A Theorem in Complex Symplectic Geometry

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ABSTRACT. We prove that two simple, closed, real-analytic curves in $\mathbf{C}^{2n}$ that are polynomially convex are equivalent under the group of symplectic holomorphic automorphisms of $\mathbf{C}^{2n}$ if and only if the two curves have the same action integral. Every two simple real-analytic arcs in $\mathbf{C}^{2n}$ are so equivalent.

1. Introduction

The main object of this paper is to prove a result on global holomorphic symplectic equivalence of curves in the complex symplectic space $(\mathbf{C}^{2n}, \omega)$. The analogous result in real symplectic geometry is well known, and it holds in arbitrary symplectic manifolds.

To motivate we first recall basic notions of complex symplectic algebra and geometry. It seems that this topic has been studied much less than the real symplectic geometry. Perhaps this is due to the fact that the real symplectic geometry is intimately connected to mechanics and other areas of physics (see Abraham and Marsden [1] or Arnold [4]), while this connection seems to be lost in the complex case. Nevertheless, the subject offers very interesting global problems.

On every complex vector space V of even dimension $2n$ there exists a complex, alternating, nondegenerate bilinear form $\omega\colon V \times V \to \mathbf{C}$ [12]. Every such form is called a *complex symplectic form*, and the pair (V, ω) is a *complex symplectic space*, terminology due to Weyl (1939). The group of complex linear maps $V \to V$ preserving ω is called the *complex symplectic group* on V relative to ω, denoted $\mathrm{Sp}(V, \omega)$. Every two symplectic forms on V are linearly equivalent. In particular, every complex symplectic space (V, ω) is isomorphic to the space $(\mathbf{C}^{2n}, \omega)$ with the standard symplectic form

$$\omega(z, w) = \sum_{j=1}^{n} z_j w_{n+j} - z_{n+j} w_j, \qquad z, w \in \mathbf{C}^{2n}. \tag{1}$$

We denote by $\mathrm{Sp}(n, \mathbf{C}) \subset \mathrm{GL}(2n, \mathbf{C})$ the symplectic group associated to this form, consisting of maps $A \in \mathrm{GL}(2n, \mathbf{C})$ which preserve ω. The corresponding complex symplectic differential form on $\mathbf{C}^{2n}$ is

$$\omega = \sum_{j=1}^{n} dz_j \wedge dz_{n+j}. \tag{2}$$

The natural object to study on $(\mathbf{C}^{2n}, \omega)$ are mappings $F: D \to \mathbf{C}^{2n}$ from domains $D \subset \mathbf{C}^{2n}$ such that the derivative $DF(z)$ belongs to $\mathrm{Sp}(n, \mathbf{C})$ for each $z \in D$. Of course such a map is necessarily holomorphic. We call such maps *symplectic holomorphic*. An equivalent definition is

Definition 1. A holomorphic mapping $F: D \subset \mathbf{C}^{2n} \to \mathbf{C}^{2n}$ is called *symplectic holomorphic* if $F^*\omega = \omega$, where ω is the form (2).

We denote by $\mathrm{Aut}_{\mathrm{sp}} \mathbf{C}^{2n}$ the group of all symplectic holomorphic automorphisms of $(\mathbf{C}^{2n}, \omega)$. Such maps are also called "symplectomorphisms," a term that we prefer to avoid. In [10] we showed that this is a large group that contains a dense subgroup $\mathcal{S}_{sp}(n)$ consisting of finite compositions of *symplectic shears*. These are maps of the form

$$F(z) = z + f(\omega(z, v))v, \qquad z \in \mathbf{C}^{2n},$$

where ω is the form (1), v is a fixed vector in $\mathbf{C}^{2n}$, and f is an entire function of one variable.

In order to formulate our result we introduce the following notion of global symplectic equivalence in $\mathbf{C}^{2n}$. This should be compared to the notion of $\mathbf{C}^n$-equivalence introduced in [8].

Definition 2. (a) Two compact subsets $K_0, K_1 \subset \mathbf{C}^{2n}$ are $\mathrm{Aut}_{\mathrm{sp}} \mathbf{C}^{2n}$-equivalent if there exist an open neighborhood U of K_0 in $\mathbf{C}^{2n}$ and a biholomorphic map $F: U \to F(U) \subset \mathbf{C}^{2n}$ such that $F(K_0) = K_1$, and F is the uniform limit in U of a sequence $F_j \in \mathrm{Aut}_{\mathrm{sp}} \mathbf{C}^{2n}$ of symplectic holomorphic automorphisms.

(b) Two maps $f_0, f_1: K \to \mathbf{C}^{2n}$ from a compact space K are $\mathrm{Aut}_{\mathrm{sp}} \mathbf{C}^{2n}$-equivalent if there exist an open neighborhood $U \subset \mathbf{C}^{2n}$ of $f_0(K)$ and a biholomorphic map $F: U \to F(U) \subset \mathbf{C}^{2n}$ such that $F \circ f_0 = f_1$, and F is a limit in U of a sequence $F_j \in \mathrm{Aut}_{\mathrm{sp}} \mathbf{C}^{2n}$.

