Holomorphic Automorphisms of C^n: A Survey

Franc Forstneric

Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA

0. Introduction

In this paper we survey some recent results on holomorphic automorphism groups of the complex Euclidean space $\mathbf{C}^n$, with emphasis on question of approximation of biholomorphic maps and approximation of diffeomorphisms on certain classes of submanifolds in $\mathbf{C}^n$ by automorphisms of $\mathbf{C}^n$. We also consider flows generated by holomorphic vector fields, especially by holomorphic Hamiltonian fields, and the global behavior of their orbits. Finally we collect the known classification results for holomorphic flows on the plane $\mathbf{C}^2$.

We do not consider questions of holomorphic dynamics of automorphisms, except for flows of Hamiltonian holomorphic vector fields (sect. 9); for dynamics in several variables we refer the reader to the recent survey [16] by Fornæss and Sibony. For the algebraic aspects of this theory we refer the reader to the survey in preparation by H. Kraft [26].

We denote by $\mathrm{Aut}\mathbf{C}^n$ the group of all holomorphic automorphisms of $\mathbf{C}^n$, and by $\mathrm{Aut}_1\mathbf{C}^n$ its subgroup consisting of automorphisms with Jacobian one ($JF = 1$). While the group $\mathrm{Aut}\mathbf{C}^1$ consists only of affine linear maps $z \mapsto az + b$ ($a \in \mathbf{C}^* = \mathbf{C}\backslash\{0\}$, $b \in \mathbf{C}$), these automorphism groups are very large and complicated when $n > 1$ which we shall assume throughout this paper. In particular, when $n > 1$, both groups are infinite dimensional Lie groups in the topology of uniform convergence on compacts in $\mathbf{C}^n$.

An important study of these groups, and especially of their actions on countable subsets of $\mathbf{C}^n$, was done by Rosay and Rudin [32]. They proved that for any two countable *dense* subsets $X, Y \subset \mathbf{C}^n$ there is an $F \in \mathrm{Aut}\mathbf{C}^n$ such that $F(X) = Y$. On the other hand, they showed that the situation is very different for countable *discrete* subsets $E \subset \mathbf{C}^n$: Every such subset can be mapped onto the standard arithmetic progression in $\mathbf{C}\times\{0\}^{n-1} \subset \mathbf{C}^n$ by an injective holomorphic map $F\colon \mathbf{C}^n \to \mathbf{C}^n$ with $JF = 1$, but in general not by any automorphism of $\mathbf{C}^n$. (Recall that an injective holomorphic map $F\colon \mathbf{C}^n \to \mathbf{C}^n$ whose image is a proper subset of $\mathbf{C}^n$ is called a Fatou-Bieberbach map, and its image $F(\mathbf{C}^n) \subset \mathbf{C}^n$ is a Fatou-Bieberbach domain.) If we call two countable discrete sets $E_0, E_1 \subset \mathbf{C}^n$ equivalent when $F(E_0) = E_1$ for some $F \in \mathrm{Aut}\mathbf{C}^n$, then there exist infinitely many equivalence classes, and there exist discrete sets $E \subset \mathbf{C}^n$ such that the only automorphism of $\mathbf{C}^n$ mapping E onto E is the identity. The paper [32] also contains many interesting results on Fatou-Bieberbach maps, but we shall not consider this topic in the present paper.

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The main tool used by Rosay and Rudin were special automorphisms of $\mathbf{C}^n$, called *shears* (resp. generalized shears); see sect. 1 below. They raised the question whether the subgroup of $\mathrm{Aut}\mathbf{C}^n$ consisting of finite compositions of (generalized) shears is dense in $\mathrm{Aut}\mathbf{C}^n$ or perhaps even equal to it.

Both of these questions were answered by E. Andersén [4] in the volume preserving case and by Andersén and Lempert [5] in the general case: The shear subgroup of $\mathrm{Aut}\mathbf{C}^n$ resp. $\mathrm{Aut}_1\mathbf{C}^n$ is dense in, but not equal to the whole groups (Corollary 2.2 below). The main step in the proof of the density result in [4,5] is a decomposition theorem, Proposition 4.1 below, to the effect that every polynomial holomorphic vector field on $\mathbf{C}^n$ is a finite sum of *complete* polynomial fields whose flows consist of (generalized) shears.

This approach was further developed by J.-P. Rosay and the author in [21]. An automorphism $F \in \mathrm{Aut}\mathbf{C}^n$ is viewed as the time one map in the flow $F_t(z)$ $(0 \leq t \leq 1)$ of a time-dependent entire holomorphic vector field X_t. Explicitly, $F_t(z)$ is the solution at time $t \in [0, 1]$ of an ordinary differential equation

$$\dot{Z} = X_t(Z), \qquad Z(0) = z,$$

where X_t is a holomorphic vector field on $\mathbf{C}^n$ for each fixed t which is of class C^1 in both variables (t, z). We can approximate X_t on short time intervals $[j/N, (j+1)/N] \subset [0, 1]$ by time independent polynomial vector fields Y_j. Applying the result of Andersén and Lempert on decomposition of polynomial holomorphic vector fields into finite sums of complete shear fields, one concludes that the flow of each Y_j (and therefore the flow of the original time dependent field X_t) can be approximated by compositions of (generalized) shears. This shows in fact that the flow of any globally defined holomorphic vector field X_t on $\mathbf{C}^n$, wherever it is defined, is approximable by automorphisms of $\mathbf{C}^n$. In [19] we extended these results to symplectic holomorphic automorphisms of $\mathbf{C}^{2n}$ by considering flows of holomorphic Hamiltonian vector fields.

In [21] and in the subsequent papers [17,19,20,22] we used this technique, together with new results on generic polynomial convexity of totally real submanifolds in $\mathbf{C}^n$, to study question of approximation of smooth mappings $F: M \to \mathbf{C}^n$ on compact, totally real, polynomially convex submanifolds $M \subset \mathbf{C}^n$ by holomorphic automorphisms of $\mathbf{C}^n$, as well as the problem of global $\mathcal{G}$-equivalence of such submanifolds in $\mathbf{C}^n$ with respect to various automorphism groups $\mathcal{G} \subset \mathrm{Aut}\mathbf{C}^n$.

In another direction we survey results on dynamics of holomorphic vector fields on $\mathbf{C}^n$, with emphasis on global questions such as completeness in real and complex time, analysis of complex orbits, abundance or nonabundance of bounded resp. exploding orbits, classification of flows induced by complete holomorphic vector fields, etc.

The paper is organized as follows. In section 1 we introduce the shear groups and the holomorphic symplectic groups. In section 2 we collect results on approximation of biholomorphic mappings between domains in $\mathbf{C}^n$ by automorphisms of $\mathbf{C}^n$. In sections 3–4 we recall some properties of flows of holomorphic vector fields and indicate proofs of results of section 2. In sections 5–7 we survey results on approximation by automorphisms on certain classes of real submanifolds in $\mathbf{C}^n$. In section 8 we mention some results on complex orbits of holomorphic vector fields. In section 9 we survey results on dynamics of holomorphic Hamiltonian vector fields on $\mathbf{C}^{2n}$. Section 10 contains results on classification of one-parameter subgroups in the automorphism group $\mathrm{Aut}\mathbf{C}^2$ of the plane.

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1. Automorphism groups of $\mathbf{C}^n$

Let $z = (z_1, \ldots, z_n)$ be the complex coordinates on $\mathbf{C}^n$. Recall that $\mathrm{Aut}\mathbf{C}^n$ is the group of all holomorphic automorphisms of $\mathbf{C}^n$ and $\mathrm{Aut}_1\mathbf{C}^n$ is the group of all automorphisms with Jacobian one. These preserve the complex volume form $\Omega = dz_1 \wedge \cdots \wedge dz_n$, in the sense that $F^*\Omega = \Omega$.

On the even dimensional spaces $\mathbf{C}^{2n}$ $(n \geq 1)$ we also have the group of *symplectic* holomorphic automorphisms. Let ω be the complex symplectic form on $\mathbf{C}^{2n}$:

$$\omega = \sum_{j=1}^{n} dz_j \wedge dz_{n+j}. \tag{1}$$

A holomorphic map $F\colon D \subset \mathbf{C}^{2n} \to \mathbf{C}^{2n}$ is said to be *symplectic holomorphic* if $F^*\omega = \omega$. The symplectic holomorphic automorphism group of $\mathbf{C}^{2n}$ is

$$\mathrm{Aut}_{\mathrm{sp}}\mathbf{C}^{2n} = \{F \in \mathrm{Aut}\mathbf{C}^{2n} \colon F^*\omega = \omega\}. \tag{2}$$

Since ω^n is a constant multiple of the volume form Ω on $\mathbf{C}^{2n}$, every symplectic holomorphic map has Jacobian one. On $\mathbf{C}^2$ these two classes of maps coincide. Thus

$$\mathrm{Aut}_{\mathrm{sp}}\mathbf{C}^{2n} \subset \mathrm{Aut}_1\mathbf{C}^{2n} \subset \mathrm{Aut}\mathbf{C}^{2n}, \qquad \mathrm{Aut}_{\mathrm{sp}}\mathbf{C}^2 = \mathrm{Aut}_1\mathbf{C}^2.$$

Each of these groups contains the corresponding linear subgroup:

$$GL(n,\mathbf{C}) \subset \mathrm{Aut}\mathbf{C}^n, \quad SL(n,\mathbf{C}) \subset \mathrm{Aut}_1\mathbf{C}^n, \quad Sp(n,\mathbf{C}) \subset \mathrm{Aut}_{\mathrm{sp}}\mathbf{C}^{2n}.$$

The automorphism groups introduced above contain the following complex one parameter subgroups (with complex parameter $t \in \mathbf{C}$):

$$F_t(z) = z + tf(\Lambda z)v, \qquad z \in \mathbf{C}^n, \tag{3}$$

$$G_t(z) = z + \big(e^{tg(\Lambda z)} - 1\big)\langle z, v\rangle v, \qquad z \in \mathbf{C}^n, \tag{4}$$

$$S_t(z) = z + th(\omega(z,v))v, \qquad z \in \mathbf{C}^{2n}. \tag{5}$$

Here $v \in \mathbf{C}^n$ is a fixed vector of length one, $\Lambda\colon \mathbf{C}^n \to \mathbf{C}^k$ is a $\mathbf{C}$-linear map for some $k < n$ satisfying $\Lambda v = 0$, $\langle z, v\rangle = \sum z_j \bar{v}_j$, f and g are entire functions on $\mathbf{C}^k$, h is an entire function on $\mathbf{C}$, and ω is the symplectic form (1).

Every map of the form (3) or (5) (for a fixed $t \in \mathbf{C}$) is called a *shear*, and maps (4) are *generalized shears*. This terminology was introduced by Rosay and Rudin [32], although automorphisms of this type have been used before. Notice that (5) is a special case of (3).

One can easily verify that in all three cases we have $F_s \circ F_t = F_{s+t}$ for $s, t \in \mathbf{C}$, which means that the above are complex one parameter subgroups of the respective automorphism groups.

Write $z = (z', z_n)$, where $z' = (z_1, \ldots, z_{n-1}) \in \mathbf{C}^{n-1}$.

