Global Holomorphic Equivalence of Smooth Submanifolds in $\mathbb{C}^n$

Franc Forstneric & Erik Løw

Abstract. Let M be a smooth compact manifold and $n \geq 2$. Given a smooth isotopy of embeddings $f_t: M \hookrightarrow \mathbb{C}^n$ $(0 \leq t \leq 1)$ such that $f_t(M) \subset \mathbb{C}^n$ is a totally real and polynomially convex submanifold of $\mathbb{C}^n$ for each fixed t, we construct a sequence Φ_j of holomorphic automorphisms of $\mathbb{C}^n$ such that $\Phi_j \circ f_0$ converges to f_1 and $\Phi_j^{-1} \circ f_1$ converges to f_0 in $\mathcal{C}^\infty(M)$ as $j \to \infty$. We also obtain results on approximating flows of asymptotically holomorphic vector fields by holomorphic automorphisms of $\mathbb{C}^n$.

1. Introduction. We denote by $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) the real (resp. complex) n-dimensional Euclidean space. Throughout the paper M will denote a compact, smooth real manifold, with or without boundary. Our main results, summarized in Theorems 1.1, 1.3, 2.1, and 6.1, concern global holomorphic equivalence of smooth totally real embeddings of such manifolds into $\mathbb{C}^n$ and certain other Stein manifolds. We also prove a result on approximation of flows of asymptotically holomorphic vector fields by automorphisms (Theorem 5.1).

For a compact set $K \subset \mathbb{C}^n$, we denote by $\hat{K}$ its polynomially convex hull [18]; K is polynomially convex if $\hat{K} = K$. An embedding $f: M \hookrightarrow \mathbb{C}^n$ of a compact manifold is polynomially convex if $f(M)$ is a polynomially convex subset of $\mathbb{C}^n$, and is totally real if for each $x \in M$ the real vector space $f_*(T_xM) \subset T_{f(x)}\mathbb{C}^n$ contains no complex line.

An *isotopy* of embeddings of M into $\mathbb{C}^n$ is a smooth map $f: [0,1] \times M \to \mathbb{C}^n$ such that $f_t = f(t, \cdot): M \hookrightarrow \mathbb{C}^n$ is an embedding for each $t \in [0,1]$. Such an isotopy is totally real (resp. polynomially convex) if, for each fixed $t \in [0,1]$, the embedding f_t is totally real (resp. polynomially convex). Two embeddings f_0, $f_1: M \hookrightarrow \mathbb{C}^n$ are *isotopic* if there exists an isotopy of embeddings $f_t: M \hookrightarrow \mathbb{C}^n$ $(0 \leq t \leq 1)$ connecting f_0 and f_1.

Our main result is the following.

Indiana University Mathematics Journal ©, Vol. 46, No. 1 (1997)

Theorem 1.1. *Let M be a compact smooth manifold. Given a smooth isotopy of embeddings $f_t\colon M \hookrightarrow \mathbb{C}^n$ $(0 \le t \le 1)$ which is totally real and polynomially convex, there exists a sequence $\Phi_j \in \operatorname{Aut}\mathbb{C}^n$ $(j = 1,2,3,\ldots)$ such that $\Phi_j \circ f_0$ converges to f_1 in $\mathcal{C}^\infty(M)$ as $j \to \infty$ and at the same time $\Phi_j^{-1} \circ f_1$ converges to f_0 in $\mathcal{C}^\infty(M)$ as $j \to \infty$.*

More precisely, if the manifold M and the isotopy $\{f_t\}$ as in Theorem 1.1 are of class $\mathcal{C}^r$, $r \ge (\dim M + 5)/2$, then we can choose a sequence $\{\Phi_j\} \in \operatorname{Aut}\mathbb{C}^n$ such that, with $k = r - (\dim M + 3)/2$, we have

$$\lim_{j\to\infty} \|\Phi_j \circ f_0 - f_1\|_{\mathcal{C}^k(M)} = 0, \quad \lim_{j\to\infty} \|\Phi_j^{-1} \circ f_1 - f_0\|_{\mathcal{C}^k(M)} = 0 \tag{1.1}$$

(Theorem 2.1 in Section 2). The result could also be formulated for isotopies of diffeomorphisms $g_t\colon M_0 \to M_t$ $(0 \le t \le 1)$ between totally real, polynomially convex submanifolds $M_t \subset \mathbb{C}^n$, with g_0 the identity on M_0 (take $M_t = f_t(M)$ and $g_t = f_t \circ f_0^{-1}$).

Theorem 1.1 extends several earlier results. The first result in this direction is due to Rosay [23] who proved that every smooth embedded arc $C \subset \mathbb{C}^n$ $(n > 1)$ can be approximately mapped onto a straight arc in $\mathbb{C}^n$ by automorphisms of $\mathbb{C}^n$. Forstneric and Rosay [14] proved that, for a real-analytic isotopy as in Theorem 1.1, there is a sequence Φ_j of automorphisms of $\mathbb{C}^n$ that converges in a neighborhood $U_0 \subset \mathbb{C}^n$ of $M_0 = f_0(M)$ to a biholomorphic map $\Phi = \lim_{j\to\infty} \Phi_j\colon U_0 \to U_1 \subset \mathbb{C}^n$ such that $\Phi \circ f_0 = f_1$ [14, Theorem 3.1]. It immediately follows that the sequence Φ_j^{-1} then converges to Φ^{-1} on U_1 as $j \to \infty$, and (1.1) follows from the Cauchy estimates. A crucial ingredient in these and later results on the topic is the work of Andersén and Lempert [3].

It is not at all trivial to extend this result to smooth isotopies. The first such extension for a general manifold M was obtained in [9] by one of the authors who proved that, under the assumptions of Theorem 1.1 above, the diffeomorphism $g = f_1 \circ f_0^{-1}\colon M_0 \to M_1$ of class $\mathcal{C}^r$ $(2 \le r < \infty)$ can be approximated in the $\mathcal{C}^r(M_0)$ norm by restrictions $\Phi|_{M_0}$ of holomorphic automorphisms $\Phi \in \operatorname{Aut}\mathbb{C}^n$. Note that in the smooth case one cannot expect the convergence of the approximating sequence $\Phi_j \in \operatorname{Aut}\mathbb{C}^n$ in any neighborhood of M_0 as this would entail that the diffeomorphism $g = f_1 \circ f_0^{-1}\colon M_0 \to M_1$ extends holomorphically to a neighborhood of M_0.

The crucial difference between [9] and the present paper is that here we obtain a two-sided approximation result. Given a diffeomorphism $g\colon M_0 \to M_1$ between submanifolds of $\mathbb{C}^n$ as above, we construct a sequence $\Phi_j \in \operatorname{Aut}\mathbb{C}^n$ such that $\Phi_j|_{M_0} \to g$ and at the same time $\Phi_j^{-1}|_{M_1} \to g^{-1}$ as $j \to \infty$ (in the smooth norms on M_0 and M_1 respectively). To accomplish this, we develop a different method by considering the isotopy $g_t = f_t \circ f_0^{-1}$ as the flow of a time-dependent, asymptotically holomorphic vector field X on $\mathbb{C}^n$. By solving a certain $\bar\partial$-equation, using Hörmander's L_2-methods, we approximate X by (time-dependent) entire holomorphic vector fields Y^ε in thin tubular neighborhoods $S(\varepsilon)$ of the trace of the isotopy g_t in the extended phase space. The rate of approximation of X by Y^ε increases as the size ε of the tube shrinks to 0. We then prove a lemma on perturbation of the flow under small perturbation of the vector field in a family of shrinking neighborhoods (Lemma 4.1). Finally, we complete the proof by invoking results of Andersén and Lempert to the effect that the flow of an entire holomorphic vector field on $\mathbb{C}^n$ is approximable on compact sets by automorphisms of $\mathbb{C}^n$.

We believe that the method developed in this paper is very flexible, and we hope that it will be useful in other global approximation problems. We indicate two such extensions in Sections 5 and 6.

Remark 1. The loss of $(\dim M + 3)/2$ derivatives in Theorem 1.1 is largely due to the method that we use for solving the $\bar\partial$-equation in tubular neighborhoods of totally real submanifolds. By unpublished results of N. Øvrelid on solving such $\bar\partial$-equations, using integral kernels, one can very likely get rid of the loss of derivatives.

Remark 2. Theorem 1.1 has the following converse: *If* $f_0,\ f_1\colon M \hookrightarrow \mathbb{C}^n$ *are totally real, polynomially convex embeddings such that* $\lim_{j\to\infty} \Phi_j \circ f_0 = f_1$ *in* $\mathcal{C}^2(M)$ *for some sequence* $\Phi_j \in \mathrm{Aut}\,\mathbb{C}^n$*, then* f_0 *can be connected to* f_1 *by a totally real, polynomially convex isotopy* $f_t\colon M \hookrightarrow \mathbb{C}^n$, $0 \le t \le 1$. (See the main theorem in [9].) This follows from connectedness of the group $\mathrm{Aut}\,\mathbb{C}^n$ and from the fact that the class of totally real, polynomially convex embeddings $M \hookrightarrow \mathbb{C}^n$ is stable under small $\mathcal{C}^2$ perturbations [8].

