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EQUIVALENCE OF REAL SUBMANIFOLDS UNDER VOLUME-PRESERVING HOLOMORPHIC AUTOMORPHISMS OF $\mathbf{C}^n$

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1. Introduction. The main theme of this paper is the global equivalence of certain types of real submanifolds in the complex euclidean space $\mathbf{C}^n$ $(n > 1)$ under the group of all volume-preserving holomorphic automorphisms of $\mathbf{C}^n$. Let Ω be the complex volume form on $\mathbf{C}^n$:

$$\Omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n. \tag{1}$$

A holomorphic mapping $F\colon D \subset \mathbf{C}^n \to \mathbf{C}^n$ is said to be *volume-preserving* if $F^*\Omega = \Omega$. Since $(F^*\Omega)(z) = JF(z)\cdot\Omega$, where JF is the complex Jacobian of F, this is equivalent to $JF(z) = 1$, $z \in D$. We denote by $\mathrm{Aut}\,\mathbf{C}^n$ the group of all holomorphic automorphisms of $\mathbf{C}^n$ and by $\mathrm{Aut}_1\mathbf{C}^n \subset \mathrm{Aut}\,\mathbf{C}^n$ the group of all volume-preserving automorphisms of $\mathbf{C}^n$.

Definition 1. Let $\mathscr{G}$ be any group of holomorphic automorphisms of $\mathbf{C}^n$.

(a) Two compact subsets $M_0, M_1 \subset \mathbf{C}^n$ are $\mathscr{G}$-equivalent if there exist a neighborhood U of M_0 in $\mathbf{C}^n$ and a biholomorphic mapping $F\colon U \to F(U) \subset \mathbf{C}^n$ such that $F(M_0) = M_1$, and F is the uniform limit in U of a sequence $F_j \in \mathscr{G}$.

(b) Let M be a compact topological space. Continuous maps $f_0, f_1\colon M \to \mathbf{C}^n$ are $\mathscr{G}$-equivalent if there exist a neighborhood U of $f_0(M)$ in $\mathbf{C}^n$ and a biholomorphic mapping $F\colon U \to F(U) \subset \mathbf{C}^n$ such that $F \circ f_0 = f_1$, and F is the uniform limit in U of a sequence $F_j \in \mathscr{G}$.

Several observations and remarks are in order. For $\mathscr{G} = \mathrm{Aut}\,\mathbf{C}^n$, our Definition 1 agrees with the definition of $\mathbf{C}^n$-equivalence as introduced in [8] (Definition 2). The same definition was used in [7] for the group $\mathrm{Aut}_{\mathrm{sp}}\mathbf{C}^{2n}$ of symplectic holomorphic automorphisms of $\mathbf{C}^{2n}$. If $\mathscr{G} = \mathrm{Aut}_1\mathbf{C}^n$, it follows that the limit map $F\colon U \to \mathbf{C}^n$ satisfying Definition 1 is itself volume-preserving. Further, the maximum principle shows that a sequence of holomorphic maps which converges on a neighborhood of a set $K \subset \mathbf{C}^n$ also converges on a neighborhood of the polynomially convex hull $\hat{K}$. Therefore $\mathscr{G}$-equivalence of sets $K_0, K_1 \subset \mathbf{C}^n$ implies $\mathscr{G}$-equivalence of their polynomial hulls. Finally, if $\mathscr{G}' \subset \mathscr{G}$ are holomorphic automorphism groups on $\mathbf{C}^n$ such that $\mathscr{G}'$ is dense in $\mathscr{G}$ (in the topology of uniform

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convergence on compacts), then two sets in $\mathbf{C}^n$ (or maps into $\mathbf{C}^n$) are $\mathscr{G}'$-equivalent if and only if they are $\mathscr{G}$-equivalent. For instance, the group $\mathrm{Aut}_1\mathbf{C}^n$ contains a dense subgroup $\mathscr{S}_1^n$, generated by *shears* (E. Andersén [2]). These are automorphisms of the form

$$F(z) = z + f(\Lambda z)v, \qquad z \in \mathbf{C}^n,$$

where $v \in \mathbf{C}^n$, Λ is a complex linear form on $\mathbf{C}^n$ satisfying $\Lambda v = 0$, and f is an entire function of one variable. Polynomial maps of this type also form a group $\mathscr{S}_a^n$ (the algebraic shear group) which is dense in $\mathrm{Aut}_1\mathbf{C}^n$. Thus any result on $\mathrm{Aut}_1\mathbf{C}^n$-equivalence is at the same time a result on $\mathscr{S}_a^n$-equivalence.

Two other examples of such pairs are the group $\mathscr{S}^n$, generated by the *generalized shears*, which is dense in $\mathrm{Aut}\,\mathbf{C}^n$ (Andersén and Lempert [3]), and the group $\mathscr{S}_{\mathrm{sp}}^n$ of *symplectic shears* on $\mathbf{C}^{2n}$ which is dense in the group $\mathrm{Aut}_{\mathrm{sp}}\mathbf{C}^{2n}$ of *symplectic holomorphic automorphisms* of $\mathbf{C}^{2n}$ [6]. An important earlier work on the questions of equivalence and approximation by shears is the paper by Rosay and Rudin [14].

In order to formulate our main result, we choose a holomorphic $(n-1)$-form β on $\mathbf{C}^n$ satisfying $d\beta = \Omega$. To be specific we will take

$$\beta(z) = \frac{1}{n}\sum_{j=1}^{n} (-1)^{j-1} z_j \, dz_1 \wedge \cdots \widehat{dz_j} \cdots \wedge dz_n,$$

where the hat indicates that the corresponding entry is deleted. If M is a smooth, compact, oriented manifold of real dimension $n-1$ and $f\colon M \to \mathbf{C}^n$ is a smooth map, we set

$$\mathscr{B}(f) = \int_M f^*\beta. \tag{2}$$

If M is closed, Stokes' theorem implies that *any two embeddings $f_0, f_1\colon M \to \mathbf{C}^n$ which are* $\mathrm{Aut}_1\mathbf{C}^n$*-equivalent satisfy* $\mathscr{B}(f_0) = \mathscr{B}(f_1)$ (see Section 3). Conversely, we will show that in the class of real-analytic, totally real, polynomially convex embeddings $M \hookrightarrow \mathbf{C}^n$, the integral (2) is the only additional invariant when passing from $\mathrm{Aut}\,\mathbf{C}^n$-equivalence to $\mathrm{Aut}_1\mathbf{C}^n$-equivalence. There are no additional invariants for lower-dimensional manifolds, or for manifolds of dimension $n-1$ with vanishing top cohomology group. The precise result is this.