1.1 Proposition. *(i) The group (3) is $SU(n, \mathbf{C})$-conjugate to a group*

$$\tilde{F}_t(z) = (z', z_n + t\tilde{f}(z')), \qquad \tilde{F}_t \in \mathrm{Aut}_1\mathbf{C}^n.$$

(ii) The group (4) is $U(n)$-conjugate to a group

$$\tilde{G}_t(z) = (z', \exp(t\tilde{g}(z'))z_n), \qquad \tilde{G}_t \in \mathrm{Aut}\mathbf{C}^n.$$

(iii) The group (5) is $Sp(n, \mathbf{C})$-conjugate to a group

$$\tilde{S}_t(z) = (z_1, \ldots, z_n, z_{n+1} + t\tilde{h}(z_1), z_{n+2}, \ldots, z_{2n}), \qquad \tilde{S}_t \in \mathrm{Aut}_{\mathrm{sp}}\mathbf{C}^{2n}.$$

The proof is straightforward; see the appendix in [17] and section 5 in [5]. It follows that the groups (3), (4), and (5) belong to, respectively, $\mathrm{Aut}_1\mathbf{C}^n$, $\mathrm{Aut}\mathbf{C}^n$, and $\mathrm{Aut}_{\mathrm{sp}}\mathbf{C}^{2n}$. The shears of type (5) are called *symplectic*.

1.2 Definition. *(i) $\mathcal{S}_1(n)$ is the subgroup of $\mathrm{Aut}_1\mathbf{C}^n$ consisting of finite compositions of shears (3);*

(ii) $\mathcal{S}(n)$ is the subgroup of $\mathrm{Aut}\mathbf{C}^n$ consisting of finite compositions of (generalized) shears (3) and (4);

(iii) $\mathcal{S}_{sp}(n)$ is the subgroup of $\mathrm{Aut}_{\mathrm{sp}}\mathbf{C}^{2n}$ consisting of finite compositions of symplectic shears (5).

It is easily verified that

$$GL(n, \mathbf{C}) \subset \mathcal{S}(n), \quad SL(n, \mathbf{C}) \subset \mathcal{S}_1(n), \quad Sp(n, \mathbf{C}) \subset \mathcal{S}_{sp}(n).$$

In fact, the group $SL(n, \mathbf{C})$ is generated by the elementary matrices with diagonal entries one and only one other nonzero entry. Clearly such a matrix represents a linear shear (3) in a coordinate direction. To get the group $GL(n, \mathbf{C})$ one has to add to these the dilations in coordinate directions, and these are compositions of linear generalized shears (4). Finally, the linear symplectic group $Sp(n, \mathbf{C})$ is generated by symplectic shears (5) with $h(\zeta) = c\zeta$.

Another group of special interest is the group $\mathcal{P}^n$ of all polynomial holomorphic automorphisms of $\mathbf{C}^n$ and its subgroup $\mathcal{P}^n_1$ of polynomial automorphisms with Jacobian one. Since the Jacobian of every polynomial automorphism is constant, we have $\mathcal{P}^n = \mathcal{P}^n_1 \times \mathbf{C}^*$. On $\mathbf{C}^{2n}$ one also have the symplectic polynomial group $\mathcal{P}^{2n}_{sp} \subset \mathrm{Aut}_{\mathrm{sp}}\mathbf{C}^{2n}$.

Notice that every shear of type (3) and (5) can be approximated by polynomial shears of the same type by approximating the function f resp. h by polynomials. Thus the polynomial shear groups generated by polynomial automorphisms of type (3) resp. (5) are dense in the corresponding shear groups. This is not the case for generalized shears (4) with nonconstant Jacobian.

2. Approximation by shears

Recall that a domain $D \subset \mathbf{C}^n$ (not necessarily pseudoconvex) is Runge if every holomorphic function in D is a limit of polynomials, uniformly on compacts in D. Every convex domain in $\mathbf{C}^n$ is Runge, and so is every starshaped domain [12]. (The last result is already mentioned in Behnke and Thullen [9], p.130, who attribute it to B. Almer.)

We are interested in the problem of approximating a biholomorphic map $F: D \to D'$ between Runge domains in $\mathbf{C}^n$ by automorphisms of $\mathbf{C}^n$. The following result of fundamental importance is due to Andersén [4] and Andersén and Lempert [5] (case (c) was established by the author in [18].) The topology is that of uniform convergence on compact sets in D.

2.1 Theorem. *Let $F: D \to F(D) \subset \mathbf{C}^n$ ($n \geq 2$) be a biholomorphic mapping from a convex (or starshaped) domain $D \subset \mathbf{C}^n$ onto a Runge domain $F(D)$. Then*

(a) F is the limit of a sequence $\Phi_j|_D$, $\Phi_j \in \mathcal{S}(n)$;

(b) if $F^*\Omega = \Omega$ ($JF = 1$), then F is the limit of a sequence $\Phi_j \in \mathcal{S}_1(n)$;

(c) if $n = 2m$ and if $F^*\omega = \omega$, then F is the limit of a sequence $\Phi_j \in \mathcal{S}_{sp}(m)$.

2.2 Corollary. *(i) The group $\mathcal{S}_1(n)$ is dense in $\mathrm{Aut}_1\mathbf{C}^n$ for $n \geq 2$;*

(ii) the group $\mathcal{S}(n)$ is dense in $\mathrm{Aut}\mathbf{C}^n$ for $n \geq 2$;

(iii) the group $\mathcal{S}_{sp}(n)$ is dense in $\mathrm{Aut}_{\mathrm{sp}}\mathbf{C}^{2n}$ for $n \geq 1$.

It is known [4,5] that the inclusions in (i) and (ii) are proper, i.e., there exist automorphisms which are not finite compositions of shears. Most likely the same is true in case (iii). It follows that the group of polynomial shears (3) resp. (5) in also dense in $\mathrm{Aut}_1\mathbf{C}^n$ resp. $\mathrm{Aut}_{\mathrm{sp}}\mathbf{C}^{2n}$.

2.3 Corollary. *Every injective holomorphic map (a Fatou-Bieberbach map) $F: \mathbf{C}^n \to \mathbf{C}^n$ onto a Runge domain $F(\mathbf{C}^n) \subset \mathbf{C}^n$ is a limit of automorphisms of $\mathbf{C}^n$.*

It is an open question whether every Fatou-Bieberbach domain in $\mathbf{C}^n$ is Runge.

The following example from [21] shows that, in general, Theorem 2.1 does not hold for an arbitrary Runge domain $D \subset \mathbf{C}^n$. Further examples can be found in [5] and [21].

Example: The curve

$$M = \{(e^{i\theta}, e^{-i\theta}): \theta \in \mathbf{R}\} \subset \mathbf{C}^2$$

is polynomially convex. The mapping

$$F(z,w) = \big(z, w + (1 - zw)(1/z - 1)\big)$$

is biholomorphic near M and fixes M pointwise. There is a basis of strongly pseudoconvex Runge neighborhoods D of M in $\mathbf{C}^2$ such that $F(D)$ is also Runge in $\mathbf{C}^2$. However, the Jacobian of F equals $JF(z,w) = z$, and its winding number along M is one. Hence F can not be approximated by automorphisms of $\mathbf{C}^2$ (or even by diffeomorphisms of $\mathbf{C}^2$!) in any open neighborhood of M.

The following generalization of Theorem 2.1 to 'isotopies' of biholomorphic maps is very useful in applications, and it can be proved by the same methods. Case (i) was proved in [21], case (ii) in [20], and case (iii) in [19].

2.4 Theorem: *Let $D \subset \mathbf{C}^n$ $(n \geq 2)$ be a Runge domain and let $F_t: D \to \mathbf{C}^n$ be a biholomorphic mapping for each $t \in [0,1]$, of class $\mathcal{C}^2$ in $(t,z) \in [0,1] \times D$, such that F_0 is the identity on D and the domain $F_t(D)$ is Runge in $\mathbf{C}^n$ for each $0 \leq t \leq 1$. Then*

(i) F_t is a limit of automorphisms of $\mathbf{C}^n$ for each $t \in [0,1]$.

(ii) If $JF_t = 1$ for each $t \in [0,1]$ and if D is a domain of holomorphy satisfying $H^{n-1}(D;\mathbf{C}) = 0$, then F_t is a limit of volume preserving automorphisms of $\mathbf{C}^n$ for each $t \in [0,1]$.

(iii) If n is even, if $F_t^*\omega = \omega$ for each $t \in [0,1]$, and if D is a domain of holomorphy satisfying $H^1(D;\mathbf{C}) = 0$, then F_t is a limit of symplectic holomorphic automorphisms of $\mathbf{C}^n$ for each t.

Examples in [19] and [20] show that the conclusions in (ii) and (iii) do not hold without the respective cohomology condition on D. See also section 6 below. A sharper version of the last result which includes regularity with respect to the parameter $x \in \mathbf{R}^k$ was proved in [17, Theorem 1.1]. We only consider the group $\mathrm{Aut}\mathbf{C}^n$.

2.5 Theorem. *Let $\Omega \subset \mathbf{C}^n$ be a Runge domain, and let B be the closed unit ball in $\mathbf{R}^k$. Assume that $F: B \times \Omega \to \mathbf{C}^n$ is a mapping of class $\mathcal{C}^p$ $(0 \leq p < \infty)$ such that for each $x \in B$, $F_x = F(x,\cdot): \Omega \to \mathbf{C}^n$ is a biholomorphic mapping onto a Runge domain $\Omega_x \subset \mathbf{C}^n$, and the map F_0 is the identity on Ω. Then for each compact set $K \subset \Omega$ and each $\epsilon > 0$ there exists a smooth map $\Phi: B \times \mathbf{C}^n \to \mathbf{C}^n$ such that $\Phi_x = \Phi(x,\cdot)$ is a holomorphic automorphism of $\mathbf{C}^n$ for every $x \in B$ and $||F - \Phi||_{\mathcal{C}^p(B\times K)} < \epsilon$.*

We will indicate the proof of these results at the end of section 4 below.

3. Flows of holomorphic vector fields

In this section we recall some basic notions concerning flows of vector fields.

Suppose that for each $t \in [0,1]$, X_t is a holomorphic vector field on a domain Ω_t in a complex manifold $\mathcal{M}$. The flow $F_{t,s}$ is the solution of the ordinary differential equation

$$\frac{d}{dt}F_{t,s}(z) = X_t\big(F_{t,s}(z)\big), \quad F_{s,s}(z) = z \in \Omega_s.$$

Assume that there is a domain $D_0 \subset \Omega_0$ such that the flow $F_t(z) = F_{t,0}(z)$ exists for all $0 \leq t \leq 1$ and all $z \in D_0$. Then $F_t: D_0 \to F_t(D_0) = D_t \subset \Omega_t$ is a biholomorphic map for each $t \in [0,1]$, F_0 is the identity on D_0, and $F_{s,t} = F_t \circ F_s^{-1}$ on D_s.

If the field $X_t = X$ is time independent, its flow F_t is a local one parameter automorphism group, i.e., $F_{t+s} = F_t \circ F_s$ where both sides are defined. In particular, $F_{t,s} = F_{t-s}$.

3.1 Definition. *A holomorphic vector field X on a complex manifold $\mathcal{M}$ is said to be complete in real (resp. complex) time if for every point $z \in \mathcal{M}$ the flow $F_t(z)$ of X exists for all real (resp. complex) values of t.*

The flow of a complete holomorphic vector field is a real (resp. complex) one parameter subgroup of the automorphism group $\mathrm{Aut}\mathcal{M}$.

The Lie derivative of a tensor α with respect to a vector field X_s is defined by

$$L_{X_s}\alpha = \frac{d}{dt}(F^*_{t,s}\alpha)\big|_{t=s}.$$

For any pair t, s we have

$$\frac{d}{dt}F^*_{t,s}\alpha = F^*_{t,s}(L_{X_t}\alpha).$$

If α is a differential form, then

$$L_{X_t}\alpha = d(X_t\rfloor\alpha) + X_t\rfloor(d\alpha),$$

where $\rfloor$ denotes the contraction (see [1, p.121]). Combining the last two identities we get

3.2 Proposition. *Let $F_{t,s}$ be the flow of a time dependent vector field X_t as above, and let α be a closed differential form on $\mathcal{M}$. The following are equivalent:*

(a) $F^*_{t,s}\alpha = \alpha$ for all $0 \le s, t \le 1$;

(b) the form $X_t\rfloor\alpha$ is closed on D_t for all $0 \le t \le 1$.