Notice that a map F satisfying Definition 2 is necessarily symplectic holomorphic. We also introduce the form

$$\theta = \sum_{j=1}^{n} z_{n+j} dz_j, \qquad -d\theta = \omega. \tag{3}$$

Denote by I the interval $[0, 1] \subset \mathbf{R}$ and by T the circle. For a smooth map $g: T \to \mathbf{C}^{2n}$ we define the *action integral*

$$\mathcal{A}(g) = - \int_T g^*\theta. \tag{4}$$

Main Theorem. *Let* $g_0, g_1: T \to \mathbf{C}^{2n}$ *be two real-analytic embeddings of the circle into* $\mathbf{C}^{2n}$. *If* g_0 *and* g_1 *are* $\mathrm{Aut}_{sp} \mathbf{C}^{2n}$-*equivalent, then* $\mathcal{A}(g_0) = \mathcal{A}(g_1)$. *Conversely we have*

(a) *if* $\mathcal{A}(g_0) = \mathcal{A}(g_1) \neq 0$, *then* g_0 *and* g_1 *are* $\mathrm{Aut}_{sp} \mathbf{C}^{2n}$-*equivalent;*

(b) *if* $\mathcal{A}(g_0) = \mathcal{A}(g_1) = 0$, *then* g_0 *and* g_1 *are* $\mathrm{Aut}_{sp} \mathbf{C}^{2n}$-*equivalent provided that the curves* $g_0(T), g_1(T) \subset \mathbf{C}^{2n}$ *are both polynomially convex;*

(c) *every two smooth, embedded, real-analytic arcs* $g_0, g_1: [0, 1] \to \mathbf{C}^{2n}$ *are* $\mathrm{Aut}_{sp} \mathbf{C}^{2n}$-*equivalent.*

Recall that a compact set $K \subset \mathbf{C}^m$ is polynomially convex if for every point $z \in \mathbf{C}^m \setminus K$ there is a holomorphic polynomial f such that $|f(z)| > \sup_K |f|$. A smooth, simple, closed curve $C \subset \mathbf{C}^m$ that is not polynomially convex bounds a one-dimensional analytic variety A according to Stolzenberg [13] (and Wermer [14] in the real-analytic case), and $\overline{A} = A \cup C$ is polynomially convex. The Stokes theorem implies $\int_C \alpha = \int_A d\alpha = 0$ for every holomorphic 1-form α on $\mathbf{C}^m$. Since θ (3) is such a form, the condition $\mathcal{A}(g) \neq 0$ ensures that the curve $g(T) \subset \mathbf{C}^{2n}$ is polynomially convex.

The conclusion of Theorem 1.1 does not hold in general for curves that are not polynomially convex, since equivalence of curves clearly implies equivalence of their polynomial hulls. For instance, a curve $C \subset \mathbf{C}^m$ that bounds a smooth (nonsingular) analytic disc is not $\mathrm{Aut}\,\mathbf{C}^m$-equivalent to a curve that bounds an analytic disc with at least one singularity, even though both curves have the action integral (4) equal to zero.

Remark 1. The real analogue of Theorem 1.1 is well known and holds in arbitrary symplectic manifold M under the additional assumption that the two simple closed curves $C_0, C_1 \subset M$ are isotopic in M. In this case there exists a symplectic diffeomorphism of M which carries C_0 onto C_1 and which is supported in a neighborhood of the isotopy C_t ($0 \leq t \leq 1$) connecting C_0 to C_1.

Remark 2. One could introduce a more restrictive notion of symplectic equivalence of sets $K_0, K_1 \subset \mathbf{C}^{2n}$ by requiring that there exist an $F \in \mathrm{Aut}_{sp} \mathbf{C}^{2n}$ satisfying $F(K_0) = K_1$. However, this notion of equivalence, which is a perfectly natural one in real symplectic geometry, is much too strong in the holomorphic category and it would not give any interesting results.

Remark 3. Since ω^n is a complex volume form on $\mathbf{C}^{2n}$, a constant multiple of the form $dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{2n}$, every symplectic holomorphic map is complex volume preserving, i.e., it has Jacobian one. On $\mathbf{C}^2$ the two classes of mappings coincide.

The paper is organized as follows. In Section 2 we recall some standard results of symplectic geometry which are needed in the proof, as well as results from [10] on decomposition of polynomial Hamiltonian vector fields. In Section 3 we construct isotopies of embedded, polynomially convex curves in $\mathbf{C}^{2n}$ with constant action integral. In Section 4 we show that every such isotopy can be realized by a Hamiltonian isotopy, consisting of symplectic holomorphic maps (Theorem 4.1). At the end of Section 4 we prove the main theorem.

I thank Y.-G. Oh for stimulating conversations on this subject. I learned from him about the analogue of the main theorem in real symplectic geometry. The present paper, as well as the paper [9], would not exist without our previous joint work [8] with J.-P. Rosay. I thank him for everything that he has contributed.

2. Holomorphic Hamiltonian vector fields and flows

In this section we first recall some standard results of symplectic geometry which will be needed in the proof of the main theorem. We refer the reader to Abraham and Marsden [1] for the details and proofs. At the end of the section we also recall some results of [10].

The symplectic form ω (2) induces an isomorphism between the tangent and cotangent bundles of type $(1, 0)$ on $\mathbf{C}^{2n}$ via

$$X \in \mathrm{T}_z^{(1,0)}\mathbf{C}^{2n} \mapsto i_X(\omega) \in \mathrm{T}_z^{*(1,0)}\mathbf{C}^{2n}, \qquad z \in \mathbf{C}^{2n}.$$

Here, i_X denotes the contraction (interior product) by X. Thus to each holomorphic vector field X on a domain $D \subset \mathbf{C}^{2n}$ corresponds the holomorphic 1-form $\alpha = i_X(\omega)$ on D.

Definition 3. A holomorphic vector field X on a domain $D \subset \mathbf{C}^{2n}$ is said to be *locally Hamiltonian* if the holomorphic differential 1-form $\alpha = i_X(\omega)$ on D is closed $(d\alpha = 0)$. The field X is *(exact) Hamiltonian* if the form $i_X(\omega)$ is exact on D, i.e., $i_X(\omega) = dH$ for some function H on D.

If X is exact Hamiltonian and $i_X(\omega) = dH$, then the function H is called the *energy function* of X, and we write $X = X_H$. The terminology comes from Hamiltonian mechanics where H represents the total energy of the system. Since the form $i_X\omega$ is of type $(1, 0)$, H is necessarily holomorphic, and it is determined up to a constant if D is connected. The explicit expression for X_H is

$$X_H(z) = \sum_{j=1}^{n} \frac{\partial H(z)}{\partial z_{n+j}} \frac{\partial}{\partial z_j} - \frac{\partial H(z)}{\partial z_j} \frac{\partial}{\partial z_{n+j}}.$$

Remark. One could take this as the definition of an exact Hamitonian vector field, where H is a smooth function, possibly real-valued. It is easily seen that X_H is holomorphic if and only if H is pluriharmonic [7]. This point of view does not seem to be the most natural one since the

antiholomorphic part of a pluriharmonic function H does not contribute anything to the Hamiltonian vector field X_H.