MAIN THEOREM. *Let M be a compact, connected, real-analytic manifold of real dimension m, and let $f_0, f_1\colon M \to \mathbf{C}^n$ be real-analytic embeddings* $(1 \leqslant m \leqslant n-1)$ *such that the submanifolds $f_j(M) = M_j \subset \mathbf{C}^n$* $(j = 0, 1)$ *are totally real and polynomially convex. Suppose that f_0 and f_1 are* $\mathrm{Aut}\,\mathbf{C}^n$*-equivalent. Then f_0 and f_1 are also* $\mathrm{Aut}_1\mathbf{C}^n$*-equivalent, provided that any one of the following conditions holds:*

(i) $m \leqslant n-2$;
(ii) $m = n-1$ *and* $H^{n-1}(M; \mathbf{R}) = 0$;
(iii) $m = n-1$, *the manifold M is closed and orientable, and* $\mathscr{B}(f_0) = \mathscr{B}(f_1) \neq 0$.

Remark. We believe that the same conclusion holds if, in the context of (iii), we have $\mathscr{B}(f_0) = \mathscr{B}(f_1) = 0$. However, there are considerable technical difficulties in treating this exceptional case (see Section 3), and we do not pursue the matter here. The same remark applies to Corollary 2 below. We also do not treat the more delicate problem of $\mathrm{Aut}_1\mathbf{C}^n$-equivalence of n-dimensional submanifolds in $\mathbf{C}^n$.

Recall from [8] (Theorem 3.1) that *two real-analytic embeddings* $f_0, f_1 \colon M \to \mathbf{C}^n$, *whose images* $f_0(M)$, $f_1(M) \subset \mathbf{C}^n$ *are totally real and polynomially convex, are* $\mathrm{Aut}\,\mathbf{C}^n$*-equivalent if and only if there exists a smooth one-parameter family* (*isotopy*) *of embeddings* $f_t \colon M \to \mathbf{C}^n$ $(0 \leqslant t \leqslant 1)$ *such that the manifold* $f_t(M) = M_t \subset \mathbf{C}^n$ *is totally real and polynomially convex for each t. Such an isotopy always exists when* $\dim_{\mathbf{R}} M \leqslant 2n/3$ ([5], Corollary 1). Together with the main theorem this implies the following.

COROLLARY 1. *Let M be a compact, connected, real-analytic manifold of dimension* $m \geqslant 1$. *If* $n \geqslant \max\{m+2, 3m/2\}$, *then every two real-analytic embeddings* f_0, $f_1 \colon M \to \mathbf{C}^n$ *whose images* $f_0(M)$ *and* $f_1(M)$ *are totally real and polynomially convex are* $\mathrm{Aut}_1\mathbf{C}^n$*-equivalent.*

Recall from [5] (Theorem 4.1) that the image of a generic smooth embedding $M \hookrightarrow \mathbf{C}^n$ is totally real and polynomially convex when $2n \geqslant 3 \dim M$. The condition $n \geqslant m+2$ in Corollary 1 is needed to exclude the cases $m = 1$, $n = 2$ and $m = 2$, $n = 3$ (curves in $\mathbf{C}^2$ and surfaces in $\mathbf{C}^3$) when the integral (2) is an invariant for volume-preserving automorphisms and their limits. The corresponding result in these cases is the following. We denote by T the circle.

COROLLARY 2. (a) *Two real-analytic embeddings* $f_0, f_1 \colon T \to \mathbf{C}^2$, *with* $f_0(T), f_1(T) \subset \mathbf{C}^2$ *polynomially convex, are* $\mathrm{Aut}_1\mathbf{C}^2$*-equivalent if and only if* $\mathscr{B}(f_0) = \mathscr{B}(f_1)$.

(b) *Let M be a closed orientable surface, and let* $f_0, f_1 \colon M \to \mathbf{C}^3$ *be real-analytic embeddings whose images* $f_0(M), f_1(M) \subset \mathbf{C}^3$ *are totally real and polynomially convex. If* $\mathscr{B}(f_0) = \mathscr{B}(f_1) \neq 0$, *then* f_0 *and* f_1 *are* $\mathrm{Aut}_1\mathbf{C}^3$*-equivalent.*

Notice that on $\mathbf{C}^2$ the volume form (1) coincides with the standard complex symplectic form, and hence part (a) of Corollary 2 is a special case of the main theorem in [7].

In relation to Definition 1 of $\mathscr{G}$-equivalence, we emphasize that the stronger requirement $F(M_0) = M_1$ for some $F \in \mathscr{G}$ would not give interesting results for manifolds. For instance, if $\Gamma_0, \Gamma_1 \subset \mathbf{C}^n$ are planar curves contained in a complex line $\Lambda \subset \mathbf{C}^n$, and if $F \in \mathrm{Aut}\,\mathbf{C}^n$ satisfies $F(\Gamma_0) = \Gamma_1$, then $F(\Lambda) = \Lambda$ and therefore the restriction of F to Λ is a linear map. Thus, unless the two curves are linearly equivalent, there is no automorphism of $\mathbf{C}^n$ sending one curve onto the other. For instance, a segment and a circular arc in $\mathbf{C}^n$ cannot be mapped one onto the other by an automorphism.

Of course this is in sharp contrast with the situation for smooth diffeomorphisms. On the other hand, this stronger notion of equivalence is suitable for certain countable subsets of $\mathbf{C}^n$; see Rosay and Rudin [14].

In the present paper, as well as in the earlier papers [8] and [7], we only consider equivalence of real-analytic submanifolds. In order to study the same problem in the class of *smooth* submanifolds in $\mathbf{C}^n$, one must adopt a weaker notion of equivalence to get meaningful results. A possible definition was suggested by Rosay in [13], but the concept is not well understood yet. For another work in this direction, see [5].