From now on let $\mathcal{M} = \mathbf{C}^n$. Let Ω the standard volume form and ω the standard symplectic form (1). Recall that the divergence $\mathrm{div}X$ of a holomorphic vector field X with respect to a volume form Ω is the unique holomorphic function satisfying $d(X\rfloor\Omega) = (\mathrm{div}\, X)\Omega$.

3.3 Definition. *(a) A holomorphic vector field X on a domain $D \subset \mathbf{C}^{2n}$ is Hamiltonian (resp. exact Hamiltonian) if the 1-form $X\rfloor\omega$ is closed (resp. exact) on D.*

(b) A holomorphic vector field X on $D \subset \mathbf{C}^n$ is divergence free (resp. exact divergence free) if the $(n-1)$-form $X\rfloor\Omega$ is closed (resp. exact) on D.

3.4 Corollary. *Let X_t $(0 \le t \le 1)$ be a time dependent holomorphic vector field on $D_t \subset \mathbf{C}^n$, and let $F_{t,s}\colon D_s \to D_t$ be its flow as above. The following are equivalent:*

(a) The map $F_{t,s}$ is symplectic (resp. volume preserving) for each $0 \le s, t \le 1$;

(b) The vector field X_t is Hamiltonian (resp. divergence free) on D_t for each $0 \le t \le 1$.

In the first case (X_t Hamiltonian) we assume of course that n is even. For later reference we recall some basic notions of Hamiltonian mechanics, refering the reader to [1] and [6] for details. We denote the coordinates in $\mathbf{C}^{2n}$ by (z, w), with $z, w \in \mathbf{C}^n$. Then $\omega = \sum_{j=1}^n dz_j \wedge dw_j$. A holomorphic vector field X on a domain $D \subset \mathbf{C}^{2n}$ is exact Hamiltonian if

$$X\rfloor\omega = dH \tag{6}$$

for some holomorphic function H on D. If D is a domain of holomorphy satisfying $H^1(D;\mathbf{C}) = 0$, then every Hamiltonian holomorphic vector field on D is exact Hamiltonian. The function H, which is determined up to an additive constant (when D is connected), is called the *energy function* (or simply the Hamiltonian) of X. Conversely, every $H \in \mathcal{O}(D)$ determines an exact Hamiltonian holomorphic vector field $X = X_H$ on D by (6). It is easily verified that

$$X_H = \sum_{j=1}^{n} \frac{\partial H}{\partial w_j}\frac{\partial}{\partial z_j} - \frac{\partial H}{\partial z_j}\frac{\partial}{\partial w_j} = (H_w, -H_z). \tag{7}$$

In particular, every entire functions H on $\mathbf{C}^{2n}$ can be regarded as a holomorphic Hamiltonian, giving rise to a Hamiltonian vector field X_H on $\mathbf{C}^{2n}$.

The (local) flow F_t of the field X_H remains in the level sets of H and it preserves the symplectic form ω ($F_t^*\omega = \omega$) on the set where F_t is defined. This holds both for real as well as complex values of time t.

4. Decomposition of polynomial vector fields into complete shear fields

Simple examples of holomorphic vector fields on $\mathbf{C}^n$ which are complete in complex time are the *shear vector fields* which generate the shear subgroups (3)–(5):

$$X(z) = f(\Lambda z)v, \qquad z \in \mathbf{C}^n, \tag{8}$$
$$Y(z) = g(\Lambda z)\langle z, v\rangle v, \qquad z \in \mathbf{C}^n, \tag{9}$$
$$W(z) = h(\omega(z, v))v, \qquad z \in \mathbf{C}^{2n}. \tag{10}$$

Fields of type (8) have divergence zero, and fields of type (10) are Hamiltonian.

The following decomposition result plays a central role in proof of the approximation theorems in section 2. Part (ii) is due to Andersén [4], part (i) to Andersén and Lempert [5], and part (iii) to the author [18]. (See the appendix in [17] for a short proof of (i) and (ii).)

4.1 Proposition: *Let X be a polynomial holomorphic vector field on $\mathbf{C}^n$.*

(i) X is a finite sum of shear vector fields of type (8) and (9).

(ii) If $\mathrm{div}X = 0$, then X is a finite sum of shear vector fields of type (8).

(iii) If $n = 2m$ and X is Hamiltonian, then X is a finite sum of Hamiltonian shear vector fields of type (10).

Of course there exist other types of complete holomorphic vector fields on $\mathbf{C}^n$. A particular example on $\mathbf{C}^2$ is $X(z, w) = (z^2w, -zw^2)$, with the flow

$$F_t(z, w) = \left(ze^{tzw}, we^{-tzw}\right), \quad t \in \mathbf{C}.$$

Andersén proved [4] that for $t \neq 0$ the automorphism F_t is not a finite composition of shears. For further examples of complete holomorphic vector fields see [2,6,34] and section 10 below.

The following result is standard; see e.g. [1, p.92]:

4.2 Proposition. *Let X be a vector field of class $\mathcal{C}^1$ on a manifold M which is a finite sum $X = \sum_{j=1}^m X_j$ of $\mathcal{C}^1$ fields X_j. Denote by F_t^j the flow of X_j and by F_t the flow of X. Then*

$$F_t(x) = \lim_{N\to\infty} \left(F^m_{t/N} \circ \cdots \circ F^1_{t/N}\right)^N (x).$$

The convergence is uniform on every compact set $K \subset M$ such that $F_t(x)$ is defined for all $x \in K$.

Combining the above two propositions and observing that the flow of every shear vector field (8)–(10) on $\mathbf{C}^n$ consists of shear automorphisms (3)–(5) we get

4.3 Proposition: *Let X be a holomorphic vector field defined on all of $\mathbf{C}^n$. Let D be open subset of $\mathbf{C}^n$ and let $t_0 > 0$. Assume that the flow $F_t(z)$ is defined for all $0 \le t \le t_0$ with arbitrary initial condition $F_0(z) = z \in D$. Then the following hold for each t, $0 \le t \le t_0$:*

(i) $F_t \colon D \to F_t(D) \subset \mathbf{C}^n$ is a biholomorphic map which can be approximated, uniformly on compact sets in D, by automorphisms $\Phi \in \mathcal{S}(n)$.

(ii) If $\operatorname{div}X = 0$, then F_t can be approximated by automorphisms $\Phi \in \mathcal{S}_1(n)$.

(iii) If $n = 2m$ and X is Hamiltonian, then F_t can be approximated by symplectic automorphisms $\Phi \in \mathcal{S}_{sp}(n)$.

The same is true for time dependent holomorphic vector fields on $\mathbf{C}^n$.

Sketch of proof of Theorem 2.4: We consider the family of biholomorphic maps $F_t \colon D \to D_t$ $(0 \le t \le 1)$ as the flow of a time dependent holomorphic vector field X_t, defined on the domain D_t for each $t \in [0,1]$. Since D_t is Runge in $\mathbf{C}^n$ for each t, we can approximate X_t on D_t by a polynomial holomorphic vector field Y_t. Similarly, if X_t is exact divergence zero (resp. exact Hamiltonian) on D_t, we can approximate X_t on D_t by polynomial divergence zero (resp. Hamiltonian) vector fields; see [20] resp. [19]. Such approximation is always possible if D_t is pseudoconvex and $H^{n-1}(D_t;\mathbf{C}) = 0$ (resp. $H^1(D_t;\mathbf{C}) = 0$). Finally, by splitting the time interval into short subintervals and approximating Y_t on every subinterval by a time independent field it suffices to consider the case when Y is time independent. The result now follows from Proposition 4.3.

Theorem 2.1 is proved by first reducing it to the case $F(0) = 0$, $DF(0) = I$, and applying Theorem 2.4 to the isotopy $F_t(z) = F(tz)/t$ for $0 < t \le 1$, $F_0(z) = z$. It is easily verified that, since $F(D)$ is Runge in $\mathbf{C}^n$, the domain $F_t(D)$ is Runge for every t.

5. Global holomorphic equivalence of real-analytic submanifolds

The motivation for results in this section came from a result of J.-P. Rosay [31] to the effect that one can approximately straighten an arbitrary smooth arc in $\mathbf{C}^n$ $(n \ge 2)$ by automorphisms of $\mathbf{C}^n$. He observed that, for real-analytic arcs, this is a consequence of Theorem 2.1 above by Andersén and Lempert. The notion of global holomorphic equivalence was developed further in the papers by Rosay and the author [21] and in [19,20]. In this section we consider the real-analytic manifolds; the case of smooth manifolds is postponed to section 7.

5.1 Definition. *Let $\mathcal{G} \subset \mathrm{Aut}\mathbf{C}^n$ be any group of holomorphic automorphisms of $\mathbf{C}^n$. A real-analytic diffemorphism $F: M_0 \to M_1$ between compact, embedded, real-analytic submanifolds $M_0, M_1 \subset \mathbf{C}^n$ is said to be a $\mathcal{G}$-equivalence if F extends to a biholomorphic mapping in a neighborhood U of M_0 which can be approximated, uniformly on U, by automorphisms $\Phi \in \mathcal{G}$. Two real-analytic embeddings $f_0, f_1: M \hookrightarrow \mathbf{C}^n$ of a compact real-analytic manifold M into $\mathbf{C}^n$ are $\mathcal{G}$-equivalent if the diffeomorphism $F = f_1 \circ f_0^{-1}$ is a $\mathcal{G}$-equivalence.*

Remark 1: Observe that, if a sequence $\Phi_j \in \mathcal{G}$ converges to a biholomorphic map F in a neighborhood of M_0, then by the maximum principle (applied to sequences $\{\Phi_j\}$ and $\{\Phi_j^{-1}\}$) the same sequence converges to a biholomorphic map in a neighborhood of the polynomial hull $\hat{M}_0$. The extended map takes the hull $\hat{M}_0$ biholomorphically onto $\hat{M}_1$. In particular, M_0 is polynomially convex if and only if M_1 is.

Remark 2: If $\mathcal{G}' \subset \mathcal{G}$ are automorphism groups such that $\mathcal{G}'$ is dense in $\mathcal{G}$, then every $\mathcal{G}$-equivalence is at the same time a $\mathcal{G}'$-equivalence.

The following basic result for the case $\mathcal{G} = \mathrm{Aut}\mathbf{C}^n$ is due to Rosay and the author [21, Theorem 3.1]:

5.2 Theorem. *Let $M_0, M_1 \subset \mathbf{C}^n$ $(n \geq 2)$ be compact, embedded real-analytic submanifolds (with or without boundary) which are totally real and polynomially convex. A real-analytic diffeomorphism $F: M_0 \to M_1$ is an $\mathrm{Aut}\mathbf{C}^n$-equivalence if and only if there exists a one parameter family of diffeomorphisms $F_t: M_0 \to M_t \subset \mathbf{C}^n$ $(0 \leq t \leq 1)$, of class $\mathcal{C}^2$ in $(t, z) \in [0,1] \times M_0$, such that F_0 is the identity on M_0, $F_1 = F$, and $M_t = F_t(M_0)$ is totally real and polynomially convex for every $t \in [0,1]$.*

A short proof can be found in [17]. Here is the main idea of the proof. By approximation we may assume that $\{F_t\}$ is real-analytic also in t. Its infinitesimal generator X_t is a vector field of type $(1,0)$ on $\mathbf{C}^n$, defined and real-analytic along M_t. Since M_t is totally real and polynomially convex, X_t can be extended to a holomorphic vector field in a neighborhood of M_t which is a uniform limit of polynomial holomorphic vector fields. The result now follows from Proposition 4.3.

