

Actions of $(\mathbf{R}, +)$ and $(\mathbf{C}, +)$ on complex manifolds

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0 Introduction

In this paper we study actions of the groups $(\mathbf{R}, +)$ and $(\mathbf{C}, +)$ on \mathbf{C}^n and on other complex manifolds by holomorphic automorphisms. We denote by $\text{Aut } M$ be the group of all holomorphic automorphisms of a complex manifold M . For $M = \mathbf{C}^n$ we denote by $\text{Aut}_1 \mathbf{C}^n$ the group of all holomorphic automorphisms with Jacobian one. Recall that an action of $(\mathbf{R}, +)$ on M is a smooth mapping $\phi : \mathbf{R} \times M \rightarrow M$ such that for each fixed $t \in \mathbf{R}$, $\phi_t = \phi(t, \cdot) \in \text{Aut } M$, and $\phi_{s+t} = \phi_s \circ \phi_t$ for all $s, t \in \mathbf{R}$. An action of $(\mathbf{C}, +)$ on M is a holomorphic map $\phi : \mathbf{C} \times M \rightarrow M$ with the same properties. We shall occasionally delete the $+$ sign when the group operation is clear. There is a one-to-one correspondence between actions of \mathbf{R} (resp. \mathbf{C}) on M and holomorphic vector fields V on M which are complete in real (resp. complex) time (Sect. 1). The vector field V arises as the infinitesimal generator of ϕ , and ϕ is obtained by integrating V . For this reason actions are also called *flows*.

Although the theory of continuous dynamical systems is one of the most established and developed parts of mathematics, very little seems to be known about basic global questions on *holomorphic continuous dynamical systems*. By continuous systems we mean flows of holomorphic vector fields, as opposed to iterations of holomorphic mappings which have attracted most attention in recent years.

Here is the outline of the paper. In Sects. 1–3 we collect some basic properties of actions and of the associated \mathbf{R} -complete holomorphic vector fields. In Sect. 1 we define the *complex orbits* of a holomorphic vector field on a complex manifold M (Definition 3). In Sect. 2 and 3 we show that for \mathbf{R} -complete fields (i.e., for actions of \mathbf{R}) the nontrivial complex orbits are of six possible types (Propositions 2.1 and 3.1).

Assuming that the manifold M is Stein, we show that the fundamental domain $\tilde{M} \subset \mathbf{C} \times M$ of an \mathbf{R} -complete holomorphic vector field on M is itself

Stein (Proposition 2.1). If M is such that every negative plurisubharmonic function on M is constant, then every \mathbf{R} -complete holomorphic vector field on M is also \mathbf{C} -complete (Corollary 2.2). This holds in particular when $M = \mathbf{C}^n$ or when $M = \mathbf{C}^n \setminus A$ for some complex hypersurface $A \subset \mathbf{C}^n$. Equivalently, every action of \mathbf{R} by holomorphic automorphisms on such a manifold extends to an action of \mathbf{C} . The relevant observation that this follows immediately from Proposition 2.1 was made by Manuel Flores.

In Sect. 3 we show that, if the manifold M is Stein, there exists a generic orbit type of an \mathbf{R} -action by $\text{Aut } M$ (Theorem 3.3). For actions of $(\mathbf{C}, +)$ on Stein manifolds the existence of a generic orbit type was proved by M. Suzuki [32]. (For analogous results see Richardson [27] and the references therein.) If the generic orbit is either \mathbf{C} or \mathbf{C}^* , then the action extends to an action of \mathbf{C} on M .

In Sect. 4 we show that for \mathbf{R} -complete holomorphic vector fields on Stein manifolds, the complex orbits of certain types have at most one limit point (Theorem 4.1). The closure of such an orbit is a pure one dimensional complex subvariety. This extends a result of Suzuki [32, 33].

In Sect. 5–7 we obtain new results on symplectic holomorphic automorphisms and symplectic actions on \mathbf{C}^{2n} . In Sect. 5 we show that the group $\mathcal{S}_{sp}^n \subset \text{Aut } \mathbf{C}^{2n}$ generated by *symplectic shears* is dense in the symplectic holomorphic automorphism group $\text{Aut}_{sp} \mathbf{C}^{2n}$. This is analogous to results of Andersén [3] and Andersén and Lempert [4] for the groups $\text{Aut}_1 \mathbf{C}^n$ and $\text{Aut } \mathbf{C}^n$.

In Sect. 6 (Theorem 6.1) we prove that every action of \mathbf{C} on \mathbf{C}^2 by symplectic holomorphic automorphisms is conjugate to one of the following:

$$\begin{aligned}\phi_t(z, w) &= (z, w + tf(z)), \\ \psi_t(z, w) &= (ze^{\lambda(zw)t}, we^{-\lambda(zw)t}),\end{aligned}$$

where $t \in \mathbf{C}$ and f resp. λ is an entire function of one variable. There are some further equivalencies, described in Propositions 6.2 and 6.3. Our result relies on the classification of entire functions of type \mathbf{C} or \mathbf{C}^* on \mathbf{C}^2 , due to Nishino [25] and Saito [29]. (See also Suzuki [33].) In this connection we recall that several other types of flows on \mathbf{C}^2 have been classified; we refer the reader to the papers [2], [7], [32], [33], [36].

In Sect. 7 we find obstructions to completeness of certain holomorphic Hamiltonian (i.e., divergence zero) vector fields on \mathbf{C}^2 . We show in particular that a Hamiltonian vector field of classical mechanics, $X = (w, f(z))$, with f an entire function of one variable, is complete on \mathbf{C}^2 only if f is linear (Proposition 7.3). This implies that for every nonlinear entire function f on \mathbf{C} , the second order conservative ordinary differential equation

$$\ddot{z} = f(z), \quad z(0) = z_0, \quad \dot{z}(0) = \dot{z}_0$$

cannot be solved for all $t \in \mathbf{R}$ and all initial data $(z_0, \dot{z}_0) \in \mathbf{C}^2$ (Corollary 7.4). This is in sharp contrast with the situation on \mathbf{R}^2 where the equation is often solvable for all t and all initial data. We also give a criterium for non-completeness of polynomial Hamiltonian fields on \mathbf{C}^2 (Proposition 7.2). The

obstructions are essentially of topological nature. The proofs rely on Proposition 3.1 and on elementary Morse theory.

In Sect. 8 we give a simpler proof (in the special case of \mathbf{C}^n) of a result of Suzuki [33] to the effect that for every action ϕ of \mathbf{R} or \mathbf{C} on \mathbf{C}^n by polynomial automorphisms (quasi-algebraic action) there is a proper polynomial embedding $F : \mathbf{C}^n \rightarrow \mathbf{C}^m$ for some $m > n$ and a linear action ψ on \mathbf{C}^m such that $\psi(t, F(z)) = F(\phi(t, z))$ for all $z \in \mathbf{C}^n$ and all t .

In Sect. 9 we show that complements $\Omega = \mathbf{C}^n \setminus A$ of ‘tame’ analytic subvarieties $A \subset \mathbf{C}^n$ of codimension at least two are homogeneous, in the sense that the group of $F \in \text{Aut}_1 \mathbf{C}^n$ which fix A pointwise acts transitively on Ω . This extends a result of Winkelmann [34] for algebraic varieties. Stronger results for discrete sets in \mathbf{C}^n were proved by Rosay and Rudin [28].

After the completion of this manuscript we received preprints from Fornæss and Sibony [13] and Fornæss and Grellier [14] with results on the global behavior of flows generated by (noncomplete) holomorphic Hamiltonian vector fields.

Several questions which were raised in the original version of this manuscript have been solved by the time of this revision. First, the question whether all holomorphic vector fields on \mathbf{C}^n which are complete in real time are also complete in complex time has been resolved in the positive (Corollary 2.2). Secondly, Buzzard and Fornæss have constructed holomorphic vector fields on \mathbf{C}^2 which cannot be approximated by complete ones, uniformly on compacts in \mathbf{C}^2 . In fact, the set of non approximable fields is dense in the set of all holomorphic vector fields in \mathbf{C}^n [37], [38].

1 Vector fields, flows, and foliations

In this section we recall some general results on holomorphic vector fields, their phase flows and the resulting foliations.

Let V be a smooth vector field on an n -dimensional manifold M . According to the local existence and regularity theory for systems of ordinary differential equations (see for instance [1], [6], or [20]), every point $p \in M$ has an open neighborhood $U \subset M$ of p and an $\varepsilon > 0$ such that the equation

$$\dot{x} = V(x), \quad x(0) = x^0 \tag{1.1}$$

has a unique solution $x(t)$ for every $x^0 \in U$ and every $|t| < \varepsilon$. Denote by $\phi(t, x^0) = \phi_t(x^0)$ the solution of (1.1) at time t . This map is called the (local) flow of V . Each time forward map ϕ_t is a diffeomorphism on the open subset of M where it is defined, and we have $\phi_t \circ \phi_s = \phi_{t+s}$ where both sides are defined.

From now on M is going to be a complex manifold. Let $\mathbf{T}M \otimes \mathbf{C} = \mathbf{T}^{(1,0)}M \oplus \mathbf{T}^{(0,1)}M$ be the usual splitting of the complexified tangent bundle to M . Denote by $J : \mathbf{T}M \rightarrow \mathbf{T}M$ the almost complex structure operator induced by the complex structure on M . Then we have an isomorphism $\mathbf{T}M \approx \mathbf{T}^{(1,0)}M$ given by $V \mapsto \tilde{V} = (V - iJV)/2$, so $V = 2\Re \tilde{V}$ and $JV = 2\Im(i\tilde{V})$. We will

say that a (real) vector field V on M is *holomorphic* if the corresponding $(1, 0)$ -field \tilde{V} is holomorphic (i.e., if \tilde{V} is a holomorphic section of the bundle $\mathbf{T}^{(1,0)}M$). If V is holomorphic, we can consider the holomorphic differential equation associated to (1.1):

$$\dot{x} = \tilde{V}(x), \quad x(0) = x^0. \quad (1.2)$$

Here \dot{x} denotes the complex derivative of x with respect to a complex variable $\zeta \in \mathbf{C}$. Again this equation has a unique local holomorphic solution $\zeta \mapsto \phi(\zeta, x^0)$, depending holomorphically on x^0 . The equation (1.2) is equivalent to the system of equations

$$\partial x / \partial t = V(x), \quad \partial x / \partial s = JV(x), \quad \zeta = t + is, \quad x(0) = x^0.$$

Definition 1. A vector field V on M is **R**-complete if the equation (1.1) has a solution $x(t) \in M$ for all initial data $x^0 \in M$ and all $t \in \mathbf{R}$. If M is a complex manifold and V is holomorphic, then V is **C**-complete if (1.2) has a solution $x(\zeta) \in M$ for all $x^0 \in M$ and all $\zeta \in \mathbf{C}$.

In the sequel we shall frequently identify the real vector field V with the field \tilde{V} , and we shall not distinguish between (1.1) and (1.2). This should not cause any confusion.

If a holomorphic vector field V is **R**-complete (resp. **C**-complete), its flow ϕ_t is an action of $(\mathbf{R}, +)$ (resp. $(\mathbf{C}, +)$) on M by holomorphic automorphisms. Conversely, one associates to every action $\phi : \mathbf{R} \times M \rightarrow M$ by holomorphic automorphisms of M the vector field

$$V_\phi(x) = \left. \frac{d}{dt} \phi(t, x) \right|_{t=0},$$

called the infinitesimal generator of ϕ . The field V_ϕ is holomorphic and **R**-complete on M , with the flow ϕ . Thus there is one to one correspondence between **R**-complete (resp. **C**-complete) holomorphic vector fields on M and actions of $(\mathbf{R}, +)$ (resp. $(\mathbf{C}, +)$) on M by holomorphic automorphisms.

Definition 2. Two actions $\phi, \tilde{\phi} : \mathbf{R} \times M \rightarrow M$ are equivalent if there exists an $F \in \text{Aut } M$ such that $F(\phi(t, x)) = \tilde{\phi}(t, F(x))$ for all $t \in \mathbf{R}$. Holomorphic vector fields V, \tilde{V} on M are equivalent if $DF \cdot V = \tilde{V} \circ F$ for some $F \in \text{Aut } M$.

Recall that the actions $\phi, \tilde{\phi}$ are equivalent if and only if their infinitesimal generators are equivalent.

We now define the complex orbit of a holomorphic vector field V . Fix $x \in M$ and let $\phi^x = \phi(\cdot, x)$ be the local holomorphic solution of (1.1) satisfying the initial condition $\phi^x(0) = x$. By analytic continuation we can extend this solution to a maximal connected Riemann domain R_x spread over \mathbf{C} . More precisely, there exists a maximal connected open Riemann surface R_x and a holomorphic immersion $\pi_x : R_x \rightarrow \mathbf{C}$ such that the local solution of (1.1) continues to a holomorphic mapping $\phi^x : R_x \rightarrow X$ satisfying $\phi^x(0) = x$ and $D\phi^x(\zeta) \frac{\partial}{\partial \zeta} = V(\phi^x(z))$. Here, $\partial/\partial\zeta$ is the pullback by π_x of the coordinate vector field on \mathbf{C} . In general R_x is not single sheeted over \mathbf{C} .