Some of the methods for global approximation by automorphisms, used in the present paper as well as in the earlier papers [8], [5], [7], originated in the works of Andersén [2] and Andersén and Lempert [3].

The paper is organized as follows. In Section 2 we recall some standard results on flows of (divergence-zero) holomorphic vector fields, as well as results of [2] and [8] on approximation of their flows by (volume-preserving) holomorphic automorphisms of $\mathbf{C}^n$ (Proposition 2.3 and Corollary 2.4). In Section 3 we prove the main theorem. In the appendix we sketch a proof of a result on extending closed (resp. exact) holomorphic forms from closed submanifolds in Stein manifolds. This result, which is needed in the proof of the main theorem, is most likely not original, but we do not know an explicit reference.

2. Divergence-zero holomorphic vector fields and flows. In this section we recall some standard results on vector fields and their flows which will be needed in the proof of the main theorem. We refer to Abraham and Marsden [1] for the details. In the last part we also recall results of [2] and [8] on approximation of flows of divergence-zero holomorphic vector fields by the volume-preserving automorphisms of $\mathbf{C}^n$ (Proposition 2.3 and Corollary 2.4).

Recall that the (local) flow F_t of a holomorphic vector field X on a complex manifold $\mathcal{M}$ is the solution of the ordinary differential equation

$$\frac{d}{dt}F_t(z) = X(F_t(z)), \qquad F_0(z) = z \in \mathcal{M}.$$

For each compact set $K \subset\subset \mathcal{M}$ there is a number $T > 0$ such that the flow $F_t(z)$ is defined for all $z \in K$ and $|t| < T$. Each map F_t is injective holomorphic where defined. The connection between the Lie derivative L_X and the flow F_t of X is given by [1, page 92]

$$\frac{d}{dt}F_t^*\alpha = F_t^*(L_X\alpha),$$

where α is any tensor field on $\mathcal{M}$. In fact, at time $t = 0$, this is just the definition of the Lie derivative $L_X\alpha$. If the vector field X is holomorphic, then L_X maps holomorphic tensors (vector fields, forms, ...) to holomorphic tensors.

We will have to consider flows of time-dependent vector fields $X_t(z)$ which are holomorphic in z for each fixed t, and of class $\mathscr{C}^1$ in (t, z); see [1, page 92]. Such a field is usually defined on a domain $\tilde{D} \subset \mathbf{R} \times \mathscr{M}$ of the form

$$\tilde{D} = \bigcup_{t \in J} \{t\} \times D_t,$$

where each D_t is a domain in the complex manifold $\mathscr{M}$ and J is an interval in $\mathbf{R}$ containing 0. The flow $F_{t,s}$ of such a field is defined as the solution of the equation

$$\frac{d}{dt} F_{t,s}(z) = X_t(F_{t,s}(z)), \qquad F_{s,s}(z) = z.$$

Again, every time-forward map $F_{t,s}$ is injective holomorphic where defined. The basic connection between the Lie derivative and the time dependent flow is

$$\frac{d}{dt} F_{t,s}^* \alpha = F_{t,s}^*(L_{X_t} \alpha), \tag{3}$$

which holds for any time-independent tensor α on $\mathscr{M}$ [1, page 92].

Suppose now that Ω is a holomorphic volume form on a complex manifold $\mathscr{M}$, i.e., a nonvanishing holomorphic form of top degree $n = \dim \mathscr{M}$. The *divergence* $\operatorname{div}_\Omega X$ of a holomorphic vector field X with respect to Ω is the unique holomorphic function satisfying

$$L_X \Omega = (\operatorname{div}_\Omega X) \cdot \Omega. \tag{4}$$

We will write $\operatorname{div} X$ when it is clear which volume form is meant. When Ω is the standard volume form (1) on $\mathbf{C}^n$ and $X = \sum_{j=1}^n X_j(\partial/\partial z_j)$, we have

$$\operatorname{div} X = \sum_{j=1}^n \frac{\partial X_j}{\partial z_j}. \tag{5}$$

Combining (3) and (4), we get

$$\frac{d}{dt} F_{t,s}^* \Omega = F_{t,s}^*(\operatorname{div} X_t \cdot \Omega).$$

This identity implies the following lemma.

LEMMA 2.1. *Let $\mathscr{M}$ be a complex manifold with a holomorphic volume form Ω. If X_t is a time-dependent holomorphic vector field, defined on an open subset of $\mathbf{R} \times \mathscr{M}$, such that the flow $F_{t,0}(z)$ is defined for $z \in D \subset \mathscr{M}$ and $0 \leqslant t \leqslant T$ for some $T > 0$, then the following are equivalent:*

(a) $\operatorname{div}_\Omega X_t = 0$ *on* $D_t = F_{t,0}(D) \subset \mathscr{M}$ *for each* $t \in [0, T]$;
(b) $F_{t,s}^* \Omega = \Omega$ *on* D_s *for each* $t, s \in [0, T]$.

The following result plays a central role in the proof of the main theorem. This is essentially due to Andersén [2]; see also Lemma 1.4 in [8].

PROPOSITION 2.2. *Let X_t be a divergence-zero holomorphic vector field on $\mathbf{C}^n$ for each $t \in [0, 1]$ such that $(t, z) \mapsto X_t(z)$ is of class $\mathscr{C}^1$. Let D be an open set in $\mathbf{C}^n$ and let $0 < T \leqslant 1$. Assume that the flow $F_t(z) = F_{t,0}(z)$ exists for $0 \leqslant t \leqslant T$ and $z \in D$. Then F_t for $0 \leqslant t \leqslant T$ is a volume-preserving biholomorphic map of D onto $F_t(D)$ which can be approximated, uniformly on compact sets in D, by volume-preserving automorphisms of $\mathbf{C}^n$.*

This was proved in [2] and, in a more explicit form, in [8] (Lemma 1.4). The proof depends on the fact (Andersén [2]) that every polynomial (time-independent) divergence-zero vector field on $\mathbf{C}^n$ is a finite sum of complete fields of the form

$$X(z) = f(\Lambda z)v, \qquad z \in \mathbf{C}^n,$$

where $v \in \mathbf{C}^n$, Λ is a complex linear form on $\mathbf{C}^n$ satisfying $\Lambda v = 0$, and f is a holomorphic polynomial in one variable. (See also the appendix in [5].) The flow of this field is given by

$$G_t(z) = z + tf(\Lambda z)v, \qquad z \in \mathbf{C}^n, t \in \mathbf{C},$$

and it belongs to the shear group $\mathscr{S}_1^n \subset \operatorname{Aut}_1 \mathbf{C}^n$. The proof of Proposition 2.2 for time-dependent fields follows by approximating the field X_t for t on short time intervals $[k/N, (k+1)/N] \subset [0, 1]$ by the time-independent divergence-zero field $X_{k/N}$. We refer the reader to [5] or [8] for the details.