Remark 3. Our result can be viewed as a holomorphic analogue of the *isotopy extension theorem:* For any isotopy of smooth embeddings $f_t\colon M \hookrightarrow \mathbb{R}^N$ $(0 \le t \le 1)$ there exists a smooth *diffeotopy* $\Phi_t\colon \mathbb{R}^N \to \mathbb{R}^N$ (a one-parameter family of diffeomorphisms of $\mathbb{R}^N$) such that $\Phi_t \circ f_0 = f_t$ [16, p. 180]. In particular, if the embeddings $f_0, f_1\colon M \hookrightarrow \mathbb{R}^N$ are isotopic, then there is a $\Phi \in \mathrm{Diff}\,\mathbb{R}^N$ such that $\Phi \circ f_0 = f_1$. Of course this holds for embeddings into other manifolds as well.

One may ask how natural and how restrictive are the conditions in Theorem 1.1. To answer this question we first recall a relevant result from Section 4 in [9] on the existence of isotopies as in Theorem 1.1. For $1 \le r \le \infty$ let $\mathcal{C}^r(M, \mathbb{C}^n)$ be the Banach (resp. Fréchét) space of all $\mathcal{C}^r$ maps $M \to \mathbb{C}^n$. We denote by $\mathcal{E}^r(M, \mathbb{C}^n)$ the subset of $\mathcal{C}^r(M, \mathbb{C}^n)$ consisting of all totally real and polynomially convex embeddings $M \hookrightarrow \mathbb{C}^n$. For a totally real embedding (or immersion) $f: M \to \mathbb{C}^n$ we denote by ν_f its complex normal bundle, i.e., the quotient $f^*(T\mathbb{C}^n)/(\mathbb{C} \otimes TM)$. It can be realized as a subbundle of $f^*(T\mathbb{C}^n)$ so that $(\mathbb{C} \otimes TM) \oplus \nu_f = f^*(T\mathbb{C}^n)$.

Proposition 1.2 ([9], Section 4.). *Let $2 \le r \le \infty$ and let M be a compact $\mathcal{C}^r$ manifold.*

(a) *$\mathcal{E}^r(M, \mathbb{C}^n)$ is an open subset of $\mathcal{C}^r(M, \mathbb{C}^n)$.*
(b) *If $\dim M \le 2n/3$, then $\mathcal{E}^r(M, \mathbb{C}^n)$ is open, connected, and dense in $\mathcal{C}^r(M, \mathbb{C}^n)$.*
(c) *If $f: M \hookrightarrow \mathbb{C}^n$ is a totally real embedding whose complex normal bundle ν_f admits a section over M with only isolated zeros, then there exist arbitrarily small $\mathcal{C}^r$ perturbations $\tilde{f}$ of f with $\tilde{f} \in \mathcal{E}^r(M, \mathbb{C}^n)$.*

In global differential topology a very important notion is that of *stability.* A property of an embedding is stable if it persists under small perturbations of the embedding. With this in mind, and in light of Theorem 1.1, we introduce the following notion of (stable) $\operatorname{Aut}\mathbb{C}^n$-equivalence. (For results on equivalence of *real-analytic* embeddings with respect to various holomorphic automorphism groups we refer the reader to the papers [10], [11], and [14].)

Definition 1. *Let $f_0, f_1: M \hookrightarrow \mathbb{C}^n$ be $\mathcal{C}^k$ embeddings of a compact manifold M in $\mathbb{C}^n$.*

(a) *f_0 and f_1 are* $\operatorname{Aut}\mathbb{C}^n$*-equivalent to order k if there is a sequence $\{\Phi_j\} \subset$* $\operatorname{Aut}\mathbb{C}^n$ *satisfying* (1.1).
(b) *f_0 and f_1 are stably* $\operatorname{Aut}\mathbb{C}^n$*-equivalent to order k if for every pair of sufficiently small $\mathcal{C}^k$ perturbations $\tilde{f}_j$ of f_j ($j = 0, 1$), $\tilde{f}_0$ and $\tilde{f}_1$ are* $\operatorname{Aut}\mathbb{C}^n$*-equivalent.*

Theorem 1.1 and Proposition 1.2 show that any two embeddings $f_0, f_1 \in \mathcal{E}^\infty(M, \mathbb{C}^n)$ which are $\operatorname{Aut}\mathbb{C}^n$-equivalent are also stably equivalent, and that $\operatorname{Aut}\mathbb{C}^n$-equivalence is indeed an equivalence relation for totally real and polynomially convex embeddings. On the other hand, it turns out that *stable* $\operatorname{Aut}\mathbb{C}^n$*-equivalence is possible only in the category of totally real and polynomially convex embeddings.* This shows that the assumptions in Theorem 1.1 are entirely natural. We summarize these observations in the following theorem.

Theorem 1.3. *Let $2 \le k \le \infty$. The following are equivalent for $\mathcal{C}^k$ embeddings $f_0, f_1: M \hookrightarrow \mathbb{C}^n$ of a compact manifold into $\mathbb{C}^n$:*

(a) *f_0 and f_1 belong to the same connected component of $\mathcal{E}^k(M, \mathbb{C}^n)$;*
(b) *$f_0, f_1 \in \mathcal{E}^k(M, \mathbb{C}^n)$ and they are* $\operatorname{Aut}\mathbb{C}^n$*-equivalent to order k;*
(c) *f_0 and f_1 are stably* $\operatorname{Aut}\mathbb{C}^n$*-equivalent to order k.*

Proof. The equivalence of (a) and (b), as well as the implication (a)⇒(c), are the content of Theorem 1.1 (including Remark 1 following it) and Proposition 1.2. For the remaining implication (c)⇒(b) let $f_0,\ f_1\colon M \hookrightarrow \mathbb{C}^n$ be embeddings which are stably $\mathrm{Aut}\,\mathbb{C}^n$-equivalent. Set $M_j = f_j(M)$ for $j = 0,1$ and $g = f_1 \circ f_0^{-1}\colon M_0 \to M_1$. Let $e_1 = (1,0,\ldots,0)$. Choose a function $h \in \mathcal{C}^k(M_0)$. Our hypothesis implies that for each sufficiently small $\varepsilon > 0$, the map $g_\varepsilon = g + \varepsilon h e_1$ is a diffeomorphism of M_0 onto a submanifold $g_\varepsilon(M_0) \subset \mathbb{C}^n$ which can be approximated in $\mathcal{C}^k(M_0)$ by restrictions $\Phi|_{M_0}$, $\Phi \in \mathrm{Aut}\,\mathbb{C}^n$. Since the same holds for $g = g_0$, it follows that the function εh and hence h can be approximated in $\mathcal{C}^k(M_0)$ by restrictions to M_0 of entire functions on $\mathbb{C}^n$. Since this holds for all $\mathcal{C}^k$ functions h on M_0, M_0 is totally real and polynomially convex. We can apply the same argument to g^{-1} to see that M_1 is totally real and polynomially convex. □

From Theorem 1.1 and Proposition 1.2 (b) we obtain the following.

Corollary 1.4. *If M is a smooth compact manifold of dimension $m \le 2n/3$, then*

(a) *any two smooth embeddings $f_0, f_1\colon M \hookrightarrow \mathbb{C}^n$ which are totally real and polynomially convex are $\mathrm{Aut}\,\mathbb{C}^n$-equivalent;*

(b) *a smooth embedding $f_0\colon M \hookrightarrow \mathbb{R}^n \subset \mathbb{C}^n$ is $\mathrm{Aut}\,\mathbb{C}^n$-equivalent to any smooth, totally real and polynomially convex embedding $f_1\colon M \hookrightarrow \mathbb{C}^n$;*

(c) *if $2\cdot\dim M \le n$ then every smooth, totally real, polynomially convex embedding $f_0\colon M \hookrightarrow \mathbb{C}^n$ is $\mathrm{Aut}\,\mathbb{C}^n$-equivalent to an embedding $f_1\colon M \hookrightarrow \mathbb{R}^n \subset \mathbb{C}^n$.*

Part (a) follows from Theorem 1.1 and Proposition 1.2. For (b) observe that every embedding with range in $\mathbb{R}^n = \mathbb{R}^n \times \{i0\} \subset \mathbb{C}^n$ is totally real and polynomially convex. Part (c) follows from (b) and Whitney's embedding theorem.

We wish to point out an observation, made by Rosay in [23], to the effect that our notion of $\mathrm{Aut}\,\mathbb{C}^n$-equivalence of two compact sets in $\mathbb{C}^n$ implies the equivalence of their polynomial hulls:

Proposition 1.5. *Let $K_0,\ K_1 \subset \mathbb{C}^n$ be compact subsets and $f\colon K_0 \to K_1$ a homeomorphism. Assume that there is a sequence of automorphisms $\{\Phi_j\} \subset \mathrm{Aut}\,\mathbb{C}^n$ such that $\lim_{j\to\infty} \Phi_j|_{K_0} = f$ and $\lim_{j\to\infty} \Phi_j^{-1}|_{K_1} = f^{-1}$ (the convergence is uniform on K_0 and K_1 respectively). Then the sequence Φ_j also converges uniformly on $\hat{K}_0$ to a homeomorphism $\Phi\colon \hat{K}_0 \to \hat{K}_1$ which maps $\mathrm{Int}\,\hat{K}_0$ biholomorphically onto $\mathrm{Int}\,\hat{K}_1$.*