Definition 3. (Notation as above.) The set $C_x = \phi^x(R_x) \subset M$ is called the complex orbit of the vector field V (and of the associated local flow ϕ) through the point $x \in M$.

1.1 Proposition. Let V be a holomorphic vector field on M with the zero set Σ . Then for each $x \in M \setminus \Sigma$ the map $\phi^x : R_x \rightarrow C_x$ is a holomorphic covering projection. The orbit C_x is a smooth, embedded, one dimensional complex manifold (a Riemann surface) in M .

Proof. If $V_x \neq 0$, the orbit C_x does not contain any zeros of V , and therefore $\phi^x : R_x \rightarrow C_x$ is a holomorphic immersion onto C_x . The last assertion of the theorem is clear from the local theory of differential equations.

It remains to show that ϕ^x is a covering map. Choose a disc $U = U(0, \varepsilon) \subset R_x$ centered at the origin that is mapped by ϕ^x biholomorphically onto a neighborhood U_x of x in C_x . If $a \in R_x$ is any point such that $\phi^x(a) = y$ belongs to U_x , there is a point $a' \in U$ such that $\phi^x(a') = y$. Then the map $\zeta \rightarrow \phi^x(\zeta + a' - a) \in M$ is a local solution of the equation (1.1) on the disc $U(a - a'; \varepsilon) \subset \mathbf{C}$. Since it agrees with ϕ^x at the point $\zeta = a$, the two solutions agree, hence the disc $U(a' - a; \varepsilon)$ is contained in R_x and is mapped by ϕ^x biholomorphically onto U_x . Thus $(\phi^x)^{-1}(U_x)$ is a disjoint union of discs in R_x (although their projections to \mathbf{C} need not be disjoint!), and each of them is mapped by ϕ^x biholomorphically onto U_x . The same argument holds at every point of C_x , and Proposition 1.1 is proved. \square

Example. There exist holomorphic vector fields $V(z)$ on \mathbf{C}^n for $n > 1$ such that every solution $z(t)$ of the equation (1.1) goes to infinity in finite time (positive or negative), starting at any point in \mathbf{C}^n . Here is one way to get such fields for $n = 2$. I thank W. Rudin for having pointed this out to me.

Let F be a polynomial automorphism of \mathbf{C}^2 with an attracting fixed point a . Let $D(a)$ be the basin of attraction of a :

$$D(a) = \{z \in \mathbf{C}^2 : \lim_{k \rightarrow \infty} F^{(k)}(z) = a\}.$$

Here, $F^{(k)}$ is the k -th iterate of F . If $D(a)$ is not all of \mathbf{C}^2 then $D(a)$ is a Fatou–Bieberbach domain [9], [28]. Bedford and Smillie proved in [8] that in this case $D(a)$ intersects every algebraic curve $A \subset \mathbf{C}^2$ in a nonempty bounded set. In particular, the intersection $A \cap D(a)$ with every affine complex line is a bounded domain in A , and Runge’s approximation theorem implies that each connected component of $A \cap D(a)$ is biholomorphic to the disc. If $\psi : \mathbf{C}^2 \rightarrow D(a)$ is the Fatou–Bieberbach map and if W is any nonzero constant vector field on \mathbf{C}^2 , then the pull-back vector field $V = \psi_*^{-1}W$ satisfies the indicated property. The complex orbits of V are (biholomorphically equivalent to) discs which form a nonsingular holomorphic foliation of \mathbf{C}^2 . A specific example is provided by Hénon maps

$$F(x, y) = (y, p(y) - cx),$$

where p is a polynomial and $c \in \mathbf{C}_*$. The point $(0, 0)$ is an attracting fixed point of F if $p(0) = 0$, $p'(0) = 0$, and $0 < |c| < 1$. The fact that the intersections of its basin of attraction with complex lines are nonempty bounded sets was observed by Rosay and Rudin [28] in the case $p(y) = y^2$. The dynamical properties of Hénon maps has been studied extensively in recent years; see for instance [8, 12, 19] and the references therein. \square

We conclude this section by recalling that every holomorphic vector field V on a complex manifold M of dimension $n \geq 2$ determines a holomorphic foliation $\mathcal{F}(V)$ of $M \setminus \sigma$, where σ is an analytic subset of codimension at least two in M . To obtain $\mathcal{F}(V)$ we begin with the foliation $\mathcal{F}_0(V)$ of $M \setminus \Sigma$, where $\Sigma = \{V = 0\}$, whose leaves are the complex orbits of V . It is possible to extend $\mathcal{F}_0(V)$ to a neighborhood of each point $p \in \Sigma$ such that in a neighborhood of p we have $V = fV'$ for some holomorphic function f and a holomorphic vector field V' satisfying $V'(p) \neq 0$. The leaves of $\mathcal{F}(V)$ near p are the local orbits of V' (which coincide with the orbits of V outside Σ). The exceptional set σ consists of all points at which there is no such factorization; these are called singular points of V . It is not hard to see that σ is contained in the union of the singular locus of Σ and the irreducible components of Σ of dimension at most $\dim M - 2$.

2 Complex orbits of $(\mathbf{R}, +)$ actions

Let $\phi : \mathbf{R} \times M \rightarrow M$ be an action of $(\mathbf{R}, +)$ on a complex manifold M , with the infinitesimal generator V . Let Σ be the zero set of V . Denote by $\phi^x : R_x \rightarrow C_x$ the complex orbit of ϕ through a point $x \in M$ (Def. 3).

2.1 Proposition. *If ϕ is an action of $(\mathbf{R}, +)$ on a complex manifold M by holomorphic automorphisms, then for each $x \in M$ the Riemann surface R_x is a strip in \mathbf{C} of the form*

$$R_x = \{\zeta = t + is \in \mathbf{C} : -b(x) < s < a(x)\}, \quad a(x) > 0, \quad b(x) > 0.$$

For each $x \in M \setminus \Sigma$ the map $\phi^x : R_x \rightarrow C_x$ is the universal covering projection, and C_x is isomorphic to the quotient R_x/G_x , where G_x is a discrete subgroup of \mathbf{R} or \mathbf{C} (the group of deck transformations). The functions a and b are lower semicontinuous on M . If the manifold M is Stein then the functions $-a$ and $-b$ are plurisubharmonic on M (or identically $-\infty$), and the fundamental domain

$$\tilde{M} = \{(\zeta, x) : x \in M, \zeta \in R_x\} \subset \mathbf{C} \times M \tag{2.1}$$

is a Stein manifold. The action ϕ extends to an action of $(\mathbf{C}, +)$ on M if and only if $\tilde{M} = \mathbf{C} \times M$.

Proof. We consider the infinitesimal generator V of ϕ as a real vector field on M , and set $W = (V - iJV)/2$. The flow ϕ_t of V is by hypothesis defined

for all $t \in \mathbf{R}$. Let ψ_s ($s \in \mathbf{R}$) be the local flow of the real vector field JV on M . Since the field W is holomorphic, we have

$$i[V, JV] = [W + \overline{W}, \overline{W} - W] = 2[W, \overline{W}] = 0,$$

and therefore the flows ϕ_t and ψ_s of V resp. JV commute.

We fix an $x \in M$ and denote by $I(x) = (-b(x), a(x)) \subset \mathbf{R}$ the maximal interval on which the flow $\psi_s(x)$ of JV is defined. We claim that the map

$$\zeta = t + is \mapsto \phi_t \circ \psi_s(x)$$

is a solution $\phi^x(\zeta)$ of the Eq. (1.2) for ζ in the strip $R_x = \mathbf{R} \oplus iI(x)$. Clearly its t -derivative equals V since ϕ_t is the flow of V . If t and s are small, we have $\phi_t \circ \psi_s(x) = \psi_s \circ \phi_t(x)$, and thus its s -derivative equals JV there. Since the map is real-analytic on R_x , the same holds on all of R_x and the claim is established. In particular it follows that the flow $\psi_s(\phi_t(x))$ is defined for all $t + is \in R_x$.

We claim that R_x is the maximal domain of ϕ^x . If not, the map $\phi^x : R_x \rightarrow M$ extends holomorphically to a neighborhood of some point $\zeta_0 = t_0 + is_0$ in the boundary of R_x . Thus, if we set $y = \phi_{t_0}(x) \in M$, the maximal interval $I(y)$ on which the flow $\psi_s(y)$ is defined is strictly larger than $I(x)$. We can now extend the map ϕ^x to the larger strip $R_y = \mathbf{R} \oplus iI(y)$ by setting

$$\phi^x(t + is) = \phi_{t-t_0} \circ \psi_s \circ \phi_{t_0}(x).$$

This definition agrees with the previously defined ϕ^x for $\zeta = t + is \in R_x$ since the flows ϕ_t and ψ_s commute there. Thus we have an extension of ϕ^x to the larger strip R_y . Therefore the interval $I(x)$ is not maximal for the flow $\psi_s(x)$, a contradiction.

Since R_x is simply connected, $\phi^x : R_x \rightarrow C_x$ is the universal covering of C_x (Proposition 1.1). Therefore $C_x \simeq R_x/G_x$, where $G_x \subset \text{Aut } R_x$ is the group of deck transformations.

The set \tilde{M} (2.1) is an open domain in $\mathbf{C} \times M$ by general properties of flows, hence the functions a and b are lower semicontinuous on M . Suppose now that the manifold M is Stein. We will prove that the functions $-a$ and $-b$ are plurisubharmonic on M . If $-a$ fails to be plurisubharmonic at a point $x_0 \in M$, there exist a closed embedded analytic disc $D \subset M$ containing x_0 and a smooth function $u : D \rightarrow \mathbf{C}$, harmonic in the interior of D , such that $0 \leq u(x) < a(x)$ for all $x \in bD$ but $u(x_0) > a(x_0)$. Let v be a harmonic conjugate of u on D and set $h = i(u + iv) = -v + iu$.

Consider the family of analytic discs $D_s = \{(sh(x), x) : x \in D\} \subset \mathbf{C} \times M$ for $0 \leq s \leq 1$. When s is small, D_s is entirely contained in \tilde{M} . By construction the union of boundaries $\bigcup_{0 \leq s \leq 1} bD_s$ is compactly contained in \tilde{M} , but the disc D_1 is not contained in \tilde{M} . Let s_1 be the smallest value of $s \in (0, 1)$ for which $D_s \cap b\tilde{M}$ is not empty, and choose a point $(\zeta_1, x_1) \in D_{s_1} \cap b\tilde{M}$. Then $\zeta_1 = s_1 h(x_1)$. Since the domain \tilde{M} is translation invariant with respect to the real part of the first variable, we can change $v = \Re h$ if necessary so that $v(z_1) = 0$ and $\zeta_1 = is_1 u(x_1)$.

Since M is Stein, it admits a holomorphic embedding $F: M \rightarrow \mathbf{C}^N$ into some Euclidean space [21]. Set $G = F \circ \phi: \tilde{M} \rightarrow \mathbf{C}^N$, where $\phi: \tilde{M} \rightarrow M$ is the flow of V . By the Kontinuitätssatz the map G admits analytic continuation along the family of discs D_s for $0 \leq s \leq 1$. This extends the flow $s \rightarrow \phi(isu(x_1), x_1)$ to all values $0 \leq s \leq 1$ in contradiction to the maximality of the domain \tilde{M} . This proves that $-a$ is plurisubharmonic on M ; the analogous proof applies to $-b$. Thus \tilde{M} is a pseudoconvex domain in the Stein manifold $\mathbf{C} \times M$ and therefore it is itself a Stein manifold. Proposition 2.1 is proved. \square

The following consequence of Proposition 2.1 was observed by Manuel Flores.

2.2 Corollary. *If M is a Stein manifold such that every plurisubharmonic function on M which is bounded from above is constant, then every action of $(\mathbf{R}, +)$ by holomorphic automorphisms on M extends to an action of $(\mathbf{C}, +)$. This holds in particular when $M = \mathbf{C}^n$ or when $M = \mathbf{C}^n \setminus A$ for some complex hypersurface $A \subset \mathbf{C}^n$.*

Equivalently, every holomorphic vector field on such a manifold M which is \mathbf{R} -complete is also \mathbf{C} -complete. In particular, every \mathbf{R} -complete holomorphic vector field on \mathbf{C}^n is \mathbf{C} -complete. The same holds on $\mathbf{C}^k \times (\mathbf{C}^*)^m$ for $k, m \geq 0$.

Proof. The functions $-a$ and $-b$ from Proposition 2.1 are negative plurisubharmonic on M (or $-\infty$), hence constant by our assumption on M . Thus the flow of the field JV exists for time $|t| < \min(a, b)$ when starting at any point $x \in M$. The group property of the flow then implies that the flow of JV exists for all times t , i.e., the field JV is \mathbf{R} -complete. Thus $a = b = \infty$ and the field V is \mathbf{C} -complete.