In Proposition 2.2 it is very important that the field X_t is defined globally on $\mathbf{C}^n$ for each t. Alternatively, it suffices to assume that X_t is defined on $D_t = F_t(D)$ and it can be approximated, uniformly on compacts in D_t, by divergence-zero fields defined on $\mathbf{C}^n$.

In order to get the precise result that we need, we recall the connection between vector fields and forms of degree $n - 1$. Let $\mathscr{M}$ be a complex manifold of dimension n with a holomorphic volume form Ω. To each holomorphic vector field X on $\mathscr{M}$, one associates the holomorphic $(n-1)$-form $\alpha = i_X \Omega$, where i_X denotes the contraction by X. Since Ω is nondegenerate, this defines an isomorphism between holomorphic vector fields and holomorphic $(n-1)$-forms on every domain $D \subset \mathscr{M}$. If Ω is the standard volume form (1) on $\mathbf{C}^n$ and if $X = \sum_{j=1}^n X_j(\partial/\partial z_j)$, then

$$i_X \Omega = \sum_{j=1}^n (-1)^{j-1} X_j \, dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n.$$

Recall [1, page 121] that for any differential form α we have

$$L_X\alpha = d(i_X\alpha) + i_X(d\alpha).$$

Applying this to Ω (for which $d\Omega = 0$) and using (4) we get

$$(\operatorname{div} X)\cdot\Omega = L_X\Omega = d(i_X\Omega).$$

Hence $\operatorname{div} X = 0$ if and only if $d(i_X\Omega) = 0$. Of course this can be verified immediately in the case when Ω is the volume form (1) on $\mathbf{C}^n$ and $\operatorname{div} X$ is given by (5).

Recall that a domain $D \subset \mathbf{C}^n$ is Runge in $\mathbf{C}^n$ if every holomorphic function on D is a limit of holomorphic polynomials, uniformly on compacts in D.

PROPOSITION 2.3. *Let X_t be a time-dependent holomorphic vector field on an open subset of $\mathbf{R} \times \mathbf{C}^n$. Let $D \subset \mathbf{C}^n$ be a domain of holomorphy such that the flow $F_t(z) = F_{t,0}(z)$ of X_t is defined for all $z \in D$ and for $0 \leqslant t \leqslant T$ for some $T > 0$. Set $D_t = F_t(D) \subset \mathbf{C}^n$. Assume that for each $t \in [0, T]$ the holomorphic $(n-1)$-form $\alpha_t = i_{X_t}\Omega$ on D_t is exact, and D_t is Runge in $\mathbf{C}^n$. Then for each $t \in [0, T]$ the map F_t is a limit of volume-preserving holomorphic automorphisms of $\mathbf{C}^n$, uniformly on compacts in D.*

Remark. It is reasonable to call a holomorphic vector field X, for which the form $\alpha = i_X\Omega$ is exact holomorphic, an *exact divergence-zero vector field*, and its flow an *exact volume-preserving flow*. This is in complete analogy with the situation in symplectic geometry where one distinguishes between *locally Hamiltonian* versus *exact Hamiltonian* vector fields and flows (see [1] and [7]).

Proof of Proposition 2.3. Recall that on a domain of holomorphy $D \subset \mathbf{C}^n$ (and on any Stein manifold D), the de Rham cohomology group $H^p(D; \mathbf{C})$ for each $p \in \mathbf{Z}_+$ coincides with the group of closed holomorphic p-forms modulo exact holomorphic p-forms [11, page 58]. Since D and therefore $D_t = F_t(D)$ is a domain of holomorphy for $t \in [0, T]$, the assumption implies that $\alpha_t = d\beta_t$ for some holomorphic $(n-2)$-form β_t on D_t. Since D_t is assumed to be Runge in $\mathbf{C}^n$, we can approximate the coefficients of β_t, which are holomorphic functions on D_t, by holomorphic polynomials. Thus we obtain $(n-2)$-forms $\tilde{\beta}_t$ on $\mathbf{C}^n$ which approximate β_t uniformly on compacts in D_t. The exact $(n-1)$-forms $\tilde{\alpha}_t = d\tilde{\beta}_t$ then approximate α_t on compacts in D_t. The holomorphic vector field Y_t on $\mathbf{C}^n$, defined by $\tilde{\alpha}_t = i_{Y_t}\Omega$, has divergence zero, and it approximates X_t uniformly on a chosen compact in D_t. It is possible to choose Y_t such that it is smooth in t, and such that $|Y - X|$ is arbitrarily small on a given compact set $\tilde{K} \subset \bigcup_{0 \leqslant t \leqslant T} \{t\} \times D_t$. (See [5], Lemma 1.2). Since Y_t is globally defined on $\mathbf{C}^n$, the flow of Y, which by construction approximates the flow F_t of X on D, is itself a limit of volume-preserving holomorphic automorphisms of $\mathbf{C}^n$ according to Proposition 2.2. This proves Proposition 2.3. ∎

COROLLARY 2.4. *Let $D \subset \mathbf{C}^n$ be a domain of holomorphy satisfying $H^{n-1}(D; \mathbf{C}) = 0$, and let $F_t\colon D \to D_t \subset \mathbf{C}^n$ $(0 \leqslant t \leqslant 1)$ be the flow of a time-dependent holomor-*

phic vector field X such that F_0 is the identity on D. Assume that for each $t \in [0, 1]$, div $X_t = 0$ *on D_t, and the domain D_t is Runge in $\mathbf{C}^n$. Then for each $t \in [0, 1]$ the map $F_t: D \to D_t$ can be approximated by volume-preserving holomorphic automorphisms of $\mathbf{C}^n$, uniformly on compacts in D.*

