Hence

$$\phi_t(x,y) = \left(e^{d\lambda t} \left(x + ty^d + \sum_{k=d+mr \neq d} b_k y^k c_k(t) \right), e^{\lambda t} y \right), \qquad e^\lambda = \beta. \quad (21)$$

As in the proof of Theorem 4.1 we see that the flow (21) is conjugate in Aut $\mathbf{C}^2$ to the flow (20).

It remains to consider the case when E is of type (c) and β is not a root of one. We will first prove that

$$G(y) = \lambda y, \qquad e^\lambda = \beta.$$

The commutation relations (6) are

$$\begin{aligned} \beta^d F(x,y) + \beta^d d y^{d-1} G(x,y) &= F(\beta^d(x + y^d), \beta y), \\ \beta G(x,y) &= G(\beta^d(x + y^d), \beta y). \end{aligned} \quad (22)$$

Iterating the second equation we have

$$\beta^k G(x,y) = G(\beta^{kd}(x+ky^d),\beta^k y). \tag{23}$$

Replacing $E = \phi_1$ by $E^{-1} = \phi_{-1}$ if necessary we may assume that $|\beta| \leq 1$. Inserting $x = y = 0$ in (23) gives $G(0,0) = 0$.

CASE 1 $|\beta| < 1$ Write

$$G(x,y) = \sum_{r+s\geq 1} G_{r,s}x^r y^s.$$

Inserting this expansion into (23) and dividing by β^k we get

$$G(x,y) = \beta^{-k} \sum_{r+s\geq 1} G_{r,s}\beta^{kdr}(x+ky^d)^r \beta^{ks} y^s$$

$$= G_{0,1}y + G_{1,0}\beta^{k(d-1)}(x+ky^d) + \sum_{r+s\geq 2} G_{r,s}\beta^{k(dr+s-1)}(x+ky^d)^r y^s.$$

In the last sum we have $dr + s - 1 \geq 1$, and therefore the sum converges to 0 for each fixed (x,y) as $k \to \infty$. If $d > 1$, the same is true for the second term on the right. If $d = 1$, the second term on the right is not bounded as $k \to \infty$ unless $G_{1,0} = 0$. Hence we must have $G_{1,0} = 0$, and in the limit as $k \to \infty$ we get

$$G(x,y) = \lambda y, \qquad \lambda = G_{0,1}.$$

CASE 2 $|\beta| = 1$ Fix an $\epsilon \in \mathbf{C}$ and pick a sequence $y_k \in \mathbf{C}$ such that $ky_k^d = \epsilon$. Then $y_k \to 0$ as $k \to \infty$. Choose a subsequence $k_j \in \mathbf{Z}_+$ such that $\beta^{k_j} \to 1$ as $k_j \to \infty$. By expanding $G(x,y) = \sum_{r\geq 0} G_r(x)y^r$ and inserting into (23) we get

$$\sum_{r\geq 1} G_r(x)y^r = \sum_{r\geq 1} G_r(\beta^{kd}(x+ky^d))\beta^{k(r-1)}y^r. \tag{24}$$

Passing to the limit along the subsequence y_{k_j} we get

$$G_0(x) = G_0(x+\epsilon).$$

Since this holds for every ϵ, we conclude that G_0 is a constant, and $G(0,0) = 0$ gives $G_0 = 0$. Thus the terms with $r = 0$ cancel out of the equation (24). We divide the new equation by y and repeat the same argument to get that G_1 is constant. Again we cancel out the terms with $r = 1$, divide by y and repeat the same argument to conclude that G_2 is constant. Comparing the terms with $r = 2$ in (24) we have $G_2 = G_2\beta^k$ and hence $G_2 = 0$. Continuing inductively we show that $G_r = 0$ for all $r \geq 2$, and hence $G(x,y) = \lambda y$ ($\lambda = G_1$).