Combining Theorem 5.2 with a result on genericity of polynomial convexity of low dimensional totally real submanifolds [21,17] we obtain ([17], Corollary 1):

5.3 Corollary. *Let M be a compact real-analytic manifold and let $f_0, f_1: M \to \mathbf{C}^n$ be real-analytic, totally real, polynomially convex embeddings. If $\dim M \leq 2n/3$, then f_0 and f_1 are $\mathrm{Aut}\mathbf{C}^n$-equivalent.*

Explicit examples to which Corollary 5.3 applies are curves in $\mathbf{C}^n$ for $n \geq 2$ and surfaces in $\mathbf{C}^n$ for $n \geq 3$. In this case we have the following more precise results (see [21], Theorem 4.2 and Corollary 6.3):

5.4 Theorem. *Let T be the circle. If $f_0, f_1: T \to \mathbf{C}^n$ $(n \geq 2)$ are real-analytic embeddings such that both curves $f_0(T)$ and $f_1(T)$ are polynomially convex, then f_0 and f_1 are $\mathrm{Aut}\mathbf{C}^n$-equivalent. Moreover, given an embedded, real-analytic, polynomially convex curve $\Gamma \subset \mathbf{C}^n$, a biholomorphic map $F: D \to \mathbf{C}^n$ defined in a neighborhood of Γ can be* *approximated by automorphisms of $\mathbf{C}^n$ in some smaller neighborhood of Γ if and only if $F(\Gamma)$ is polynomially convex and the winding number of the Jacobian $J(F)$ along Γ equals zero.*

For a stronger result on equivalence of curves see Theorem 6.1 below.

5.5 Theorem. *Let $M \subset \mathbf{C}^n$ ($n \geq 3$) be a compact, embedded, real-analytic surface that is totally real and polynomially convex. A biholomorphic mapping $F: U \to \mathbf{C}^n$, defined in a neighborhood of M, can be approximated by automorphisms of $\mathbf{C}^n$ near M if and only if $F(M)$ is polynomially convex and the Jacobian $J(F): M \to \mathbf{C}^*$ is homotopic to a constant on M. In particular, if M is a two dimensional sphere, then F can be approximated by automorphisms of $\mathbf{C}^n$ near M if and only if $F(M)$ is polynomially convex.*

The following result [21, Corollary 4.1] follows directly from Theorem 2.1.

5.6 Corollary. *All manifolds are assumed to be embedded and real-analytic.*

(a) *Any two totally real, polynomially convex, k-dimensional discs in $\mathbf{C}^n$ ($k \leq n$) are* $\mathrm{Aut}\mathbf{C}^n$*-equivalent.*

(b) *Any two arcs in $\mathbf{C}^n$ are* $\mathrm{Aut}\mathbf{C}^n$*-equivalent.*

(c) *Any two embedded analytic discs in $\mathbf{C}^n$ are* $\mathrm{Aut}\mathbf{C}^n$*-equivalent.*

Moreover, if $M \subset \mathbf{C}^n$ is as in (b) or (c), then every biholomorphic map $F: D \to \mathbf{C}^n$ in a neighborhood D of M in $\mathbf{C}^n$ can be approximated in a smaller neighborhood of M by automorphisms of $\mathbf{C}^n$. If M is a k-disc as in (a), then F can be so approximated near M if and only if the image $F(M)$ is polynomially convex in $\mathbf{C}^n$.

6. Global symplectic and volume preserving equivalence

In this section we recall the results of [19] and [20] on $\mathcal{G}$-equivalence in the cases when $\mathcal{G}$ is either $\mathrm{Aut}_{\mathrm{sp}}\mathbf{C}^{2n}$ or $\mathrm{Aut}_1\mathbf{C}^n$.

Choose a $(1,0)$-form θ on $\mathbf{C}^{2n}$ such that $d\theta = -\omega$; to be specific we will take

$$\theta = \sum_{j=1}^{n} z_{n+j} dz_j.$$

Denote by T the circle. For a smooth map $g: T \to \mathbf{C}^{2n}$ we define the *action integral*

$$\mathcal{A}(g) = -\int_T g^*\theta.$$

The following is the main result of [19].

6.1 Theorem. *Let $g_0, g_1: T \to \mathbf{C}^{2n}$ be two real-analytic embeddings of the circle into $\mathbf{C}^{2n}$. If g_0 and g_1 are* $\mathrm{Aut}_{\mathrm{sp}}\mathbf{C}^{2n}$*-equivalent, then $\mathcal{A}(g_0) = \mathcal{A}(g_1)$. Conversely we have*

(a) *if $\mathcal{A}(g_0) = \mathcal{A}(g_1) \neq 0$, then g_0 and g_1 are* $\mathrm{Aut}_{\mathrm{sp}}\mathbf{C}^{2n}$*-equivalent;*

(b) *if $\mathcal{A}(g_0) = \mathcal{A}(g_1) = 0$, then g_0 and g_1 are* $\mathrm{Aut}_{\mathrm{sp}}\mathbf{C}^{2n}$*-equivalent provided that the curves $g_0(T), g_1(T) \subset \mathbf{C}^{2n}$ are both polynomially convex;*

(c) *every two smooth, embedded, real-analytic arcs* $g_0, g_1: [0,1] \to \mathbf{C}^{2n}$ *are* $\mathrm{Aut}_{\mathrm{sp}}\mathbf{C}^{2n}$-*equivalent.*

Example 1: The following example shows that in part (b) of Theorem 6.1 the conclusion may fail if the curves are not polynomially convex. Let $g_0, g_1: T \to \mathbf{C}^2$ be the embeddings

$$g_0(\theta) = \left(e^{i\theta}, 0\right),$$
$$g_1(\theta) = \left(e^{2i\theta}, e^{3i\theta}\right).$$

The curve $C_0 = g_0(T)$ bounds the smooth analytic disc $\{(z,0): |z| \le 1\}$, while $C_1 = g_1(T)$ bounds the analytic curve $\{(z,w): z^3 = w^2,\ |z| \le 0\}$ with a singularity at the origin. Thus $\mathcal{A}(g_0) = 0 = \mathcal{A}(g_1)$ although the curves C_0 and C_1 are not $\mathrm{Aut}\mathbf{C}^2$-equivalent (see remark 1 following Def. 5.1).

Example 2: The map $F_t(z,w) = (z, w + t/z)$ is a volume preserving automorphism of $\mathbf{C}^* \times \mathbf{C}$ for all $t \in \mathbf{C}$. The circle $T = \{(z,\bar{z}) \in \mathbf{C}^2: |z| = 1\}$ is polynomially convex, hence it has pseudoconvex tubular neighborhoods Ω which are Runge in $\mathbf{C}^2$. The same is true for circles $F_t(T)$ for $t \ne -1$. Theorem 5.4 implies that for such t, F_t is a limit of automorphisms of $\mathbf{C}^2$ in a neighborhood of T. However, F_t for $t \ne 0$ is not the limit of symplectic automorphisms of $\mathbf{C}^2$ in any neighborhood of T. This is because the action integral

$$\mathcal{A}(F_t) = \int_{F_t(T)} w\,dz = 2\pi i(1+t)$$

depends on t, while Stokes' theorem shows that this integral is preserved by symplectic holomorphic automorphism of $\mathbf{C}^2$ (and therefore by their limits).

Sketch of proof of Theorem 6.1: We first construct a real-analytic isotopy of embeddings $g_t: T \to \mathbf{C}^{2n}$ such that g_0 and g_1 are the given maps, the curve $C_t = g_t(T)$ is polynomially convex for every t, and the action integral $\mathcal{A}(g_t)$ is independent of t. We then show that the flow $g_t \circ g_0^{-1}: C_0 \to C_t$ can be extended to a flow of a (time dependent) exact Hamiltonian vector field X_t, defined in a neighborhood of C_t for each $t \in [0,1]$. This extension is done in complete analogy to the real case. Let H_t be a holomorphic Hamiltonian (the energy function) of X_t, defined in a tube around C_t. Since C_t is polynomially convex, we can approximate H_t near C_t by holomorphic polynomials. The corresponding polynomial Hamiltonian vector field Y_t then approximates X_t near C_t. The flow of Y_t, which by construction approximates the flow of X_t and hence the original flow $g_t \circ g_0^{-1}: C_0 \to C_t$, is itself approximable (at each time t) by compositions of symplectic automorphisms of $\mathbf{C}^{2n}$ according to Proposition 4.3 (iii).

We now consider the problem of $\mathrm{Aut}_1\mathbf{C}^n$-equivalence of real-analytic submanifolds in $\mathbf{C}^n$. Let β be the $(n-1,0)$-form

$$\beta(z) = \frac{1}{n}\sum_{j=1}^{n}(-1)^{j-1} z_j dz_1 \wedge \cdots \widehat{dz_j} \cdots \wedge dz_n,$$

where the hat indicates that the corresponding entry is deleted. Then $d\beta = \Omega$ is the complex volume form on $\mathbf{C}^n$. If M is a smooth, compact, oriented manifold of real dimension

$n-1$ and $f: M \to \mathbf{C}^n$ is a smooth map, we set

$$\mathcal{B}(f) = \int_M f^* \beta.$$

If M is closed, Stokes's theorem implies that *any two embeddings* $f_0, f_1: M \to \mathbf{C}^n$ *which are* $\mathrm{Aut}_1 \mathbf{C}^n$*-equivalent satisfy* $\mathcal{B}(f_0) = \mathcal{B}(f_1)$. The following results were proved in [20].

6.2 Theorem. *Let M be a compact, connected, real-analytic manifold of real dimension m, and let $f_0, f_1: M \to \mathbf{C}^n$ be real-analytic embeddings ($1 \le m \le n-1$) such that the submanifolds $f_j(M) = M_j \subset \mathbf{C}^n$ ($j = 0, 1$) are totally real and polynomially convex. Suppose that f_0 and f_1 are* $\mathrm{Aut}\mathbf{C}^n$*-equivalent. Then f_0 and f_1 are also* $\mathrm{Aut}_1 \mathbf{C}^n$*-equivalent provided that any one of the following conditions holds:*

(i) $m \le n-2$;

(ii) $m = n-1$ *and* $\mathrm{H}^{n-1}(M; \mathbf{R}) = 0$;

(iii) $m = n-1$, *the manifold M is closed and orientable, and* $\mathcal{B}(f_0) = \mathcal{B}(f_1) \ne 0$.

Remark: We believe that the conclusion in part (iii) holds also when $\mathcal{B}(f_0) = \mathcal{B}(f_1) = 0$.

The proof of Theorem 6.2 is analogous to that of Theorem 6.1, except that we replace Hamiltonian flows by flows of exact divergence zero vector fields. Combining Theorem 6.2 with Corollary 5.3 we get

6.3 Corollary. *Let M be a compact, connected, real-analytic manifold of dimension $m \ge 1$. If $n \ge \max\{m+2, 3m/2\}$, then every two real-analytic embeddings $f_0, f_1: M \to \mathbf{C}^n$ whose images $f_0(M)$ and $f_1(M)$ are totally real and polynomially convex are* $\mathrm{Aut}_1 \mathbf{C}^n$*-equivalent.*

Recall that T is the circle.