Proof. The condition div $X_t = 0$ implies that the holomorphic $(n-1)$-form $\alpha_t = i_{X_t}\Omega$ is closed. Since $H^{n-1}(D_t; \mathbf{C}) = H^{n-1}(D; \mathbf{C}) = 0$, it follows that α_t is exact, and therefore Proposition 2.3 applies. ∎

Remark. This result was stated incorrectly in [8] (part of Theorem 1.1), where the cohomology condition $H^{n-1}(D; \mathbf{C}) = 0$ was missing. The following example shows that the result fails without this condition. (See also the forthcoming correction to [8] in *Inventiones Mathematicae.*)

Example. The map $F_t(z, w) = (z, w + t/z)$ is a volume-preserving automorphism of $\mathbf{C}_* \times \mathbf{C}$ for all $t \in \mathbf{C}$. The circle $T = \{(z, \bar{z}) \in \mathbf{C}^2 : |z| = 1\}$ is polynomially convex, hence it has pseudoconvex tubular neighborhoods Ω which are Runge in $\mathbf{C}^2$. The same is true for the curve $T_t = F_t(T)$ for each t. All conditions in Corollary 2.4, except the cohomological one, are satisfied. However, F_t for $t \neq 0$ is not the limit of volume-preserving automorphisms of $\mathbf{C}^2$ in any neighborhood of T. This can be seen by calculating the integral of the form $\theta = w\,dz$ on T_t: The integral equals $2\pi i(1 + t)$ and so it depends on t, while this quantity is preserved by the uniform limits of volume preserving automorphisms of $\mathbf{C}^2$ (see Section 3). On the other hand, each F_t is a limit of (non-volume-preserving) automorphisms of $\mathbf{C}^2$ in a neighborhood of T according to Theorem 1.1 in [8]. Of course it is far from obvious how to get such an explicit approximation.

An example of this type exists for every $n \geqslant 2$. Let S be the $(n-1)$-dimensional sphere, embedded as a hypersurface in $\mathbf{R}^n \subset \mathbf{C}^n$. There exists a closed, but nonexact holomorphic $(n-1)$-form α in a tubular neighborhood of S such that $\int_S \alpha \neq 0$. The flow F_t of the divergence-zero vector field X defined by $\alpha = i_X\Omega$ is a family of volume-preserving mappings near S such that F_t cannot be approximated by volume-preserving automorphisms of $\mathbf{C}^n$ when t is small but not 0.

3. Construction of exact volume-preserving isotopies. In this section we prove the main theorem. Let $f_0, f_1: M \to \mathbf{C}^n$ be real-analytic embeddings which are Aut$\mathbf{C}^n$-equivalent by a biholomorphic map F, defined in a neighborhood of $f_0(M)$ (see Definition 1). We begin by showing the necessity of the condition $\mathscr{B}(f_0) = \mathscr{B}(f_1)$ for Aut$_1\mathbf{C}^n$-equivalence in case (ii). To this end we suppose that there is an F as above such that $F = \lim_{\nu\to\infty} F_\nu$ in a neighborhood of $M_0 = f_0(M) \subset \mathbf{C}^n$, where $F_\nu \in \text{Aut}_1\mathbf{C}^n$ for all $\nu \in \mathbf{Z}_+$. The closed orientable manifold M_0 bounds a compact n-cycle $\tilde{M}_0$ in $\mathbf{C}^n$ in the homological sense. (We can take $\tilde{M}_0$ to be a piecewise smooth manifold with boundary M_0.) Since $d\beta = \Omega$ and $F_\nu^*\Omega = \Omega$, the Stokes theorem gives

$$\int_{F_\nu(M_0)} \beta = \int_{M_0} F_\nu^* \beta = \int_{\tilde{M}_0} d(F_\nu^* \beta) = \int_{\tilde{M}_0} F_\nu^* \Omega = \int_{\tilde{M}_0} \Omega = \int_{M_0} \beta .$$

Letting $\nu \to \infty$, we get $\int_{M_0} \beta = \int_{F(M_0)} \beta = \int_{M_1} \beta$, which means $\mathscr{B}(f_0) = \mathscr{B}(f_1)$.

In the remainder of the section we prove that the conditions in the main theorem are sufficient for $\mathrm{Aut}_1 \mathbf{C}^n$-equivalence of f_0 and f_1. Recall from ([8], Theorem 3.1) that real-analytic, totally real, polynomially convex embeddings $f_0, f_1 \colon M \to \mathbf{C}^n$ are $\mathrm{Aut}\,\mathbf{C}^n$-equivalent if and only if there exists an isotopy of embeddings $f_t \colon M \to \mathbf{C}^n$ $(0 \leqslant t \leqslant 1)$, connecting f_0 to f_1, such that the submanifold $M_t = f_t(M) \subset \mathbf{C}^n$ is totally real and polynomially convex for each t. By approximation we ensure that the map

$$f \colon [0, 1] \times M \to \mathbf{C}^{1+n}, \qquad f(t, x) = (t, f_t(x)),$$

is real-analytic. If M is closed, oriented, of dimension $n - 1$, and if $\mathscr{B}(f_0) = \mathscr{B}(f_1) \neq 0$, we can ensure in addition that

$$\mathscr{B}(f_t) = \mathscr{B}(f_0), \qquad 0 \leqslant t \leqslant 1. \tag{6}$$

This can be done by first noting that, since the integrals $\mathscr{B}(f_t)$ are complex-valued, a generically chosen isotopy f_t $(0 \leqslant t \leqslant 1)$ will satisfy $\mathscr{B}(f_t) \neq 0$ for all $t \in [0, 1]$. Let $\Theta_c(z_1, z_2, \ldots, z_n) = (cz_1, z_2, \ldots, z_n)$. Then $\Theta_c^* \beta = c\beta$, and therefore $\mathscr{B}(\Theta_c \circ f) = c\mathscr{B}(f)$ for every map $f \colon M \to \mathbf{C}^n$. Replacing the original isotopy f_t by $\Theta_{c_t} \circ f_t$, with $c_t = \mathscr{B}(f_0)/\mathscr{B}(f_t)$, gives (6). Clearly, the rescaling by Θ_c preserves the other relevant properties of the embedding such as real-analyticity, total reality and polynomial convexity.