The quotient space of locally Hamiltonian modulo exact Hamiltonian fields on D is isomorphic to the cohomology group of closed holomorphic 1-forms on D modulo the exact ones. If D is a domain of holomorphy (or, more generally, a Stein manifold), this group is isomorphic to $H^1(D; C)$ [11, p. 58].

Recall that the flow F_t of a vector field X on D is the solution of the ordinary differential equation

$$\frac{d}{dt} F_t(z) = X\big(F_t(z)\big), \qquad F_0(z) = z \in D.$$

For each compact set $K \subset\subset D$ there is a number $T > 0$ such that the flow $F_t(z)$ is defined for all $z \in K$ and $|t| < T$. If X is holomorphic, then each F_t is injective holomorphic where defined.

Recall from [1, p. 91] that the Lie derivative L_X with respect to a vector field X satisfies the following identity for any tensor field α:

$$\frac{d}{dt} F_t^* \alpha = F_t^*(L_X \alpha).$$

In fact, at time $t = 0$, this may be taken as the definition of the Lie derivative. If X is holomorphic, then L_X maps holomorphic tensors (vector fields, forms, ...) to holomorphic tensors. If α is a differential form, we have [1, p. 121]

$$L_X \alpha = d(i_X \alpha) + i_X(d\alpha). \tag{5}$$

Applying this to the symplectic form ω for which $d\omega = 0$ we get

$$\frac{d}{dt} F_t^* \omega = F_t^*(L_X \omega) = F_t^* d(i_X \omega).$$

This identity implies that a holomorphic vector field X on a domain $D \subset C^{2n}$ is locally Hamiltonian if and only if the flow F_t preserves ω, i.e., every time map F_t is symplectic holomorphic where defined.

We must also consider time-dependent vector fields $X_t(z)$ which are holomorphic in z for each fixed t, and of class C^1 in (t, z); see [1, p. 92]. Such a field is usually defined on a domain in $\tilde{D} \subset R \times C^m$ of the form

$$\tilde{D} = \bigcup_{t \in J} \{t\} \times D_t,$$

where each D_t is a domain in C^m and J is an interval in R containing 0. The flow $F_{t,s}$ of such a field is defined as the solution of

$$\frac{d}{dt} F_{t,s}(z) = X_t\big(F_{t,s}(z)\big), \qquad F_{s,s}(z) = z.$$

Thus s represents the initial time of the flow. Again every time map $F_{t,s}$ is injective holomorphic where defined.

The most important observation for us is that such time-dependent holomorphic vector fields correspond to one-parameter families (isotopies) of biholomorphic maps $\Phi_t : D \to D_t$ between domains in $\mathbf{C}^m$ which depend smoothly (at least of class C^2) on $(t, z) \in J \times D$ for some time interval $J \subset \mathbf{R}$. In fact, if we fix an $s \in J$, then $\Phi_t \circ \Phi_s^{-1} : D_s \to D_t$ is the flow of the vector field

$$X_t(z) = \frac{d}{d\tau}\Phi_\tau(\Phi_t^{-1}(z))\Big|_{\tau=t}, \qquad t \in J, \ z \in D_t,$$

with initial condition at time $t = s$ given by the identity on D_s. The field X_t is called the infinitesimal generator of the family Φ_t. Since every holomorphic automorphism $\Phi \in \mathrm{Aut}\,\mathbf{C}^m$ can be joined to the identity on $\mathbf{C}^m$ by such an isotopy Φ_t for $0 \le t \le 1$ [8], Φ is the time one map of a time-dependent holomorphic vector field on $[0, 1] \times \mathbf{C}^m$.

For any tensor α we have the identity

$$\frac{d}{dt}F_{t,s}^*\alpha = F_{t,s}^*(L_{X_t}\alpha)$$

on the set where these quantities are defined [1, p. 92]. Applying this to the form ω and using (5) we get

$$\frac{d}{dt}F_{t,s}^*\omega = F_{t,s}^*(d(i_{X_t}\omega)).$$

This implies the following lemma.

Lemma 2.1. *Let X_t be a time-dependent holomorphic vector field, defined on an open subset of $\mathbf{R} \times \mathbf{C}^{2n}$, such that the flow $F_{t,0}(z)$ is defined for $z \in D \subset \mathbf{C}^{2n}$ and $0 \le t \le T$ for some $T > 0$. Then the following are equivalent:*

(a)   *the field X_t is locally Hamiltonian on the domain $D_t = F_{t,0}(D)$ for each $t \in [0, T]$;*

(b)   *the map $F_{t,s} : D_s \to D_t$ is symplectic holomorphic when $t, s \in [0, T]$.*

The following result plays a central role in the proof of the main theorem.