It is well known that there are no nonconstant negative plurisubharmonic functions on \mathbf{C}^n [35, p. 338]. The same holds for $\mathbf{C}^n \setminus A$, where A is an analytic hypersurface in \mathbf{C}^n , since a bounded plurisubharmonic function extends across A . \square

2.3 Corollary. *If ϕ is an action of $(\mathbf{R}, +)$ by holomorphic automorphisms on a Stein manifold M such that $R_x = \mathbf{C}$ for all points x in a non-pluripolar set in M , then ϕ extends to an action of $(\mathbf{C}, +)$ on M .*

3 Types of orbits of an $(\mathbf{R}, +)$ action

Let $\phi: \mathbf{R} \times M \rightarrow M$ be an action of $(\mathbf{R}, +)$ on a complex manifold M with the fixed point set $\Sigma \subset M$. We extend ϕ to the maximal domain $\tilde{M} \subset \mathbf{C} \times M$ (2.1) defined in Sect. 2.

3.1 Proposition. *For each $x \in M \setminus \Sigma$ the complex orbit C_x of ϕ containing x is biholomorphic to one of the following Riemann surfaces:*

- (a) the complex line \mathbf{C} ;
- (b) the punctured line $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$;

- (c) a torus;
- (d) the disc $U = \{z \in \mathbf{C} : |z| < 1\}$;
- (e) the punctured disc $U^* = U \setminus \{0\}$;
- (f) an annulus $A(r) = \{z \in \mathbf{C} : 1 < |z| < r\}$.

Actions of $(\mathbf{C}, +)$ on M only have orbits of types (a)–(c).

Remark. If M is Stein then ϕ has no toral orbits. If M is a hyperbolic manifold then ϕ has only orbits of types (d)–(e), and there are no actions of \mathbf{C} on M .

Proof. Recall that $C_x \simeq R_x/G_x$, where R_x is a strip in \mathbf{C} (Proposition 2.1) and G_x is a discrete group of automorphisms of R_x (the deck group). Fix a $g \in G_x$. We claim that $g(\zeta + t) = g(\zeta) + t$ for all $\zeta \in R_x$ and $t \in \mathbf{R}$, and therefore $g(\zeta) = \zeta + a$ for some $a \in \mathbf{C}$. To see this, recall that by definition of ϕ^x we have $D\phi^x(\zeta)(\partial/\partial\zeta) = V(\phi^x(\zeta))$. Therefore ϕ^x maps the flow $(t, \zeta) \rightarrow \zeta + t$ of the field $\partial/\partial\zeta$ on the strip R_x to the flow ϕ of the field $V: \phi^x(\zeta + t) = \phi_t(\phi^x(\zeta))$ for each $\zeta \in R_x$ and $t \in \mathbf{R}$. Since $g \in G_x$ satisfies $\phi^x \circ g = \phi^x$, we obtain

$$\phi^x(g(\zeta) + t) = \phi_t \circ \phi^x(g(\zeta)) = \phi_t \circ \phi^x(\zeta) = \phi^x(\zeta + t) = \phi^x(g(\zeta + t)).$$

Since ϕ^x is a covering projection, it follows that $g(\zeta + t) = g(\zeta) + t$ as claimed.

This shows that G_x is a discrete subgroup of $(\mathbf{C}, +)$ which is necessarily contained in \mathbf{R} if $R_x \neq \mathbf{C}$. If $R_x = \mathbf{C}$, G_x has either zero, one, or two generators. The orbit $C_x = \mathbf{C}/G_x$ is then isomorphic to, respectively, \mathbf{C} , \mathbf{C}^* , or a torus. If $R_x \neq \mathbf{C}$ then G_x is a discrete subgroup of \mathbf{R} , hence it has either zero or one generator. In the first case C_x is isomorphic to the disc; in the second case it is isomorphic either to an annulus or to the punctured disc U^* , depending on whether both numbers $a(x), b(x)$ determining the strip R_x are finite or one of them is infinite. This proves Proposition 3.1. \square

The proof given above actually gives more. Let V be a holomorphic vector field on a complex manifold M , and let ϕ_t be its local flow.

Definition 4. A complex orbit C of V is **R-complete** (resp. **C-complete**) if $\phi_t(x)$ exists for all $x \in C$ and all $t \in \mathbf{R}$ (resp. for all $t \in \mathbf{C}$).

Observe that a holomorphic vector field V is **R-complete** (resp. **C-complete**) if every complex orbit of V is **R-complete** (resp. **C-complete**).

3.2 Proposition. Every nontrivial, **R-complete** complex orbit is isomorphic to one of the surfaces in Proposition 3.1.

In other words, if $C \subset M$ is a complex orbit of V which is not isomorphic to any of the surfaces in Proposition 3.1, then there exists a point $x \in C$ and a finite $t_0 \in \mathbf{R}$ such that $\phi_t(x)$ leaves every compact set of M as t approaches t_0 . The proof of Proposition 3.2 is exactly the same as that of Proposition 3.1.

Richardson [27] proved that for every holomorphic action of a reductive complex Lie group G on a connected Stein manifold M there exists a connected, dense open subset M_0 of M such that the orbits C_x are isomorphic to each other for all $x \in M_0$. We now prove a similar result for the complex orbits of an action of \mathbf{R} on a connected Stein manifold M .

3.3 Theorem. *If ϕ is an action of $(\mathbf{R}, +)$ by holomorphic automorphisms on a connected Stein manifold M , there exists a ϕ -invariant pluripolar set $E \subset M$ (E containing the fixed point set Σ of ϕ) such that every complex orbit C_x for $x \in M \setminus E$ is of the same type (a), (b), or (d)–(e). If the generic orbit is U^* or an annulus, then ϕ has a period $\lambda > 0$, and it factors through an action of the circle group $(S, +)$ on M . ϕ extends to an action of $(\mathbf{C}, +)$ on M if and only if the generic complex orbit is either \mathbf{C} or \mathbf{C}^* ; this is always the case when $M = \mathbf{C}^n$.*

Remark. For actions of $(\mathbf{C}, +)$ on Stein spaces the existence of a generic orbit type (\mathbf{C} or \mathbf{C}^*) was proved by Suzuki [32, Proposition 2]. His proof shows that when the generic orbit type is \mathbf{C}^* then C_x is isomorphic to \mathbf{C}^* for every x outside a closed analytic subset $A \subset M$. (A is the union of zero and polar sets of a meromorphic function λ^2 on M such that $\lambda(x)$ is a generator of the isotropy group G_x .)

Proof. We shall only consider actions of $(\mathbf{R}, +)$ which do not extend to actions of $(\mathbf{C}, +)$, since in the later case the result has been proved in [32]. We define pluripolar sets $E_0, E_1 \subset M$ as follows. If $a(x)$ is not identically equal to $+\infty$ on M , we set $E_0 = \{x \in M : a(x) = +\infty\}$; otherwise we set $E_0 = \emptyset$. (Recall that $-a$ and $-b$ are plurisubharmonic.) Similarly, if $b(x)$ is not identically $+\infty$, we set $E_1 = \{x : b(x) = +\infty\}$, and we set $E_1 = \emptyset$ otherwise. By the hypothesis on ϕ at least one of these two sets is nonempty, and $R_x \neq \mathbf{C}$ for all $x \in M \setminus (E_0 \cup E_1)$. Set

$$G = \{(\zeta, x) \in \tilde{M} : \phi(\zeta, x) = x\},$$

$$G_x = \{\zeta \in R_x : \phi(\zeta, x) = x\}.$$

G is a closed, proper analytic subvariety of \tilde{M} , and G_x is a discrete subset of R_x (the isotropy subgroup at x). We can write $G = G^0 \cup G'$, where $G^0 = \{0\} \times M$ and G' is the union of the remaining irreducible components of G . Denote by $\pi : \mathbf{C} \times M \rightarrow M$ the projection $\pi(\zeta, x) = x$. Let $E_2 \subset M$ be the projection in M of the union of all irreducible components of G of dimension less than n . Then E_2 is contained in at most a countable union of analytic sets in M of dimension $< n$ [10] and hence it is pluripolar.

Set $E = E_0 \cup E_1 \cup E_2$; this set is also pluripolar. If $\dim G' < n$, G_x is trivial for all $x \in M \setminus E_2$, and hence the orbit C_x is isomorphic to the disc for each $x \in M \setminus E$.

Consider now the case $\dim G' = n$. Choose a regular n -dimensional point $g_0 = (\lambda_0, x_0) \in G'$ which is a regular point of $\pi|_{G'}$. Such points exist on each irreducible n -dimensional component of G' since the fibers G_x of G are discrete. We claim that G' contains the set $\{\lambda_0\} \times M$ as an irreducible component. To prove this we observe that locally near g_0 , G is the graph of a holomorphic function $\lambda(x)$ defined near $x_0 \in M$. Since $G_x \subset \mathbf{R}$ for most $x \in M$, λ is real valued and thus constant, $\lambda(x) = \lambda_0 \in \mathbf{R}$. Hence G' contains $\{\lambda_0\} \times M$ as claimed.

We repeat the same proof for every n -dimensional irreducible component of G . The conclusion is that the n -dimensional part of G equals $\lambda_0 \mathbf{Z} \times M$, where $\lambda_0 > 0$ is the smallest period of the action ϕ . Since $\phi(\lambda_0, x) = x$ for all $x \in M$, ϕ factors through an action of the circle group on M . We have $G_x = \lambda_0 \mathbf{Z}$ for all $x \in M \setminus E_2$. For each $x \in M \setminus E$ the orbit $C_x = R_x / G_x$ is of the same type (e) or (f) (Proposition 3.1). If both E_0 and E_1 are nonempty, these orbits are annuli. If one of the sets E_0 resp. E_1 is empty, the generic orbit is U^* . If both E_0 and E_1 are empty, ϕ extends to an action of \mathbf{C} on M . This completes the proof of Theorem 3.3. \square

We conclude this section by a list of examples, showing that all types of orbits listed in Proposition 3.1, except type (c), arise as generic orbits of $(\mathbf{R}, +)$ -actions on Stein manifolds.

Example 1. The following list of actions of \mathbf{C} on \mathbf{C}^2 is taken out of Suzuki [33]:

$$t \circ (x, y) = (x, y + tf(x))$$

$$t \circ (x, y) = (x, e^{t\lambda(x)}(y - b(x)) + b(x))$$

$$t \circ (x, y) = (xe^{n\lambda t}, ye^{m\lambda t}), \quad n, m \in \mathbf{Z}_+, \lambda \in \mathbf{C}^*$$

$$t \circ (x, y) = (xe^{n\lambda(z)t}, ye^{-m\lambda(z)t}), \quad m, n \in \mathbf{Z}_+, z = x^m y^n$$

$$t \circ (x, y) = (xe^{\lambda t}, e^{n\lambda t}(y + tx^n)), \quad n \in \mathbf{Z}_+, \lambda \in \mathbf{C}^*.$$

Denote by $C_{(x,y)}$ the complex orbit through $(x, y) \in \mathbf{C}^2$. For the first action we have $C_{(x,y)} = \{x\} \times \mathbf{C}$ if $f(x) \neq 0$; hence the generic orbit type is \mathbf{C} unless f is identically zero. In the second action the functions λ and λb are holomorphic on \mathbf{C} (hence b is meromorphic). If x is not a pole of b , then $(x, b(x))$ is a fixed point of the action, and $C_{(x,y)} \simeq \mathbf{C}^*$ for $y \neq b(x)$. Hence the generic orbit type is \mathbf{C}^* . The third action factors through an action of \mathbf{C}^* on \mathbf{C}^2 (since it has a period $t_0 \in \mathbf{C}^*$ independent of (x, y)), and every orbit $C_{(x,y)}$ except $C_{(0,0)}$ is of type \mathbf{C}^* . The fourth action also has a nontrivial period $t_0(x, y)$ which depends on (x, y) , and the generic orbit is \mathbf{C}^* . For the last action we have $C_{(x,y)} \simeq \mathbf{C}$ for $x \neq 0$, and its limit set is $\omega(C_{(x,y)}) = \{0\} \times \mathbf{C}$. (See Sect. 4 for the definition of the limit set.)

Example 2. Let $B \subset \mathbf{C}^n (n \geq 2)$ be the unit ball. For each $n \times n$ matrix A satisfying $A + \bar{A}^t = 0$ we have the unitary action of \mathbf{R} on B

$$\phi_t(z) = \exp(At)z, \quad t \in \mathbf{R}.$$

If $A = (id_j)$ is a diagonal matrix, with $d_j \in \mathbf{R}$, we have

$$\exp(At)z = (e^{id_1 t} z_1, \dots, e^{id_n t} z_n).$$

If the quotients d_j/d_k are all rational, there is a nontrivial period $t_0 > 0$, the orbits are closed in $B \setminus \{0\}$, and the generic orbit type is U^* . If one of the

numbers d_j/d_k in irrational, there is no nontrivial period and the generic orbit type is U . In this case most complex orbits have large limit sets in B . Every action on B with a fixed point $a \in B$ is conjugate in $\text{Aut } B$ to an action of this form. Of course there are no \mathbf{C} -actions on B since B is hyperbolic.