6.4 Corollary. *(a) Two real-analytic embeddings $f_0, f_1: T \to \mathbf{C}^2$ with polynomially convex images $f_0(T), f_1(T)$ are* $\mathrm{Aut}_1 \mathbf{C}^2$*-equivalent if and only if $\mathcal{B}(f_0) = \mathcal{B}(f_1)$.*

(b) Let M be a closed orientable surface, and let $f_0, f_1: M \to \mathbf{C}^3$ be real-analytic embeddings whose images $f_0(M), f_1(M) \subset \mathbf{C}^3$ are totally real and polynomially convex. If $\mathcal{B}(f_0) = \mathcal{B}(f_1) \ne 0$, then f_0 and f_1 are $\mathrm{Aut}_1 \mathbf{C}^3$*-equivalent.*

Example [20]: Let S be the $(n-1)$-dimensional sphere, embedded as a hypersurface in $\mathbf{R}^n \subset \mathbf{C}^n$. It is easy to find a closed, but non-exact holomorphic $(n-1)$-form α in a tubular neighborhood U of S such that $\int_S \alpha \ne 0$. Let X be the divergence zero holomorphic vector field in U defined by the formula $\alpha = X \rfloor \Omega$, where Ω is the complex volume form on $\mathbf{C}^n$. Its flow $\{F_t\}$ is a family of volume preserving biholomorphic mappings near S such that F_t for each sufficiently small t can be approximated by automorphisms of $\mathbf{C}^n$ near S (Theorem 5.2), but not by volume preserving automorphisms of $\mathbf{C}^n$ when $t \ne 0$ is small. The reason is that

$$\frac{d}{dt}\mathcal{B}(F_t)|_{t=0} = \int_S \alpha \ne 0$$

(see [20]) and hence the integral $\mathcal{B}(F_t)$ depends on t.

7. Approximation by automorphisms on smooth submanifolds of $\mathbf{C}^n$

In this section we consider the question of approximation of smooth mappings $F: M \to \mathbf{C}^n$ on smooth submanifolds $M \subset \mathbf{C}^n$ by restrictions to M of holomorphic automorphisms of $\mathbf{C}^n$.

Let M be a manifold of class $\mathcal{C}^p$, $p \geq 1$. A $\mathcal{C}^p$ *isotopy of embeddings* of M into $\mathbf{C}^n$ is a map $F: [0,1] \times M \to \mathbf{C}^n$ of class $\mathcal{C}^p$ such that for each fixed $t \in [0,1]$, $F_t = F(t,\cdot): M \to \mathbf{C}^n$ is an embedding. The isotopy $\{F_t\}$ is totally real (resp. polynomially convex) if the submanifold $F_t(M) \subset \mathbf{C}^n$ is totally real (resp. polynomially convex) for every $t \in [0,1]$.

The following result was proved in [17].

7.1 Theorem. *If $M \subset \mathbf{C}^n$ $(n \geq 2)$ is a compact, totally real, polynomially convex submanifold of class $\mathcal{C}^p$ $(2 \leq p < \infty)$ and $F: M \to \mathbf{C}^n$ is a $\mathcal{C}^p$ mapping, then the following are equivalent:*

(i) For each $\epsilon > 0$ there exists a $\Phi \in \mathrm{Aut}\mathbf{C}^n$ such that $||F - \Phi|_M||_{\mathcal{C}^p(M)} < \epsilon$.

(ii) For each $\epsilon > 0$ there exists a totally real, polynomially convex isotopy $F_t: M \to \mathbf{C}^n$ $(t \in [0,1])$ of class $\mathcal{C}^p$ such that F_0 is the identity on M and $||F_1 - F||_{\mathcal{C}^p(M)} < \epsilon$.

Remark: If $M \subset \mathbf{C}^n$ is a compact, totally real, polynomially convex submanifold of class $\mathcal{C}^p$ $(p \geq 1)$, then the set of restrictions to M of holomorphic polynomials on $\mathbf{C}^n$ is dense in the $\mathcal{C}^p(M)$. Thus Theorem 7.1 can be viewed as an analogue of this result for mappings $M \mapsto \mathbf{C}^n$.

The implication (i)$\Rightarrow$(ii) in Theorem 7.1 is a trivial consequence of connectedness of $\mathrm{Aut}\mathbf{C}^n$. The main implication (ii)$\Rightarrow$(i) is proved in a similar way as Theorem 5.2 by constructing an isotopy of biholomorphic mappings in a neighborhood of M which approximates the given isotopy F_t and then applying Theorem 2.4.

Recall from [17] that for compact manifolds M of class $\mathcal{C}^p$ $(p \geq 2)$ of real dimension at most $2n/3$, the set of $\mathcal{C}^p$ embeddings $M \hookrightarrow \mathbf{C}^n$ which are totally real and polynomially convex is open and dense in the space $\mathcal{C}^p(M)^n$ of all $\mathcal{C}^p$ mappings $M \mapsto \mathbf{C}^n$ (in the $\mathcal{C}^p$ topology). This implies

7.2 Corollary. *Let $M \subset \mathbf{C}^n$ be a compact, totally real, polynomially convex submanifold of class $\mathcal{C}^p$, $2 \leq p < \infty$, and of dimension at most $2n/3$. Then the set*

$$\mathrm{Aut}\mathbf{C}^n|_M = \{F|_M : F \in \mathrm{Aut}\mathbf{C}^n\}$$

of restrictions to M of holomorphic automorphisms of $\mathbf{C}^n$ is dense in $\mathcal{C}^p(M)^n$.

Consider now a diffeomorphism $F: M_0 \to M_1$ of class $\mathcal{C}^p$ $(1 \leq p \leq \infty)$ between submanifolds $M_0, M_1 \subset \mathbf{C}^n$. One may ask whether there exists a sequence $\Phi_j \in \mathrm{Aut}\mathbf{C}^n$ such that Φ_j converges to F on M_0 (in the $\mathcal{C}^p$ topology) and, at the same time, the sequence of inverses Φ_j^{-1} converges to the inverse $F^{-1}: M_1 \to M_0$ on M_1. A result of this type was first proved by J.-P. Rosay [31] for $\mathcal{C}^\infty$ arcs in $\mathbf{C}^n$. Unlike in the real-analytic case (section 5), one can not expect the convergence of the approximating sequence Φ_j in any neighborhood of M_0.

The following result was proved by E. Løw and the author [22].

7.3 Theorem: *If $F\colon M_0 \to M_1$ is a smooth ($\mathcal{C}^\infty$) diffeomorphism between smooth, compact submanifolds $M_0, M_1 \subset \mathbf{C}^n$ which are totally real and polynomially convex, then the following are equivalent:*

(i) There exists a totally real, polynomially convex isotopy of embeddings $F_t\colon M_0 \to \mathbf{C}^n$ such that F_0 is the identity on M_0 and $F_1 = F$;

(ii) there exists a sequence $\Phi_j \in \mathrm{Aut}\mathbf{C}^n$ ($j \in \mathbf{Z}_+$) such that $\lim_{j\to\infty} \Phi_j|_{M_0} = F$ and $\lim_{j\to\infty} \Phi_j^{-1}|_{M_1} = F^{-1}$ in the $\mathcal{C}^\infty$ topology on the respective manifolds.

We expect to obtain a sharp version of the last result for $\mathcal{C}^p$ diffeomorphisms between $\mathcal{C}^p$ submanifolds for $3 \le p < \infty$. We believe that in this case the approximating sequence Φ_j can be chosen so that the convergence in Theorem 7.3 takes place in the $\mathcal{C}^p$ topology on M_0 resp. M_1.

In the proof of Theorem 7.3 we must solve a certain $\overline{\partial}$-problem in small tubular neighborhoods around totally real submanifolds in $\mathbf{C}^n$. One possible approach is to use the L^2-methods as in Hörmander and Wermer [24] (see also [37]). This method has an inherent loss of derivatives; it can be used to prove Theorem 7.3, but it does not give the sharp result in case of finite smoothness. In that case we intend to use a result of N. Øvrelid (in preparation) on solution of the $\overline{\partial}$-problem in tubes around totally real manifolds by means of an integral kernel.

8. Complex orbits of holomorphic vector fields

Let X be a holomorphic vector field on a complex manifold $\mathcal{M}$. Recall that its local flow $F_t(z)$ ($t \in \mathbf{R}$) is the solution at time t of the differential equation

$$\dot{Z} = X(Z), \qquad Z(0) = z \in \mathcal{M}. \tag{11}$$

There exists a maximal open interval $J(z) = \big(-\beta(z), \alpha(z)\big) \subset \mathbf{R}$, containing the origin, such that $F_t(z)$ is defined for $t \in J(z)$. The functions α and $-\beta$ with values in $(0,\infty]$ are lower semicontinuous on $\mathcal{M}$. If $\alpha(z) < \infty$ then $F_t(z)$ leaves every compact set in $\mathcal{M}$ as $t \in J$ approaches $\alpha(z)$, and similarly for $-\beta(z)$. The set $\{F_t(z)\colon t \in J(z)\}$ is called the *real orbit* of X through z.

We recall the notion of *complex orbits* of a holomorphic vector field; see [18] for details. Locally near $t = 0$ one can solve the equation (11) for complex values of t. This gives a holomorphic mapping $Z(t)$, defined in a neighborhood of 0 in the complex plane, with values in the manifold $\mathcal{M}$. By analytic continuation we can extend the local solution to a maximal global solution $Z\colon R_z \to \mathcal{M}$, which is defined and holomorphic on a connected Riemann domain R_z spread over $\mathbf{C}$ and which can not be analytically continued to any larger Riemann domain over $\mathbf{C}$. Its image $C_z = Z(R_z) \subset \mathcal{M}$ is called the complex orbit of X through z.

We say that the complex orbit C_z of a point $z \in \mathcal{M}$ is *complete in real time* if $J(z') = (-\infty, +\infty)$ for every point $z' \in C_z$, i.e., the equation (11) can be solved for all real values of time when starting at any point in C_z. The orbit C_z is said to be nontrivial if $C_z \neq \{z\}$; this is true if and only if $X(z) \neq 0$.

A proof of the following simple result can be found in [18] (Proposition 3.2). It depends on the observation that for every complex orbit $C_z \subset \mathcal{M}$ of X which is complete in real time, the Riemann surface R_z is a strip in $\mathbf{C}$ of the form

$$R_z = \{t + is \in \mathbf{C}: -b(z) < s < a(z)\} \subset \mathbf{C},$$

with $a(z), b(z) \in (0, \infty]$.

8.1 Proposition. *Every nontrivial complex orbit of a holomorphic vector field which is complete in real time is isomorphic to one of the following Riemann surfaces:*

(a) the complex line $\mathbf{C}$;

(b) the punctured complex line $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$;

(c) a torus;

(d) the disc $U = \{z \in \mathbf{C}: |z| < 1\}$;

(e) the punctured disc $U^* = U\backslash\{0\}$;

(f) an annulus $A(r) = \{z \in \mathbf{C}: 1 < |z| < r\}$.

Remark: If the manifold $\mathcal{M}$ is Stein then there are no toral orbits. If $\mathcal{M}$ is hyperbolic, then all nontrivial orbits are of types (d)–(e) since the surfaces (a)–(c) are nonhyperbolic. Notice that the fundamental group of C_z has at most one generator unless C_z is a torus.