Proposition 2.2. *Let X_t be a holomorphic Hamiltonian vector field on $\mathbf{C}^{2n}$ for each $t \in [0, 1]$ such that $(t, z) \mapsto X_t(z)$ is of class C^1. Let D be an open set in $\mathbf{C}^{2n}$ and let $0 < T \le 1$. Assume that the flow $F_t(z) = F_{t,0}(z)$ exists for $0 \le t \le T$ and $z \in D$. Then F_t for $0 \le t \le T$ is a symplectic biholomorphic map from D to $F_t(D)$ which can be approximated, uniformly on compact sets in D, by symplectic automorphisms in $\mathrm{Aut}_{\mathrm{sp}}\,\mathbf{C}^{2n}$.*

For time-independent fields this was proved in [10], Proposition 5.4. The proof depends on the fact that every polynomial (time-independent) Hamiltonian vector field on C^{2n} is a finite sum of complete Hamiltonian fields of the form

$$X(z) = f(\omega(z, v))v, \qquad z \in C^{2n},$$

where $v \in C^{2n}$, ω is the symplectic form (1), and f is a holomorphic polynomial in one variable. (See [10], Proposition 5.2.) The flow of this field, given by

$$F_t(z) = z + tf(\omega(z, v))v, \qquad z \in C^{2n}, \ t \in C,$$

consists of *symplectic shears*. In the case when f is linear, $f(\zeta) = \lambda\zeta$ for some $\lambda \in C^*$, such maps are also called *symplectic transvections* [12]. For similar approximation results for holomorphic and divergence zero vector fields see [2], [3], and Lemma 1.4 in [8]. The proof of Proposition 2.2 for time-dependent Hamiltonian fields then follows as in [10] by approximating the given time-dependent field X_t for t on short time intervals $[k/N, (k + 1)/N] \subset [0, 1]$ by the time-independent field $X_{k/N}$. See also [8].

In Proposition 2.2 it is very important that the field X_t is defined globally on C^{2n} for each t. Instead it suffices to assume that X_t is defined on $D_t = F_t(D)$ and it can be approximated by global Hamiltonian fields defined on C^{2n}, uniformly on compacts in D_t. The following proposition will be useful to us. Recall that a domain $D \subset C^m$ is Runge in C^m if every holomorphic function on D is a limit of holomorphic polynomials, uniformly on compacts in D.

Proposition 2.3. *Let $X_t(z)$ be a time-dependent holomorphic vector field on an open subset of $R \times C^{2n}$, and let $D \subset C^{2n}$ be a domain such that the flow $F_t(z) = F_{t,0}(z)$ of X_t is defined for all $z \in D \subset C^{2n}$ and $0 \leq t \leq T$ for some $T > 0$. Set $D_t = F_t(D)$. Assume that for each $t \in [0, T]$, X_t is exact Hamiltonian on D_t, and the domain D_t is Runge in C^{2n}. Then each map F_t is the limit of symplectic holomorphic automorphisms of C^{2n}, uniformly on compacts in D.*

Proof. The assumption on X_t is that $X_t = X_{H_t}$ for some holomorphic function H_t on D_t. Since D_t is Runge in C^{2n}, H_t can be approximated by holomorphic polynomials P. The corresponding Hamiltonian vector field Y_t, defined by $dP = i_{Y_t}\omega$, is globally defined, and it approximates X_t on compacts in D_t. The result follows as above. $\square$

Remark. If the domain $D \subset C^{2n}$ is pseudoconvex, and if $H^1(D; C) = 0$, then every locally Hamiltonian holomorphic vector field on D is exact Hamiltonian. If, in addition, D is Runge in C^{2n}, it follows that every locally Hamiltonian holomorphic vector field on D is a limit of polynomial Hamiltonian vector fields on C^{2n}, uniformly on compacts in D. If $F_t: D \to D_t$ is an isotopy of symplectic biholomorphic mappings from such a domain D such that F_0 is the identity on D and $F_t(D)$ is Runge in C^{2n} for each t, Proposition 2.3 implies that each F_t is a limit of symplectic holomorphic automorphisms of C^{2n}, uniformly on compacts in D. However, we can not use this

result directly in the proof of the main theorem, since tubular neighborhoods of closed curves have the first cohomology group isomorphic to **Z**.

3. Isotopies of curves with constant action

In this section we prove the following result. Recall that T is the circle, and for a map $g: T \to$ $\mathbf{C}^{2n}$ the action integral $\mathcal{A}(g)$ is defined by (4). An isotopy of embeddings $g_t: T \to \mathbf{C}^{2n}, t \in [0, 1]$, is piecewise real-analytic if g is continuous on $[0, 1] \times T$, and $[0, 1]$ is the union of finitely many closed subintervals I_m such that the map $(t, s) \in I_m \times T \to g_t(s)$ is real-analytic for each I_m.

Proposition 3.1. *If* $g_0, g_1: T \to \mathbf{C}^{2n}$ *are real-analytic embeddings such that* $\mathcal{A}(g_0) = \mathcal{A}(g_1)$, *then there exists a piecewise real-analytic isotopy of embeddings* $g_t: T \to \mathbf{C}^{2n}, 0 \le t \le 1$, *such that* $\mathcal{A}(g_t)$ *is independent of* t. *If, in addition, the curves* $g_0(T), g_1(T) \subset \mathbf{C}^{2n}$ *are polynomially convex, we can choose the isotopy as above such that* $g_t(T)$ *is polynomially convex for each* $t \in [0, 1]$.

Proof. We identify T with the circle $\{e^{is}: s \in \mathbf{R}\}$. We represent a real-analytic embedding $g: T \to \mathbf{C}^{2n}$ by a Fourier series

$$g(s) = \sum_{k \in \mathbf{Z}} a^k e^{iks}, \qquad a^k = (a_j^k)_{j=1}^{2n} \in \mathbf{C}^{2n}. \tag{6}$$

The series for g and also for its derivatives are converging uniformly on T. By a preliminary isotopy involving translations we can eliminate the constant term a^0 and consider only series with $a^0 = 0$.