We have shown that $G(x,y) = \lambda y$. It follows that F is linear in $x : F(x,y) = a(y)x + b(y)$ (Section 2). Inserting this into the commutation relation (22) for F we get

$$\beta^d(a(y)x + b(y)) + \beta^d dy^{d-1}\lambda y = a(\beta y)(\beta^d(x+y^d)) + b(\beta y).$$

Equating the coefficients of x gives $a(y) = a(\beta y)$. Since β is not a root of one, it follows that $a(y) = a$ is a constant. We then have

$$\beta^d b(y) + \beta^d \lambda d y^d = a\beta^d y^d + b(\beta y),$$
$$\beta^d b(y) - b(\beta y) = \beta^d y^d (a - d\lambda).$$

Expanding $b(y) = \sum_{j\geq 0} b_j y^j$ we get

$$\sum_{j\geq 0} b_j y^j (\beta^d - \beta^j) = \beta^d y^d (a - d\lambda).$$

Since β is not a root of 1, we conclude that $b_j = 0$ unless $j = d$, and therefore

$$a = d\lambda, \qquad b(y) = by^d \quad (b = b_d).$$

Thus the infinitesimal generator is

$$V(x,y) = (d\lambda x + by^d, \lambda y),$$

and the flow ϕ_t is equal to (20) (with $b = 1$). This completes the proof of Theorem 4.2. ■

Combining Theorem 3.1 and Theorem 4.2 we obtain

COROLLARY 4.3 *Every flow $\phi_t \in \mathrm{Aut}\,\mathbf{C}^2$ whose time one map $\phi_1 = E$ is linear is conjugate in $\mathrm{Aut}\,\mathbf{C}^2$ to a linear flow.*

5. FLOWS WHOSE TIME ONE MAP IS AFFINE

THEOREM 5.1 *Let $\phi_t \in \mathrm{Aut}\,\mathbf{C}^2$ be a flow whose infinitesimal generator $V = (F,G)$ is polynomial and whose time one map $\phi_1 = E$ is of type* (b) *in Theorem* 3:

$$E(x,y) = (x+1, \beta y).$$

(a) *If β is not a root of* 1, *then*

$$\phi_t(x,y) = (x+t, e^{\lambda t} y), \qquad e^\lambda = \beta. \tag{25}$$

(b) *If β is a primitive rth root of* 1, *then ϕ_t is of the form*

$$\phi_t(x,y) = \left(x + t + \sum_{n\geq 1} b_n (y-y_0)^{nr} c_{nr}(t), e^{\lambda t}(y - y_0) + y_0 \right), \tag{26}$$

where $e^\lambda = \beta$, $y_0 = 0$ unless $\beta = 1$, $b_n \in \mathbf{C}$, and $c_k(t) = (e^{k\lambda t} - 1)/k\lambda$. This flow is conjugate in $\mathrm{Aut}\,\mathbf{C}^2$ to the flow (25).

Proof The commutation relation (6) for $V = (F,G)$ is

$$F(x,y) = F(x+1, \beta y),$$
$$\beta G(x,y) = G(x+1, \beta y).$$

CASE 1 $\beta = 1$ We have

$$F(x,y) = F(x+1,y), \qquad G(x,y) = G(x+1,y).$$

Since F and G are assumed to be polynomials, it follows that they only depend on y. Thus $G(y) = \lambda(y - y_0)$ and $F(y) = \sum_k b_k(y - y_0)^k$. Integrating the system (5) and comparing the time one map with E gives a flow of the form (26) with $r = 1$ and $e^\lambda = 1$. This flow is conjugate to (25) by the automorphism

$$h(x,y) = \left(x - \sum_{k\geq 1} b_k(y-y_0)^k/k\lambda, y - y_0 \right).$$

CASE 2 $\beta^r = 1$ *for some* $r > 1$ Choose r so that β is a primitive rth root of 1. Then $E^r(x,y) = (x + r, y)$. By the previous case the flow ϕ_t is of the form (26), except that we now have $e^\lambda = \beta$ and $y_0 = 0$. This is completely analogous to the reduction in the proof of Theorem 4.2.