Example 3. The domain $\Omega = \{w = (w_1, w') \in \mathbf{C}^n : \Im w_1 > |w'|^2\}$ is the unbounded realization of the ball (the Siegel upper half space). Consider the actions of \mathbf{R} on Ω

$$\begin{aligned}\phi_t(w) &= (w_1 + t, w'), \\ \psi_t(w) &= (e^{2t}w_1, e^t w').\end{aligned}$$

They have no fixed points in Ω . The first one has the only fixed point at ∞ , while the second one has fixed points 0 and ∞ . These actions are conjugate to actions on B which have one resp. two fixed points in bB and no fixed points in B . The generic orbit type is U in the first case and U^* in the second case.

Example 4. Let $D \subset \mathbf{C}^n$ be a bounded circular domain which does not contain the origin. Then the orbits C_z of the action $\phi_t(z) = e^{it}z$ ($t \in \mathbf{R}$) are annuli whose holomorphic type may depend on the point $z \in D$.

4 Limit sets of complex orbits

Let ϕ be an action of \mathbf{R} on a complex manifold M , and let $\phi^x: R_x \rightarrow C_x$ be the complex orbit of ϕ passing through x (Def. 3). Pick any increasing sequence of compacts $\{K_j\}_{j=1}^\infty$ in R_x whose union equals R_x . The limit set $\omega(C_x) \subset M$ of the orbit C_x is defined by

$$\omega(C_x) = \bigcap_{j=1}^\infty \overline{\phi^x(R_x - K_j)}.$$

Clearly $\omega(C_x)$ is a closed subset of M invariant under the action of ϕ (hence it is itself a union of \mathbf{R} -orbits of ϕ), and $\overline{C_x} = C_x \cup \omega(C_x)$.

The action is said to be *proper* if the limit set of every complex orbit C is empty or discrete; the closure \overline{C} is then a closed analytic subvariety of M . Proper actions of \mathbf{C} on \mathbf{C}^2 have been classified by Suzuki [33, Theorem 4].

The following result was proved for actions of $(\mathbf{C}, +)$ on Stein manifolds by Suzuki [32, Theorem 1], [33].

4.1 Theorem. *Let ϕ be an action of $(\mathbf{R}, +)$ by holomorphic automorphisms on a Stein manifold M , and let C be a nontrivial complex orbit of ϕ . If C is isomorphic to an annulus, then $\omega(C) = \emptyset$. If C is isomorphic to the punctured disc U^* , or if C is isomorphic to the punctured plane \mathbf{C}^* and ϕ extends to an action of $(\mathbf{C}, +)$ on M , then the limit set $\omega(C)$ consists of at most one point, and this point is a fixed point of the action.*

In each of these cases the closure \overline{C} is a pure one dimensional analytic subvariety of M . If \overline{C} does not intersect the singular set σ of the action, then \overline{C} is smooth.

Proof. The idea is the same as in [32]. Recall from Proposition 2.1 that the complex orbit C through a point $x_0 \in M \setminus \Sigma$ is the image of a strip $R = \{\zeta = t + is \in \mathbf{C} : -b < s < a\}$, and C is isomorphic to the quotient R/G , where G is the isotropy group of $x_0 : G = \{\zeta \in R : \phi(\zeta, x_0) = x_0\}$. G is a discrete subgroup of \mathbf{C} which is contained in \mathbf{R} if $R \neq \mathbf{C}$.

If C is of one of the types as in the theorem, then G is nontrivial. Since M is Stein, G can not have two generators, hence it has precisely one generator $\lambda \in \mathbf{C}$. The quotient $C = R/G$ is isomorphic either to \mathbf{C}^* (when $a = b = \infty$), to U^* (when one of the numbers a, b is infinite and the other one is finite), or to an annulus (when both a and b are finite).

Consider first the case when C is isomorphic to \mathbf{C}^* and ϕ extends to an action of $(\mathbf{C}, +)$ on M . Replacing ϕ by the action $(\zeta, x) \mapsto \phi(\lambda\zeta, x)$ we may assume that $\lambda = 1$. (This is the only place where we need the assumption that ϕ extends to an action of $(\mathbf{C}, +)$.) The set $M_1 = \{x \in M : \phi(1, x) = x\}$ is a closed analytic subset of M containing C . Therefore $\phi(1, x) = x$ for all $x \in \bar{C}$.

Each limit point $p \in \omega(C)$ is the limit of a sequence $\phi(\zeta_j, x_0) = x_j \in C$ such that $0 \leq \Re \zeta_j < 1$ and $|\Im \zeta_j| \rightarrow \infty$. Passing to a subsequence we may assume without loss of generality that $s_j = \Im \zeta_j \rightarrow +\infty$. Let $L_j = \{\phi(t + is_j, x_0) : t \in \mathbf{R}\}$ be the \mathbf{R} -orbit of x_j . These orbits are compact since $\phi(1, x) = x$ for $x \in \bar{C}$, and the continuity of ϕ they converge to the compact \mathbf{R} -orbit $L = \{\phi(t, p) : t \in \mathbf{R}\}$ of the limit point p . Hence the union $K = \bigcup_{j \geq 1} L \cup L_j$ is compact in M .

Choose a holomorphic embedding $F : M \rightarrow \mathbf{C}^N$ such that $|F| < 1$ on a neighborhood of K in M [21]. Since the mapping $\zeta \rightarrow F(\phi(\zeta, x_0))$ is periodic with period one, it can be written as a holomorphic map $H(w)$ of $w = \exp(2\pi i \zeta)$. The line $\Im \zeta = s_j$ corresponds to the circle $|w| = e^{-2\pi s_j}$. By construction we have $|H| < 1$ on each of these circles for $j \geq j_0$. Since $s_j \rightarrow \infty$, the maximum principle implies that $|H| < 1$ in a neighborhood of the origin in \mathbf{C} . Therefore H extends holomorphically to the origin, and the sets $\{H(w) : |w| = e^{-2\pi s_j}\} = F(L_j)$ converge to the point $H(0)$ as $j \rightarrow \infty$. Since $F(L_j)$ also converge to $F(L)$, it follows that $F(L) = H(0)$, and therefore $L = \{p\}$. Thus p is a fixed point of ϕ . The proof also shows that $\lim_{\Im \zeta \rightarrow +\infty} \phi(\zeta, x_0) = p$.

This shows that the limit set of the orbit $C \simeq \mathbf{C}^*$ at each of its two ends is either empty or else a point p which is a fixed point of ϕ . Adding this point to C compactifies C at this end, and $C \cup \{p\} \simeq \mathbf{C}$. We can not compactify C at both ends at the same time since this would yield a compact Riemann surface (sphere) in the Stein manifold M . This proves the theorem for orbits of type \mathbf{C}^* .

Consider now the case $C \simeq U^*$. This happens when one of the numbers a, b is finite and the other infinite; say $b < \infty$ and $a = \infty$. The isotropy group G is a discrete subgroup of \mathbf{R} with one generator $\lambda > 0$. The quotient R/G is biholomorphic to the punctured disc U^* via $\zeta \rightarrow w = e^{2\pi(i\zeta - b)/\lambda}$, with the end $\Im \zeta \rightarrow +\infty$ corresponding to the origin $w = 0$. The same proof as above shows that C is either closed at this end, or else its closure is obtained by adding to C a fixed point of ϕ .

Next we consider the end of C corresponding to $\Im\zeta \rightarrow -b$. Since R is the maximal domain on which $\phi(\cdot, x_0)$ is defined, it follows by standard arguments that the sequence $\phi(\zeta_j, x_0)$ goes out of every compact subset of M for any sequence $\zeta_j \in R$ converging to a compact subset of $bR = \{t - ib : t \in \mathbf{R}\}$. Since $\phi(\cdot, x_0)$ is λ -periodic, we conclude that C has no limit points in M at this end.

Finally, if C is an annulus, both a and b are finite and the group G is nontrivial, generated by a $\lambda > 0$. The same argument as in the case $C \simeq U^*$ shows that C is closed in M . This proves Theorem 4.1. \square

Remark. Suzuki pointed out in [32, Theorem 2] that for actions of $(\mathbf{C}, +)$ on a two dimensional Stein manifold M , with generic orbit of type \mathbf{C}^* , the closure in M of every nontrivial orbit is an irreducible, one dimensional analytic subset of M . The reason is that the orbits of type \mathbf{C} only occur for x in a closed subvariety $A \subset M$. Since $\dim M = 2$, A is an analytic curve, and $C_x \subset A$ for $x \in A$. Hence \bar{C}_x is an analytic subvariety of M according to a theorem of Remmert and Stein [10].

5 Symplectic holomorphic automorphisms of \mathbf{C}^{2n}

In this section we consider symplectic holomorphic automorphisms of \mathbf{C}^{2n} . We refer the reader to [1] and [5] for motivation and basic results of (real) symplectic geometry.

Let $(z, w) = (z_1, \dots, z_n, w_1, \dots, w_n)$ be the complex coordinates on \mathbf{C}^{2n} . We will also use the notation $(z, w) = x = (x_1, \dots, x_{2n})$, where $x_j = z_j$, $x_{n+j} = w_j$ for $1 \leq j \leq n$. We denote by $\{e^j : 1 \leq j \leq 2n\}$ the standard basis of \mathbf{C}^{2n} . The alternating bilinear form

$$\omega(u, v) = \sum_{j=1}^n u_j v_{n+j} - u_{n+j} v_j, \quad u, v \in \mathbf{C}^{2n}, \tag{5.1}$$

is called the *standard symplectic form* on \mathbf{C}^{2n} [23]. The group $\text{Sp}(n, \mathbf{C})$, consisting of all linear maps $A \in \text{GL}(2n, \mathbf{C})$ preserving this form, is called the *linear symplectic group*. This group is best described by introducing the linear operator

$$Je^k = -e^{n+k}, \quad Je^{n+k} = e^k, \quad 1 \leq k \leq n.$$

In the standard basis J has the matrix $\begin{pmatrix} 0 & I^n \\ -I^n & 0 \end{pmatrix}$, where I^n is the $n \times n$ identity matrix. Then

$$\omega(u, v) = u^t J v, \quad u, v \in \mathbf{C}^{2n}.$$

Here u^t is the transpose of the column vector u . Thus

$$\text{Sp}(n, \mathbf{C}) = \{A \in \text{GL}(2n, \mathbf{C}) : A^t J A = J\}. \tag{5.2}$$

Recall [23, p. 373] that $\text{Sp}(n, \mathbf{C})$ is generated by the linear *symplectic transvections*

$$A_{\lambda, v}(x) = x + \lambda \omega(x, v)v, \quad \lambda \in \mathbf{C}^*, \quad v \in \mathbf{C}^{2n}. \tag{5.3}$$

The symplectic differential form on \mathbf{C}^{2n} corresponding to (5.1) is

$$\omega = \sum_{j=1}^n dz_j \wedge dw_j = \sum_{j=1}^n dx_j \wedge dx_{n+j}. \quad (5.4)$$

We shall use the same letter ω for both (5.1) and (5.4) since it will always be clear from the context which form is meant. The $(2,0)$ -form (5.4) is holomorphic, closed, nondegenerate, and

$$\frac{(-1)^{[n/2]}}{n!} \omega^n = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{2n} = \Omega.$$

The $(2n, 0)$ -form Ω is called the *canonical form* (or the holomorphic volume form) on \mathbf{C}^{2n} . Notice that $\omega = \Omega$ on \mathbf{C}^2 .

Definition 5. A holomorphic mapping $F: D \subset \mathbf{C}^{2n} \rightarrow \mathbf{C}^{2n}$ is *symplectic holomorphic* if $F^*\omega = \omega$.

A symplectic holomorphic map also satisfies $F^*\Omega = \Omega$, which means that F has Jacobian one; in particular, F is an immersion. Clearly the composition of symplectic holomorphic maps is again symplectic holomorphic.

We denote the group of symplectic holomorphic automorphisms of \mathbf{C}^{2n} by $\text{Aut}_{\text{sp}} \mathbf{C}^{2n}$. Recall that $\text{Aut}_1 \mathbf{C}^n$ is the group of automorphisms with Jacobian one. We have

$$\text{Sp}(n, \mathbf{C}) \subset \text{Aut}_{\text{sp}} \mathbf{C}^{2n} \subset \text{Aut}_1 \mathbf{C}^{2n} \subset \text{Aut} \mathbf{C}^{2n}, \quad \text{Aut}_{\text{sp}} \mathbf{C}^2 = \text{Aut}_1 \mathbf{C}^2.$$

We shall now consider a subgroup $\mathcal{S}_{\text{sp}}^n \subset \text{Aut}_{\text{sp}} \mathbf{C}^{2n}$ generated by *symplectic shears*. Recall from [28] that a shear on \mathbf{C}^n is an automorphism of the form

$$F(z) = z + f(\Lambda z)v, \quad v \in \mathbf{C}^n,$$

where $\Lambda: \mathbf{C}^n \rightarrow \mathbf{C}^k$ ($k < n$) is a linear map such that $\Lambda v = 0$, and f is an entire function of k variables. Linear maps of this type are also called ‘transvections’ [23].