We denote by $\mathcal{P}$ the vector space of all Fourier series (6) with coefficients $a^k \in \mathbf{C}^{2n}, a^0 = 0$, which represent real-analytic maps $T \mapsto \mathbf{C}^{2n}$. For each $N \in \mathbf{Z}_+$ we denote by $\mathcal{P}_N$ the subspace of $\mathcal{P}$ satisfying $a^k = 0$ for $|k| > N$. In other words, $\mathcal{P}_N$ contains all trigonometric polynomials of degree at most N with coefficients in $\mathbf{C}^{2n}$, and with no constant term. $\mathcal{P}_N$ is a complex vector space of dimension $4nN$, with complex coordinates a_j^k ($|k| \le N, k \ne 0, 1 \le j \le 2n$). If g is given by (6), the action integral equals

$$\mathcal{A}(g) = -\sum_{j=1}^{n} \int_T g^*(z_{n+j} \, dz_j)$$

$$= -\sum_{j=1}^{n} \int_0^{2\pi} \left(\sum_{k \in \mathbf{Z}} a_{n+j}^k e^{iks} \right) \left(\sum_{l \in \mathbf{Z}} a_j^l i l e^{ils} \right) ds$$

$$= 2\pi i \sum_{j=1}^{n} \sum_{k \in \mathbf{Z}} k a_{n+j}^k a_j^{-k}.$$

Lemma 3.2. *Given a real-analytic embedding* $g: T \to \mathbf{C}^{2n}$, *there exists an integer* $N > 0$ *and a real-analytic isotopy of embeddings* $g_t: \mathbf{T} \to \mathbf{C}^{2n}$ $(0 \le t \le 1)$ *such that* $g_0 = g, g_1 \in \mathcal{P}_N$,

and $\mathcal{A}(g_t)$ is independent of t. If $g(T)$ is polynomially convex, we can choose the isotopy such that $g_t(T)$ is polynomially convex for each $t \in [0, 1]$.

Proof. For $c \in \mathbf{C}$ we define

$$h(c, s) = \left(\sqrt{c}e^{-is}, 0, \ldots, 0, \sqrt{c}e^{is}/2\pi i, 0, \ldots, 0\right), \qquad s \in \mathbf{R},$$

where the second entry is on the $(n + 1)$st spot. Then $\mathcal{A}\big(h(c, \cdot)\big) = c$. (It does not matter which square root of c we take.) We write $g = g_N + \tilde{g}$, where $g_N \in \mathcal{P}_N$ and $\tilde{g}$ contains only terms e^{iks} with $|k| > N$. Set

$$g_t(s) = g_N(s) + (1 - t)\tilde{g}(s) + th\big(\mathcal{A}(\tilde{g}), s\big), \qquad 0 \le t \le 1, \ s \in \mathbf{R}.$$

Then $g_0 = g$, $g_1 = g_N + h(\mathcal{A}(\tilde{g}), \cdot) \in \mathcal{P}_N$, and $\mathcal{A}(g_t) = \mathcal{A}(g)$ for all t. Moreover, if we choose N sufficiently large, the functions $\tilde{g}$ and $h(\mathcal{A}(\tilde{g}), \cdot)$ are small in the C^2 norm, and hence g_t is a small C^2 deformation of $g = g_0$. Thus, if g is an embedding and $g(T)$ is polynomially convex, the same is true for every g_t ($0 \le t \le 1$), provided that N is chosen large enough. This proves Lemma 3.2. $\qquad\qquad\square$

Lemma 3.2 reduces the proof of Proposition 3.1 to the case when $g_0, g_1 \in \mathcal{P}_N$. We define the following subsets in $\mathcal{P}_N$:

$$A = \{g \in \mathcal{P}_N : \mathcal{A}(g) = 0\},$$

$$\Sigma(s) = \{g \in \mathcal{P}_N : g'(s) = 0\}, \qquad s \in \mathbf{R},$$

$$\Pi(s, t) = \{g \in \mathcal{P}_N : g(s) = g(t)\}, \qquad s, t \in \mathbf{R}, \ s \ne t \ (\mathrm{mod}\, 2\pi).$$

The set A is defined by the homogeneous quadratic equation

$$\sum_{j=1}^{n} \sum_{|k| \le N} k a_{n+j}^k a_j^{-k} = 0.$$

Hence A is an irreducible conical complex hypersurface in $\mathcal{P}_N$ with vertex at the origin, and $A \backslash \{0\}$ is a smooth, connected complex manifold. For each $s, t \in [0, 2\pi)$, with $s \ne t$, the sets $\Sigma(s)$ and $\Pi(s, t)$ are complex subspaces of $\mathcal{P}_N$ of complex codimension $2n$ (since we have $2n$ independent equations, one for each component g_j of g). Set

$$\Pi = \bigcup_{s,t} \Sigma(s) \cup \Pi(s, t),$$

where the union is over all $s, t \in [0, 2\pi]$ such that $s \ne t$ modulo 2π. Since this is a union of complex subspaces of real codimension $4n$ in $\mathcal{P}_N$, depending analytically on two additional real parameters, its Hausdorff codimension in $\mathcal{P}_N$ is $4n - 2 \ge 2$. (See Appendix 6 in [6] for a survey of Hausdorff measures and Hausdorff dimension.) Therefore the Hausdorff codimension of $A \cup \Pi$ in $\mathcal{P}_N$ is two, and hence $A \cup \Pi$ does not disconnect $\mathcal{P}_N$. Moreover, as $t \ne s$ converge

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to some $u \in \mathbf{R}$, the normalized defining equations for $\Pi(s, t)$ converge to those of $\Sigma(u)$. Hence $\lim_{s,t \to u} \Pi(s, t) \subset \Sigma(u)$. This implies that the sets Π and $A \cup \Pi$ are closed in $\mathcal{P}_N$.

We can now prove Proposition 3.1 in the case $\mathcal{A}(g_0) = \mathcal{A}(g_1) \neq 0$. This means that g_0 and g_1 belong to the connected open set $\mathcal{B} = \mathcal{P}_N \backslash (A \cup \Pi)$, and hence we can connect them within $\mathcal{B}$ by a real-analytic path $\tilde{g}_t$, $0 \leq t \leq 1$. By construction, this path defines a real-analytic isotopy of embeddings $\tilde{g}_t : T \to \mathbf{C}^{2n}$ such that $\mathcal{A}(\tilde{g}_t) \neq 0$ for each t. It remains to modify $\tilde{g}_t$ to make the action constant. This is achieved by the (nonhomogeneous) dilations on $\mathbf{C}^{2n}$

$$R(c)(z_1, \ldots, z_{2n}) = (z_1, \ldots, z_n, cz_{n+1}, \ldots, cz_{2n}).$$

Clearly $\mathcal{A}(R(c) \circ g) = c\mathcal{A}(g)$ for each map $g : T \to \mathbf{C}^{2n}$. Set $c_t = \mathcal{A}(g_0)/\mathcal{A}(\tilde{g}_t)$ and $g_t = R(c_t) \circ \tilde{g}_t$. Then we have $\mathcal{A}(g_t) = \mathcal{A}(g_0)$ for all t. Since the dilations $R(c)$ for $c \neq 0$ preserve all other properties of the path (embedding, polynomial convexity), the new isotopy g_t satisfies Proposition 3.1.