Proof. To see this, observe that, by the maximum principle, any sequence Φ_j of entire mappings which converges on K_0 also converges on $\hat{K}_0$ to a continuous map $\Phi: \hat{K}_0 \to \mathbb{C}^n$ which is holomorphic on $\operatorname{Int} \hat{K}_0$. Similarly, the sequence Φ_j^{-1} converges uniformly on $\hat{K}_1$ to a continuous map $\Psi: \hat{K}_1 \to \mathbb{C}^n$ which is holomorphic in $\operatorname{Int} \hat{K}_1$. Our assumtion implies that $\Phi|_{K_0} = f$ and $\Psi|_{K_1} = f^{-1}$. It is immediate that $\Phi(\hat{K}_0) \subset \hat{K}_1$ and $\Psi(\hat{K}_1) \subset \hat{K}_0$. Since $\Psi \circ \Phi: \hat{K}_0 \to \hat{K}_0$ (which is a limit of entire maps) is the identity on K_0, it is also the identity on $\hat{K}_0$, and likewise $\Phi \circ \Psi$ is the identity on $\hat{K}_1$. This proves Proposition 1.5. □

Outline of the proof of Theorem 1.1. Let $\{f_t\}$ be an isotopy as in Theorem 1.1. Set $M_t = f_t(M) \subset \mathbb{C}^n$. The isotopy of diffeomorphisms $g_t = f_t \circ f_0^{-1}: M_0 \to M_t$ $(0 \le t \le 1)$ is the flow of a certain time-dependent holomorphic vector field X, defined initially only along the trace $S = \bigcup_{0 \le t \le 1} \{t\} \times M_t$ of the isotopy $\{g_t\}$ in the extended phase space $[0,1] \times \mathbb{C}^n \subset \mathbb{C}^{n+1}$. Since S is totally real, we can extend X to a smooth vector field (of type (1,0)) on $\mathbb{C}^n$ such that the extension is $\bar{\partial}$-flat on S (Section 2).

The next step is to approximate the extended field X sufficiently well by entire vector fields $Y^\varepsilon: \mathbb{C}^{n+1} \to \mathbb{C}^n$ in ε-tubes $S(\varepsilon)$ around S for all sufficiently small $\varepsilon > 0$. This is done in Section 3 by standard $\bar{\partial}$-methods. Here we also need the polynomial convexity of each M_t and hence of S in order to get an approximation with entire vector fields Y^ε. We solve the equation $\bar{\partial} W^\varepsilon = \bar{\partial} X$ on $\mathbb{C}^{n+1}$ with $\mathcal{C}^k$ estimates in the tube $S(\varepsilon)$. Then $Y^\varepsilon = X - W^\varepsilon: \mathbb{C}^{n+1} \to \mathbb{C}^n$ is entire and it approximates X in $S(\varepsilon)$. When solving the $\bar{\partial}$-equation we follow the approach developed by Hörmander and Wermer [19] and used later by Nirenberg and Wells [20] to get a good L^2 estimate for W^ε in a larger tube $S(2\varepsilon)$. However, when estimating the derivatives of $W^\varepsilon = X - Y^\varepsilon$, we depart from [20] (where the Sobolev embedding lemma was used) and obtain the $\mathcal{C}^k$ estimate on a smaller tube $S(\varepsilon)$ directly from the L^2 estimate of W^ε and the sup-norm estimates of the derivatives of $\bar{\partial} W^\varepsilon = \bar{\partial} X$ on the larger tube $S(2\varepsilon)$. Our Lemma 3.2, which extends Lemma 4.4 in [19], is proved by a simple application of the Bochner-Martinelli integral formula. Our loss of derivatives is smaller than the one in [20].

In Section 4 we show that, if the approximation of X by Y^ε is sufficiently close on $S(\varepsilon)$ (and getting better as ε decreases to 0), then the flow ψ_t^ε of Y^ε approximates the flow g_t of X in a smaller tube $S(\varepsilon/\eta_0)$ for some $\eta_0 > 1$ (independent of ε). For this we use methods from the theory of ordinary differential equations. We hope that our Lemma 4.1 on perturbation of flows will be of independent interest.

We conclude the proof by invoking the result of Andersén [2] and Andersén and Lempert [3] (see also Lemma 1.4 in [14]) to the effect that each time-t

map ψ_t^ε of an entire (time-dependent) vector field Y^ε on $\mathbb{C}^n$ is approximable by automorphisms of $\mathbb{C}^n$. This gives a sequence $\{\Phi_j\} \subset \mathrm{Aut}\,\mathbb{C}^n$ converging to $g = g_1$ on M_0. To see that Φ_j^{-1} converges to g^{-1} on M_1 we simply reverse the time and apply the same arguments to the backward flows of X and Y^ε.

In Section 5 we extend our method to the following situation. We consider a smooth, time dependent vector field X on $\mathbb{C}^n$. Let $K \subset \mathbb{C}^n$ be a compact set and let $t_0 > 0$ be such that the flow $\varphi_t(z)$ of X with the initial condition $\varphi_0(z) = z$ exists for all $z \in K$ and $0 \le t \le t_0$. Denote by $S = S_{K,t_0}$ the trace of the flow, i.e., the subset of the extended phase space consisting of the integral curves $(t, \varphi_t(z))$, $z \in K$, $t \in [0, t_0]$. If $\bar\partial X$ vanishes to a high order on S and if S has a regular basis of pseudoconvex Runge neighborhoods, then each time-t map φ_t for $0 \le t \le t_0$ can be approximated in a smooth norm on K by automorphisms of $\mathbb{C}^n$ (Theorem 5.1).

The reader may wonder why we don't use the local approximation technique of Baouendi and Treves [4] (by convolution with the complex Gaussian kernel) in Section 3. This is essentially what was done in [9] (Section 2). The problem we are facing here is that we must approximate a given $\bar\partial$-flat vector field X not just on the submanifold S but also in a tube $S(\varepsilon)$ around S. We have not been able to do this by methods of [4].

2. Proof of the Main Theorem. Theorem 1.1 in Section 1 is a special case (with $K = \varnothing$) of the following result.

Theorem 2.1. *Let* $n \ge 2$ *and* $r \ge 3$. *Let* $K \Subset \mathbb{C}^n$ *be a compact polynomially convex set and* $U \subset \mathbb{C}^n$ *an open set containing* K. *For each* $t \in [0,1]$ *let* $M_t \subset \mathbb{C}^n$ *be a totally real submanifold of class* $\mathcal{C}^r$ *and* $f_t\colon M_0 \to M_t$ *a* $\mathcal{C}^r$ *diffeomorphism such that*

(i) $K \cup M_t$ *is polynomially convex for all* $0 \le t \le 1$,
(ii) $f_0(z) = z$ *for all* $z \in M_0$,
(iii) $f_t(z) = z$ *for all* $z \in M_0 \cap U$ *and* $0 \le t \le 1$ *(so* $M_t \cap U = M_0 \cap U$ *for* $t \in [0,1]$*), and*
(iv) f_t *and* $\partial f_t/\partial t$ *are of class* $\mathcal{C}^r$ *in* $(t,z) \in [0,1] \times M_0$.

Let $m = \dim M$ *and assume* $1 \le k \le r - (m+3)/2$. *Then for each* $\varepsilon > 0$ *there is a* $\Phi \in \mathrm{Aut}\,\mathbb{C}^n$ *satisfying*

(a) $\|\Phi|_{M_0} - f_1\|_{\mathcal{C}^k(M_0)} < \varepsilon$,
(b) $\|\Phi^{-1}|_{M_1} - f_1^{-1}\|_{\mathcal{C}^k(M_1)} < \varepsilon$,
(c) $\|\Phi - \mathrm{Id}\|_{\mathcal{C}^k(K)} < \varepsilon$.

(Here Id *is the identity map.) If in addition* E *is a finite subset of* $K \cup M_0$, *we can choose* Φ *such that it agrees with the identity to order* k *at each point of* $E \cap K$ *and* $\Phi|_{M_0}$ *agrees with* f_1 *to order* k *at each point of* $E \cap M_0$.

In part (c), $\mathcal{C}^k(K)$ denotes the k-jet norm on a compact set $K \subset \mathbb{C}^n$, i.e., if f is a $\mathcal{C}^k$ function in a neighborhood of K, then

$$\|f\|_{\mathcal{C}^k(K)} = \max\{|D^\alpha f(z)| : z \in K,\ |\alpha| \le k\}. \tag{2.1}$$

The maximum is over all multiindices $\alpha = (\alpha_1, \ldots, \alpha_{2n}) \in (\mathbf{Z}_+)^{2n}$ of degree $|\alpha| = \sum_{j=1}^{2n} \alpha_j \le k$, and $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial y_1^{\alpha_2} \cdots \partial y_n^{\alpha_{2n}}$ is the partial derivative with respect to the real coordinates $(x_1, y_1, \ldots, x_n, y_n)$ on $\mathbb{R}^{2n} = \mathbb{C}^n$.

We consider separately the case when M_0 and M_1 are smooth arcs attached to K at a single point of K. The following result was proved in [13] for $k = 0$. For a recent application to proper holomorphic embeddings $\mathbb{C} \hookrightarrow \mathbb{C}^n$ see [7].

Corollary 2.2 (Combing hair smoothly by automorphisms)**.** *Let $n \ge 2$. Let $K \subset \mathbb{C}^n$ be a compact polynomially convex set and let $M_0, M_1 \subset \mathbb{C}^n$ be embedded arcs of class $\mathcal{C}^r$, $3 \le r < \infty$, (diffeomorphic images of $[0,1] \subset \mathbb{R}$) which intersect K at a single point. Suppose that U is an open neighborhood of K and $f: K \cup M_0 \to K \cup M_1$ is a $\mathcal{C}^r$ diffeomorphism such that f is the identity on $(K \cup M_0) \cap U$. Then for each $k \le r - 2$, $\varepsilon > 0$, and each finite set $E \subset K \cup M_0$ there is a $\Phi \in \mathrm{Aut}\,\mathbb{C}^n$ satisfying the conclusion of* Theorem 2.1.