Let $v \in \mathbf{C}^{2n}$, and let f be an entire function of one variable. The map

$$F_{f,v}(x) = x + f(\omega(x, v))v, \quad x \in \mathbf{C}^{2n}, \quad (5.5)$$

is a symplectic holomorphic automorphism of \mathbf{C}^{2n} with the inverse $F_{f,v}^{-1} = F_{-f,v}$. We will call maps of this form *symplectic shears*. The derivative of $F = F_{f,v}$ equals

$$DF(x)u = u + f'(\omega(x, v))\omega(u, v)v,$$

and it follows immediately that $\omega(DF(x)u, DF(x)u') = \omega(u, u')$ for all $u, u' \in \mathbf{C}^{2n}$. Thus $F^*\omega = \omega$ as claimed. Notice that every shear on \mathbf{C}^2 is symplectic.

Here is the main result of this section. Analogous results for the groups $\text{Aut} \mathbf{C}^n$ and $\text{Aut}_1 \mathbf{C}^n$ ($n \geq 2$) have been obtained in [3] and [4].

5.1 Theorem. For each $n \geq 1$ the group $\mathcal{S}_{\text{sp}}^n$, consisting of finite compositions of symplectic shears (5.5), is dense in $\text{Aut}_{\text{sp}}\mathbb{C}^{2n}$ in topology of uniform convergence on compact sets in \mathbb{C}^{2n} .

Notice that $\mathcal{S}_{\text{sp}}^n$ contains the linear symplectic group $\text{Sp}(n, \mathbb{C})$ since every linear symplectic transvection (5.3) belongs to $\mathcal{S}_{\text{sp}}^n$, and $\text{Sp}(n, \mathbb{C})$ is generated by symplectic transvections. The result analogous to Theorem 5.1 holds also for smooth symplectic maps on \mathbb{R}^{2n} , but it is much less interesting there.

Remark 1. Andersén [3] proved that the group $\mathcal{S}_1^n \subset \text{Aut}_1\mathbb{C}^n$, consisting of finite compositions of shears on \mathbb{C}^n , is dense in $\text{Aut}_1\mathbb{C}^n$, but not equal to $\text{Aut}_1\mathbb{C}^n$. (See also [4].) In particular, if f is a nonconstant entire function of one variable, the map $(z, w) \mapsto (ze^{f(zw)}, we^{-f(zw)})$, which belongs to $\text{Aut}_1\mathbb{C}^2 = \text{Aut}_{\text{sp}}\mathbb{C}^2$, is not a finite composition of shears on \mathbb{C}^2 . We expect that the two groups in Theorem 5.1 are different in every dimension.

Remark 2. Although Theorem 5.1 is stated only for global symplectic maps, the same technique gives approximation of certain isotopies of symplectic holomorphic maps. We refer the reader to the papers [17] and [18] in which we study invariants of symplectic and volume preserving holomorphic maps.

Proof of Theorem 5.1. The proof follows from the ideas outlined in the proof of Theorem 1.1 in [15]. We must recall the notion of complex Hamiltonian vector fields. We state without proof several well known results of symplectic algebra and geometry; we refer to [1, 5, 23].

Since the form ω (5.4) is nondegenerate, it induces an isomorphism between the holomorphic tangent and cotangent bundles to \mathbb{C}^{2n} :

$$v \in T_x\mathbb{C}^{2n} \mapsto \omega_x(v, \cdot) \in T_x^*\mathbb{C}^{2n}.$$

To each holomorphic function $H(z, w)$ (Hamiltonian) on a domain $D \subset \mathbb{C}^{2n}$ we associate the holomorphic Hamiltonian vector field X_H on D by the equation

$$dH = \omega(X_H, \cdot) = i_X\omega.$$

Here, $dH = \sum_{j=1}^n H_{z_j} dz_j + H_{w_j} dw_j$. In coordinates (z, w) we have

$$X_H = \sum_{j=1}^n H_{w_j} \frac{\partial}{\partial z_j} - H_{z_j} \frac{\partial}{\partial w_j} = (H_w, -H_z).$$

Let ϕ_t^H be the local flow of X_H . The chain rule implies that $\frac{d}{dt}H(\phi_t^H(x)) = 0$ for all t , hence H is constant on the orbits of ϕ^H . Classically this is referred to as the ‘conservation of energy’. Each time map ϕ_t^H is symplectic where it is defined. In particular, if $H \in \mathcal{O}(\mathbb{C}^{2n})$ is an entire function such that the Hamiltonian field X_H is complete, then the flow ϕ_t^H is a one parameter subgroup of $\text{Aut}_{\text{sp}}\mathbb{C}^{2n}$. Conversely, if ϕ is an action of \mathbb{R} or \mathbb{C} on \mathbb{C}^{2n} such that $\phi_t \in \text{Aut}_{\text{sp}}\mathbb{C}^{2n}$ for each t , then the infinitesimal generator $X(\phi)$ is a complex Hamiltonian vector field on \mathbb{C}^{2n} .

Each symplectic automorphism $F \in \text{Aut}_{\text{sp}} \mathbf{C}^{2n}$ maps Hamiltonian vector fields to Hamiltonian fields:

$$DF \cdot X_{H \circ F} = X_H \circ F .$$

Hence F maps the flow ϕ'_t of the field $X_{H \circ F}$ to the flow ϕ_t of X_H :

$$F \circ \phi'_t(x) = \phi_t(F(x)), \quad x = (z, w) \in \mathbf{C}^{2n} .$$

The symplectic map (5.5) is the time one map of the symplectic action

$$F_t(x) = x + tf(\omega(x, v))v, \quad t \in \mathbf{C} .$$

Its infinitesimal generator is the *Hamiltonian shear field*

$$X_{f, v}(x) = f(\omega(x, v))v . \tag{5.6}$$

Let h be a holomorphic primitive of $-f(h'(\zeta) = -f(\zeta))$. Then

$$H(x) = h(\omega(x, v)), \quad x \in \mathbf{C}^{2n} ,$$

is the energy function of the field (5.6).

The following result plays a central role in the proof of Theorem 5.1. This should be compared with Sect. 5 in [3] and Proposition 3.8 in [4]. (See also Lemma 1.3 in [15].)

5.2 Proposition. *Every polynomial Hamiltonian vector field X on \mathbf{C}^{2n} is a finite sum of Hamiltonian shear fields (5.6), with f a polynomial in one variable.*

We will need the following lemma whose proof can be found in [4].

5.3 Lemma. *For every holomorphic polynomial $p(z)$ on \mathbf{C}^n there exist a finite number of polynomials q_1, q_2, \dots of one variable, and linear forms ℓ_1, ℓ_2, \dots on \mathbf{C}^n such that*

$$p(z) = q_1(\ell_1 z) + q_2(\ell_2 z) + \dots$$

Proof of Proposition 5.2. Let $X = (X_1, \dots, X_{2n})$ be a polynomial vector field on \mathbf{C}^{2n} with the polynomial energy function $H(x)$. Using Lemma 5.3 we write

$$H(x) = \sum_{j=1}^m q_j(\ell_j x) = \sum_{j=1}^m H_j(x) ,$$

where q_j is a polynomial of one variable and

$$\ell_j x = \sum_{k=1}^{2n} x_k a_{j,k} .$$

Let $a_j \in \mathbf{C}^{2n}$ be the vector with components $a_{j,k}$. Set $v_j = -Ja_j$. Then

$$\omega(x, v_j) = x \cdot Jv_j = x \cdot a_j = \ell_j x .$$

Hence the function

$$H_j(x) = q_j(\ell_j x) = q_j(\omega(x, v_j)), \quad x \in \mathbb{C}^{2n},$$

is the Hamiltonian energy function of the symplectic shear field

$$X_j(x) = -q'_j(\omega(x, v_j))v_j.$$

Since $H = \sum_{j=1}^m H_j$, we have $X = \sum_{j=1}^m X_j$. This proves Proposition 5.2. \square

Proposition 5.2 implies the following approximation result. (Compare with Lemma 1.4 in [15].)

5.4 Proposition. *Let X be a holomorphic Hamiltonian vector field on \mathbb{C}^{2n} . Let D be an open set in \mathbb{C}^{2n} and let $t_0 > 0$. Assume that the ordinary differential equation $dR/dt = X(R(t))$ can be integrated for $0 \leq t \leq t_0$ with arbitrary initial condition $R(0) = x \in D$. Set $F_t(x) = R(t)$. Then F_t for $0 \leq t \leq t_0$ is a symplectic biholomorphic map from D into \mathbb{C}^{2n} which can be approximated, uniformly on compact sets in D , by symplectic automorphisms in the group $\mathcal{S}_{\text{sp}}^n$.*

This result is proved in exactly the same way as Lemma 1.4 in [15]. We recall the main idea. According to Proposition 5.2 it suffices to prove the result in the case when X is a finite sum of symplectic shear fields $X = X_1 + \dots + X_k$. The flow of each X_j belongs to the group $\mathcal{S}_{\text{sp}}^n$. The flow of X for time t can be approximated by compositions of flows of fields X_j in the following way. Choose a large integer N and flow for time t/N along each of the fields X_1, X_2, \dots, X_k (in this order). We repeat the same procedure N times, so we flow in total for time kt . The resulting map of course belongs to the group $\mathcal{S}_{\text{sp}}^n$. As $N \rightarrow \infty$, the process converges to the flow of X for time t . A good reference is [1], pp. 76–78. \square

Theorem 5.1 is now obtained as follows. Fix an $F \in \text{Aut}_{\text{sp}} \mathbb{C}^{2n}$. Since translations on \mathbb{C}^{2n} belong to $\mathcal{S}_{\text{sp}}^n$, we may assume that $F(0) = 0$. Also, since $DF(0) \in \text{Sp}(n, \mathbb{C}) \subset \mathcal{S}_{\text{sp}}^n$, we can replace F by $DF(0)^{-1} \cdot F$ and assume that $DF(0) = I$. Set

$$F_t(x) = \begin{cases} F(tx)/t, & \text{if } 0 < t \leq 1; \\ x, & \text{if } t = 0. \end{cases}$$

This is a smooth isotopy of maps in $\text{Aut}_{\text{sp}} \mathbb{C}^{2n}$. Let $X(t, x)$ be the time dependent vector field on \mathbb{C}^{2n} such that

$$\frac{d}{dt} F_t(x) = X(t, F_t(x)), \quad x \in \mathbb{C}^{2n}, \quad t \in [0, 1].$$

Then $X_t = X(t, \cdot)$ is a Hamiltonian vector field on \mathbb{C}^{2n} for each fixed $t \in [0, 1]$. Fix a compact set $K \subset \mathbb{C}^{2n}$. Choose a large integer N and partition the time interval $[0, 1]$ into N subintervals of length $1/N$. On each subinterval $[k/N, (k+1)/N]$ we approximate the time dependent field X_t by the time independent Hamiltonian field $X_{k/N}$. If N is large enough, the approximation

is as close as we want on K , and the flow of X_t on K will be approximated by suitable composition of flows of the fields $X_{k/N}$. It remains to apply Proposition 5.4 to the flow of each Hamiltonian field $X_{k/N}$. This proves Theorem 5.1. \square

We conclude this section with an observation concerning completeness of Hamiltonian vector fields on \mathbf{C}^{2n} whose energy functions are analytically dependent. Recall that, on \mathbf{C}^N , the two notions of completeness (in real resp. complex time) coincide.

5.5 Lemma. *Let $H = h \circ H_0$, where H and H_0 are nonconstant entire functions on \mathbf{C}^{2n} and h is an entire function of one variable. Then X_H is complete if and only if X_{H_0} is.*

Remark. According to K. Stein [30], every nonconstant entire function H on \mathbf{C}^n can be factored as $h \circ H_0$, where h is an entire function of one variable and H_0 is a primitive entire function on \mathbf{C}^n which can not be further factored in this way for a nonlinear h . Lemma 5.5 shows that it suffices to study completeness of Hamiltonian vector fields associated to primitive functions.