CASE 3 β *is not a root of* 1 By inserting the expansions

$$F(x,y) = \sum_{j\geq 0} F_j(x)y^j, \qquad G(x,y) = \sum_{j\geq 0} G_j(x)y^j$$

into the commutation relation we get

$$\begin{aligned} F_j(x) &= \beta^j F_j(x+1) = b^{kj} F_j(x+k), \\ G_j(x) &= \beta^{j-1} G_j(x+1) = \beta^{k(j-1)} G_j(x+k). \end{aligned} \tag{27}$$

Recall that F_j and G_j are assumed to be polynomials.

Suppose first that $|\beta| < 1$. We get that F_0 is periodic and thus constant. Letting $k \to \infty$ we also get $F_j(x) = 0$ for $j \geq 1$. In the same way we see that G_1 is constant and $G_j(x) = 0$ for $j \neq 1$. Thus $V(x,y) = (\mu, \lambda y)$, and the flow is of the form (25). (We have used the condition $\phi_1 = E$ to get $\mu = 1$.)

The case $|\beta| > 1$ is handled in the same way by letting $k \to -\infty$. In the remaining case when $|\beta| = 1$ we see from (27) that each F_j and G_j is bounded along the sequence $\{x + k : k \in \mathbf{Z}_+\}$ and therefore constant. Thus $F = F(y)$ and $G = G(y)$. Using again the commutation relations we have

$$F(y) = F(\beta y), \qquad \beta G(y) = G(\beta y).$$

Expanding F and G in power series and using the hypothesis that β is not a root of 1 shows that F is constant and $G(y) = \lambda y$ ($e^\lambda = \beta$); hence the flow equals (25) as before. This completes the proof of Theorem 5.1. ■

EXAMPLE *The following is a counterexample to the proof of Theorem* 2.1 *in* [10]. *We integrate the system of equations*

$$\dot{x} = \lambda x, \qquad \dot{y} = \lambda y + x + bx^2$$

for some $\lambda = 2\pi i n$, $n \in \mathbf{Z}\backslash\{0\}$. *The solution is*

$$\phi_t(x,y) = (e^{\lambda t}x, e^{\lambda t}(y + tx + bx^2(e^{\lambda t} - 1)/\lambda)).$$

Then $\phi_1(x,y) = (x, y + x)$, *so* $\phi_1 \in \mathcal{A}\backslash\mathcal{E}$. *However,* $\phi_t \notin \mathcal{A} \cup \mathcal{E}$ *unless* $e^{\lambda t} = 1$. *In particular, if* $\lambda = 2\pi i$, *the map* ϕ_t *does not belong to* $\mathcal{A} \cup \mathcal{E}$ *for any* $0 < t < 1$, *and hence its representation as a reduced word has length at least three. Of course* ϕ_t *is conjugate to a flow in* $\mathcal{E}$ *by a quadratic automorphism of* $\mathbf{C}^2$, *but the point is that we have to choose the conjugating map correctly.*

Replacing the term bx^2 *in the equation for* $\dot{y}$ *above by a series* $\sum_{k\geq 2} b_k x^k$ *one gets flows of any degree, and also non-polynomial flows, whose time one map is the non-elementary affine map* $(x,y) \mapsto (x, x + y)$.

6. FLOWS WHOSE TIME ONE MAP IS A GENERALIZED SHEAR

THEOREM 6.1 *Let*

$$f(x,y) = (e^{g(y)}x + h(y), y), \tag{28}$$

where g *and* h *are entire functions on* $\mathbf{C}$ *and* g *is nonconstant. There exists a flow* $\phi_t \in \operatorname{Aut}\mathbf{C}^2$ *such that* $\phi_1 = f$ *if and only if*

$$h(y) = \frac{e^{g(y)} - 1}{g(y) + 2\pi i k} b(y) \tag{29}$$

for some entire function b *and some* $k \in \mathbf{Z}$. *If we set* $a = g + 2\pi i k$, *the flow equals*

$$\begin{aligned}\phi_t(x,y) &= (e^{ta(y)}x + (e^{ta(y)} - 1)b(y)/a(y), y)\\ &= (e^{ta(y)}(x + b(y)/a(y)) - b(y)/a(y), y).\end{aligned} \tag{30}$$

Note that the function $(e^g - 1)/(g + 2\pi i k)$ is entire and its zero set is

$$\{y \in \mathbf{C} : g(y) = 2\pi i n,\ n \in \mathbf{Z},\ n \neq -k\}.$$

If f lies in a flow, then h must vanish on this set for some $k \in \mathbf{Z}$. This shows that *most generalized shears* (28) *do not belong to any flow in* $\operatorname{Aut}\mathbf{C}^2$.