Remark. We believe that in the exceptional case $\mathscr{B}(f_0) = \mathscr{B}(f_1) = 0$ one can still choose the isotopy $f_t \colon M \to \mathbf{C}^n$ such that $\mathscr{B}(f_t) = 0$ for all $t \in [0, 1]$, and the other properties hold. For closed curves in $\mathbf{C}^2$ this is a special case of Proposition 3.1 in [7]. For higher-dimensional manifolds the required technical details seem considerable, and we will not prove this here.

Let $M_t = f_t(M) \subset \mathbf{C}^n$. We will show that each real-analytic diffeomorphism $f_t \circ f_0^{-1} \colon M_0 \to M_t$ extends to a volume-preserving biholomorphic mapping F_t, defined in a neighborhood of M_0 in $\mathbf{C}^n$, such that each map F_t, and in particular $F = F_1$, is a limit of volume-preserving automorphisms of $\mathbf{C}^n$. This will complete the proof of the main theorem.

For a fixed $t \in [0, 1]$, we let

$$\Sigma(t) = f([0, t] \times M) = \bigcup_{0 \leqslant t \leqslant 1} \{t\} \times M_t, \qquad \Sigma = \Sigma(1).$$

This is a compact real-analytic submanifold in $\mathbf{C}^{1+n}$ which is fibered over $[0, 1] \subset \mathbf{R}$ and whose fiber over $t \in [0, 1]$ is M_t. Since M_t is totally real and polynomially convex for each t, Σ is also totally real and polynomially convex. We define a vector field $X(t, z) = X_t(z)$ of type (1, 0) along Σ, with values in $\mathbf{C}^n$, as the infinitesimal generator of f_t:

$$\frac{d}{dt} f_t(x) = X_t(f_t(x)), \qquad x \in M, 0 \leqslant t \leqslant 1. \tag{7}$$

We emphasize that X is a section of $T^{(1,0)}\mathbf{C}^n$ whose coefficients are real-analytic functions on Σ. Then the map $f_t \circ f_0^{-1}\colon M_0 \to M_t$ is the flow of X which starts with the identity on M_0 at time $t = 0$.

PROPOSITION 3.1. *Keep hypotheses as above. There exists a real-analytic vector field $Y_t(z) = Y(t, z)$ with values in $\mathbf{C}^n$, defined in an open neighborhood $U = \bigcup_{0 \leqslant t \leqslant 1} \{t\} \times U_t$ of Σ in $[0, 1] \times \mathbf{C}^n$, such that*

(i) *$Y_t = Y(t, \cdot)$ is holomorphic in U_t for each fixed t,*
(ii) *$Y = X$ on Σ, and*
(iii) *the holomorphic $(n-1)$-form $\tilde{\alpha}_t = i_{Y_t}\Omega$ is exact in U_t for each $t \in [0, 1]$.*

Once the proposition is proved, we obtain the main theorem as follows. Let F_t be the flow of Y_t such that $F_0(z) = z$ for $z \in M_0$. Since $Y_t = X_t$ on M_t for each $t \in [0, 1]$, we get $F_t \circ f_0(x) = f_t(x)$ for $x \in M$ and $t \in [0, 1]$. This implies that $F_t(z)$ is defined for all z in an open neighborhood $U_0 \subset \mathbf{C}^n$ of M_0 and for all $t \in [0, 1]$. Moreover, since $\operatorname{div} Y_t = 0$ for all $t \in [0, 1]$, each F_t is a volume-preserving biholomorphic map of U_0 onto $U_t = F_t(U_0) \subset \mathbf{C}^n$. Since the manifold M_t is totally real and polynomially convex for each $t \in [0, 1]$, we can shrink the neighborhood U_0 such that U_t is pseudoconvex and Runge in $\mathbf{C}^n$ for each $t \in [0, 1]$ ([8], Lemma 2.2). Hence the conditions of Proposition 2.3 are satisfied, and therefore each map F_t for $t \in [0, 1]$ is a limit of volume-preserving holomorphic automorphisms of $\mathbf{C}^n$. The map $F = F_1$ realizes the $\mathrm{Aut}_1\mathbf{C}^n$-equivalence of the embeddings f_0 and f_1. This establishes the main theorem, provided that Proposition 3.1 holds.

Proof of Proposition 3.1. Let X_t be defined by (7), and set

$$\alpha_t = i_{X_t}\Omega, \qquad 0 \leqslant t \leqslant 1. \tag{8}$$

This is a real-analytic section of $T^{*(n-1,0)}\mathbf{C}^n$ over M_t which depends analytically on $t \in [0, 1]$.

We claim that α_t extends to an exact holomorphic $(n-1)$-form $\tilde{\alpha}_t$ in a neighborhood $U_t \subset \mathbf{C}^n$ of M_t, depending analytically on $t \in [0, 1]$, with some uniformity in the size of U_t with respect to t. Once we have such extension, we define a holomorphic vector field Y_t on U_t by $\tilde{\alpha}_t = i_{Y_t}\Omega$. Clearly this field satisfies Proposition 3.1, and the main theorem follows.

Denote by $i_t\colon M_t \hookrightarrow \mathbf{C}^n$ the inclusion map. Recall that α_t is a holomorphic form of degree $n-1$. If $H^{n-1}(M; \mathbf{C}) = 0$, the pull-back form $i_t^*\alpha_t$ is necessarily exact on M_t for each $t \in [0, 1]$. (It even vanishes when $\dim M < n - 1$.) The corollary in the appendix (in the parametrized version) implies that α_t extends to an exact holomorphic form $\tilde{\alpha}_t$ in a neighborhood of M_t such that the extension is real-analytic in all variables, including $t \in [0, 1]$.