In the proof for the remaining case $\mathcal{A}(g_0) = \mathcal{A}(g_1) = 0$ we will need the following

Lemma 3.3. *The set $A \backslash \Pi \subset \mathcal{P}_N$ is a connected complex manifold.*

Proof. We already know that $A \backslash \{0\}$ is a connected, smooth complex hypersurface in $\mathcal{P}_N$. It remains to show that each intersection $A \cap \Sigma(s)$ and $A \cap \Pi(s, t)$ is a proper complex subvariety of A. Once this is shown, we conclude as before that $A \cap \Pi$ is a closed subset of Hausdorff codimension at least 2 in A which therefore does not disconnect A.

Fix $s \in \mathbf{R}$. Both sets A and $\Sigma(s)$ are complex cones with vertex at the origin, and therefore it suffices to consider their intersections with complex hyperplanes $a_j^k = const$ for various $|k| \leq N$, $k \neq 0$, and $1 \leq j \leq 2n$. Fix such a pair (j_0, k_0). Assume that $1 \leq j_0 \leq n$ (the proof in the other case is analogous). Consider the intersections with the hyperplane $H = H(j_0, k_0) = \{a_{j_0}^{-k_0} = 1\} \subset \mathcal{P}_N$. The intersection $A \cap H$ is given by

$$a_{j_0}^{-k_0} = 1, \qquad a_{n+j_0}^{k_0} = -\frac{1}{k_0} \sum k a_{n+j}^k a_j^{-k},$$

where the summation is over $1 \leq j \leq n$, $|k| \leq N, k \neq 0$, and $(j, k) \neq (j_0, k_0)$. The equations defining $\Sigma(s)$ are

$$\sum_{|k| \leq N} ik a_j^k e^{iks} = 0, \qquad 1 \leq j \leq 2n.$$

These equations for $j \neq j_0$ and $j \neq n + j_0$ are completely independent of the defining equations for $A \cap H$, and hence they cut out of $A \cap H$ an analytic subset of complex codimension $2n - 2$. If $n > 1$ this already proves Lemma 3.3 (since the real codimension is $4n - 4 \geq 4$). In the remaining case when $n = 1$ we have $j_0 = 1$. A straightforward calculation shows that $A \cap \Sigma(s) \cap H$ is

defined by the following four equations

$$a_1^{-k_0} = 1,$$

$$a_2^{k_0} = -\frac{1}{k_0} \sum_{k \neq k_0} k a_2^k a_1^{-k},$$

$$i k_0 e^{-k_0 s} = \sum_{k \neq -k_0} i k a_1^k e^{iks},$$

$$0 = \sum_{k \neq k_0} i k a_2^k (e^{iks} - a_1^{-k}).$$

We have already used the first two equations, defining $A \cap H$, to eliminate $a_1^{-k_0}$ and $a_2^{k_0}$ from the last two equations. The third equation defines a complex hyperplane in the variables a_1^k, $k \neq -k_0$. For most points in this hyperplane the last equation defines a complex hyperplane in the remaining variables a_2^k, $k \neq k_0$. Thus the last two equations define a complex plane of codimension two in the space with variables a_j^k for $(j, k) \neq (1, -k_0)$ and $(j, k) \neq (2, k_0)$. The first two equations now define a graph over this plane in $\mathcal{P}_N$. This shows that $A \cap H \cap \Sigma(s)$ has complex codimension two in $A \cap H$. Since this is true for every $H = H(j_0, k_0)$, the result follows. Similarly one deals with intersections $A \cap \Pi(s, t)$; we leave out the details. This establishes Lemma 3.3. □

We now conclude the proof of Proposition 3.1 in the case when $\mathcal{A}(g_0) = \mathcal{A}(g_1) = 0$. This means that g_0 and g_1 are points in the connected complex manifold $A \backslash \Pi$, and therefore they can be connected within $A \backslash \Pi$ by a real-analytic path g_t, $0 \leq t \leq 1$. We must ensure in addition that the closed curve $g_t(T) \subset \mathbf{C}^{2n}$ is polynomially convex. This can be done by choosing another holomorphic 1-form α on $\mathbf{C}^{2n}$ such that $\int_T g_j^* \alpha \neq 0$ for $j = 0, 1$. In fact, most holomorphic 1-forms will satisfy this property. The set $B = \{g \in A : \int_T g^* \alpha = 0\}$ is a proper analytic subset of A that does not disconnect A and $A \backslash \Pi$. Choosing our path g_t inside the connected complex manifold $A \backslash (B \cup \Pi)$ ensures that $g_t(T)$ is polynomially convex for each $t \in [0, 1]$ (see the remark following the main theorem). □

4. Construction of holomorphic Hamiltonian isotopies

Recall that T is the circle and $I = [0, 1]$.

Theorem 4.1. *If* $g_t : T \to \mathbf{C}^{2n}$, $0 \leq t \leq 1$, *is a real-analytic isotopy of closed, embedded, real-analytic curves such that the action integral* $\mathcal{A}(g_t)$ *(4) is independent of t, then there exists a neighborhood* $U \subset \mathbf{C}^{2n}$ *of the curve* $g_0(T)$ *and an isotopy* $F_t : U \to F_t(U) \subset \mathbf{C}^{2n}$ *(0 ≤ t ≤ 1), consisting of symplectic biholomorphic mappings, such that*

    (i) $F_t \circ g_0 = g_t$ *for each* $t \in [0, 1]$,

    (ii) F_0 *is the identity on* U, *and*

    (iii) *the infinitesimal generator* X_t *of* F_t *is an exact Hamiltonian vector field on* $F_t(U) \subset \mathbf{C}^{2n}$ *for each* $t \in [0, 1]$.