Proof of Theorem 6.1 The kth iterate of f equals

$$f^k(x,y) = (e^{kg(y)}x + h(y)(e^{kg(y)} - 1)/(e^{g(y)} - 1), y).$$

Since g is nonconstant, the open set

$$\Omega = \{(x,y) \in \mathbf{C}^2 : \Re g(y) < 0\}$$

is nonempty, and we have

$$\lim_{k\to\infty} f^k(x,y) = (h(y)/(1 - e^{g(y)}), y), \qquad (x,y) \in \Omega. \tag{31}$$

Consider the commutation relations (6) for the infinitesimal generator $V = (F, G)$ of ϕ_t:

$$\begin{aligned} e^g F + (e^g g' x + h')G &= F \circ f, \\ G &= G \circ f. \end{aligned} \tag{32}$$

We iterate the second equation, $G = G \circ f^k$, and let $k \to \infty$. It follows from (31) that G is independent of x on Ω and therefore on $\mathbf{C}^2$. Thus

$$G(x,y) = \lambda(y - y_0), \qquad F(x,y) = a(y)x + b(y).$$

(See Section 2.) We insert this into the first equation in (32); for simplicity we delete the variables whenever possible:

$$e^g(ax + b) + (e^g g' x + h')\lambda(y - y_0) = a(e^g x + h) + b.$$

This is a linear equation in x. Comparing the coefficients of the linear terms we get $g'\lambda(y - y_0) = 0$ and hence $\lambda = 0$ (since g is assumed to be nonconstant). Therefore $G = 0$ and the second component of the flow is constant: $Y(t, y) = y$. The first component of the flow is

$$X(t,x,y) = e^{ta(y)}x + (e^{ta(y)} - 1)b(y)/a(y).$$

Comparing with f at time $t = 1$ we get

$$e^g = e^a, \qquad (e^a - 1)b/a = h.$$

Thus $a = g + 2\pi i k$ for some $k \in \mathbf{Z}$, h is given by (29), and the flow ϕ_t is given by (30). This proves Theorem 6.1. ■

7. FLOWS IN THE SHEAR GROUPS

We denote by $\mathcal{E}$ the set of all maps $E : \mathbf{C}^2 \to \mathbf{C}^2$ of the form

$$E(x,y) = (e^{g(y)}x + h(y), \beta y + \gamma), \tag{33}$$

where g and h are entire functions, $\beta \in \mathbf{C}^*$, and $\gamma \in \mathbf{C}$. In analogy to the polynomial case we will call such maps elementary. It is easily verified that $\mathcal{E}$ is a subgroup of $\operatorname{Aut}\mathbf{C}^2$.

We denote by $\mathcal{E}_1$ the set of maps (33) with Jacobian one; clearly this requires that g is constant, so $\alpha = e^g \in \mathbf{C}^*$, and $\alpha\beta = 1$. We denote by $\mathcal{E}_c$ the set of maps (33) with constant nonvanishing Jacobian; this requires that g is constant.

Recall that $\mathcal{A}$ is the complex affine group on $\mathbf{C}^2$. Let $\mathcal{S}(2)$ be the subgroup of $\operatorname{Aut}\mathbf{C}^2$ generated by $\mathcal{E}$ and $\mathcal{A}$. Similarly we let $\mathcal{S}_1(2)$ be the group generated by $\mathcal{E}_1$ and $\mathcal{A}_1$, where $\mathcal{A}_1$ contains all affine maps with Jacobian one. Finally let $\mathcal{S}_c(2) = \mathcal{S}_1(2) \times \mathbf{C}^*$ be the group generated by $\mathcal{E}_c$ and $\mathcal{A}$. Clearly $\mathcal{S}_c(2)$ consists of maps in $\mathcal{S}(2)$ with constant Jacobian.