Proof. The associated Hamiltonian fields satisfy $X_H = (h' \circ H_0)X_{H_0}$. Thus the orbits of X_H on which $h' \circ H_0 \neq 0$ equal to the corresponding orbits of X_{H_0} , and the remaining orbits of X_H are trivial. If X_{H_0} is complete then clearly so is X_H . Explicitly, if ϕ^0 is the flow of X_{H_0} , then the flow ϕ of X_H equals

$$\phi_t(x) = \phi_{st}^0(x), \quad s = h'(H_0(x)).$$

Observe that $s = s(x)$ is constant on each orbit of ϕ . Conversely, assume that X_H is complete. Then X_{H_0} can be integrated for every $t \in \mathbf{C}$ and every initial point in the set $\Omega = \{x \in \mathbf{C}^{2n} : h'(H_0(x)) \neq 0\}$. Thus the flow ϕ^0 is defined and holomorphic on $\mathbf{C} \times \Omega \subset \mathbf{C} \times \mathbf{C}^{2n}$. Moreover, ϕ^0 is also defined for every initial point in the analytic subvariety $A = \mathbf{C}^{2n} \setminus \Omega$ and for all sufficiently small $t \in \mathbf{C}$. It is easily seen that the envelope of holomorphy of the domain on which ϕ^0 is defined is all of $\mathbf{C} \times \mathbf{C}^{2n}$. Therefore X_{H_0} is complete. \square

5.6 Corollary. *Suppose that the entire functions H, H_0 on \mathbf{C}^{2n} satisfy $H = h \circ H_0 \circ F$ for some nonconstant entire function h of one variable and $F \in \text{Aut}_{\text{sp}} \mathbf{C}^{2n}$. If one of the Hamiltonian fields X_H, X_{H_0} is complete, then so is the other one.*

6 Classification of actions by symplectic automorphisms \mathbf{C}^2

The main result of this section, Theorem 6.1, provides a classification of complete holomorphic Hamiltonian vector fields X_H on \mathbf{C}^2 and of the corresponding symplectic actions (flows) $\phi : \mathbf{C} \times \mathbf{C}^2 \rightarrow \mathbf{C}^2$. Equivalently, since $\text{Aut}_{\text{sp}} \mathbf{C}^2 = \text{Aut}_1 \mathbf{C}^2$ (Sect. 5), Theorem 6.1 provides a classification of $(\mathbf{C}, +)$ -actions on \mathbf{C}^2 by complex volume preserving automorphisms of \mathbf{C}^2 . Notice that a holomorphic vector field $X = (X_1, X_2)$ on \mathbf{C}^2 is Hamiltonian if and only if it is divergence free: $\text{div } X = \partial X_1 / \partial z + \partial X_2 / \partial w = 0$.

There exists a classification of proper (and therefore of symplectic) actions of \mathbf{C} on \mathbf{C}^2 up to conjugation in $\text{Aut}\mathbf{C}^2$, due to Suzuki [33, Theorem 4]. This, however, does not immediately give the classification of symplectic actions since these are preserved only by symplectic holomorphic automorphisms $F \in \text{Aut}_1\mathbf{C}^2$.

In what follows, $H(x) = H(z, w)$ always denotes a nonconstant entire function on \mathbf{C}^2 (the Hamiltonian), and $X_H = (H_w, -H_z)$ is the corresponding Hamiltonian vector field. Let $\Sigma = \{x \in \mathbf{C}^2 : X_H(x) = 0\}$. For each $x_0 \in \mathbf{C}^2 \setminus \Sigma$ the maximal complex orbit C_{x_0} is the connected component of $\{x \in \mathbf{C}^2 \setminus \Sigma : H(x) = H(x_0)\}$ containing x_0 . Thus the closure of C_{x_0} is an irreducible component of the level set $H = H(x_0)$. In the language of Nishino [25], the irreducible components of level sets of H are called ‘surfaces premières’, which we shall translate as ‘primary surfaces’.

Assume now that X_H is complete, and let ϕ be the action of \mathbf{C} on \mathbf{C}^2 generated by X_H . Recall that ϕ is either of type \mathbf{C} or \mathbf{C}^* , depending on the type of its generic orbit (see Theorem 3.3). It follows that every primary surface of H is isomorphic either to \mathbf{C} or to \mathbf{C}^* , and most of them are of the same type. We will say that H is of type \mathbf{C} or \mathbf{C}^* depending on the generic type of its primary surfaces.

6.1 Theorem. *Let H be an entire function on \mathbf{C}^2 such that the associated Hamiltonian vector field $X_H = (H_w, -H_z)$ is complete, with the action (flow) $\{\phi_t\}_{t \in \mathbf{C}}$. Then one of the following two cases holds.*

Case 1: H is of type \mathbf{C} . Then there exists a volume preserving automorphism $F \in \text{Aut}_1\mathbf{C}^2$ and an entire function $h(z)$ such that $H \circ F(z, w) = h(z)$ and

$$\phi_t \circ F = F \circ \psi_t, \quad t \in \mathbf{C}, \tag{6.1}$$

where ψ_t is the action

$$\psi_t(z, w) = (z, w - th'(z)), \quad t \in \mathbf{C}. \tag{6.2}$$

Case 2: H is of type \mathbf{C}^ . Then there exists an $F \in \text{Aut}\mathbf{C}^2$ such that $H \circ F(z, w) = h(zw)$ and $JF(z, w) = g(zw)$ are entire functions of the product zw . If we set $\lambda(\zeta) = h'(\zeta)/g(\zeta)$, then (6.1) holds with*

$$\psi_t(z, w) = (ze^{\lambda(zw)t}, we^{-\lambda(zw)t}), \quad t \in \mathbf{C}. \tag{6.3}$$

Remark. Note that (6.2) is the flow of the Hamiltonian vector field $X_h = (0, -h'(z))$ with the energy function $h(z)$, and (6.3) is the flow of the Hamiltonian vector field $X_\Lambda = \lambda(zw)(z, -w)$ with the energy function $\Lambda(zw)$, where $\Lambda'(\zeta) = \lambda(\zeta)$. Clearly (6.2) is a shear for each t . On the other hand, the map (6.3) for $t \neq 0$ and λ not constant can not be expressed as a finite composition of shears according to [3]. In relation to Case 2 of the theorem, we do not know whether there exist any automorphisms of \mathbf{C}^2 whose Jacobian is a nonconstant function of the product zw .

Proof of Theorem 6.1. Case 1: H is of type C. According to Nishino [25] and Suzuki [33] there exist an $F \in \text{Aut}\mathbf{C}^2$ and an entire function h of one variable such that $H \circ F(z, w) = h(z)$. Let ψ_t be the conjugate flow defined by (6.1). Since ψ_t remains in the level sets of z , ψ_t and its infinitesimal generator has one of the following two forms (see [33]):

$$\psi_t(z, w) = (z, w + tf(z)), \quad V(z, w) = (0, f(z)),$$

$$\psi_t(z, w) = (z, e^{t\lambda(z)}(w - b(z)) + b(z)), \quad V(z, w) = (0, \lambda(z)(w - b(z))), \quad (6.4)$$

where λ is an entire function and b is a meromorphic function on \mathbf{C} such that λb is entire.

We claim that ψ_t can not be of the second type in (6.4). To see this, observe that for each $z_0 \in \mathbf{C}$ which is not a pole of b , the set $\{(z_0, w) : w \neq b(z_0)\}$ is a complex orbit of ψ_t which contains the fixed point $(z_0, b(z_0))$ in its closure. On the other hand, most complex orbits of a Hamiltonian vector field (those that correspond to the regular level sets of the energy function) are closed. Another way to see this is by a direct calculation, using the relation

$$DF \cdot V = X_H \circ F. \quad (6.5)$$

Let $F = (F^1, F^2)$. We differentiate the equation

$$H(F^1(z, w), F^2(z, w)) = h(z)$$

with respect to z and w :

$$(H_z \circ F)F_z^1 + (H_w \circ F)F_z^2 = h'(z);$$

$$(H_z \circ F)F_w^1 + (H_w \circ F)F_w^2 = 0.$$

Solving this system for $H_z \circ F$ and $H_w \circ F$ we obtain

$$X_H \circ F = (H_w, -H_z) \circ F = \frac{-h'(z)}{JF}(F_w^1, F_w^2).$$

On the other hand, writing $V = (0, V^2)$, we have

$$DF \cdot V = V^2(z, w)(F_w^1, F_w^2).$$

Thus the condition (6.5) is equivalent to

$$-\frac{h'(z)}{JF(z, w)} = V^2(z, w).$$

If V is of the second type in (6.4), we choose a point $z \in \mathbf{C}$ which is not a pole of b and such that $h'(z) \neq 0$. Then $V^2(z, b(z)) = 0$ but the left hand side is nonzero, a contradiction.

Thus $V(z, w) = (0, f(z))$, and hence $JF(z, w) = -h'(z)/f(z) = g(z)$ is a function of z . Set $F_0(z, w) = (z, g(z)w)$, and let $F = F_1 \circ F_0$. Then $JF(z, w) = g(z) = JF_0(z, w)$, hence $JF_1 = 1$. We have $H \circ F_1(z, g(z)w) = h(z)$ and

therefore $H \circ F_1(z, w) = h(z)$. The same analysis as above shows that F_1 maps the flow

$$\psi_t(z, w) = (z, w - th'(z))$$

to the flow ϕ_t of X_H . This establishes the Case 1.

Case 2: H is of type \mathbf{C}^ .* According to Saito [29] and Nishino [25] (see also Suzuki [33]) there exists an automorphism $F \in \text{Aut } \mathbf{C}^2$ such that $H \circ F = h \circ Q$, where h is an entire function of one variable and Q is a polynomial on \mathbf{C}^2 of the form

$$Q(z, w) = z^m(z^l w - P_l(z))^n,$$

where m and n are positive integers, $l \in \mathbf{Z}_+$, and $P_l(z)$ is a polynomial of degree at most $(l - 1)$, $P_0 = 0$, and $P_l(0) \neq 0$ if $l > 0$. As before there is a flow ψ_t in the level sets of Q such that (6.1) holds. We consider two cases.

Case 2.1: $l = 0$, $Q(z, w) = z^m w^n$. Every flow ψ_t in the level sets of $z^m w^n$ is of the form

$$\psi_t(z, w) = (ze^{n\lambda(z^m w^n)t}, we^{-m\lambda(z^m w^n)t}) \tag{6.6}$$

for some entire function λ on \mathbf{C} [33]. Its infinitesimal generator is

$$V(z, w) = \lambda(z^m w^n)(nz, -mw).$$

A calculation similar to the one in Case 1 shows that the condition (6.5) is equivalent to

$$JF(z, w) = z^{m-1} w^{n-1} \frac{h'(z^m w^n)}{\lambda(z^m w^n)}.$$

The left hand side is a nonvanishing entire function on \mathbf{C}^2 , and hence so is the right hand side. Due to its special form, this is possible only when $m = n = 1$ and the quotient $h'/\lambda = g$ is a nonvanishing entire function on \mathbf{C} . Thus $JF(z, w) = g(zw)$, and ψ_t is of the form (6.3), with $\lambda(\zeta) = h'(\zeta)/g(\zeta)$. This concludes the proof in Case 2.1.

Case 2.2: $l > 0$. We will show that this case does not give any complete symplectic flows. Write $H = h \circ H_0$, where h is an entire function of one variable and H_0 is primitive entire on \mathbf{C}^2 (that is, H_0 can not be further factored in this way except for a linear h). By Lemma 5.5 it suffices to show that X_{H_0} is not complete. To reach a contradiction we assume that X_{H_0} is complete, with the flow ϕ_t .

According to [29] there exists an $F \in \text{Aut } \mathbf{C}^2$ such that

$$H_0 \circ F(z, w) = az^m(z^l w - P_l(z))^n + b$$

for some $a \in \mathbf{C}^*$, $b \in \mathbf{C}$. The mapping

$$A(z, w) = (z, z^l w - P_l(z)), \quad (z, w) \in \mathbf{C}^2,$$

is an automorphism of $\mathbf{C}^* \times \mathbf{C}$, and we have $H_0 \circ F \circ A^{-1}(z, w) = az^m w^n + b$. The composition $G = F \circ A^{-1} : \mathbf{C}^* \times \mathbf{C} \rightarrow \mathbf{C}^2$ is biholomorphic onto its image.

Let $\psi_t = G^{-1} \circ \phi_t \circ G$ be the (local) flow on $\mathbf{C}^* \times \mathbf{C}$ which is conjugate to ϕ_t by G . Then

$$H_0 \circ G \circ \psi_t = H_0 \circ \phi_t \circ G = H_0 \circ G,$$

hence ψ_t remains in the level sets of $z^m w^n$. Therefore ψ_t is of the form (6.6). Moreover, since the flow

$$A^{-1} \circ \psi_t \circ A = F^{-1} \circ \phi_t \circ F$$

is holomorphic on all of \mathbf{C}^2 , a simple calculation shows that λ must vanish to order at least $\geq l/m > 0$ at the origin.

Denote by V the infinitesimal generator of ψ_t (see Case 2.1). The condition $DG \cdot V = X_{H_0} \circ G$ is equivalent to

$$JG(z, w) = az^{m-1}w^{n-1}/\lambda(z^m w^n).$$

The left hand side is nonvanishing holomorphic on $\mathbf{C}^* \times \mathbf{C}$. Since λ vanishes at the origin, the right hand side has a pole at $z = 1, w = 0$. This contradiction shows that X_{H_0} can not be complete, and Theorem 6.1 is proved. \square

The following two propositions classify symplectic actions of types (6.2) and (6.3).