If we replace T by I (i.e., we have an isotopy of embedded real-analytic arcs in $\mathbf{C}^{2n}$), the same result holds without any condition on the action integral.

Proof. Let $C_t = g_t(T) \subset \mathbf{C}^{2n}$. For $0 \leq t_1 \leq 1$ set

$$S(t_1) = \bigcup_{0 \leq t \leq t_1} \{t\} \times C_t, \qquad S = S(1).$$

S is a real-analytic surface in $[0, 1] \times \mathbf{C}^{2n}$ (a tube) that is parametrized by the map $g(t, s) = g_t(s)$, $t \in [0, 1], s \in T$. Let $Y_t(z)$ be the infinitesimal generator of the flow $g_t \circ g_0^{-1} : C_0 \to C_t$. Explicitly, Y_t is a vector field of type $(1, 0)$ on $\mathbf{C}^{2n}$, defined along the curve C_t, such that

$$\frac{d}{dt} g_t(s) = Y_t\big(g_t(s)\big), \qquad s \in T, \ 0 \leq t \leq 1. \tag{7}$$

Since the isotopy g_t is real-analytic in both variables $(t, s) \in [0, 1] \times T$, $Y_t(z)$ is real-analytic in $(t, z) \in S$.

For a fixed $t \in [0, 1]$ we define a differential $(1, 0)$-form α_t on $\mathbf{C}^{2n}$ along C_t by

$$\alpha_t(z) = i_{Y_t(z)}(\omega). \tag{8}$$

Thus α_t is a section of $T^{*(1,0)}\mathbf{C}^{2n}$ over C_t, with coefficients which are real-analytic functions of $(t, z) \in S$.

Lemma 4.2. *If $g_t : T \to \mathbf{C}^{2n}$ is a real-analytic isotopy of embeddings of the circle, and if α_t is defined by (8), then*

$$\frac{d}{dt} \mathcal{A}(g_t) = \int_T g_t^* \alpha_t, \qquad 0 \leq t \leq 1.$$

Proof. Fix $0 \leq t_1 \leq 1$. Since $d\theta = -\omega$, Stokes' theorem gives

$$\mathcal{A}(g_{t_1}) - \mathcal{A}(g_0) = -\int_{C_{t_1}} \theta + \int_{C_0} \theta$$

$$= \int_{S(t_1)} \omega$$

$$= \int_0^{t_1} dt \int_T \omega\big(\partial g/\partial t(t, s), \partial g/\partial s(t, s)\big) \, ds$$

$$= \int_0^{t_1} dt \int_T \omega\big(Y_t(g_t(s)), \partial g/\partial s(t, s)\big) \, ds$$

$$= \int_0^{t_1} dt \int_T \langle \alpha_t(g_t(s)), \partial g_t/\partial s(s) \rangle$$

$$= \int_0^{t_1} dt \int_T g_t^* \alpha_t.$$

Here $\langle \cdot, \cdot \rangle$ indicates the natural pairing between forms and vectors. The lemma follows by differentiating both sides on t_1. □

We continue with the proof of Theorem 4.1. Since the isotopy g_t is such that $\mathcal{A}(g_t)$ is independent of t, Lemma 4.2 implies $\int_{C_t} \alpha_t = 0$ for all $t \in [0, 1]$. This means that the pull-back of the form α_t to C_t is exact on C_t, i.e., $i_{C_t}^* \alpha_t = dh_t$ for some real-analytic function $h_t: C_t \to \mathbf{C}$ that is determined up to a constant. Since α_t is real-analytic in (t, z), we can adjust these constants such that the function $h(t, z) = h_t(z)$, defined on $(t, z) \in S$, is real-analytic.

Lemma 4.3. *There exists a holomorphic function $H(t, z)$ in an open neighborhood of S in $\mathbf{C}^{1+2n}$ such that $H|_S = h$, and for each $t \in [0, 1]$ the function $H_t = H(t, \cdot)$ satisfies*

$$d_z H_t(z) = \alpha_t(z), \quad z \in C_t. \tag{9}$$

Here, d_z is the exterior derivative with respect to the variables $z \in \mathbf{C}^{2n}$.

We emphasize that (9) is an identity on the cotangent bundle $T^{*(1,0)}\mathbf{C}^{2n}$ at points of C_t.

Proof. For $t \in [0, 1]$ and $z \in C_t$ we denote by $N_{(t,z)} \subset T_z\mathbf{C}^{2n} \simeq \mathbf{C}^{2n}$ the complex hyperplane orthogonal to the tangent line $T_z C_t$ to the curve C_t at z. Set $N_{(t,z)}(r) = N_{(t,z)} \cap B(z; r)$, where $B(z; r) \subset \mathbf{C}^{2n}$ is the ball of radius $r > 0$ centered at z. If r is sufficiently small, then the set

$$N = \bigcup_{(t,z) \in S} \{t\} \times \left(z + N_{(t,z)}(r)\right)$$

is a real-analytic, Levi flat, Cauchy–Riemann (C-R) submanifold in $\mathbf{C}^{1+2n}$, foliated by the complex balls $N_{(t,z)}(r)$ of dimension $2n - 1$. We will write points of N as (t, z, v), where $(t, z) \in S$ and $v \in N_{(t,z)}(r)$. Let $\pi: N \to S$ be the projection onto S with fibers $\pi^{-1}(t, z) = N_{(t,z)}(r)$. We define a function $H: N \to \mathbf{C}$ by

$$H(t, z, v) = h(t, z) + \langle \alpha_t(z), v \rangle, \qquad (t, z) \in S, \ v \in N_{(t,z)}(r). \tag{10}$$

Clearly H is real-analytic and $H|_S = h$. Moreover, since H is complex linear on the fibers $N_{(t,z)}(r)$ of N, it is a C-R function on N. Thus H extends to a holomorphic function in a neighborhood of S in $\mathbf{C}^{1+2n}$ [5, p. 141]. We denote the extended function again by H.

