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Interpolation by Holomorphic Automorphisms and Embeddings in $\mathbf{C}^n$

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ABSTRACT. *Let $n > 1$ and let $\mathbf{C}^n$ denote the complex n-dimensional Euclidean space. We prove several jet-interpolation results for nowhere degenerate entire mappings $F : \mathbf{C}^n \to \mathbf{C}^n$ and for holomorphic automorphisms of $\mathbf{C}^n$ on discrete subsets of $\mathbf{C}^n$. We also prove an interpolation theorem for proper holomorphic embeddings of Stein manifolds into $\mathbf{C}^n$. For each closed complex submanifold (or subvariety) $M \subset \mathbf{C}^n$ of complex dimension $m < n$ we construct a domain $\Omega \subset \mathbf{C}^n$ containing M and a biholomorphic map $F : \Omega \to \mathbf{C}^n$ onto $\mathbf{C}^n$ with $JF \equiv 1$ such that $F(M)$ intersects the image of any nondegenerate entire map $G : \mathbf{C}^{n-m} \to \mathbf{C}^n$ at infinitely many points. If $m = n - 1$, we construct F as above such that $\mathbf{C}^n \setminus F(M)$ is hyperbolic. In particular, for each $m \geq 1$ we construct proper holomorphic embeddings $F : \mathbf{C}^m \to \mathbf{C}^{m+1}$ such that the complement $\mathbf{C}^{m+1} \setminus F(\mathbf{C}^m)$ is hyperbolic.*

1. Introduction

Let $\mathbf{C}^n$ denote the complex Euclidean space of dimension n. We shall always assume $n > 1$. In this paper we obtain several interpolation results of Mittag–Leffler type for nowhere degenerate entire maps $\mathbf{C}^n \to \mathbf{C}^n$, for injective entire maps on $\mathbf{C}^n$, for automorphisms of $\mathbf{C}^n$, and for proper holomorphic embeddings of Stein manifolds into $\mathbf{C}^n$.

A holomorphic mapping $F : U \to \mathbf{C}^n$, defined in an open set $U \subset \mathbf{C}^n$, is *nondegenerate* at a point $z \in U$ if its complex Jacobian $JF(z) = \det F'(z)$ at z is not zero. If this holds for all $z \in U$, we say that F is *nowhere degenerate* on U. We also say that F is *nondegenerate* on U if its Jacobian JF is not identically zero on any connected component of U. F is *volume preserving* if $JF \equiv 1$. An entire map $F : \mathbf{C}^n \to \mathbf{C}^n$ is a holomorphic automorphism of $\mathbf{C}^n$ if it is one-to-one and onto $\mathbf{C}^n$; such F has a holomorphic inverse. We denote by $\mathrm{Aut}\,\mathbf{C}^n$ the group of all holomorphic automorphisms of $\mathbf{C}^n$, and by $\mathrm{Aut}_1\,\mathbf{C}^n$ the group of automorphisms $F \in \mathrm{Aut}\,\mathbf{C}^n$ with Jacobian one. We denote by B the open unit ball in $\mathbf{C}^n$.

In Section 2 we construct holomorphic automorphisms of $\mathbf{C}^n$ which have prescribed jets of finite order at a finite set of points and are close to the identity on a given polynomially convex set (Proposition 2.1). This is the basic step in the proof of results in Sections 3, 4, and 6. For interpolation

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of "volume preserving" jets at one point of $\mathbf{C}^n$ by volume preserving automorphisms (and without any additional conditions on a polynomially convex set) this result was proved by Andersén and Lempert [2, Proposition 6.3]. Buzzard proved the corresponding result for interpolation of one-jet (derivative) of the map at a finite set of points [3, Lemmas 2.3 and 2.4].

In Section 3 we construct nowhere degenerate entire maps $\mathbf{C}^n \to \mathbf{C}^n$ with prescribed Taylor expansion of any finite order (with nondegenerate linear part) at any discrete set of points in $\mathbf{C}^n$. We prove the analogous theorem for volume preserving maps. This extends Theorem 1.1 of Rosay and Rudin [17].

In Section 4 we consider the same problem for injective entire maps $\mathbf{C}^n \to \mathbf{C}^n$ and for holomorphic automorphisms of $\mathbf{C}^n$. If $\{a_j\}$, $\{b_j\} \subset \mathbf{C}^n$ are discrete sequences (without repetition) which are *tame* in the sense of Rosay and Rudin [17], there exists an automorphism $F \in \operatorname{Aut} \mathbf{C}^n$ such that $F(a_j) = b_j$ and F has a prescribed Taylor expansion of finite order m at a_j for each $j = 1, 2, 3, \dots$ (Corollary 4.2). If $\{a_j\}$ is an arbitrary discrete set in $\mathbf{C}^n$ and $b_j = (j, 0, \dots, 0)$ $(j = 1, 2, 3, \dots)$, this holds for an injective holomorphic map (Corollary 4.4), but in general not for any automorphisms of $\mathbf{C}^n$ (see [17]).