THEOREM 7.1 *Every one parameter subgroup $\{\phi_t : t \in \mathbf{R}\} \subset \mathcal{S}(2)$ is conjugate in $\mathcal{S}(2)$ to one of the following:*

(i) $\phi_t(x,y) = (e^{\mu t}x, e^{\lambda t}y)$, $\lambda, \mu \in \mathbf{C}$;
(ii) $\phi_t(x,y) = (x + tf(y), y)$, *where* f *is an entire function on* $\mathbf{C}$;
(iii) $\phi_t(x,y) = (e^{a(y)t}(x - b(y)) + b(y), y)$, *where* a *is a nonconstant entire function and* b *is a meromorphic function such that the product* ab *is entire*;
(iv) $\phi_t(x,y) = (e^{n\lambda t}(x + ty^n), e^{\lambda t}y)$, $\lambda \in \mathbf{C}^*$, $n \in \mathbf{Z}_+$.

Every subgroup of $\mathcal{S}_1(2)$ *is conjugate in* $\mathcal{S}_1(2)$ *either to a linear group* (i) *with* $\lambda + \mu = 0$ *or to a group* (ii). *Every subgroup of* $\mathcal{S}_c(2)$ *is conjugate in* $\mathcal{S}_c(2)$ *to one of the groups* (i), (ii), *or* (iv).

Observe that the groups (i) and (iv) are polynomial, and (ii) is polynomial when f is a polynomial. Comparing this with the classification of one parameter polynomial groups on $\mathbf{C}^2$ in [6] or in [22] we see that the only new type in $\mathcal{S}(2)$ is (iii). A group of type (iii) is not polynomial unless g is constant and b is polynomial; in this case (iii) is conjugate to a linear group (i) with $g = \mu$ and $\lambda = 0$.

Proof Ahern and Rudin proved in [3] that the group $\mathcal{S}(2)$ is a free product of the subgroups $\mathcal{E}$ and $\mathcal{A}$, amalgamated over their intersection $\mathcal{B} = \mathcal{A} \cap \mathcal{E}$. Earlier C. de Fabritiis proved [10, 11] that the groups $\mathcal{S}_1(2)$ resp. $\mathcal{S}_c(2)$ are amalgamated free products of their subgroups $\mathcal{E}_1$ and $\mathcal{A}_1$ resp. $\mathcal{E}_c$ and $\mathcal{A}$. ∎

We will need the following result of combinatorial group theory.

THEOREM 7.2 *Let* $\mathcal{G}$ *be a topological group which is a free product* $\mathcal{G} = \mathcal{A} * \mathcal{E}$ *of subgroups* $\mathcal{A}$ *and* $\mathcal{E}$, *amalgamated over their intersection* $\mathcal{B} = \mathcal{A} \cap \mathcal{E}$. *Assume that* $\mathcal{B}$ *is closed in* $\mathcal{G}$. *Then any topological subgroup of* $\mathcal{G}$ *which is isomorphic to* $\mathbf{R}$ *or* $\mathbf{C}$ *is conjugate in* $\mathcal{G}$ *to a subgroup of* $\mathcal{A}$ *or to a subgroup of* $\mathcal{E}$.

COROLLARY 7.3

(i) *Every real one parameter subgroup of* $\mathcal{S}(2)$ *is conjugate in* $\mathcal{S}(2)$ *to a subgroup of* $\mathcal{E}$.
(ii) *Every one parameter subgroup in* $\mathcal{S}_1(2)$ *is conjugate to a subgroup of* $\mathcal{E}_1$.
(iii) *Every one parameter subgroup of* $\mathcal{S}_c(2)$ *is conjugate to a subgroup of* $\mathcal{E}_c$.