6.2 Proposition. *The actions*

$$\psi_t(z, w) = (z, w + tf(z)), \quad \tilde{\psi}_t(z, w) = (z, w + t\tilde{f}(z))$$

are conjugate by an $F \in \text{Aut } \mathbf{C}^2$ if and only if there exist numbers $a \in \mathbf{C}^*$, $b \in \mathbf{C}$ such that the quotient $f(az + b)/f(z) = g(z)$ is a nonvanishing entire function; in this case the automorphism $F(z, w) = (az + b, g(z)w)$ satisfies $F \circ \tilde{\psi}_t = \psi_t \circ F$ ($t \in \mathbf{C}$). These two actions are conjugate by a volume preserving automorphism if and only if $\tilde{f}(z) = af(az + b)$ for some $a \in \mathbf{C}^*$, $b \in \mathbf{C}$; in this case they are conjugate by $F(z, w) = (az + b, a^{-1}w)$.

The proof is a simple calculation which we omit. \square

6.3 Proposition. *The symplectic actions*

$$\psi_t(z, w) = (ze^{\lambda(zw)t}, we^{-\lambda(zw)t}), \quad \tilde{\psi}_t(z, w) = (ze^{\tilde{\lambda}(zw)t}, we^{-\tilde{\lambda}(zw)t})$$

are conjugate in $\text{Aut } \mathbf{C}^2$ if and only if $\tilde{\lambda}(\zeta) = \lambda(a\zeta)$ for some $a \in \mathbf{C}^*$; in this case the automorphism $F(z, w) = (z, aw)$ satisfies $F \circ \tilde{\psi}_t = \psi_t \circ F$ ($t \in \mathbf{C}$). These actions are conjugate in $\text{Aut}_1 \mathbf{C}^2$ if and only if $\tilde{\lambda}(\zeta) = \lambda(\pm\zeta)$; if $\tilde{\lambda}(\zeta) = \lambda(-\zeta)$ then they are conjugate by $F(z, w) = (w, -z)$.

Proof. The infinitesimal generator X of ψ is $X(z, w) = \lambda(zw)(z, -w)$, and similarly for \tilde{X} . If $F = (F_1, F_2) \in \text{Aut}_1 \mathbf{C}^2$ conjugates $\tilde{\psi}$ to ψ , then F maps every level set $\{zw = c\}$ to another level set of the same function. This means that the product $F_1 F_2$ is a function of zw , say $r(zw)$. Since F is invertible, r is entire and one-to-one, hence linear: $r(\zeta) = a\zeta + b$. Moreover, since $\{z = 0\} \cup \{w = 0\}$

is the only level set of zw with a singularity, it is preserved by F , and therefore $b = 0$.

The conclusion is that $F_1(z, w)F_2(z, w) = azw$. Thus the zero divisor of F_1F_2 is the union of the two coordinate axes. Since F is an automorphism of \mathbf{C}^2 , each axis is the zero divisor of exactly one of the components of F . Thus F has one of the following two forms:

$$\text{i) } F(z, w) = (e^g z, ae^{-g} w),$$

$$\text{ii) } F(z, w) = (e^g w, ae^{-g} z),$$

for some entire function $g(z, w)$. We first consider the case (i). We have

$$JF = a(1 + zg_z - wg_w),$$

and the condition $DF \cdot \tilde{X} = X \circ F$ is equivalent to the single equation

$$\tilde{\lambda}(zw)(1 + zg_z - wg_w) = \lambda(azw).$$

It follows that $zg_z - wg_w$ is a function of zw . Hence g itself is a function of zw , since a term $z^k w^l$ in the Taylor expansion of g gives the term $(k - l)z^k w^l$ in the Taylor expansion of $zg_z - wg_w$. It follows that $zg_z - wg_w = 0$, so $JF = a$ and $\tilde{\lambda}(zw) = \lambda(azw)$ as claimed. The conjugation equation no longer contains g , and hence we may take $g = 1$. The corresponding map is $F(z, w) = (z, aw)$. Similarly one deals with case (ii). \square

7 Obstructions to completeness of Hamiltonian fields on \mathbf{C}^2

In spite of its intrinsic interest, the classification of complete Hamiltonian vector fields on \mathbf{C}^2 given by Theorem 6.1 is usually not very helpful if we want to determine whether a given Hamiltonian vector field X_H with the entire energy function H is complete. To detect non-completeness of X_H it is much simpler to look at the level sets of H and show that some of them are not isomorphic to any of the surfaces listed in Proposition 3.1.

In this section we obtain several results of this type. In particular, since the fundamental group of every complex orbit of a complete holomorphic vector field has at most one generator, it suffices to find connected, smooth level surfaces of H with connectivity at least two. Whenever this happens, the vector field X_H can not be complete.

Recall that, since we are in \mathbf{C}^2 , Hamiltonian fields are precisely the ones with divergence zero. We begin with a simple but useful observation about polynomial Hamiltonian fields.

7.1 Lemma. *If X is a polynomial Hamiltonian vector field X on \mathbf{C}^2 and if C is a nontrivial complex orbit of X which is \mathbf{R} -complete (Def. 4), then the closure \bar{C} in \mathbf{CP}^2 is an algebraic curve of genus 0, i.e., \bar{C} is normalized by the Riemann sphere, and C is also \mathbf{C} -complete.*

Proof. Let H be a polynomial on \mathbf{C}^2 such that $X = X_H$. Fix a point $x_0 = (z_0, w_0) \in \mathbf{C}^2 \setminus \Sigma$, where Σ is the zero set of X_H , and let C be the complex orbit of X_H through x_0 . Since C is contained in the (affine) algebraic curve $\{x \in \mathbf{C}^2 : H(x) = H(x_0)\}$, the closure \bar{C} in \mathbf{CP}^2 is a projective algebraic curve, and C is obtained by removing a finite number of points from \bar{C} . Such a curve C is called algebroid.

If C is \mathbf{R} -complete, then by Proposition 3.2 C is isomorphic to one of the surfaces \mathbf{C} , \mathbf{C}^* , U , U^* , or an annulus. Only the first two of these surfaces are algebroid, and hence C is \mathbf{C} -complete.

Let $\pi : S \rightarrow \bar{C}$ be the normalization of \bar{C} by a compact Riemann surface S . Since the orbit C is contained in the regular part of \bar{C} , S contains a biholomorphic image of C which is either \mathbf{C} or \mathbf{C}^* . The only compact Riemann surface with this property is the Riemann sphere. If $C \simeq \mathbf{C}$, then $\bar{C} = C \cup \{p\}$ for some point $p \in \mathbf{CP}^2$ in the line at infinity. If $C \simeq \mathbf{C}^*$, then $\bar{C} = C \cup \{p, q\}$, where at least one (and perhaps both) of the points p, q is at infinity. If one of the points, say p , belongs to the finite part \mathbf{C}^2 of \mathbf{CP}^2 , then p is a zero of X_H (hence a critical point of H) by Theorem 4.1. This proves Lemma 7.1. \square

Here is a useful criterium for non-completeness of polynomial Hamiltonian fields.

7.2 Proposition. *Let $H(z, w) = \sum_{j=1}^k H^j(z, w)$ be a polynomial of degree $d \geq 3$, with H^j its homogeneous part of degree j . Suppose that $(0, 0)$ is the only common zero of the following four polynomials: H^d , H^{d-1} , $\partial H^d / \partial z$, $\partial H^d / \partial w$. Then the Hamiltonian vector field $X_H = (H_w, -H_z)$ is not complete. More precisely, if ϕ_t is the local flow of X_H , then every regular level set of H contains a point p_0 such that $\phi_t(p_0)$ is not defined for all real t (p_0 is flown to infinity in finite time, either positive or negative).*

Proof. Let \mathbf{CP}^2 be the projective plane with homogeneous coordinates $[z_0 : z : w]$. Let $L_0 = \{z_0 = 0\}$ be the line at infinity, and identify \mathbf{C}^2 with $\mathbf{CP}^2 \setminus L_0$ by the embedding $(z, w) \rightarrow [1 : z : w]$. Choose a regular value c of H . The affine algebraic curve $C_0 = \{(z, w) \in \mathbf{C}^2 : H(z, w) = c\}$ is smooth in \mathbf{C}^2 , and its closure $C = \bar{C}_0 \subset \mathbf{CP}^2$ is a projective algebraic curve. Since $X_H \neq 0$ on C_0 , each connected component of C_0 is an orbit of X_H .

In order to prove the proposition we will show that C is a smooth algebraic curve, possibly disconnected, such that at least one of its connected components is not isomorphic to the Riemann sphere. Lemma 7.1 then implies that this connected component contains a point which is flown to infinity in finite time.

We first show C is smooth at every point at infinity (and therefore everywhere). The homogeneous equation for C near a point $p = [0 : a : b] \in C \cap L_0$ is

$$G(z_0, z, w) = H^d(z, w) + z_0 H^{d-1}(z, w) + O(z_0^2) = 0. \tag{7.1}$$

Since $0 = G(0, a, b) = H^d(a, b)$ and $(a, b) \neq (0, 0)$, one of the numbers $H^{d-1}(a, b)$, $H_z^d(a, b)$, $H_w^d(a, b)$ is nonzero by hypothesis. Observe that these numbers are just the partial derivatives of G with respect to the variables z_0, z, w at the point $p_0 = (0, a, b)$.

If $H^{d-1}(a, b) \neq 0$, the equation (7.1) has a solution near $(0, a, b)$ of the form $z_0 = Z_0(z, w)$. We can take either (z_0, z) or (z_0, w) as local affine coordinates on \mathbf{CP}^2 near p , depending on which of the numbers a, b is nonzero. In each case we see that C is a graph near p , hence smooth. If $H_z^d(a, b) \neq 0$, (7.1) has a local solution of the form $z = Z(z_0, w)$. In this case $b \neq 0$ (since $H^d(a, 0) = 0$ implies $H^d(z, w) = wh(z, w)$ and therefore $H_z^d(a, 0) = 0$). Thus we may take (z_0, z) as the local affine coordinates on \mathbf{CP}^2 near p , and we see again that C is smooth near p . Similarly we deal with the remaining case.

This shows that C is a smooth algebraic curve of degree d in \mathbf{CP}^2 . Let $C = \bigcup_{j=1}^m C_j$ be its decomposition into connected components. We will show that when $d \geq 3$, at least one of the components C_j has Euler number $\chi(C_j) \leq 0$, and therefore C_j is not the Riemann sphere. Thus $C_j \setminus L_0$ is an orbit of X_H which is not \mathbf{R} -complete.

In the remainder of the proof we shall no longer need the hypotheses in the Proposition, and therefore we are free to change coordinates on \mathbf{CP}^2 . We proceed as in the proof of the classical genus formula [22, p. 219]. Choose a point $p \in \mathbf{CP}^2$ not on C and a line L not containing p . After a linear change of coordinates we may take $p = [0 : 0 : 1]$ and $L = \{w = 0\}$. We may assume also that the line at infinity ($z_0 = 0$) is not tangent to C at any point of their intersection.

Let $\pi : \mathbf{CP}^2 \setminus \{p\} \rightarrow L$ be the linear projection which maps each point $q \neq p$ to the intersection of the line through p and q with L . The restriction $\pi : C \rightarrow L$ is a branched analytic cover of total degree d . Therefore the Euler characteristics of the two surfaces satisfy

$$\chi(C) = d\chi(L) - b = 2d - b,$$

where $b = \sum_{q \in C} b(q)$ is the sum of the local branching order of $\pi|_C$ [11, p.24]. By the choice of coordinates $\pi|_C$ does not branch at any point at infinity. Notice that π restricted to the finite part of \mathbf{CP}^2 is just $\pi(z, w) = z$. Therefore the local branching order $b(q)$ at every point $q \in C$ equals the local intersection number at q of C with the curve $D = \{(z, w) \in \mathbf{C}^2 : H_w(z, w) = 0\}$. Hence b is the global intersection number of C and D in \mathbf{CP}^2 , and this equals $d(d - 1)$ by Bezout's theorem [10]. Thus

$$\sum_{j=1}^m \chi(C_j) = \chi(C) = 2d - d(d - 1) = d(3 - d).$$

If $d \geq 3$, $\chi(C) \leq 0$, and therefore $\chi(C_j) \leq 0$ for at least one j . This establishes the claim and completes the proof of Proposition 7.2. \square

To motivate the next result we recall that, if $Q(x) \geq 0$ is a nonnegative smooth real function on \mathbf{R} , the Hamiltonian vector field $X(x, y) = (y, -Q'(x))$ with the energy function $H(x, y) = y^2/2 + Q(x)$ is complete on \mathbf{R}^2 . In contrast to this we have

7.3 Proposition. *If $f(z)$ is an entire function on \mathbf{C} with is not affine linear, then the vector field $X(z, w) = (w, f(z))$ on \mathbf{C}^2 is not complete. More precisely,*

every regular level set of the function $H(z, w) = w^2/2 + Q(z)$, where $Q'(z) = -f(z)$, contains a point which is flown to infinity in finite time.