In the remaining case when M is a closed, connected, orientable manifold of dimension $n - 1$, the group $H^{n-1}(M; \mathbf{C})$ is isomorphic to $\mathbf{C}$, and an $(n - 1)$-form α on M is exact if and only if $\int_M \alpha = 0$ (see [4]). To complete the proof we need the following lemma.

LEMMA 3.2. *If X_t is the vector field defined by (7) and α_t is defined by (8), then*

$$\frac{d}{dt}\mathscr{B}(f_t) = \int_{M_t} \alpha_t, \qquad 0 \leqslant t \leqslant 1.$$

Proof. This is an application of the Stokes theorem. Recall that $d\beta = \Omega$. Fix a $t_1 \in [0, 1]$. Then

$$\begin{aligned}
\mathscr{B}(f_{t_1}) - \mathscr{B}(f_0) &= \int_{M_{t_1}} \beta - \int_{M_0} \beta \\
&= \int_{\Sigma(t_1)} \Omega \\
&= \int_0^{t_1} dt \int_{M_t} \Omega(\partial f_t/\partial t, \cdot) \\
&= \int_0^{t_1} dt \int_{M_t} \Omega(X_t, \cdot) \\
&= \int_0^{t_1} dt \int_{M_t} \alpha_t .
\end{aligned}$$

The result follows by differentiating both sides on t_1. ■

Since our isotopy f_t was constructed in such a way that $\mathscr{B}(f_t)$ is independent of t, Lemma 3.2 implies that $\int_{M_t} \alpha_t = 0$ for all $t \in [0, 1]$. Hence the pull-back $i_t^*\alpha_t$ is exact for each t, and the corollary in the appendix applies as before. This proves Proposition 3.1 and therefore the main theorem. ■

APPENDIX

Extension of closed holomorphic forms

THEOREM. *If M is a closed complex submanifold in a Stein manifold X, there exists an open Stein neighborhood $\Omega \subset X$ of M with the following property. Given* *a holomorphic p-form α on Ω such that $d(i_M^*\alpha) = 0$, there exists a closed holomorphic p-form $\tilde{\alpha}$ on Ω satisfying $\tilde{\alpha}|_M = \alpha|_M$. If $i_M^*\alpha$ is exact, then $\tilde{\alpha}$ can be chosen to be exact on Ω.*

Here, $i\colon M \hookrightarrow X$ is the inclusion map, $i_M^*\alpha$ is the pull-back of α to M, and $\alpha|_M$ is the restriction of α to points of M. In fact, it suffices to assume that α is a p-form on X which is only defined and holomorphic along the submanifold M.

Proof. According to a theorem of Docquier and Grauert [10, page 257], the submanifold M has Stein neighborhoods $\Omega \subset X$ such that there exists a holomorphic retraction $\pi\colon \Omega \to M$. (In [10] the result is proved when the ambient manifold is $\mathbf{C}^n$, and the general result follows from the embedding theorem for Stein manifolds; see [11, page 132].) Fix such an Ω, and set $\alpha = \alpha_0 + \alpha_1$, where $\alpha_0 = \pi^*(i_M^*\alpha)$ and $\alpha_1 = \alpha - \alpha_0$. The holomorphic form α_0 on Ω is closed since $d\alpha_0 = \pi^*(di_M^*\alpha) = 0$. Moreover, if $i_M^*\alpha = d\beta$, then $\alpha_0 = \pi^*(d\beta) = d(\pi^*\beta)$ is exact holomorphic on Ω. It suffices now to prove that the form $\alpha_1|_M$ has an exact holomorphic extension $\tilde{\alpha}_1$ on Ω, since the form $\tilde{\alpha} = \alpha_0 + \tilde{\alpha}_1$ then satisfies the theorem.

We first show that for every point $z_0 \in M$ there is an open Stein neighborhood $U \subset \Omega$ of z_0 and a holomorphic $(p-1)$-form γ on U satisfying

$$\gamma|_{M \cap U} = 0, \qquad d\gamma|_{M \cap U} = \alpha_1|_{M \cap U}. \tag{9}$$

To do this we choose local holomorphic coordinates $z = (z', z'')$ on Ω, with $z' \in \mathbf{C}^m$ and $z'' \in \mathbf{C}^{n-m}$ ($n = \dim X$, $m = \dim M$), such that in these coordinates $z_0 = 0$ and $M = \{z'' = 0\}$. Write

$$\alpha_1(z) = \sum_{\# J = p} a_J(z)\, dz_J,$$

where $J = (j_1, \ldots, j_p) \in \mathbf{Z}_+^p$, $1 \leqslant j_1 < j_2 < \cdots < j_p \leqslant n$, and $dz_J = dz_{j_1} \wedge \cdots \wedge dz_{j_p}$. Consider a typical term $a_J(z)\, dz_J$. If $J \subset \{1, \ldots, m\}$, this term is tangential to $M = \{z'' = 0\}$. Since $i_M^*\alpha_1 = 0$ by definition of α_1, it follows that $a_J(z', 0) = 0$ for every such term. Hence we do not need to consider terms of this type since they restrict to 0 on $M \cap U$.

Suppose now that J contains an index $j_s > m$. Write $J = \{j_s\} \cup J'$. We have

$$d(a_J(z) z_{j_s}\, dz_{J'}) = (-1)^{s-1} a_J(z)\, dz_J + z_{j_s}\, da_J(z) \wedge dz_{J'}.$$

This shows that the form $\gamma_J = (-1)^{s-1} a_J(z) z_{j_s}\, dz_{J'}$ satisfies (9) if we replace α_1 by the term $a_J(z)\, dz_J$. Taking $\gamma = \sum \gamma_J$, we obtain a local form γ satisfying (9).