The corollary is a special case of Theorem 2.1 since f can be connected to the identity map $f_0(z) = z$ on M_0 by an isotopy $f_t: M_0 \to M_t$ $(0 \le t \le 1)$ satisying the hypotheses of Theorem 2.1. Note that the polynomial convexity of $K \cup M_t$ follows from Stolzenberg's theorem [24].

Proof of Theorem 2.1. For each compact set $K \subset \mathbb{C}^n$ and $\varepsilon > 0$ we denote

$$K(\varepsilon) = \{z \in \mathbb{C}^n : d(z, K) < \varepsilon\}, \tag{2.2}$$

where $d(z, K)$ is the Euclidean distance of z to K.

Let $\{f_t\}$ be an isotopy of diffeomorphisms as in Theorem 2.1. We shall think of t as time and of $\{f_t\}$ as the flow of a suitable time-dependent vector field (to be defined). It will be convenient to extend t to a complex variable. Thus, let $\mathbb{C}^{n+1} = \mathbb{C} \times \mathbb{C}^n$ be the extended phase space with complex coordinates $w = (t, z)$, with $t \in \mathbb{C}$ and $z \in \mathbb{C}^n$. We define the following sets in $\mathbb{C}^{n+1}$:

$$L_0 = [0,1] \times K = \{(t, z) : 0 \le t \le 1,\ z \in K\},$$

$$S = \bigcup_{0 \le t \le 1} \{t\} \times M_t,$$

$$L = L_0 \cup S = \bigcup_{0 \le t \le 1} \{t\} \times (K \cup M_t).$$

Since $K \cup M_t$ is polynomially convex for each $t \in [0,1]$, L is polynomially convex. Clearly $S = L \backslash L_0$ is a totally real submanifold (with boundary) of class $\mathcal{C}^r$.

We extend each f_t to $M_0 \cup U$ as the identity map on U. For each $0 \le t \le 1$ let $X_t\colon M_t \cup U \to \mathbb{C}^n$ be the velocity vector field of f_t, defined by the equation

$$\frac{\partial}{\partial t} f_t(z) = X_t\big(f_t(z)\big), \quad z \in M_0 \cup U,\ 0 \le t \le 1.$$

The map $X(t,z) = X_t(z)$ with values in $\mathbb{C}^n$ is defined on the set $S \cup ([0,1] \times U) \subset \mathbb{C}^{n+1}$. Clearly $\{f_t\}$ is the flow of X_t satisfying the initial condition $f_0(z) = z$ for $z \in M_0 \cup U$.

Property (iii) of $\{f_t\}$ implies that X vanishes on $[0,1] \times U$. Also, the restriction $X|_S$ is of class $\mathcal{C}^r(S)$. After shrinking U slightly we can extend X to a $\mathcal{C}^r$ map $X\colon \mathbb{C}^{n+1} \to \mathbb{C}^n$ with compact support such that X vanishes on $V = \mathbb{C} \times U \subset \mathbb{C}^{n+1}$ and satisfies

$$|D^\alpha(\bar\partial X)(w)| = o\big(d(w,S)^{r-1-|\alpha|}\big), \quad w \to S,\ \ 0 \le |\alpha| \le r-1. \tag{2.3}$$

(See Lemma 4.3 in [19] or Lemma 4 on p.148 in [6].) Here $\bar\partial X$ is the Cauchy-Riemann operator on $\mathbb{C}^{n+1}$ applied to each component of X, so $\bar\partial X$ is a $(0,1)$-form on $\mathbb{C}^{n+1}$ with values in $\mathbb{C}^n$. The derivative D^α is applied to each component of $\bar\partial X$, and $|D^\alpha \bar\partial X|$ denotes the Euclidean norm on the space $\mathbb{C}^{n(n+1)}$ of components of $D^\alpha \bar\partial X$.

Proposition 2.3. *Let $r \in \mathbf{Z}_+$, $r \ge (m+5)/2$, and $k = r - (m+3)/2 \ge 1$. Then for each sufficiently small $\varepsilon > 0$ there exists an entire map $Y^\varepsilon\colon \mathbb{C}^{n+1} \to \mathbb{C}^n$ such that for all multiindices $\alpha \in \mathbf{Z}_+^{2n+2}$ with $|\alpha| \le k$ we have*

$$\|D^\alpha(X - Y^\varepsilon)\|_{L^\infty(L(\varepsilon))} = o(\varepsilon^{k-|\alpha|}), \quad \varepsilon \to 0. \tag{2.4}$$

Proposition 2.3 is proved in Section 3 below. Recall that f_t is the flow of X with $f_0(z) = z$. Since X has compact support, f_t extends to a (complete) flow on $\mathbb{C}^n$. We interpret Y^ε as a time-dependent vector field on $\mathbb{C}^n$. Let ψ_t^ε be its (local) flow, the solution of the ODE

$$\frac{\partial}{\partial t} \psi_t^\varepsilon(z) = Y^\varepsilon\big(t, \psi_t^\varepsilon(z)\big), \quad \psi_0^\varepsilon(z) = z. \tag{2.5}$$

From Proposition 2.3 and Lemma 4.1 in Section 4 below we obtain

Proposition 2.4 (Asumptions as above)**.** *There is a neighborhood $V \subset \mathbb{C}^n$ of $K \cup M_0$ and an $\varepsilon_0 > 0$ such that the flow $\psi_t^\varepsilon(z)$ (2.5) exists for all $z \in V$, $t \in [0,1]$ and $0 < \varepsilon < \varepsilon_0$ and satisfies*

$$(2.6) \qquad \lim_{\varepsilon \to 0} \|f_t - \psi_t^\varepsilon\|_{\mathcal{C}^k(K \cup M_0)} = 0, \qquad \lim_{\varepsilon \to 0} \|(f_t)^{-1} - (\psi_t^\varepsilon)^{-1}\|_{\mathcal{C}^k(K \cup M_t)} = 0.$$

Here, $\mathcal{C}^k(K \cup M_0)$ denotes the k-jet norm (2.1).

Theorem 2.1 follows immediatelly from Proposition 2.4 (applied with $t = 1$) and the following lemma to the effect that for each fixed $\varepsilon > 0$, ψ_1^ε is a uniform limit of automorphisms of $\mathbb{C}^n$ in a neighborhood of $K \cup M_0$. Lemma 2.5 can be found in [14] (Lemma 1.4), and it is essentially due to Andersén [2] and Andersén and Lempert [3].

Lemma 2.5. *Let Y_t be an entire vector field on $\mathbb{C}^n$ for each $0 \le t \le t_0$, of class $\mathcal{C}^1$ in $(t,z) \in [0,t_0] \times \mathbb{C}^n$. Let Ω be an open subset of $\mathbb{C}^n$. Assume that the ordinary differential equation $dR/dt = Y_t(R(t))$ can be integrated for $0 \le t \le t_0$ with arbitrary initial condition $R(0) = z \in \Omega$. Set $G_t(z) = R(t)$ as above. Then for each $t \in [0,t_0]$, G_t is a biholomorphic map from Ω into $\mathbb{C}^n$ that can be approximated, uniformly on compact sets in Ω, by holomorphic automorphisms of $\mathbb{C}^n$.*

Combining these results, we get Theorem 2.1, except for the last statement concerning the finite set $E \subset K \cup M_0$. This follows from Proposition 1.1 in [12] to the effect that there exist automorphisms of $\mathbb{C}^n$ with prescribed finite order jets on any finite subset of $\mathbb{C}^n$, and the automorphisms can be chosen to depend continuously on the data for small perturbations of the data. Thus we can make the required small $\mathcal{C}^k$ correction at points of E by an automorphism which is close to the identity on some ball containing $K \cup M_0 \cup M_1$. This will establish Theorem 2.1 once we prove Propositions 2.3 and 2.4.

3. Solving $\bar\partial$-equation with $\mathcal{C}^k$-estimates in tubes. In this section we prove Proposition 2.3 following the approach of Hörmander and Wermer [19] (pp. 15-16). A special case of Proposition 2.3 with $m = 1$ and $k = 0$ was proved in [13] (Lemma 1). We shall use the notation introduced in Section 2. We will need the following lemma from [13] (Lemma 4).

Lemma 3.1. *There exists a continuous plurisubharmonic exhausting function $\rho \ge 0$ on $\mathbb{C}^{n+1}$ such that*
(a) *$\rho^{-1}(0) = L = L_0 \cup S$,*
(b) *$\rho(w) \le d(w,L)^2$ for w near L, and*
(c) *$\rho(w) = d(w,S)^2$ for z near $S \backslash V$.*

For $\varepsilon > 0$ set

$$\omega_\varepsilon = \{w \in \mathbb{C}^{n+1} : \rho(w) < \varepsilon^2\}.$$

Choose $\varepsilon_0 > 0$ such that $\omega_{\varepsilon_0} \subset L_{\varepsilon_0} \cup V$ and such that properties (b)–(c) in Lemma 3.1 hold for $w \in \omega_{\varepsilon_0}$. For $0 < \varepsilon \leq \varepsilon_0$ we then have

$$L(\varepsilon) \subset \omega_\varepsilon \subset L(\varepsilon) \cup V, \quad \omega_\varepsilon \backslash V = L(\varepsilon) \backslash V = S(\varepsilon) \backslash V.$$

Here we are using the notation (2.2). Recall that X has compact support in $\mathbb{C}^{n+1}$, it vanishes on V, and $\bar{\partial} X$ satisfies the estimates (2.3). Applying (2.3) with $|\alpha| = 0$, we get

$$\int_{\omega_{3\varepsilon}} |\bar{\partial} X|^2 d\lambda = o(\varepsilon^{2(r-1)}) O(\varepsilon^{2(n+1)-(m+1)}) = o(\varepsilon^{2(r+n)-m-1}), \quad \varepsilon \to 0. \tag{3.1}$$

Here $d\lambda$ is the Lebesgue measure on $\mathbb{C}^{n+1}$. The second term $O(\varepsilon^{2(n+1)-(m+1)})$ comes from the volume of the tube $S(3\varepsilon)$.