Suppose now that a holomorphic vector field X on a complex manifold $\mathcal{M}$ is $\mathbf{R}$-complete. Let

$$\tilde{\mathcal{M}} = \{(\zeta, z): z \in \mathcal{M}, \ -b(z) < \Im\zeta < a(z)\} \subset \mathbf{C} \times \mathcal{M},$$

where $a(z)$ and $b(z)$ are as above. We call $\tilde{\mathcal{M}}$ the *fundamental domain* of the complex flow of X. The functions a and b are lower semicontinuous on $\mathcal{M}$. The following was proved in [18, Proposition 2.1]:

8.2 Proposition. *If the manifold* $\mathcal{M}$ *is Stein, then the functions* $-a$, $-b$ *are (negative) plurisubharmonic on* $\mathcal{M}$, *and the fundamental domain* $\tilde{\mathcal{M}} \subset \mathbf{C} \times \mathcal{M}$ *is pseudoconvex (hence a Stein manifold).*

8.3 Corollary. *If* $\mathcal{M}$ *is a Stein manifold such that every negative plurisubharmonic function on* $\mathcal{M}$ *is constant, then every* $\mathbf{R}$*-complete holomorphic vector field on* $\mathcal{M}$ *is* $\mathbf{C}$*-complete. This holds in particular if* $\mathcal{M} = \mathbf{C}^n$ *or if* $\mathcal{M} = \mathbf{C}^n\backslash A$ *for some complex hypersurface* $A \subset \mathbf{C}^n$.

The reason is that the plurisubharmonic functions $-a$ and $-b$ defining $\tilde{\mathcal{M}}$ must be constant, and therefore the flow extends to all complex values of time by the group property of flows.

The following result from [18] (Theorem 3.3) shows that holomorphic vector fields on Stein manifolds which are complete in real time have a 'generic type' of complex orbits.

8.4 Theorem. *Let $\mathcal{M}$ be a connected Stein manifold and X a holomorphic vector field on $\mathcal{M}$ which is complete in real time (Def. 3.1). Then there exists a pluripolar set $E \subset \mathcal{M}$, invariant under the flow $\{F_t : t \in \mathbf{R}\}$ of X and containing the zero set of X, such that for every $z \in \mathcal{M} \backslash E$ the complex orbit C_z of X through z is of the same type (a), (b), or (d)–(e). If this generic orbit type is U^* or an annulus, then the flow F_t has a period $\lambda > 0$ and it factors through an action of the circle group $(S, +)$ on $\mathcal{M}$. The action $\{F_t : t \in \mathbf{R}\}$ extends to an action of $\mathbf{C}$ on $\mathcal{M}$ (i.e., X is complete in complex time) if and only if the generic complex orbit is either $\mathbf{C}$ or $\mathbf{C}^*$. This is always the case when $\mathcal{M} = \mathbf{C}^n$.*

For actions of $\mathbf{C}$ on Stein spaces the existence of a generic orbit type ($\mathbf{C}$ or $\mathbf{C}^*$) was proved by Suzuki [34, Proposition 2]. His proof shows that when the generic orbit type is $\mathbf{C}^*$, then C_z is isomorphic to $\mathbf{C}^*$ for every z outside a closed analytic subset of $\mathcal{M}$. For related results see Richardson [30].

Another result which seems important in the study of dynamical properties of complete holomorphic vector fields is that those complex orbits which are not simply connected tend to have at most one limit point, amd this point is a fixed point of the flow. The precise result is as follows ([18, Theorem 4.1], [34]):

8.5 Theorem. *Let X be an $\mathbf{R}$-complete holomorphic vector field on a Stein manifold $\mathcal{M}$, and let C be a nontrivial complex orbit of X. If C is isomorphic to an annulus, then C is closed in $\mathcal{M}$. If C is isomorphic to the punctured disc U^*, or if C is isomorphic to the punctured plane $\mathbf{C}^*$ and X is $\mathbf{C}$-complete, then the limit set of C consists of at most one point which is a critical point of X. In particular, if $\mathcal{M} = \mathbf{C}^n$, then every orbit of type $\mathbf{C}^*$ has at most one limit point.*

The proof in [18] (or in [34]) shows that, if C is a complex orbit of type $\mathbf{C}^*$ and if p is a limit point of C, then $C \cup \{p\}$ is a smooth complex manifold at p. If we had two such limit points p, q, then $C \cup \{p, q\}$ would be a holomorphically embedded Riemann sphere in $\mathcal{M}$ which is impossible since $\mathcal{M}$ is Stein. A similar result holds for orbits of type U^*.

8.6 Corollary. *Let X be a complete holomorphic vector field on $\mathbf{C}^n$ ($n > 1$) with a hyperbolic critical point p such that the unstable manifold $W^u(p)$ has complex dimension one. Then $W^u(p)$ is closed in $\mathbf{C}^n$. In particular, $W^u(p)$ does not intersect the stable manifold $W^s(q)$ of any other critical point of X.*

Proof: Since $W^u(p)$ is a complex manifold of complex dimension one (an injectively immersed $\mathbf{C}$), the set $C = W^u(p) \backslash \{p\}$ is a complex orbit of X which is necessarily of type $\mathbf{C}^*$. Since p is a limit point of C, C can not have any other limit point according to Theorem 8.5. If $W^u(p) \cap W^s(q) \neq \emptyset$, then q is a limit point of C, a contradiction.

Corollary 8.6 can be used to construct holomorphic vector fields on $\mathbf{C}^n$ which can not be approximated by complete fields. The following example is due to Buzzard and Fornæss (private communication, November 1994).

Example (Buzzard-Fornaess): Let $X(z, w) = \big(z(z-1), -w\big)$. Then $p = (1, 0)$ is a hyperbolic critical point of X whose unstable manifold $W^u(p)$ contains the segment $I = (0, 1) \times \{0\}$, and $q = (0, 0)$ is an attracting critical point which contains I in its basin of attraction. Thus $W^u(p) \cap W^s(q) \neq \emptyset$. Since this behavior is stable under small perturbations of the field, X can not be approximated by complete holomorphic fields. In fact, the closure of the set of complete holomorphic vector fields on $\mathbf{C}^n$ is nowhere dense in the set of all holomorphic vector fields (in the topology of uniform convergence on compacts).

9. Flows of holomorphic Hamiltonian vector fields

Proposition 8.1 is useful in determining whether a given holomorphic vector field is complete or not by examining its complex orbits. We shall illustrate this by looking at Hamiltonian vector fields on the plane $\mathbf{C}^2$.

Suppose that X is a Hamiltonian holomorphic vector field on $\mathbf{C}^2$ with the energy function H (see section 3). Let $\Sigma = \{p \in \mathbf{C}^2 : X(p) = 0\}$. It is easily seen that for each point $p = (z_0, w_0) \in \mathbf{C}^2 \backslash \Sigma$ the complex orbit C_p through p is the connected component of the set $\{(z,w) \in \mathbf{C}^2 \backslash \Sigma : H(z,w) = H(z_0, w_0)\}$. Thus, to show that the vector field X is not complete, it suffices to find such a component which is not isomorphic to any of the surfaces listed in Proposition 8.1.

We first consider polynomial Hamiltonians H on $\mathbf{C}^2$. In that case every level set $\{H = c\}$ is an affine algebraic curve in $\mathbf{C}^2$ which closes up to a projective algebraic curve in $\mathbf{CP}^2$. Hence every orbit of the Hamiltonian vector field X_H is obtained by removing at most a finite number of points from a compact algebraic curve. The only Riemann surfaces listed in Proposition 8.1 above which have this property are $\mathbf{C}$ and $\mathbf{C}^*$. Together with Theorem 8.2 this proves the following [18, Lemma 7.1].

9.1 Lemma. *If X is a polynomial Hamiltonian vector field on $\mathbf{C}^2$ and if C is a nontrivial complex orbit of X which is $\mathbf{R}$-complete, then the closure $\overline{C}$ in $\mathbf{CP}^2$ is an algebraic curve of genus 0 ($\overline{C}$ is normalized by the Riemann sphere), and C is also $\mathbf{C}$-complete.*

By analyzing the holomorphic type of generic level sets of H, using the Riemann-Hurwitz formula, one gets the following [18, Proposition 7.2].

9.2 Proposition. *Let $H(z,w) = \sum_{j=1}^{d} H^j(z,w)$ be a polynomial of degree $d \geq 3$ on $\mathbf{C}^2$, with H^j its homogeneous part of degree j. Suppose that $(0,0)$ is the only common zero of the following four polynomials: H^d, H^{d-1}, $\partial H^d/\partial z$, $\partial H^d/\partial w$. Then the Hamiltonian vector field $X_H = (H_w, -H_z)$ on $\mathbf{C}^2$ is not complete. In fact, every regular level set of H contains a point p such that the flow $F_t(p)$ of X_H is not defined for all real t.*

To motivate the next result we recall the well known result that, if $Q(x) \geq 0$ is a nonnegative smooth real function on $\mathbf{R}$, the Hamiltonian vector field $X(x,y) = (y, -Q'(x))$ with the energy function $H(x,y) = y^2/2 + Q(x)$ is complete on $\mathbf{R}^2$. In contrast to this we have [18, Proposition 7.3]:

9.3 Proposition. *If f is an entire function on $\mathbf{C}$ which is not affine linear, then the vector field $X(z,w) = \big(w, f(z)\big)$ on $\mathbf{C}^2$ is not complete.*

Notice that X is a Hamiltonian vector field with the energy function $H(z,w) = w^2/2 + Q(z)$, where $Q'(z) = -f(z)$. The proof in [18] uses elementary Morse theory and it shows that every regular level set of H contains a point p such that the flow $F_t(p)$ of X is not defined for all real t.

9.4 Corollary. *If f is an entire function on* $\mathbf{C}$ *which is not affine linear, there exists a point* $(z_0, \dot{z}_0) \in \mathbf{C}^2$ *such that the second order ordinary differential equation*

$$\ddot{Z} = f(Z), \qquad Z(0) = z_0, \ \dot{Z}(0) = \dot{z}_0$$

can not be integrated for all real $t \in \mathbf{R}$.

The Corollary follows from Proposition 9.3 by introducing the variable $w = \dot{z}$ and changing this second order equation to the Hamiltonian system $\dot{z} = w$, $\dot{w} = f(z)$.

In theory of Hamiltonian mechanics one basic question is whether most orbits go to infinity. This question was studied in the holomorphic case by Fornæss and Sibony [14,15] who showed that for most Hamiltonian holomorphic vector fields on $\mathbf{C}^{2n}$, most orbits go to infinity.

In order to formulate their results precisely we first recall from [14] the notion of *singular orbit* of a holomorphic vector field X on $\mathbf{C}^n$. At this point X need not be Hamiltonian. Let $\{F_t : t \in \mathbf{R}\}$ be the local flow of X.

9.5 Definition. *The orbit $F_t(z)$ of a point $z \in \mathbf{C}^n$ is said to be singular if there is a sequence $0 < t_1 < t_2 < t_3 < \ldots$ such that $\lim_{j\to\infty} |F_{t_j}(z)| = \infty$. The set of points $z \in \mathbf{C}^n$ with nonsingular orbits is denoted by K_X.*

In other words, the orbit is singular if the set $\{F_t(z) : 0 < t < T\}$ is unbounded, where $T \in (0, \infty]$ is the largest number such that the flow $F_t(z)$ is defined on $[0, T)$. If $T < \infty$, then by general properties of flows the orbit of z is necessarily singular since $|F_t(z)| \to \infty$ when $t \to T$. Thus the points $z \in \mathbf{C}^n$ with nonsingular orbits are those for which the flow $F_t(z)$ is defined and bounded for all $t \geq 0$. The set K_X is always of type F_σ, i.e., a countable union of closed sets.