We now globalize this construction in a standard way. Cover Ω by open Stein sets U_j such that for every j there exists a holomorphic $(p-1)$-form γ_j on U_j satisfying (9). Let $U_{i,j} = U_i \cap U_j$, and define

$$\gamma_{i,j} = (\gamma_i - \gamma_j)|_{U_{i,j}}.$$

This is a 1-cocycle on the cover $\mathscr{U} = \{U_j\}$ which satisfies

$$\gamma_{i,j}|_{M \cap U_{i,j}} = 0, \qquad d\gamma_{i,j}|_{M \cap U_{i,j}} = 0.$$

We denote by $\mathscr{E}^k$ the sheaf of germs of holomorphic differential k-forms on Ω and by $\mathscr{E}^k_M$ the subsheaf of $\mathscr{E}^k$ consisting of germs of k-forms satisfying $\gamma|_M = 0$ and $d\gamma|_M = 0$. If $f \in \mathscr{O}$ is a germ of a holomorphic function on Ω and $\gamma \in \mathscr{E}^k_M$, then the formula $d(f\gamma) = df \wedge \gamma + f\, d\gamma$ implies $f\gamma \in \mathscr{E}^k_M$. Hence $\mathscr{E}^k_M$ is an analytic sheaf on Ω for each $k \geqslant 0$, and $\{\gamma_{i,j}\}$ is a 1-cocycle with coefficients in the sheaf $\mathscr{E}^{p-1}_M$.

LEMMA. *The sheaf $\mathscr{E}^k_M$ is coherent analytic for each $k \geqslant 0$.*

We refer the reader to [9] or [11] for the definition and main results on (coherent) analytic sheaves.

If the lemma holds, Cartan's Theorem B (see [11, page 199] or [9, page 124]) implies $H^1(\Omega; \mathscr{E}^{p-1}_M) = 0$. This means that every Cousin-I problem with coefficients in this sheaf is solvable (after passing to a finer cover if necessary). Thus there exist holomorphic $(p-1)$-forms $\tilde{\gamma}_i$ on U_i satisfying

$$\tilde{\gamma}_i|_{M \cap U_i} = 0, \qquad d\tilde{\gamma}_i|_{M \cap U_i} = 0, \qquad \tilde{\gamma}_i - \tilde{\gamma}_j = \gamma_{i,j} = \gamma_i - \gamma_j \qquad \text{on } U_{i,j}.$$

The form γ which equals $\gamma_i - \tilde{\gamma}_i$ on U_i is then globally defined on Ω, and it satisfies $d\gamma|_M = \alpha_1|_M$. Thus $d\gamma$ is the required exact holomorphic extension of α_1. This proves the theorem, provided that the lemma holds.

The proof of the lemma is an exercise in multilinear algebra. Given a point $z_0 \in M$, we choose local holomorphic coordinates $\zeta = (z, w)$ on Ω, centered at z_0, such that $z \in \mathbf{C}^m$ ($m = \dim M$), $w \in \mathbf{C}^d$ ($d = \dim \Omega - m$), and $M = \{w = 0\}$. The result is obvious for $k = 0$ since $\mathscr{E}^0_M$ is the sheaf of germs of holomorphic functions which vanish to second order on $M = \{w = 0\}$. For $k = 1$ we first observe that the following 1-forms belong to $\mathscr{E}^1_M$:

$$w_i w_j\, dz_k,\ w_i w_j\, dw_k,\ w_i\, dw_i,\ w_i\, dw_j + w_j\, dw_i.$$

In the last case we take $i \neq j$. It is easy to show that these forms (for all possible values of i, j, k) generate each stalk of $\mathscr{E}^1_M$ as an $\mathscr{O}$-module. To get local generators for $\mathscr{E}^k_M$, $k > 1$, it suffices to multiply these 1-forms with the standard basis holomorphic $(k-1)$-forms. This shows that the sheaves $\mathscr{E}^k_M$ are locally finitely generated. Similarly one shows that the sheaf of relations between these generators is locally finitely generated; we leave out the details. Therefore the sheaf $\mathscr{E}^k_M$ is coherent analytic. ∎

COROLLARY. *Let M be a compact, real-analytic, totally real submanifold in a complex manifold X. Given a $(p, 0)$-form α on X, whose coefficients are defined and real-analytic on M, satisfying $d(i^*_M\alpha) = 0$, there exists an open Stein neighborhood $\Omega \subset X$ of M and a closed holomorphic p-form $\tilde{\alpha}$ on Ω such that $\tilde{\alpha}|_M = \alpha|_M$. If $i^*_M\alpha$ is exact, the extension $\tilde{\alpha}$ can be chosen to be exact holomorphic.*

Remark. Note that, unlike in the theorem, the size of the neighborhood Ω in the corollary depends on the form $\alpha|_M$.

Proof. Since M is totally real in X, it is the zero level set of a nonnegative, smooth, strongly plurisubharmonic function ρ, defined in a neighborhood of M (see [12], Proposition 1.3). The sublevel sets $\rho < \varepsilon$ for small $\varepsilon > 0$ are a basis of Stein neighborhoods of M in X. If Ω is sufficiently small, M is contained in a unique closed complex submanifold $\tilde{M} \subset \Omega$ (the complexification of M) of complex dimension $m = \dim M$. By shrinking Ω and $\tilde{M}$ we may assume that the real-analytic form α, defined initially on M, extends to a form $\tilde{\alpha}$, defined on $\tilde{M}$, whose coefficients (in local holomorphic coordinates) are holomorphic functions on $\tilde{M}$. The form itself has values on X, i.e., it is a section over $\tilde{M}$ of the appropriate exterior bundle on X. Since $i_M^*\alpha$ is closed (resp. exact), the pull-back of $\tilde{\alpha}$ to $\tilde{M}$ is also closed (resp. exact). The corollary now follows from the theorem above. ∎

Remark. The theorem and its corollary also hold in the *parametrized version.* We state the corollary in the case of one real parameter since this is the result that we need in Section 3.

Let $M_t \subset X$ be a family of compact, real-analytic, totally real submanifolds, depending real-analytically on the parameter $t \in [0, 1]$. Assume that for each fixed $t \in [0, 1]$ we have a $(p, 0)$-form α_t on X, defined and real-analytic along M_t, such that the dependence of α_t on t is real-analytic as well. Assume that for each $t \in [0, 1]$, α_t pulls back to a closed (resp. exact) form on M_t. Then there exists a closed (resp. exact) holomorphic p-form $\tilde{\alpha}_t$ in a neighborhood $U_t \subset X$ $(0 \leqslant t \leqslant 1)$ which depends real-analytically on $t \in [0, 1]$. This parametrized version can be proved by using the same methods.

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