For each ε satisfying $0 < \varepsilon \leq \varepsilon_0/3$ let $\varphi_\varepsilon = h_\varepsilon \circ \rho$, where $h_\varepsilon : \mathbb{R} \to \mathbb{R}_+$ is a convex increasing function chosen so that $h_\varepsilon(t) = 0$ for $t \leq 4\varepsilon^2$ and $\int_{\mathbb{C}^{n+1} \backslash \omega_{3\varepsilon}} |\bar{\partial} X|^2 e^{-\varphi_\varepsilon} d\lambda$ is no larger than the integral in (3.1). This requires that h_ε is sufficiently large on $t > 9\varepsilon^2$. For such φ_ε we have

$$\int_{\mathbb{C}^{n+1}} |\bar{\partial} X|^2 e^{-\varphi_\varepsilon} d\lambda = o(\varepsilon^{2(r+n)-m-1}), \quad \varepsilon \to 0.$$

According to Theorem 2.2.3 in [17], there is, for each ε, $0 < \varepsilon < \varepsilon_0/3$, a solution of the equation $\bar{\partial} W^\varepsilon = \bar{\partial} X$ on $\mathbb{C}^{n+1}$ satisfying

$$\int |W^\varepsilon|^2 e^{-\varphi_\varepsilon} d\lambda \leq C \int |\bar{\partial} X|^2 e^{-\varphi_\varepsilon} d\lambda = o(\varepsilon^{2(r+n)-m-1}), \quad \varepsilon \to 0,$$

where $C > 0$ only depends on the radius of the support of X. Since $\varphi_\varepsilon = 0$ on the set $\omega_{2\varepsilon} \Subset \mathbb{C}^{n+1}$, we get

$$\|W^\varepsilon\|_{L^2(\omega_{2\varepsilon})} = o(\varepsilon^{r+n-(m+1)/2}), \quad \varepsilon \to 0. \tag{3.2}$$

Set $Y^\varepsilon = X - W^\varepsilon$. Then $Y^\varepsilon : \mathbb{C}^{n+1} \to \mathbb{C}^n$ is entire for each ε. It remains to prove that Y^ε satisfies the estimates (2.4). These follow from (2.3) and from the following lemma (which extends Lemma 4.4 in [19]). We denote by $\mathbb{B}$ the open unit ball in $\mathbb{C}^n$ centered at 0 and by $\varepsilon\mathbb{B}$ the ball of radius ε.

Lemma 3.2. *For each integer $s \in \mathbf{Z}_+$ there is a constant $C_s < \infty$ such that if $f \in \mathcal{C}^{s+1}(\varepsilon\mathbb{B})$ and $\alpha \in \mathbf{Z}_+^{2n}$ with $|\alpha| = s$, then*

$$|D^\alpha f(0)| \le C_s \left(\varepsilon^{-n-s} \|f\|_{L^2(\varepsilon\mathbb{B})} + \sum_{|\beta|\le s} \varepsilon^{|\beta|+1-s} \|D^\beta \bar\partial f\|_{L^\infty(\varepsilon\mathbb{B})} \right).$$

Assume the lemma for a moment. By Lemma 3.1 we have $L(2\varepsilon) \subset \omega_{2\varepsilon}$, and hence for each point $w \in L(\varepsilon)$ we have $B(w;\varepsilon) \subset \omega_{2\varepsilon}$. Applying Lemma 3.2 to the function W^ε (with $\bar\partial W^\varepsilon = \bar\partial X$) and using the estimates (3.2) and (2.3), we get for each $w \in L(\varepsilon)$ and $|\alpha| \le k$

$$\begin{aligned}|D^\alpha W^\varepsilon(w)| &\le C_{|\alpha|} \left(\varepsilon^{-n-1-|\alpha|} \|W^\varepsilon\|_{L^2(\omega_{2\varepsilon})} + \sum_{|\beta|\le|\alpha|} \varepsilon^{|\beta|+1-|\alpha|} \|D^\beta \bar\partial X\|_{L^\infty(\omega_{2\varepsilon})} \right) \\ &= o\big(\varepsilon^{r-(m+3)/2-|\alpha|}\big) + \sum_{|\beta|\le|\alpha|} \varepsilon^{|\beta|+1-|\alpha|} o\big(\varepsilon^{r-1-|\beta|}\big) \\ &= o\big(\varepsilon^{k-|\alpha|}\big), \qquad \varepsilon \to 0.\end{aligned}$$

As $W^\varepsilon = X - Y^\varepsilon$, this proves Proposition 2.3 provided that Lemma 3.2 holds.

Proof of Lemma 3.2. By rescaling, it suffices to prove the lemma for $\varepsilon = 1$. Let $z = (z_1, \ldots, z_n)$ be coordinates on $\mathbb{C}^n$. Set $dz = dz_1 \wedge \cdots \wedge dz_n$ and let $dz[j]$ be the $(n-1)$-form obtained by omitting the j-th term dz_j from dz. Let

$$K(z,w) = c_n \sum_{j=1}^n (-1)^{j+1} \frac{\overline{z_j - w_j}}{|z-w|^{2n}} \, d\bar z[j] \wedge dz$$

be the Bochner-Martinelli kernel for the point $w \in \mathbb{C}^n$ [21, p. 154] and let

$$K_0(z) = K(z,0) = c_n \sum_{j=1}^n (-1)^{j+1} \frac{\bar z_j}{|z|^{2n}} \, d\bar z[j] \wedge dz.$$

Then the Bochner-Martinelli formula gives for any $f \in \mathcal{C}^1(\bar{\mathbb{B}})$

$$f(0) = \int_{b\mathbb{B}} f(z) K_0(z) - \int_{\mathbb{B}} \bar\partial f(z) \wedge K_0(z).$$

Let $\chi \in \mathcal{C}_0^\infty(\mathbb{B})$ be a cut-off function such that $\chi = 1$ in $\frac{1}{2}\mathbb{B}$. The formula applied to $D^\alpha(\chi f)$ (which vanishes near $b\mathbb{B}$ and equals $D^\alpha f$ on $\frac{1}{2}\mathbb{B}$) gives

$$\begin{aligned}-D^\alpha f(0) &= \int_B \bar\partial\big(D^\alpha(\chi f)\big) \wedge K_0 \\ &= \int_{\mathbb{B}} D^\alpha\big(f\bar\partial\chi + \chi\bar\partial f\big) \wedge K_0 \\ &= \int_{\mathbb{B}} D^\alpha(f\bar\partial\chi) \wedge K_0 + \int_{\mathbb{B}} D^\alpha(\chi\bar\partial f) \wedge K_0.\end{aligned}$$

In the first integral $f\bar\partial\chi$ is compactly supported away from the singularity of K_0 at 0, and we may integrate by parts to get

$$\int_{\mathbb{B}} D^\alpha(f\bar\partial\chi)\wedge K_0 = (-1)^{|\alpha|}\int_{\mathbb{B}} f\,\big(\bar\partial\chi\wedge D^\alpha K_0\big).$$

The form integrated against f is smooth and has compact support contained in the shell $\{z\in\mathbb{C}^n : \frac{1}{2}<|z|<1\}$. Hence by Cauchy-Schwarz inequality the integral is dominated by $C\|f\|_{L^2(\mathbb{B})}$ with C depending only on $|\alpha|$. The second integral is a linear combination of terms

$$\int_{\mathbb{B}} (D^\beta\bar\partial f)\wedge(D^\gamma\chi)K_0$$

with multiindices $\beta,\gamma\in\mathbf{Z}_+^{2n}$, $\beta+\gamma=\alpha$. Since K_0 and therefore $(D^\gamma\chi)K_0$ is integrable on $\mathbb{B}$, each such term is dominated by $C\|D^\beta\bar\partial f\|_{L^\infty(\mathbb{B})}$ for some constant C. This proves Lemma 3.2. □

4. Perturbation of flows. In this section we prove a result on perturbation of flows of vector fields which shows that Proposition 2.4 in Section 2 follows from Proposition 2.3.

We cast the results of this section in the context of real time-dependent vector fields. Thus, let $\mathbb{R}^{n+1}$ be the extended phase space with real coordinates (t,x), where $t\in\mathbb{R}$ is the time variable and $x\in\mathbb{R}^n$ is the space variable. Given a domain $\Omega\subset\mathbb{R}^{n+1}$, we write $\Omega_t=\{x\in\mathbb{R}^n : (t,x)\in\Omega\}$. A continuous map $X : \Omega\to\mathbb{R}^n$ will be interpreted as a time-dependent vector field on (a subset of) $\mathbb{R}^n$. We shall write $X_t=X(t,\cdot\,)$. Fix such an X and assume that there is a constant $B<\infty$ such that

$$\big|X(t,x)-X(t,y)\big|\le B|x-y|,\quad x,y\in\Omega_t,\ t\in\mathbb{R} \tag{4.1}$$

(i.e., X is Lipschitz in the space variable, uniformly in t). For $x\in\Omega_s$ we denote by $\varphi_{t,s}(x)\in\Omega_t$ the time-forward map (flow) of X, defined as the solution of the equation

$$\frac{\partial}{\partial t}\varphi_{t,s}(x)=X_t\big(\varphi_{t,s}(x)\big),\quad \varphi_{s,s}(x)=x. \tag{4.2}$$

Recall that the flow satisfies the semigroup property $\varphi_{t,u}\circ\varphi_{u,s}=\varphi_{t,s}$ and $\varphi_{s,t}=\varphi_{t,s}^{-1}$. When $s=0$, we shall write $\varphi_{t,0}=\varphi_t$.