We consider each entire function $H \in \mathcal{O}(\mathbf{C}^{2n})$ as a holomorphic Hamiltonian, giving rise to a holomorphic Hamiltonian vector field X_H (7). Denote the set of nonsingular orbits of X_H by K_H (Def. 9.5). The main result of Fornæss and Sibony [15, Theorem 3.4] is

9.6 Theorem. *There exists a dense G_δ set $\mathcal{E} \subset \mathcal{O}(\mathbf{C}^{2n})$ of holomorphic Hamiltonians such that for each $H \in \mathcal{E}$ the set K_H has empty interior.*

Fornæss and Sibony also obtained a number of interesting results for the *Fatou set* $U_X \subset K_X$ of holomorphic Hamiltonian vector fields on $\mathbf{C}^{2n}$.

9.7 Definition. *Let X be a holomorphic vector field on $\mathbf{C}^n$. For each constant $0 < c < \infty$ we denote by $U_X(c)$ the set of all points $z \in \mathbf{C}^n$ for which there is a neighborhood V of z such that*

$$\sup_{t \geq 0} |F_t(w)| < c \qquad \text{for all } w \in V.$$

The set $U_X = \bigcup_{0<c<\infty} U_X(c)$ is called the Fatou set of X. Any connected component of U_X is called a Fatou component.

Notice that U_X is contained in $\mathrm{int} K_X$, and it is the largest open set on which the flow F_t of X exists for all time $0 \leq t < \infty$ and the family $\{F_t : t \geq 0\}$ is locally bounded. By a category argument U_X is an open dense subset of the interior of K_X. It is shown in [15, Proposition 3.3] that every Fatou component is a Runge domain in $\mathbf{C}^n$.

The following result is proved in [15, Theorem 3.5].

9.8 Theorem. *Let X be a Hamiltonian holomorphic vector field on $\mathbf{C}^{2n}$ with flow F_t and Fatou set U_X. Let $W(c)$ $(0 < c < \infty)$ be a connected component of the set $U_X(c)$ (Def. 9.7). Then F_t is an automorphism of $W(c)$ for each $t \in \mathbf{R}$. The closure G of $\{F_t|_{W(c)}: t \in \mathbf{R}\}$ is a Lie subgroup of $\mathrm{Aut}W(c)$, isomorphic to a torus T^k for some $k \le n$. The vector field X is conjugate on $W(c)$ to a field $Y = (i\theta_1 z_1, \ldots, i\theta_n z_n, 0, \ldots, 0)$.*

A related question is how many orbits of a holomorphic vector field X on $\mathbf{C}^n$ go to infinity in finite time. Following [13] we say that the (real) orbit of a point $z \in \mathbf{C}^n$ *explodes* if the integral curve $\{F_t(z): t \ge 0\}$ through z is unbounded on some finite time interval $[0, t_0) \subset \mathbf{R}_+$. The following result was proved by Fornæss and Grellier [13].

9.10 Theorem. *There is a dense set $\mathcal{E}$ of entire functions on $\mathbf{C}^2$ such that for every $H \in \mathcal{E}$ the Hamiltonian vector field X_H has a dense set of points in $\mathbf{C}^2$ with exploding orbits.*

They proved a similar result for holomorphic Reeb fields on $\mathbf{C}^3$. Observe that Propositions 9.2 and 9.3 above also give exploding orbits of either X or $-X$.

Example: There exist Hamiltonian holomorphic vector fields X on $\mathbf{C}^2$ such that X is a limit of complete Hamiltonian fields, but *every real orbit of X explodes.* One way to construct such fields is as follows (see [18]). Let F be a polynomial automorphism of $\mathbf{C}^2$ with an attracting fixed point p such that the basin of attraction $D(p)$ is not all of $\mathbf{C}^2$. Then $D(p)$ is a Fatou-Bieberbach domain which intersects every complex line in a bounded set [8,32], and there is a biholomorphic map $G: \mathbf{C}^2 \to D(p)$ with Jacobian one which is a limit of automorphisms of $\mathbf{C}^2$ (and hence a limit of automorphisms G_j with Jacobian one). If Y is any constant vector field on $\mathbf{C}^2$, then $X = DG^{-1}\cdot Y$ is a Hamiltonian field which is a limit of complete Hamiltonian fields $X_j = DG_j^{-1}\cdot Y$, but every orbit of X explodes.

10. Flows on the complex plane $\mathbf{C}^2$

In this section we collect some results on classification of actions $F: \mathbf{R} \times \mathbf{C}^2 \to \mathbf{C}^2$ by holomorphic automorphisms $F_t \in \mathrm{Aut}\mathbf{C}^2$. Recall (Corollary 8.3) that every such action extends to an action of $\mathbf{C}$ on $\mathbf{C}^2$. Equivalently, we may consider such actions as flows of the complete holomorphic vector fields on $\mathbf{C}^2$, and also as one parameter subgroups of $\mathrm{Aut}\mathbf{C}^2$.

The following types of holomorphic flows on $\mathbf{C}^2$ have been classified:

1. polynomial flows (Theorem 10.1 below);
2. flows in the shear groups $\mathcal{S}(2)$, $\mathcal{S}_1(2)$, and $\mathcal{S}_1(2) \times \mathbf{C}^*$ (Theorem 10.4 below);
3. proper flows (Theorem 10.5 below);
4. symplectic flows (Theorem 10.6 below);
5. most flows whose time one map is polynomial.

Recall that $\mathcal{P}^2$ is the polynomial automorphism group of $\mathbf{C}^2$. Let (x, y) be the complex coordinates on $\mathbf{C}^2$. The following classification of polynomial flows is due to Suzuki [35] and, independently, to Bass and Meisters [7].

10.1 Theorem. *Every one parameter subgroup $\{F_t : t \in \mathbf{R}\}$ of the polynomial automorphism group $\mathcal{P}^2$ is conjugate in $\mathcal{P}^2$ to one of the following:*

$$\phi_t(x,y) = \left(e^{\mu t}x, e^{\lambda t}y\right), \quad \lambda, \mu \in \mathbf{C}, \tag{12}$$

$$\phi_t(x,y) = \left(x + tf(y), y\right), \quad f \text{ a polynomial}, \tag{13}$$

$$\phi_t(x,y) = \left(e^{n\lambda t}(x + ty^n), e^{\lambda t}y\right), \quad \lambda \in \mathbf{C}^*,\ n \in \mathbf{Z}_+. \tag{14}$$

Previously R. Rentschler has proved [29] that the every algebraic action of $\mathbf{C}$ on $\mathbf{C}^2$ is algebraically conjugate to an action (13).

The methods used in [7] and [35] are quite different from each other. Suzuki first showed that every one parameter subgroup $\{\phi_t : t \in \mathbf{R}\} \subset \mathcal{P}^2$ has bounded degree, i.e., $\deg\phi_t \le N$ for some N independent of $t \in \mathbf{R}$. This implies that there exist a polynomial embedding $H : \mathbf{C}^n \to \mathbf{C}^N$ for some N and an $N \times N$ matrix A such that

$$H \circ F_t(z) = \exp(tA) \cdot H(z), \quad z \in \mathbf{C}^n,\ t \in \mathbf{R}.$$

Suzuki then analysed several cases to obtain the classification.

The proof of Bass and Meisters has two main ingredients. One is the theorem of Van der Kulk [36] and Rentschler [29] (see also Jung [25] and Friedland and Milnor [23]) to the effect that the polynomial automorphism group $\mathcal{P}^2$ is an amalgamated free product $\mathcal{P}^2 = \mathcal{A} * \mathcal{E}$ of the affine automorphism group $\mathcal{A}$ of $\mathbf{C}^2$ and the group $\mathcal{E}$ of all elementary polynomial transformations

$$E(x,y) = \left(\alpha x + p(y), \beta y + \gamma\right),$$

where $\alpha, \beta \in \mathbf{C}^*$, $\gamma \in \mathbf{C}$, and p is a polynomial. This means that every element $g \in \mathcal{P}^2$ which does not belong to $\mathcal{A}$ or $\mathcal{E}$ can be represented by a word $g = \cdots e_r a_r e_{r-1} a_{r-1} \cdots$ of finite length, in which the e_j's belong to $\mathcal{E} \backslash \mathcal{A}$ and the a_j's belong to $\mathcal{A} \backslash \mathcal{E}$. The elements in $\mathcal{A} \cap \mathcal{E}$ are treated as units in this representation.

The second main ingredient is an algebraic theorem of Serre to the effect that every subgroup $\mathcal{H} \subset \mathcal{A} * \mathcal{E}$ in an amalgamated free product group whose elements have uniformly bounded length with respect to the given amalgamated free product structure is conjugate to a subgroup in $\mathcal{A}$ or in $\mathcal{E}$. Since every one parameter subgroup of $\mathcal{P}^2$ has bounded degree, it has bounded length and therefore Serre's theorem aplies. One then identifies and classifies all one parameter subgroups of $\mathcal{A}$ and $\mathcal{E}$.

It is known [26] that the analogous decomposition of the polynomial automorphism group $\mathcal{P}^n$ does not hold when $n \ge 3$. On the other hand, it has been proved recently that the shear groups on $\mathbf{C}^2$ (see section 1) also admit an amalgamated free product decomposition. Let $\mathcal{E} \subset \mathcal{S}(2)$ be the subgroup consisting of all automorphisms of the form

$$E(x,y) = \left(e^{a(y)}x + b(y), \beta y + \gamma\right), \tag{15}$$

where a and b are entire functions on $\mathbf{C}$, $\beta \in \mathbf{C}^*$, and $\gamma \in \mathbf{C}$. Maps of this form will be called *elementary* in analogy to the polynomial case when a is a constant and b is a polynomial (see [23]). Ahern and Rudin proved in [3] that the group $\mathcal{S}(2)$ is an amalgamated free product $\mathcal{S}(2) = \mathcal{A} * \mathcal{E}$, where $\mathcal{A}$ is the affine automorphism group on $\mathbf{C}^2$ and $\mathcal{E}$ is the group (15). Analogous result holds for the groups $\mathcal{S}_1(2)$ (Jacobian one) and $\mathcal{S}_1(2) \times \mathbf{C}^*$ (Jacobian constant but not necessarily one); see de Fabritiis [10,11]. In these cases the group $\mathcal{E}$ consists of automorphisms (15) for which a is constant and, in the volume preserving case, $e^a\beta = 1$.

In order to obtain the analogue of Theorem 10.1 for the shear groups one uses the following result of combinatorial group theory.

10.2 Theorem. *Let $\mathcal{G}$ be a topological group which is a free product $\mathcal{G} = \mathcal{A} * \mathcal{E}$ of subgroups $\mathcal{A}$ and $\mathcal{E}$, amalgamated over their intersection $\mathcal{B} = \mathcal{A} \cap \mathcal{E}$. Suppose that $\mathcal{B}$ is closed in $\mathcal{G}$. Then any topological subgroup of $\mathcal{G}$ which is isomorphic to $\mathbf{R}$ or $\mathbf{C}$ is conjugate in $\mathcal{G}$ to a subgroup in $\mathcal{A}$ or in $\mathcal{E}$.*

Theorem 10.2 follows from a result of Moldavanski ([27] or [38], Theorem 0.3); we refer the reader to the forthcoming paper [2] for the details. The same result holds for connected abelian Lie subgroups in $\mathcal{A} * \mathcal{E}$.

10.3 Corollary. *Every one parameter subgroup of the shear group $\mathcal{S}(2)$ is conjugate in $\mathcal{S}(2)$ to a subgroup of $\mathcal{E}$ (15). The analogous result holds for one parameter subgroups in $\mathcal{S}_1(2)$ and $\mathcal{S}_1(2) \times \mathbf{C}^*$.*

It is immediate that every subgroup in $\mathcal{A}$ is linearly conjugate to a subgroup in $\mathcal{E}$ (by conjugating the relevant matrix to its Jordan form).