It is immediate that every subgroup in $\mathcal{A}$ is linearly conjugate to a subgroup in $\mathcal{E}$ (by conjugating the relevant matrix to its Jordan form).

Proof of Theorem 7.2 The proof is the same in the real and the complex case, so we will consider a real subgroup $\{\phi_t : t \in \mathbf{R}\} \subset \mathcal{G}$. First we show that for every $t \in \mathbf{R}$ the element ϕ_t is conjugate in $\mathcal{G}$ to an element in $\mathcal{A}$ or in $\mathcal{E}$. Let g be the element in the conjugacy class of ϕ_t whose reduced word with respect to the amalgamated product structure in $\mathcal{G}$ has the shortest length. If g does not belong to either $\mathcal{A}$ or $\mathcal{E}$, then g must have even length $2r \geq 2$, since otherwise the two end elements in the word representing g belong to the same group $\mathcal{A}, \mathcal{E}$, and hence g could be shortened by conjugating it with one of the end elements. (See [15] for the details.) If $g = h^m$ for some $h \in \mathcal{G}$ and $m \in \mathbf{N}$, it follows that h also has even length $2k$ and $mk = r$. In particular, g cannot have roots of order larger than r with respect to composition, and hence it cannot lie in a flow, a contradiction.

Next we want to show that we can choose an $h \in \mathcal{G}$ such that $h^{-1} \circ \phi_t \circ h$ belongs to the same group $\mathcal{A}$ or $\mathcal{E}$ for all $t \in \mathbf{R}$. C. de Fabritiis [11] observed that one can apply the following theorem of Moldavanski ([18] or [25, Theorem 0.3]):

THEOREM *If $\mathcal{H}$ is an abelian subgroup of an amalgamated free product group $\mathcal{G} = \mathcal{A} * \mathcal{E}$, then precisely one of the following holds:*

(i) *$\mathcal{H}$ is conjugate in $\mathcal{G}$ to a subgroup of $\mathcal{A}$ or $\mathcal{E}$;*
(ii) *$\mathcal{H}$ is not conjugate to any subgroup of $\mathcal{A}$ or $\mathcal{E}$, but $\mathcal{H} = \bigcup_{j=1}^{\infty} \mathcal{H}_j$, where $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \cdots$ is a nested chain of subgroups such that each $\mathcal{H}_j$ is conjugate in $\mathcal{G}$ to a subgroup of $\mathcal{B} = \mathcal{A} \cap \mathcal{E}$;*
(iii) *$\mathcal{H} = F \times (g)$ is the product of a subgroup F conjugate to a subgroup of $\mathcal{B}$ and a subgroup (g) generated by an element $g \in \mathcal{G}$ which is not conjugate to any element of $\mathcal{A}$ or $\mathcal{E}$.*

We continue with the proof of Theorem 7.2 as in [11]. Let $\mathcal{H} = \{\phi_t : t \in \mathbf{R}\} \subset \mathcal{G}$. We want to show that $\mathcal{H}$ is of type (i) in the theorem above. Clearly type (iii) is impossible since every ϕ_t is conjugate to an element of $\mathcal{A}$ or $\mathcal{E}$. Suppose now that $\mathcal{H}$ is of type (ii). Let $C_j = \{t \in \mathbf{R} : \phi_t \in \mathcal{H}_j\}$, so $\mathbf{R} = \bigcup_{j=1}^{\infty} C_j$. Let $h_j \in \mathcal{G}$ be an element which conjugates the subgroup $\mathcal{H}_j$ into $\mathcal{B} : h_j^{-1} \circ \phi_t \circ h_j \in \mathcal{B}$ for every $t \in C_j$. Since $\mathcal{B}$ is closed in $\mathcal{G}$ and the group operations are continuous, the same is true for every $t \in \overline{C}_j$. Since $\mathbf{R} = \bigcup_j \overline{C}_j$, the Baire category theorem implies that one of the sets $\overline{C}_j$ has a nonempty interior. But then the group property implies that $\overline{C}_j = \mathbf{R}$ and therefore the element $h = h_j$ conjugates $\mathcal{H}$ into $\mathcal{B}$ in contradiction to our assumption. Thus $\mathcal{H}$ is of type (i) as claimed. This completes the proof of Theorem 10.2. ■

Remark The proof of Theorem 7.2 clearly extends to the case when $\mathcal{H}$ is a connected abelian Lie group, for instance, $\mathcal{H} = T^k \times \mathbf{R}^m$, where T^k is a torus.