Assume now that $\Omega_0\ne\emptyset$ and fix a compact subset $K\subset\Omega_0$. Choose a number $t_0>0$ such that the flow $\varphi_t(x)$ exists and remains in Ω_t when $x\in K$ and $0\le t\le t_0$. Set $K_t=\varphi_t(K)\subset\Omega_t$. For any $\varepsilon>0$ we let

$$S(\varepsilon)=\{(t,x) : 0\le t\le t_0,\ d(x,K_t)<\varepsilon\}.$$

Set

$$\eta_0=(1+t_0)e^{Bt_0}>0,$$

where B is the Lipschitz constant (4.1). Choose $\varepsilon_0>0$ sufficiently small such that $S(\varepsilon_0\eta_0)\Subset\Omega$.

Lemma 4.1. (Notation as above.) *Assume that for some* ε, $0 < \varepsilon < \varepsilon_0$, *we have a continuous map* $Y^\varepsilon : \Omega \to \mathbb{R}^n$ *(a time-dependent vector field) satisfying*

$$\|X - Y^\varepsilon\|_{L^\infty(S(\varepsilon\eta_0))} \le \varepsilon.$$

Then any flow $\psi_t^\varepsilon(x)$ *of* Y_t^ε *(a solution of (2.5)) exists for all* $x \in K(\varepsilon)$ *and* $0 \le t \le t_0$ *and it satisfies*

$$\|\varphi_t - \psi_t^\varepsilon\|_{L^\infty(K(\varepsilon))} \le t_0 e^{Bt} \|X - Y^\varepsilon\|_{L^\infty(S(\varepsilon\eta_0))}, \quad 0 \le t \le t_0. \tag{4.3}$$

Assume in addition that we have a Y^ε *as above for every small* $\varepsilon > 0$ *such that* Y^ε *is of class* $\mathcal{C}^k$ *for some* $k \ge 1$ *and satisfies*

$$\|D^\alpha(X - Y^\varepsilon)\|_{L^\infty(S(\varepsilon\eta_0))} = o\big(\varepsilon^{k-|\alpha|}\big), \quad \varepsilon \to 0 \tag{4.4}$$

for all multiindices $\alpha \in \mathbf{Z}_+^n$ *with* $|\alpha| \le k$. *Then*

$$\|D^\alpha(\varphi_t - \psi_t^\varepsilon)\|_{L^\infty(K(\varepsilon))} = o\big(\varepsilon^{k-|\alpha|}\big), \quad \varepsilon \to 0 \tag{4.5}$$

for $|\alpha| \le k$ *and* $0 \le t \le t_0$.

The lemma, applied with $\varphi_t = f_t$ on the set $K \cup \bar{U} \subset \mathbb{C}^n$, shows that (2.4) implies the first estimate in (2.6). The second estimate in (2.6) follows by applying the same results to the backward flows $\varphi_{0,t} = \varphi_{t,0}^{-1} = f_t^{-1}$ and $\psi_{0,t}^\varepsilon = (\psi_{t,0}^\varepsilon)^{-1}$ on the set $K_t \cup \bar{U} = \varphi_t(K \cup \bar{U})$. (Recall that $X = 0$ on $\mathbb{C} \times U$, and hence $\varphi_t(z) = z$ for $z \in U$.)

Proof of Lemma 4.1. Let $A(\varepsilon) = \|X - Y^\varepsilon\|_{L^\infty(S(\varepsilon\eta_0))}$. Fix $x \in K(\varepsilon)$ and set

$$f(t) = \big|\varphi_t(x) - \psi_t^\varepsilon(x)\big|, \quad t \ge 0,$$

where ψ_t^ε is some local flow of Y^ε with $\psi_0^\varepsilon(x) = x$. We have $f(0) = 0$ and

$$\begin{aligned} f(t) &= \left| \int_0^t (X_s(\varphi_s(x)) - Y_s^\varepsilon(\psi_s^\varepsilon(x))\, ds \right| \\ &\le \int_0^t \big|X_s(\varphi_s(x)) - Y_s^\varepsilon(\psi_s^\varepsilon(x))\big|\, ds \\ &\le \int_0^t \big|X_s(\varphi_s(x)) - X_s(\psi_s^\varepsilon(x))\big|\, ds + \int_0^t \big|X_s(\psi_s^\varepsilon(x)) - Y_s^\varepsilon(\psi_s^\varepsilon(x))\big|\, ds \\ &\le B \int_0^t f(s) ds + t_0 A(\varepsilon). \end{aligned}$$

The last line holds for all $0 \le t \le t_0$ such that $\psi_t^\varepsilon(x) \in K_t(\varepsilon\eta_0)$. For such t Gronwall's lemma [1, p. 63] implies

$$\left|\varphi_t(x) - \psi_t^\varepsilon(x)\right| = f(t) \le A(\varepsilon) t_0 e^{Bt}.$$

For $x \in K(\varepsilon)$ Gronwall's inequality also gives $\varphi_t(x) \in K_t(\varepsilon e^{Bt})$. Thus we have

$$\begin{aligned} d\big(\psi_t^\varepsilon(x), K_t\big) &\le |\psi_t^\varepsilon(x) - \varphi_t(x)| + d(\varphi_t(x), K_t) \\ &\le A(\varepsilon) t_0 e^{Bt} + \varepsilon e^{Bt} \\ &\le \varepsilon(t_0+1)e^{Bt_0} = \varepsilon\eta_0. \end{aligned}$$

We used the assumption $A(\varepsilon) \le \varepsilon$ and the definition of η_0. This shows that $\psi_t^\varepsilon(x)$ remains in $K_t(\varepsilon\eta_0)$ for all $0 \le t \le t_0$, so the estimate (4.3) holds.

It remains to prove the estimates (4.5) for the derivatives, assuming that (4.4) holds. We will use the notation of [1], Chapter 1.2. In particular, $L^p(\mathbb{R}^n, \mathbb{R}^n)$ denotes the set of multilinear maps from $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ (p factors) to $\mathbb{R}^n$ and for a map $f\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ the p-th order derivative is denoted by $D^p f(x) \in L^p(\mathbb{R}^n, \mathbb{R}^n)$. If we differentiate the equation defining φ_t

$$\frac{\partial}{\partial t}\varphi_t(x) = X_t\big(\varphi_t(x)\big), \quad \varphi_0(x) = x$$

with respect to x, we get the equation of variation:

$$\frac{\partial}{\partial t} D\varphi_t(x) = DX_t\big(\varphi_t(x)\big) \circ D\varphi_t(x), \quad D\varphi_0(x) = I.$$

This is a linear equation in $D\varphi_t(x)$ whose solution is

$$D\varphi_t(x) = e^{\int_0^t DX_s(\varphi_s(x))ds}$$

We also have

$$D\psi_t^\varepsilon(x) = e^{\int_0^t DY_s^\varepsilon(\psi_s^\varepsilon(x))ds}.$$

(4.3) and (4.4) now give the following matrix norm estimate :

$$\begin{aligned} &\|DX_t(\varphi_t) - DY_t^\varepsilon(\psi_t^\varepsilon)\|_{L^\infty(K(\varepsilon))} \\ &\quad\le \|DX_t(\varphi_t) - DX_t(\psi_t^\varepsilon)\|_{L^\infty(K(\varepsilon))} + \|DX_t(\psi_t^\varepsilon) - DY_t^\varepsilon(\psi_t^\varepsilon)\|_{L^\infty(K(\varepsilon))} \\ &\quad\le \|D^2X\|_{L^\infty(S(\varepsilon\eta_0))}\|\varphi_t - \psi_t^\varepsilon\|_{L^\infty(K(\varepsilon))} + \|D(X - Y^\varepsilon)\|_{L^\infty(S(\varepsilon\eta_0))} \\ &\quad= o(\varepsilon^{k-1}), \quad \varepsilon \to 0. \end{aligned}$$

In case $k = 1$ the estimate follows directly from the second line. Hence the integral formulas above give the same estimate on the flows:

$$\|D\varphi_t - D\psi_t^\varepsilon\|_{L^\infty(K(\varepsilon))} = o(\varepsilon^{k-1}), \qquad \varepsilon \to 0.$$

Repeated differentiation gives that $D^p\varphi_t(x)$ satisfies the nonhomogenous linear equation

$$\frac{\partial}{\partial t} D^p\varphi_t(x) = DX_t\big(\varphi_t(x)\big) \circ D^p\varphi_t(x) + H_X^p(t,x), \quad D^p\varphi_0(x) = 0.$$

Here $H_X^p(t,x) \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ is a sum of terms involving derivatives of the vector field X and derivatives of order less than p of the flow φ_t. By induction, (4.4) and (4.5) imply

$$\|H_X^p - H_{Y^\varepsilon}^p\|_{L^\infty(S(\varepsilon\eta_0))} = o(\varepsilon^{k-p}), \quad \varepsilon \to 0.$$

The equations for $D^p\varphi_t(x)$ and $D^p\psi_t^\varepsilon(x)$ may be solved to find

$$D^p\varphi_t(x) = e^{\int_0^t DX_s(\varphi_s(x))ds} \circ \int_0^t e^{-\int_0^s DX_u(\varphi_u(x))du} \circ H_X^p(s,x)\, ds.$$

Similarly we can compute $D^p\psi_t^\varepsilon$:

$$D^p\psi_t^\varepsilon(x) = e^{\int_0^t DY_s^\varepsilon(\psi_s^\varepsilon(x))ds} \circ \int_0^t e^{-\int_0^s DY_u^\varepsilon(\psi_u^\varepsilon(x))du} \circ H_Y^p(s,x)\, ds.$$

These integral formulas immediately imply that the same estimate will hold for the flows:

$$\|D^p\varphi_t - D^p\psi_t^\varepsilon\|_{L^\infty(K(\varepsilon))} = o(\varepsilon^{k-p}), \quad \varepsilon \to 0.$$

5. Asymptotically holomorphic flows. In this section we prove a result on approximation of flows of *asymptotically holomorphic vector fields* on certain subsets of $\mathbb{C}^n$ by automorphisms of $\mathbb{C}^n$ (Theorem 5.1).