One can describe one parameter subgroups $\{F_t : t \in \mathbf{R}\} \subset \mathcal{E}$ in the elementary group $\mathcal{E}$ (15) by using the methods in section 2 of [2]. The infinitesimal generator $V = (V_1, V_2)$ of F_t has the form

$$V_1(x,y) = a(y)x + b(y), \qquad V_2(y) = \lambda y + \gamma.$$

An automorphism Φ of $\mathbf{C}^2$ conjugates V to the field $\tilde{V}$ satisfying $D\Phi \cdot V = \tilde{V} \circ \Phi$. A preliminary linear change of coordinates in the y variable conjugates V to a field in which V_2 is either λy ($\lambda \in \mathbf{C}$) or $V_2 = 1$. Conjugating V with shears $\Phi(x,y) = \left(x + g(y), y\right)$ and generalized shears $\Phi(x,y) = \left(xe^{g(y)}, y\right)$ (and taking into account Corollary 10.3) one obtains the following classification result [2, Theorem 7.1]:

10.4 Theorem. *Every real one parameter subgroup $\{F_t : t \in \mathbf{R}\}$ of the generalized shear group $\mathcal{S}(2)$ is conjugate in $\mathcal{S}(2)$ to one of the following:*

(i) $\phi_t(x,y) = \left(x + tf(y), y\right)$, where f is an entire function on $\mathbf{C}$;

(ii) $\phi_t(x,y) = \big(e^{a(y)t}(x - b(y)) + b(y), y\big)$, *where a is a nonconstant entire function and b is a meromorphic function such that the product ab is entire;*

(iii) $\phi_t(x,y) = \big(e^{\mu t}x, e^{\lambda t}y\big)$, $\lambda, \mu \in \mathbf{C}$;

(iv) $\phi_t(x,y) = \big(e^{n\lambda t}(x + ty^n), e^{\lambda t}y\big)$, $\lambda \in \mathbf{C}^*$, $n \in \mathbf{Z}_+$.

Every subgroup $\{F_t\} \subset \mathcal{S}_1(2) \times \mathbf{C}^*$ $(t \in \mathbf{R})$ is conjugate in $\mathcal{S}_1(2) \times \mathbf{C}^*$ to one of the groups (i), (iii), or (iv). Every subgroup $\{F_t\} \subset \mathcal{S}_1(2)$ $(t \in \mathbf{R})$ is conjugate in $\mathcal{S}_1(2)$ either to a group (i) or to a linear group (iii) with $\lambda + \mu = 0$.

Observe that the groups (iii) and (iv) are polynomial, and (i) is polynomial when f is a polynomial. Comparing this with the classification of one parameter polynomial groups (Theorem 10.1) we see that the only group of new type in $\mathcal{S}(2)$ is (ii).

In the paper [2] we attempted to describe all real one parameter subgroups $F_t \in \mathrm{Aut}\mathbf{C}^2$ whose time one map F_1 is a polynomial automorphism $E \in \mathcal{P}^2$. The maps F_t for non-integer values of t need not be polynomial. We succeeded in all cases except when the time one map E is conjugate to an affine aperiodic map. In the last case we identified all flows with polynomial infinitesimal generator. We will not state the results of [2] here since there are too many different cases to consider. In [2] we also identified all generalized shears on $\mathbf{C}^2$ which belong to a flow; it turns out that most do not belong to any flow. This should be compared with results of Palis [28].

In [35] Suzuki classified *proper flows* $\{F_t : t \in \mathbf{C}\}$ on the plane $\mathbf{C}^2$. Recall that a flow is called proper if the complex orbit $C_z = \{F_t(z) : t \in \mathbf{C}\}$ of every point has at most a discrete set of limit points in $\mathbf{C}^2$; hence its closure $\overline{C}_z$ is an analytic curve in $\mathbf{C}^2$. The classification is as follows [35, Theorem 4].

10.5 Theorem: *Every proper holomorphic flow on $\mathbf{C}^2$ is conjugate in* $\mathrm{Aut}\mathbf{C}^2$ *to one of the following flows:*

(i) linear flow $\phi_t(x,y) = \big(e^{n\lambda t}x, e^{m\lambda t}y\big)$, *where $\lambda \in \mathbf{C}^*$ and $m, n \in \mathbf{N} = \{1, 2, 3, \ldots\}$;*

(ii) shear flow $\phi_t(x,y) = \big(x + tf(y), y\big)$, *where f is an entire function on $\mathbf{C}$;*

(iii) $\phi_t(x,y) = \big(e^{\lambda(y)t}(x - b(y)) + b(y), y\big)$, *where b is a meromorphic function and λ, λb are entire functions on $\mathbf{C}$;*

(iv) $\phi_t(x,y) = \big(e^{n\lambda(z)t}x, e^{-m\lambda(z)t}y\big)$, *where $m, n \in \mathbf{N}$, $z = x^m y^n$, and λ is an entire function on $\mathbf{C}$;*

(v) flows of the form $\alpha^{-1} \circ \rho_t \circ \alpha$, where $\alpha(x,y) = \big(x, x^l y + P_l(x)\big)$, $l \in \mathbf{N}$, P_l is a polynomial of degree $\le l - 1$ such that $P_l(0) \ne 0$, and ρ_t is a flow of type (iv) above such that $\lambda(z)$ has a zero of order $\ge l/m$ at $z = 0$.

The proof is based on a previous result of Suzuki [33] to the effect that every proper action of $\mathbf{C}$ on $\mathbf{C}^2$ has a meromorphic first integral H. Recall (Theorem 8.2) that every complex orbit of a $\mathbf{C}$-action is isomorphic to either $\mathbf{C}$ or $\mathbf{C}^*$, and most orbits are of the same type. Outside the fixed point set of the action the complex orbits correspond to connected components of the level sets of H; hence most of these are isomorphic to either $\mathbf{C}$ or $\mathbf{C}^*$. Results of Nishino and Saito imply that for such a function H there is an automorphism Φ of $\mathbf{C}^2$ such that $Q = H \circ \Phi$ is a rational function of a special type. Clearly Φ conjugates the flow F_t to a flow whose first integral is Q. Suzuki then identified all flows on $\mathbf{C}^2$ whose first integral is one of these special functions Q.

Every symplectic holomorphic flow on $\mathbf{C}^2$ is proper since the energy function (Hamiltonian) of the infinitesimal generator is a holomorphic first integral. Using this observation we proved in [18] (Theorem 6.1):

10.6 Theorem. *Every symplectic flow $\{F_t : t \in \mathbf{C}\} \subset \mathrm{Aut}_1\mathbf{C}^2$ is conjugate in $\mathrm{Aut}\mathbf{C}^2$ to one of the following:*

$$\phi_t(x,y) = \big(x, y + th(x)\big), \tag{16}$$
$$\psi_t(x,y) = \left(e^{t\lambda(xy)}x, e^{-t\lambda(xy)}y\right), \tag{17}$$

where h resp. λ is an entire function on $\mathbf{C}$. In the first case the flow is conjugated to a flow (16) by a symplectic automorphism of $\mathbf{C}^2$.

We see by inspection that (16) and (17) are the only symplectic flows from the list of equivalence classes of proper flows given by Theorem 10.5. However, since conjugation by non-symplectic automorphisms does not preserve symplectic flows, additional work is needed to prove Theorem 10.6. We don't know whether in the second case the flow is conjugate to (17) by a symplectic automorphism of $\mathbf{C}^2$.

11. Open problems

Problem 1: Classify one parameter subgroups of $\mathrm{Aut}\mathbf{C}^2$. Partial results are given in section 10 above.

Problem 2: Classify one parameter subgroups of the polynomial automorphism group on $\mathbf{C}^3$ (or $\mathbf{C}^n$ for $n \geq 3$).

Problem 3: Which properties of a holomorphic vector field make it nonapproximable by complete fields ? See Corollary 8.6 and the example following it.

Problem 4: Is every Fatou-Bieberbach domain in $\mathbf{C}^n$ Runge ? Equivalently, is every injective holomorphic mapping $F: \mathbf{C}^n \to \mathbf{C}^n$ (Fatou-Bieberbach map) a limit of automorphisms of $\mathbf{C}^n$? This question has already been raised by Rosay and Rudin [32], but it remains unsolved.

Problem 5: Recall that the shear group $\mathcal{S}(n)$ is dense in $\mathrm{Aut}\mathbf{C}^n$. The proof of Theorem 1.1 in [18] implies the following stronger result: *If M is a closed ball in $\mathbf{R}^k$ for some $k \in \mathbf{Z}_+$ and $\Phi: M \times \mathbf{C}^n \to \mathbf{C}^n$ is a map of class $\mathcal{C}^p$ $(0 \leq p \leq \infty)$ satisfying $\Phi_x = \Phi(x, \cdot\,) \in \mathrm{Aut}\mathbf{C}^n$ for every $x \in M$, then Φ can be approximated in the $\mathcal{C}^p$ topology (uniformly on compacts in $M \times \mathbf{C}^n$) by maps $\Phi': M \times \mathbf{C}^n \to \mathbf{C}^n$ such that $\Phi'_x \in \mathcal{S}(n)$ for $x \in M$.* For which other manifolds M is this true ? In particular, does this hold when M is the circle, a sphere, a torus ?

Problem 6: On $\mathbf{C}^n$ one can define $(n-1)$ shear groups $\mathcal{S}^k(n)$ for $1 \leq k \leq n-1$ by letting $\mathcal{S}^k(n)$ be the group generated by all shears of the form (3) and (4), where Λ is a linear projection into $\mathbf{C}^m$ for some $m \leq k$. Are these groups all distinct ?

Problem 7: Let $D \subset \mathbf{C}^n$ be a smoothly bounded, strongly pseudoconvex Runge domain diffeomorphic to the ball, and let $F: D \to \mathbf{C}^n$ be a biholomorphic map onto a Runge domain $F(D) \subset \mathbf{C}^n$. Is F a limit of automorphisms of $\mathbf{C}^n$?

Problem 8: Suppose that Φ is an automorphism of $\mathbf{C}^2$ whose Jacobian $J\Phi$ is a function of the product xy, where (x, y) are complex coordinates on $\mathbf{C}^2$. Does it follow that $J\Phi$ is constant ? If so, then every flow in Theorem 10.5 which is conjugate in $\mathrm{Aut}\mathbf{C}^2$ to a flow of the form (16) is already conjugate to such a flow in $\mathrm{Aut}_1\mathbf{C}^2$. More generally, which nonvanishing functions are the Jacobians of automorphisms of $\mathbf{C}^2$ ($\mathbf{C}^n$) ?

Problem 9: Suppose that $\Phi \in \mathrm{Aut}\mathbf{C}^2$ fixes pointwise both coordinate axes and the point $(1,1)$. Does it follow that $J\Phi(1,1) = 1$? If there exists a Φ as above such that $(1,1)$ is an attracting fixed point (this requires $|J\Phi(1,1)| < 1$), then its basin of attraction is a Fatou-Bieberbach domain in $(\mathbf{C}^*)^2$ [32].

Problem 10: Suppose Φ is an automorphism of $\mathbf{C}^n$ with finite period (i.e., Φ^m is the identity for some $m \in \mathbf{N}$). Is Φ linearizable ? Is every finite subgroup of $\mathrm{Aut}\mathbf{C}^n$ linearizable ? For partial results see [3] and [26].

Problem 11: Find characterizations of $\mathbf{C}^n$ in terms of its automorphism group.

Several other problems concerning automorphisms of $\mathbf{C}^n$ and Fatou-Bieberbach maps are mentioned in papers [21], [26], and [32].

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