In Section 5 we prove a result on convergence of sequences of compositions

$$\Phi_m = \Psi_m \circ \Psi_{m-1} \circ \cdots \circ \Psi_1 ,$$

where Ψ_j is an automorphism of $\mathbf{C}^n$ which is close to the identity on a given compact set K_j for each $j = 1, 2, 3, \dots$, and the sets K_j exhaust a domain $D \subset \mathbf{C}^n$ as $j \to \infty$. We show that, under rather weak conditions on K_j and Ψ_j, there is a domain $\Omega \subset \mathbf{C}^n$ such that Φ_m converges on Ω (as $m \to \infty$) to a biholomorphic map from Ω onto D (Proposition 5.1). Results of this kind (when $D = \mathbf{C}^n$) are implicitly contained in the recent papers [4, 14], and possibly in others.

Combining Proposition 5.1 and the main result of the paper [12] by Globevnik et al., we prove (Corollary 5.4) that *every discrete set in a connected pseudoconvex Runge domain $D \subset \mathbf{C}^n$ $(n > 1)$ is contained in one leaf of a nonsingular holomorphic foliation of D by simply connected curves (i.e., each leaf of the foliation is biholomorphic to either $\mathbf{C}$ or to the disc $\{\zeta \in \mathbf{C}: |\zeta| < 1\}$).*

In Section 6 we prove an interpolation theorem for embeddings of Stein manifolds and Stein spaces into $\mathbf{C}^n$. If $M \subset \mathbf{C}^n$ is a closed complex submanifold (or subvariety) of $\mathbf{C}^n$, and if $\{p_j\} \subset \mathbf{C}^n$ is a discrete set in $\mathbf{C}^n$, there exists a Runge domain $\Omega \subset \mathbf{C}^n$ containing M and a biholomorphic map $F\colon \Omega \to \mathbf{C}^n$ *onto* $\mathbf{C}^n$, with $JF(z) = 1$ for each $z \in \Omega$, such that $p_j \in \operatorname{Reg} F(M)$ for each j and $F(M)$ has a prescribed jet of any finite order at each point p_j (Theorem 6.1). In particular, we can prescribe the tangent space $T_{p_j} F(M)$ at each p_j. This extends the result in [11].

Further, we construct maps F as above such that if $d = n - \dim M$, *every entire map $G\colon \mathbf{C}^d \to \mathbf{C}^n$ of rank d intersects the image $F(M)$ at infinitely many points. When $d = 1$, F can be chosen such that $\mathbf{C}^n \backslash F(M)$ is Kobayashi hyperbolic.* By rank of G we mean its generic (maximal) rank.

The following special case is of particular interest (see Corollary 6.3): *for each pair of integers $m, d \geq 1$ there exists a proper holomorphic embedding $F\colon \mathbf{C}^m \hookrightarrow \mathbf{C}^n$, with $n = m + d$, such that every holomorphic map $G\colon \mathbf{C}^d \to \mathbf{C}^n$ of rank d intersects $F(\mathbf{C}^m)$ infinitely many times; and there exist embeddings $F\colon \mathbf{C}^m \hookrightarrow \mathbf{C}^{m+1}$ for every $m \geq 1$ such that the complement $\mathbf{C}^{m+1} \backslash F(\mathbf{C}^m)$ is Kobayashi hyperbolic.* This extends the result of Buzzard and Fornæss [4] for one-dimensional complex curves in $\mathbf{C}^2$.

Here is an interesting consequence of Corollary 6.3 (see Corollary 6.4 and the remark following it). We identify $\mathbf{C}^m$ with $\mathbf{C}^m \times \{0\} \subset \mathbf{C}^{m+1}$. *For each pair of integers (m, n) such that $1 \leq m < n \leq 2m + 1$, there exists a proper holomorphic embedding $F\colon \mathbf{C}^m \hookrightarrow \mathbf{C}^n$ which does not extend to* *an injective holomorphic map of* $\mathbf{C}^{m+1}$ *into* $\mathbf{C}^n$. On the other hand, when $n > 2m+1$, it is easily seen that every proper holomorphic embedding $F: \mathbf{C}^m \hookrightarrow \mathbf{C}^n$ extends to an injective holomorphic immersion $\tilde{F}: \mathbf{C}^{m+1} \to \mathbf{C}^n$.

For the reader who is mainly interested in results on holomorphic embeddings in Section 6, we wish to point out that the only results from the rest of the paper which are used there are Propositions 2.1 and 5.1. In fact, all the tools needed in the proof of Theorem 6.1, except for the precise jet-interpolation result at points of the given discrete set, have been developed in the earlier papers [2, 11], and [13].

For a survey of some recent results on Aut $\mathbf{C}^n$ we refer the reader to [10].