Recall that an open set $D \subset \mathbb{C}^n$ is *Runge* if every holomorphic function on D is a limit of holomorphic polynomials, uniformly on compacts in D [18]. For a compact set $K \subset \mathbb{C}^n$ let $K(\varepsilon)$ denote its ε-neighborhood (2.2).

Definition 2. *Let $h \in \mathbb{R}$, $h \geq 1$. A compact set $K \subset \mathbb{C}^n$ is h-regular if there are a basis of pseudoconvex Runge neighborhoods D_j of K, a constant $C > 0$ and a sequence $\varepsilon_j > 0$ with $\lim_{j\to\infty} \varepsilon_j = 0$ such that*

$$K(C\varepsilon_j^h) \subset D_j \subset K(\varepsilon_j), \quad j = 1,2,3,\ldots \tag{5.1}$$

An h-regular set K is polynomially convex since it is an intersection of pseudoconvex Runge sets. Conversely, every polynomially convex set $K \subset \mathbb{C}^n$ is the zero set of a smooth plurisubharmonic exhausting function $\rho\colon \mathbb{C}^n \to [0,\infty)$. If ρ can be chosen such that

$$c_1 d(z,K)^a \leq \rho(z) \leq c_2 d(z,K)^b \tag{5.2}$$

for some constants $c_1, c_2 > 0$ and $0 < b \leq a$ and for all points z near K, then K is h-regular with $h = a/b \geq 1$ since the sublevel sets $D_\varepsilon = \{\rho < c_1\varepsilon^a\}$ are pseudoconvex and Runge [18] and they satisfy

$$K\big((c_1/c_2)^{1/h}\varepsilon^h\big) \subset D_\varepsilon \subset K(\varepsilon)$$

for all small $\varepsilon > 0$. This holds in particular if K is the closure of a strongly pseudoconvex Runge domain or if K is a totally real, polynomially convex submanifold of $\mathbb{C}^n$; in both cases K is 1-regular.

Let $r \in \mathbf{Z}_+$. Let $K \subset \mathbb{C}^n$ be a compact set and $D \subset \mathbb{C}^n$ an open set containing K.

Definition 3. *A function $f \in \mathcal{C}^{r+1}(D)$ is $\bar\partial$-flat to order r on K if*

$$|D^\alpha \bar\partial f(z)| = o\big(d(z,K)^{r-|\alpha|}\big), \quad z \to K,\ |\alpha| \leq r. \tag{5.3}$$

If this holds for all $r \in \mathbf{Z}_+$, we say that f is $\bar\partial$-flat on K.

We shall consider time dependent vector fields on $\mathbb{C}^n$. Let $\mathbb{C}^{n+1} = \mathbb{C} \times \mathbb{C}^n$ be the extended phase space with the time variable $t \in \mathbb{C}$ and the space variable $z = (z_1, \ldots, z_n)$. For a domain $\Omega \subset \mathbb{C} \times \mathbb{C}^n$ and $t \in \mathbb{C}$ we write

$$\Omega_t = \{z \in \mathbb{C}^n \colon (t,z) \in \Omega\} \subset \mathbb{C}^n.$$

Let $X = (X_1, \ldots, X_n)\colon \Omega \to \mathbb{C}^n$ be a smooth time-dependent vector field. Assume that $\Omega_0 \neq \emptyset$ and let $K \subset \Omega_0$ be a compact set. Choose a $t_0 > 0$ such that the flow $\varphi_t(z)$ of X satisfying

$$\dot\varphi_t(z) = \frac{\partial}{\partial t}\varphi_t(z) = X\big(t, \varphi_t(z)\big), \quad \varphi_0(z) = z, \tag{5.4}$$

exists and remains in Ω_t for all $z \in K$ and $0 \leq t \leq t_0$. Set $K_t = \varphi_t(K) \subset \Omega_t$ and

$$S_{K,t_0} = \bigcup_{0 \leq t \leq t_0} \{t\} \times K_t \subset \Omega. \tag{5.5}$$

S_{K,t_0} is called the *trace* of the flow $\{\varphi_t\colon 0 \leq t \leq t_0\}$ on K.

Theorem 5.1 (Notation as above). *Let $X:\Omega \to \mathbb{C}^n$ be a time-dependent vector field of class $\mathcal{C}^{r+1}$, $K \subset \Omega_0$ a compact set and $t_0 > 0$ such that the flow $\varphi_t(z)$* (5.4) *is defined for all $z \in K$ and all $0 \le t \le t_0$. Assume that the trace $S = S_{K,t_0}$* (5.5) *is h-regular for some $h \ge 1$* (Definition 2) *and X is $\bar{\partial}$-flat to order r on S* (Definition 3). *If $1 \le k \le r/h - (n+1)$ then for each fixed $t \in [0,t_0]$ there exists a sequence $\Phi_j \in \operatorname{Aut}\mathbb{C}^n$ such that*

$$\lim_{j\to\infty} \|\Phi_j - \varphi_t\|_{\mathcal{C}^k(K)} = 0, \qquad \lim_{j\to\infty} \|\Phi_j^{-1} - \varphi_t^{-1}\|_{\mathcal{C}^k(K_t)} = 0.$$

(*Here $\mathcal{C}^k(K)$ is the k-jet norm* (2.1).)

Proof. We may assume that X has compact support in $\mathbb{C}^{n+1}$. Let $D_j \subset \mathbb{C}^{n+1}$ be a sequence of neighborhoods of $S = S_{K,t_0}$ satisfying (5.1). The estimate (5.3) with $\alpha = 0$ gives $\|\bar{\partial}X\|_{L^2(D_j)} = o(\varepsilon_j^r)$. By Theorem 2.2.3 of Hörmander [17] there is a $W^j: D_j \to \mathbb{C}^n$ such that $\bar{\partial}W^j = \bar{\partial}X$ and

$$\|W^j\|_{L^2(D_j)} \le c\|\bar{\partial}X\|_{L^2(D_j)} = o(\varepsilon_j^r)$$

for some constant $c > 0$ independent of j. Then $Y^j = X - W^j: D_j \to \mathbb{C}^n$ is holomorphic and satisfies

$$\|X - Y^j\|_{L^2(D_j)} = o(\varepsilon_j^r). \tag{5.6}$$

Since D_j is assumed to be Runge in $\mathbb{C}^{n+1}$, we can approximate Y^j uniformly on compacts in D_j by entire maps $\mathbb{C}^{n+1} \to \mathbb{C}^n$. Thus, after shrinking D_j a little so that (5.2) still holds with C replaced by $C/2$, we may assume that Y^j is entire and satisfies (5.6).

Using the left inclusion in (5.1) (with K replaced by S) and Lemma 3.2 in Section 3 (with ε replaced by ε_j^h), we get the following estimates in a smaller tube $S_j = S(c_1\varepsilon_j^h)$ (where $c_1 > 0$ is a constant independent of j):

$$\|D^\alpha(X - Y^j)\|_{L^\infty(S_j)} = o\big(\varepsilon_j^{r-h(n+1)-h|\alpha|}\big).$$

These hold for all multiindices $\alpha \in (\mathbf{Z}_+)^{2n+2}$ with $|\alpha| \le r/h - (n+1)$ (so that the exponent on the right hand side above is nonnegative).