It remains to show that for each $t \in [0, 1]$, the function $H_t = H(t, \cdot)$ satisfies (9). Fix $(t, z) \in S$ and a tangent vector $v \in T_z^{(1,0)}\mathbf{C}^{2n}$. If v is tangent to the curve C_t at z then $\langle dH_t(z), v \rangle = \langle dh_t(z), v \rangle = \langle \alpha_t(z), v \rangle$ since $H|_S = h$ and $dh_t = i_{C_t}^*(\alpha_t)$ on C_t. If on the other hand $v \in N_{(t,z)}$,

we observe that the first term on the right-hand side in (10) is constant and the second is linear in the direction v, and hence $\langle dH_t(z), v \rangle = \langle \alpha_t(z), v \rangle$. This proves Lemma 4.3. $\square$

Choose a neighborhood $U_t \subset \mathbf{C}^{2n}$ of C_t for $0 \leq t \leq 1$ such that the function H_t satisfying Lemma 4.3 is defined and holomorphic in U_t, and such that $\bigcup_{0 \leq t \leq 1} \{t\} \times U_t$ is an open neighborhood of S in $[0, 1] \times \mathbf{C}^{2n}$. We define a holomorphic vector field X_t on U_t by the equation

$$dH_t(z) = i_{X_t(z)}(\omega), \qquad z \in U_t, \ t \in [0, 1].$$

In other words, X_t is the exact Hamiltonian vector field on U_t with the energy function H_t. Since $d_z H_t = \alpha_t$ at points of C_t, it follows that $X_t(z) = Y_t(z)$ for $z \in C_t$, where Y_t is defined by (7). The flow $F_t(z)$ of X_t, which is defined for all $0 \leq t \leq 1$ and for all z in a smaller neighborhood $U \subset U_0$ of the curve C_0 in $\mathbf{C}^{2n}$, satisfies the conclusions of Theorem 4.1. This completes the proof for closed curves.

In case of arcs (when T is replaced by $I = [0, 1]$) the differential form α_t defined by (8) is automatically exact on the real-analytic arc $g_t(I) \subset \mathbf{C}^{2n}$. The rest of the proof is exactly the same as above. This completes the proof of Theorem 4.1. $\square$

Proof of the main theorem.   We will consider the case of closed curves; it will be obvious that the same proof applies to arcs. Let $g_0, g_1 \colon T \to \mathbf{C}^{2n}$ be closed, embedded, real-analytic curves such that $\mathcal{A}(g_0) = \mathcal{A}(g_1)$. Assume also that both curves are polynomially convex; this holds automatically if this common number is nonzero. By Lemma 3.1 there exists a piecewise real-analytic isotopy of embedding $g_t \colon T \to \mathbf{C}^{2n}$, $t \in [0, 1]$, connecting g_0 to g_1, such that $\mathcal{A}(g_t)$ is independent of t, and the curve $g_t(T)$ is polynomially convex for each t.

Consider first the case when the isotopy is real-analytic on all of $[0, 1]$. Let F_t be the isotopy as in Theorem 4.1, consisting of symplectic biholomorphic mappings, defined in a neighborhood $U \subset \mathbf{C}^{2n}$ of C_0, such that $F_t \circ g_0 = g_t$. Recall that the infinitesimal generator X_t of F_t is exact Hamiltonian, associated to a holomorphic function H_t. Since every curve C_t is polynomially convex, we can choose the neighborhood U such that $U_t = F_t(U)$ is Runge in $\mathbf{C}^{2n}$ for each $t \in [0, 1]$ [8]. Hence H_t can be approximated by holomorphic polynomials, uniformly on compacts in U_t. Therefore X_t is the limit of polynomial Hamiltonian vector fields on $\mathbf{C}^{2n}$, uniformly on compacts in U_t. By Proposition 2.3 each map F_t, and in particular the time one map $F = F_1$, is the limit of symplectic holomorphic automorphisms of $\mathbf{C}^{2n}$ in some neighborhood of C_0. Thus F satisfies the conclusion of the main theorem.

Consider now a piecewise real-analytic isotopy $g_t, 0 \leq t \leq 1$. Let $0 = t_0 < t_1 < \cdots < t_m < 1$ be such that the isotopy is real-analytic on each interval $t_{j-1} \leq t \leq t_j$. Let F_j for $1 \leq j \leq m$ be a symplectic biholomorphic map, defined near $g_{t_{j-1}}(T)$ and satisfying $F_j \circ g_{t_{j-1}} = g_{t_j}$, which is a limit of symplectic holomorphic automorphisms of $\mathbf{C}^{2n}$. Such maps exist by what was already proved. Then the map $F = F_m \circ F_{m-1} \circ \cdots \circ F_1$ satisfies the conclusion of the main theorem.

In the case of arcs, recall that every real-analytic arc in $\mathbf{C}^m$ is polynomially convex [14]. Since the extension result in Lemma 4.3 holds without any further condition, the proof given above applies and gives a map F as in the main theorem.   □

Remark.   In the case of real symplectic manifolds one can extend a local Hamiltonian isotopy F_t, defined in a tube around the curve C_0, to the entire manifold by simply multiplying the Hamiltonian energy function H_t of the infinitesimal genarator X_t with a smooth cut-off function which vanishes outside a suitable neighborhood of the curve C_t. This gives a global Hamiltonian isotopy, consisting of symplectomorphisms of M, which is compactly supported. Of course this is impossible in the holomorphic case, and the best one can do is to approximate F_t as above.

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