Proof. X is a Hamiltonian field with the energy function $H(z, w) = w^2/2 + Q(z)$, where $Q'(z) = -f(z)$. The critical set of H (which equals the zero set of X) is $\Sigma = \{(z, 0) : Q'(z) = 0\}$. Let $c \in \mathbf{C}$ be a regular value of Q such that the equation $Q(z) = c$ has at least three solutions. Since f is nonlinear, Q is either a polynomial of degree at least three or an entire function, and therefore most values of c satisfy these properties. Under these conditions we will show that the level set $L = \{(z, w) \in \mathbf{C}^2 : H(z, w) = c\}$ is a smooth, connected, one dimensional complex submanifold of \mathbf{C}^2 (a Riemann surface) whose fundamental group has at least two generators. Since L is a complex orbit of the vector field X_H , Proposition 3.1 implies that L contains at least one point which is flown to infinity in finite time. In particular, X_H is not complete.

Let $\psi : L \rightarrow \mathbf{C}$ be the restriction of the first coordinate projection $(z, w) \rightarrow z$ to L . Clearly ψ is nondegenerate at every point $(z, w) \in L$ where $w \neq 0$ since z serves as a local coordinate on L at such points. Near points $(a, 0) \in L$ we have $Q(a) = c$ and $Q'(a) \neq 0$. Hence the equation $w^2/2 + Q(z) = c$ has a local solution near $(a, 0)$ of the form $z = a + h(w^2)$, where $h(0) = 0$ but $h'(0) \neq 0$. If we take w as the local coordinate on L near $(a, 0)$, the map ψ is given in these coordinates by $w \rightarrow a + h(w^2)$. Thus ψ has branching order two at every point $(a, 0) \in L$.

Let $\{a_j\} \subset \mathbf{C}$ be all solutions of $Q(z) = c$; then $(a_j, 0) \in L$ are all the branch points of $\psi : L \rightarrow \mathbf{C}$. Composing ψ by a suitable translation of \mathbf{C} we may assume that $Q(0) \neq c$ and that $|a_j|$ are all distinct and ordered: $0 < |a_1| < |a_2| < |a_3| < \dots$. The function $|\psi|^2 : L \rightarrow \mathbf{R}_+$ is then a Morse function on L . Each branch point $(a_j, 0) \in L$ of ψ is a critical point of $|\psi|^2$ with Morse index one (because the branching order of ψ equals two). The two points $(0, \pm w_0) \in L$ over 0, with $w_0^2 = 2(c - Q(0))$, are critical points with Morse index zero. There are no other critical points of $|\psi|^2$.

For each $r > 0$ we set $L(r) = \{(z, w) \in L : |z| < r\} = \{|\psi|^2 < r^2\}$. The fundamental result of Morse theory [24] implies that the topological type of $L(r)$ for $r > 0$ changes only when r passes a critical value $|a_j|$. The set $L(r)$ for $|a_j| < r < |a_{j+1}|$ is obtained from $L(r)$ for $|a_{j-1}| < r < |a_j|$ by attaching to the latter a handle of type $(I \times I, I \times \{0, 1\})$ ($I = [0, 1]$).

For small $r > 0$ the set $L(r)$ is the disjoint union of two discs. After attaching to these two discs the first handle (when r passes $|a_1|$) we get a connected disc. After attaching the second handle at $r = |a_2|$ we get a surface with fundamental group \mathbf{Z} . In general, every time we attach a handle, the fundamental group of $L(r)$ gains another generator, and we never introduce any relations between them. Thus $\pi_1(L(r))$ is a free group on $k - 1$ generators, where $k = k(r)$ is the number of points a_j in the disc $|z| < r$.

If Q is a polynomial of degree k , it follows that $\pi_1(L)$ is a free group on $k - 1$ generators. Thus, if $k \geq 3$, $\pi_1(L)$ has at least two generators and hence L can not be any of the surfaces listed in Proposition 3.1. The same

is true if Q is entire since $\pi_1(L)$ then has infinite connectivity. This proves Proposition 7.3. \square

7.4 Corollary. *If $f(z)$ is an entire function of z which is not affine linear, there exists a point $(z_0, \dot{z}_0) \in \mathbf{C}^2$ such that the second order differential equation*

$$\ddot{z} = f(z), \quad z(0) = z_0, \quad \dot{z}(0) = \dot{z}_0$$

can not be integrated for all real $t \in \mathbf{R}$.

Proof. By introducing the variable $w = \dot{z}$ we change this equation to the Hamiltonian system $\dot{z} = w, \dot{w} = f(z)$ with the Hamiltonian function $H(z, w) = w^2/2 + Q(z)$, where Q is a holomorphic primitive of $-f : Q'(z) = -f(z)$. Thus Corollary 7.4 follows from Proposition 7.3. \square

8 Quasi-algebraic actions on \mathbf{C}^n

An action $\phi : G \times \mathbf{C}^n \rightarrow \mathbf{C}^n$ of a group G on \mathbf{C}^n is said to be *quasi-algebraic* if $\phi_g = \phi(g, \cdot)$ is a polynomial automorphism of \mathbf{C}^n for all $g \in G$. The action ϕ is linear if $\phi_g \in GL(n, \mathbf{C})$ for every $g \in G$. The following Proposition is due to Suzuki [33] for actions of the group $(\mathbf{C}, +)$ on algebraic manifolds. We give a very simple proof in the case when the manifold is \mathbf{C}^n .

8.1 Proposition. *Let G be a connected, real-analytic Lie group, and let ϕ be a real-analytic, quasi-algebraic action of G on \mathbf{C}^n . Then there exist an integer $m > n$, an algebraic embedding $F : \mathbf{C}^n \rightarrow \mathbf{C}^m$, and a linear action $\psi : G \times \mathbf{C}^m \rightarrow \mathbf{C}^m$ such that*

$$\psi(g, F(z)) = F(\phi(g, z)), \quad z \in \mathbf{C}^n, \quad g \in G. \tag{8.1}$$

Proof. Let

$$\phi_g(z) = \sum_{\alpha \in \mathbf{Z}_+^n} c_\alpha(g) z^\alpha$$

be the Taylor expansion of $\phi_g = \phi(g, \cdot)$. Since the action is real-analytic, the coefficients $c_\alpha(g)$ are real-analytic functions of $g \in G$. The assumption is that ϕ_g is polynomial for each $g \in G$. We first show that there is an integer $d \in \mathbf{Z}_+$ such that the degree of ϕ_g is at most d for all $g \in G$. For each $d \in \mathbf{Z}_+$ we set

$$G_d = \{g \in G : c_\alpha(g) = 0 \text{ for all } |\alpha| > d\}.$$

Clearly G_d is closed in G , and $G = \bigcup_{d=1}^\infty G_d$. Since G is a manifold and thus a Baire space, one of the sets G_d has nonempty interior. Thus the function c_α vanishes on a nonempty open set in G whenever $|\alpha| > d$. The identity principle implies that these functions are identically zero on G . Thus ϕ_g is polynomial of degree at most d on \mathbf{C}^n for all $g \in G$.

Let \mathcal{P} be the complex vector space of polynomials on \mathbf{C}^n generated by the components of all maps $\phi_g, g \in G$. Since $\deg P \leq d$ for each $P \in \mathcal{P}$, \mathcal{P} is finite dimensional. Let $m = \dim \mathcal{P}$. The group g acts linearly on \mathcal{P} by

$(g, P) \rightarrow P \circ \phi_g$. For each $z \in \mathbf{C}^n$ we denote by $\delta_z \in \mathcal{P}^*$ the evaluation at $z : \delta_z(P) = P(z)$. Let ψ be the dual linear action of G on the dual space $\mathcal{P}^* \simeq \mathbf{C}^m$, given by

$$\psi(g, \lambda)(P) = \lambda(P \circ g), \quad g \in G, \quad \lambda \in \mathcal{P}^*, \quad P \in \mathcal{P}.$$

Finally let F be defined by $F(z) = \delta_z \in \mathcal{P}^* \simeq \mathbf{C}^m, z \in \mathbf{C}^n$. It is immediate that Proposition 8.1 holds.

9 Complements of tame analytic sets

Our last result, Proposition 9.1 below, extends Proposition 1 of Winkelman [34]. For a closed analytic subvariety $A \subset \mathbf{C}^n$ we set

$$\omega(A) = \{ \lim_{j \rightarrow \infty} [a^j] : a^j \in A, |a^j| \rightarrow \infty \} \subset \mathbf{CP}^{n-1}.$$

Clearly $\omega(A)$ is a closed subset of \mathbf{CP}^{n-1} . If A has pure dimension p , then A is algebraic if and only if $\omega(A)$ is an analytic (hence an algebraic) subset of \mathbf{CP}^{n-1} of dimension $p-1$ [10]. If A has dimension $n-1$ and is not algebraic, then $\omega(A) = \mathbf{CP}^{n-1}$.

Definition 6. *An analytic subset $A \subset \mathbf{C}^n$ is tame if $\omega(A) \neq \mathbf{CP}^{n-1}$.*

Every algebraic subset $A \subset \mathbf{C}^n$ of codimension at least two is tame, but there exist tame analytic subset of codimension ≥ 2 which are not algebraic.

9.1 Proposition. *If $A \subset \mathbf{C}^n$ is a tame analytic subset of dimension at most $(n-2)$, then the group*

$$\{F \in \text{Aut}_1 \mathbf{C}^n : F(z) = z \text{ for all } z \in A\} \quad (9.1)$$

acts transitively on the domain $\mathbf{C}^n \setminus A$.

Proof. For each $v \in \mathbf{C}_*^n = \mathbf{C}^n \setminus \{0\}$ we denote by $\pi_v : \mathbf{C}^n \rightarrow v^\perp \simeq \mathbf{C}^{n-1}$ the orthogonal projection with kernel $\mathbf{C}v$. We claim that the restriction $\pi_v : A \rightarrow v^\perp$ is proper when $[v] \notin \omega(A)$. By a linear change of coordinates on \mathbf{C}^n it suffices to consider the case $v = e_n = (0, \dots, 0, 1)$ and $\pi(z) = z' = (z_1, \dots, z_{n-1})$. If $\pi|_A$ is not proper, there exists a sequence $\{a^j\} \subset A$ such that $|a^j| \rightarrow \infty$ but $a'^j = \pi(a^j)$ is bounded. Thus $|a_n^j| \rightarrow \infty$ and therefore $[a^j] \rightarrow [e_n]$ as $j \rightarrow \infty$. Hence $[v] = [e_n] \in \omega(A)$. This established the claim.

Fix $v \in \mathbf{C}^n$ such that $[v] \in \mathbf{CP}^{n-1} \setminus \omega(A)$. The proper mapping theorem [10] implies that $\pi_v(A) = A_v \subset v^\perp$ is an analytic subvariety of v^\perp of dimension $p = \dim A \leq n-2$. Since v^\perp has dimension $n-1$, there exist holomorphic functions f on $v^\perp \simeq \mathbf{C}^{n-1}$ which vanish on A_v but do not vanish at a given point not in A_v . The map

$$F_{f,v}(z) = z + f(\pi_v(z))v \quad (9.2)$$

is an automorphism of \mathbf{C}^n (a shear) which preserves the set $\pi_v^{-1}(A_v) = A + \mathbf{C}v$ pointwise. If $z^0, z^1 \in \mathbf{C}^n$ satisfy $\pi_v(z^0) = \pi_v(z^1) \notin A_v$, we can choose an f as above such that $F_{f,v}$ maps z^0 to z^1 .

It is now clear that we can map every $z^0 \notin A$ to any other $z^1 \notin A$ by a composition of maps (9.2). In fact, since $\mathbf{C}\mathbf{P}^{n-1} \setminus \omega(A)$ is open and nonempty, we can choose n linearly independent vectors $v^1, \dots, v^n \in \mathbf{C}^n$ such that $\pi_{v^j}(z^i) \notin A_{v^j}$ for $1 \leq j \leq n$ and $i = 0, 1$. Then we can find a finite sequence of points $w^0 = z^0, w^1, \dots, w^n = z^1$ such that $\pi_{v^j}(w^{j-1}) = \pi_{v^j}(w^j)$ for $1 \leq j \leq n$. (Write $z^1 - z^0 = \sum_{j=1}^n \lambda_j v^j$ and take $w^0 = z^0, w^1 = z^0 + \lambda_1 v^1, w^2 = w^1 + \lambda_2 v^2$, etc.) By the first step we can map every w^j to w^{j+1} by an automorphism (9.2). Hence we can map z^0 to z^1 by a composition of n shears (9.2). \square

Remark. If A is algebraic of dimension $\leq n - 2$, then A is tame. Moreover, each proper projection $\pi_v(A) \subset v^\perp$ is also algebraic. Hence the proof given above shows that the group of polynomial shears (9.2) on \mathbf{C}^n which restrict to the identity on A acts transitively on $\mathbf{C}^n \setminus A$ (Winkelman [34]). Stronger results for certain discrete sets in \mathbf{C}^n are due to Rosay and Rudin [28].

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