To complete the proof of Theorem 7.1 it now suffices to classify the one parameter subgroups in the groups $\mathcal{E}$ (33) and its subgroups $\mathcal{E}_1, \mathcal{E}_c$. Since the second component of (33) every elementary map (33) is a linear automorphism of the y-axis which does not depend on x, it follows as in Section 2 that the infinitesimal generator $V = (V_1, V_2)$ has the form

$$V_1(x,y) = a(y)x + b(y), \qquad V_2(y) = \lambda y + \gamma. \tag{34}$$

We can simplify the field V further by conjugating it with automorphisms Φ of $\mathbf{C}^2$. The conjugate field $\tilde{V}$ is determined by the relation $D\Phi \cdot V = \tilde{V} \circ \Phi$.

Conjugating V by linear automorphisms $\Phi(x,y) = (x, \alpha y + \beta)$ we get a field as above in which the second component is either $V_2(y) = \lambda y$ $(\lambda \in \mathbf{C})$ or $V_2(y) = 1$. We consider several cases.

CASE 1 $V_2(y) = \lambda y$, $\lambda \neq 0$ We can change the function a to a constant by conjugating V by a generalized shear $\Phi(x,y) = (e^{g(y)}x, y)$, where g is an entire function to be determined later. The conjugation equation is

$$D\Phi \cdot V(x,y) = \begin{pmatrix} e^g & g'e^g x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} ax+b \\ \lambda y \end{pmatrix}$$
$$= \begin{pmatrix} e^g(ax+b) + \lambda xyg'e^g \\ \lambda y \end{pmatrix}$$
$$= \begin{pmatrix} (a+\lambda yg')(e^g x) + be^g \\ \lambda y \end{pmatrix}.$$

This shows that

$$\tilde{V}_1(x,y) = (a(y) + \lambda yg'(y))x + b(y)e^{g(y)}, \qquad \tilde{V}_2(x,y) = \lambda y.$$

Let $a(y) = \sum_{k\geq 0} a_k y^k$ and $g(y) = \sum_{k\geq 0} g_k y^k$. Then $a + \lambda yg' = \sum (a_k + k\lambda g_k)y^k$. By choosing $g_k = -a_k/\lambda k$ for $k \geq 1$ we get $a + \lambda yg' = a_0$. Denote this constant by μ. Thus the conjugated field (which we again denote by V) is of the form

$$V_1(x,y) = \mu x + b(y), \qquad V_2(x,y) = \lambda y.$$

To simplify b we conjugate by a shear $\Phi(x,y) = (x + q(y), y)$:

$$D\Phi \cdot V(x,y) = \begin{pmatrix} 1 & q'(y) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu x + b \\ \lambda y \end{pmatrix} = \begin{pmatrix} \mu x + b + \lambda yq' \\ \lambda y \end{pmatrix}.$$

The new first component is $\tilde{V}_1(x,y) = \mu + \tilde{b}(y)$, where

$$\tilde{b}(y) = b(y) + \lambda yq'(y) - \mu q(y) = \sum_{k\geq 0} (b_k + (k\lambda - \mu)q_k)y^k.$$

CASE 1.1 $k\lambda \neq \mu$ for all $k \in \mathbf{Z}_+$ Set $q_k = b_k/(\mu - k\lambda)$. Clearly $q(y) = \sum q_k y^k$ converges on the entire plane $\mathbf{C}$. The corresponding automorphism Φ conjugates V to the linear field $\tilde{V}(x,y) = (\lambda x, \mu y)$ whose flow is type (i) in Theorem 7.1.