Denote by ψ_t^j the flow of Y^j, a solution of

$$\frac{\partial}{\partial t}\psi_t^j(z) = Y^j\big(t, \psi_t^j(z)\big), \quad \psi_0^j(z) = z.$$

This flow exists for all z near K and $t \in [0, t_0]$. If k satisfies $1 \le k \le r/h - (n+1)$, we can apply Lemma 4.1 (with ε replaced by ε_j^h) to conclude that for each fixed $t \in [0, t_0]$ we have

$$\|D^\alpha(\varphi_t - \psi_t^j)\|_{L^\infty(K)} = o\big(\varepsilon_j^{h(k-|\alpha|)}\big)$$

for all multiindices $|\alpha| \le k$. By reversing the time we also get

$$\|D^\alpha(\varphi_{-t} - \psi_{-t}^j)\|_{L^\infty(K_t)} = o\big(\varepsilon_j^{h(k-|\alpha|)}\big)$$

for $|\alpha| \le k$, where $K_t = \varphi_t(K) \subset \mathbb{C}^n$. Since ψ_t^j is the flow of an entire vector field, Lemma 2.5 implies that ψ_t can be approximated, uniformly in a neighborhood of K, by holomorphic automorphisms of $\mathbb{C}^n$. This gives a sequence $\Phi_j \in \mathrm{Aut}\,\mathbb{C}^n$ satisfying Theorem 5.1. □

6. Gobal holomorphic equivalence in other Stein manifolds. One may ask how much of what has been done remains true for isotopies $f_t: M \hookrightarrow \mathcal{M}$ in complex manifolds $\mathcal{M}$ other than $\mathbb{C}^n$. The only place where special properties of $\mathbb{C}^n$ have been used in an essential way was Lemma 2.5 (the Andersén-Lempert theorem) to the effect that flows of globally defined holomorphic vector fields on $\mathbb{C}^n$ can be approximated (uniformly on compacts) by automorphisms of $\mathbb{C}^n$. This holds on any complex manifold $\mathcal{M}$ satisfying the following property:

Definition 4. *A complex manifold $\mathcal{M}$ satisfies the* density property (DP) *if the Lie algebra generated by all complete holomorphic vector fields on $\mathcal{M}$ is everywhere dense in the Lie algebra of all holomorphic vector fields on $\mathcal{M}$ (in the topology of uniform convergence on compacts in $\mathcal{M}$).*

Recall that a vector field on $\mathcal{M}$ is complete (in real time) if its flow $\varphi_t(z)$ (satisfying $\varphi_0(z) = z$) exists for all $t \in \mathbb{R}$ and for all $z \in \mathcal{M}$. Andersén and Lempert [3] proved that $\mathbb{C}^n$ has DP when $n > 1$. The terminology is due to Varolin [25], who established several results about such manifolds. He proved, for instance, that for every complex Stein Lie group G of positive dimension, $G \times \mathbb{C}$ has DP; in particular, the spaces $\mathbb{C}^n \times (\mathbb{C}^*)^k$ for $n \ge 1$ and $n + k \ge 2$ have DP.

Granted the density property, Lemma 2.5 remains in effect as is clear from the proof of Lemma 1.4 in [14], and we get the following extension of Theorem 1.1.

Theorem 6.1. *Let M be a compact smooth manifold and let $\mathcal{M}$ be a Stein manifold satisfying the density property* (Definition 4). *Given a smooth isotopy of embeddings $f_t: M \hookrightarrow \mathcal{M}$ $(0 \le t \le 1)$ such that $f_t(M)$ is totally real and holomorphically convex in $\mathcal{M}$ for each $t \in [0,1]$, there exists a sequence Φ_j of holomorphic automorphisms of $\mathcal{M}$ such that $\Phi_j \circ f_0$ converges to f_1 and $\Phi_j^{-1} \circ f_1$ converges to f_0 in the $\mathcal{C}^\infty$ topology as $j \to \infty$. If the isotopy f_t is real-analytic, the sequence Φ_j can be chosen such that it converges uniformly in a neighborhood U of $f_0(M)$ to a biholomorphic map $\Phi: U \to \Phi(U) \subset \mathcal{M}$ satisfying $\Phi \circ f_0 = f_1$.*

The proof given for the case $\mathcal{M} = \mathbb{C}^n$ can be adapted without difficulties to this more general setting. The condition that $\mathcal{M}$ be Stein allows one to solve the $\bar{\partial}$-equation in Section 3 following Hörmander [17, 18]. One can also adapt Lemma 4.1 (with some obvious changes) to flows in any real manifold. The proof given in [9] for the real-analytic case and for $\mathcal{M} = \mathbb{C}^n$ extends to any Stein manifold $\mathcal{M}$ with the density property.

7. A final remark. If we only assume that the isotopy of embeddings f_t is totally real (but not necessarily polynomially convex resp. holomorphically convex), we can only get as far as Proposition 2.4 in Section 2. This gives a sequence of biholomorphic maps $\Phi_j : U_j \to V_j$ from open neighborhoods $U_j \subset \mathcal{M}$ of $f_0(M) = M_0$ onto open neighborhoods $V_j \subset \mathcal{M}$ of $f_1(M) = M_1$, satisfying (1.1). In general, as $j \to \infty$, the sets U_j and V_j shrink to M_0 and M_1 respectively. On the other hand, if the isotopy f_t is totally real and real-analytic, there is a fixed neighborhood $U \subset \mathcal{M}$ of M_0 and a sequence of biholomorphic maps $\Phi_j : U \to \Phi_j(U) \subset \mathcal{M}$ which converges, uniformly on U, to a biholomorphic map $\Phi : U \to \Phi(U) \subset \mathcal{M}$ satisfying $\Phi \circ f_0 = f_1$. Results in this direction (for $\mathcal{M} = \mathbb{C}^n$ and also for unimodular biholomorphic maps) have been recently obtained by Gong [15].

Acknowledgements. This work was begun when the first author was visiting the University of Oslo and was finished when both authors were visiting Purdue University. We express our sincere thanks to both institutions. We wish to thank J.-P. Rosay and L. Lempert for useful discussions on this subject. The first author wishes to acknowledge partial support by the NSF grant DMS-9322766 and by a grant from the Ministry of Science of the Republic of Slovenia.

REFERENCES

[1] R. ABRAHAM & J.E. MARSDEN, *Foundations of Mechanics,* 2nd. ed., Reading: Benjamin, 1987.

[2] E. ANDERSÉN, *Volume-preserving automorphisms of* $\mathbb{C}^n$, Complex Var. **14** (1990), 223-235.

[3] E. ANDERSÉN & L. LEMPERT, *On the group of holomorphic automorphisms of* $\mathbb{C}^n$, Invent. Math. **110** (1992), 371–388.

[4] M.S. BAOUENDI & F. TREVES, *A property of the functions and distributions annihilated by a locally integrable system of complex vector fields*, Annals of Math. **113** (1981), 387–421.

[5] B. BERNDTSSON, *Integral kernels and approximation on totally realsubmanifolds of* $\mathbb{C}^n$, Math. Ann. **243** (1979), 125–129.

[6] A. BOGGESS, *CR Manifolds and the Tangential Cauchy-Riemann Complex*, Boca Raton: CRC Press, 1991.

[7] G. BUZZARD & F. FORSTNERIC, *A Carleman type theorem for proper holomorphic embeddings*, Arkiv Mat. **35** (1997), 157–169.

[8] F. FORSTNERIC, *Stability of polynomial convexity of totally real sets*, Proc. Amer. Math. Soc. **96** (1986), 489–494.

[9] F. Forstneric, *Approximation by automorphisms on smooth submanifolds of* $\mathbb{C}^n$, Math. Ann. **300** (1994), 719–738.

[10] F. Forstneric, *A theorem in complex symplectic geometry*, J. Geom. Anal. **5** (1995), 379–393.

[11] F. Forstneric, *Equivalence of real submanifolds under volume preserving holomorphic automorphisms of* $\mathbb{C}^n$, Duke Math. J. **77** (1995), 431–445.

[12] F. Forstneric, *Interpolation by holomorphic automorphisms and embeddings in* $\mathbb{C}^n$., J. Geom. Anal., to appear.

[13] F. Forstneric, J. Globevnik, & B. Stensønes, *Embedding holomorphic discs through discrete sets*, Math. Ann. **305** (1995), 559–569.

[14] F. Forstneric & J.-P. Rosay, *Approximation of biholomorphic mappings by automorphisms of* $\mathbb{C}^n$, Invent. Math **112**, (1993), 323-349; Erratum, Invent. Math **118**, (1994), 573–574.

[15] X. Gong, *On totally real spheres in complex space.* Preprint, (1996).

[16] M. Hirsch, *Differential Topology.*, Jour Grad. Texts in Math. **33**, New York: Springer 1976.

[17] L. Hörmander, L^2 *estimates and existence theorems for the* $\bar{\partial}$ *operator*, Acta Math. **113** (1965), 89–152.

[18] L. Hörmander, *An Introduction to Complex Analysis in Several Variables,* 3rd ed, Amsterdam: North Holland, 1990.

[19] L. Hörmander & J. Wermer, *Uniform approximation on compact sets in* $\mathbb{C}^n$, Math. Scand. **23** (1968), 5–21.

[20] R. Nirenberg & R.O. Wells, Jr., *Approximation theorems on differentiable submanifolds of a complex manifold*, Trans. Amer. Math. Soc. **142** (1969), 15–35.

[21] M. R. Range, *Holomorphic Functions and Integral Representations in Several Complex Variables*, New York: Springer, 1986.

[22] R.M. Range & Y.T. Siu, $\mathcal{C}^k$ *approximation by holomorphic functions and* $\bar{\partial}$*-closed forms on* $\mathcal{C}^k$ *submanifolds of a complex manifold*, Math. Ann. **210** (1974), 105–122.

[23] J.-P. Rosay, *Straightening of arcs*, Astérisque **217** (1993), 217–226.

[24] G. Stolzenberg, *Polynomially and rationally convex sets*, Acta Math. **109** (1963), 259–289.

[25] D. Varolin, *Ph. D. Dissertation*, University of Wisconsin–Madison, 1997.

Department of Mathematics
University of Wisconsin
Madison, WI 53706, USA

Department of Mathematics
University of Oslo
P.O.Box 1053, Blindern
N-0315 Oslo

Received: October 15th, 1996; revised: December 23rd, 1996.