CASE 1.2 $n\lambda = \mu$ for some (unique) integer $n \in \mathbf{Z}_+$ The only term in $\tilde{b}$ which we can not eliminate as above is $b_n y^n$. If $b_n = 0$, we have a linear flow (i). If $b_n \neq 0$, we conjugate $\tilde{V}$ further by $(x,y) \mapsto (x/b_n, y)$ and get the field

$$W_1(x,y) = \mu x + y^n, \qquad W_2(x,y) = \lambda y$$

whose flow is given by (iv) in Theorem 7.1.

CASE 2 $V_2 = 0$ This case was treated in Section 6 above. If a is not identically zero, the flow is given by (30). This flow is of type (iii) in Theorem 7.1 if a is nonconstant. If $a = \mu$ is constant, the function b/μ in (30) is entire and hence the flow is conjugate (by a shear) to the linear flow (i) in Theorem 7.1. If $a = 0$, we have a shear flow (ii).

CASE 3 $V_2 = 1$ As in Case 1 we conjugate V by a generalized shear $\Phi(x,y) = (xe^{g(y)}, y)$:

$$\begin{pmatrix} e^g & xe^g g' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} ax+b \\ 1 \end{pmatrix} = \begin{pmatrix} (a+g')e^g x + be^g \\ 1 \end{pmatrix}.$$

If we choose g such that $a(y) + g'(y) = 0$, then the conjugated field is of the form $\tilde{V}(x,y) = (\tilde{b}(y), 1)$. Conjugating this field by a shear $\Phi(x,y) = (x + q(y), y)$ such that $q'(y) = -\tilde{b}(y)$ we get the field $(0,1)$ whose flow is $\phi_t(x,y) = (x, y+t)$. The map $(x,y) \mapsto (y,x)$ conjugates it into the flow $\psi_t(x,y) = (x+t, y)$ of type (ii). This completes the classification of flows in $\mathcal{S}(2)$.

Next we consider one parameter subgroups in $\mathcal{S}_c(2)$. By Corollary 7.3 we may assume that the group is contained in $\mathcal{E}_c$, and hence its infinitesimal generator is of the form (34) with $a = \mu =$ constant. After the initial conjugation as above we may assume that $V_2 = \lambda y$ or $V_2 = 1$. In the first case we can complete the classification as above by conjugating with shears and dilations. In the second case when $V_2 = 1$ we cannot eliminate μ as in Case 3 above since that would require conjugation by a generalized shear. Instead we conjugate by a shear $\Phi(x,y) = (x + q(y), y)$:

$$D\Phi \cdot V(x,y) = \begin{pmatrix} 1 & q' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu x + b \\ 1 \end{pmatrix} = \begin{pmatrix} \mu x + b + q' \\ 1 \end{pmatrix}.$$

The conjugate field is $\tilde{V}(x,y) = (\mu x + \tilde{b}, 1)$ where

$$\tilde{b}(y) = b(y) + q'(y) - \mu q(y).$$

If we let C be the entire function satisfying $C'(y) = -b(y)e^{-\mu y}$ and $q(y) = C(y)e^{\mu y}$, we get $\tilde{b} = 0$, and the new field is $(\mu x, 1)$. Its flow is $\phi_t(x,y) = (e^{\mu t}x, y+t)$. The map $(x,y) \mapsto (y,x)$ conjugates it either to (iv) (with $n = 0$ and $\mu = \lambda$) or to (ii) (with $f = 1$).

The analysis in the remaining case $\mathcal{S}_1(2)$ is similar to $\mathcal{S}_c(2)$. After the initial conjugation the infinitesimal generator V is of the form (34), with $a = \mu =$ constant and $\lambda + \mu = 0$. If $\lambda, \mu \neq 0$, we conjugate V by a shear $\Phi(x,y) = (x + q(y), y)$ as in Case 1 above to eliminate b; hence the flow is conjugate to (i). If $\lambda = \mu = 0$, we get the shear flow (ii) with $f = b$. This completes the proof of Theorem 7.1. ■

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