2. Jet interpolation by automorphisms on a finite set

We denote by $\mathbf{Z}$ the set of all integers, by $\mathbf{Z}_+$ the set of nonnegative integers, and by $\mathbf{N} = \{1, 2, 3, \ldots\}$ the set of natural numbers.

By degree of a holomorphic polynomial map $P = (P_1, \ldots, P_n): \mathbf{C}^n \to \mathbf{C}^n$ we mean the maximal degree of the components P_j. We say that P is homogeneous of degree k if each component P_j is such.

Recall that a compact subset $K \subset \mathbf{C}^n$ is polynomially convex if for each point $p \in \mathbf{C}^n \backslash K$ there exists a holomorphic polynomial $f: \mathbf{C}^n \to \mathbf{C}$ such that $|f(p)| > \sup\{|f(z)|: z \in K\}$.

Proposition 2.1. *Let* $n > 1$. *Assume that*

(a) $K \subset \mathbf{C}^n$ *is a compact polynomially convex set,*
(b) $\{a_j\}_{j=1}^{s} \subset K$ *is a finite subset of* K,
(c) p *and* q *are arbitrary points in* $\mathbf{C}^n \backslash K$ *(not necessarily distinct),*
(d) N *is a nonnegative integer, and*
(e) $P: \mathbf{C}^n \to \mathbf{C}^n$ *is a holomorphic polynomial map of degree at most* $m \geq 1$ *with* $P(0) = 0$ *and* $JP(0) \neq 0$.

Then for each $\epsilon > 0$ *there exists an automorphism* $F \in \mathrm{Aut}\,\mathbf{C}^n$ *satisfying*

(i) $F(p) = q$ *and* $F(z) = q + P(z-p) + O(|z-p|^{m+1})$ *as* $z \to p$,
(ii) $F(z) = z + O(|z-a_j|^N)$ *as* $z \to a_j$ *for each* $j = 1, 2, \ldots, s$, *and*
(iii) $|F(z) - z| + |F^{-1}(z) - z| < \epsilon$ *for each* $z \in K$.

If, in addition, the polynomial map P *satisfies*

(e') $JP(z) = 1 + O(|z|^m)$ *as* $z \to 0$,

we may choose F *to be a polynomial automorphism with Jacobian one.*

Corollary 2.2. *Let* $n > 1$. *Given finite subsets* $\{a_j\}_{j=1}^{s}$ *and* $\{b_j\}_{j=1}^{s}$ *of* $\mathbf{C}^n$ *(without repetition), and for each* $j = 1, \ldots, s$ *a polynomial* $P_j: \mathbf{C}^n \to \mathbf{C}^n$ *of degree at most* $m_j \geq 1$ *satisfying* $P_j(0) = 0$ *and* $JP_j(0) \neq 0$, *there exists an automorphism* $F \in \mathrm{Aut}\,\mathbf{C}^n$ *such that for each* $j = 1, 2, \ldots, s$ *we have* $F(a_j) = b_j$ *and*

$$F(z) = b_j + P_j(z - a_j) + O\left(|z - a_j|^{m_j+1}\right), \quad z \to a_j. \tag{2.1}$$

If in addition $JP_j(z) = 1 + O(|z|^{m_j})$ *as* $z \to 0$ *for* $j = 1, 2, \ldots, s$, *we may choose* F *to be a polynomial automorphism with Jacobian one.*

In the case of a single point and P satisfying the last condition in Corollary 2.2, this result was proved by Andersén and Lempert [2, Proposition 6.3].

Proof of Corollary 2.2. By Proposition 2.1 there is for each $j \in \{1, 2, \ldots, s\}$ an automorphism $F_j \in \mathrm{Aut}\,\mathbf{C}^n$ satisfying

$$\begin{aligned} F_j(z) &= b_j + P_j\left(z - a_j\right) + O\left(\left|z - a_j\right|^{m_j+1}\right), \quad z \to a_j\,, \\ F_j(z) &= z + O\left(|z - b_k|^{m_k+1}\right), \quad z \to b_k, \quad 1 \le k \le j-1\,, \\ F_j(z) &= z + O\left(|z - a_k|^{m_k+1}\right), \quad z \to a_k, \quad j+1 \le k \le s\,. \end{aligned}$$

The composition $F = F_s \circ F_{s-1} \circ \cdots \circ F_1$ then satisfies Corollary 2.2. □

The following was proved by Buzzard [3].

Corollary 2.3. *Let* $n > 1$. *Given finite sets* $\{a_j\}, \{b_j\} \subset \mathbf{C}^n$ *as in Corollary 2.2, and linear maps* $A_j \in GL(n, \mathbf{C})$, *there exists* $F \in \mathrm{Aut}\,\mathbf{C}^n$ *satisfying*

$$F\left(a_j\right) = b_j, \qquad F'\left(a_j\right) = A_j, \qquad j = 1, 2, \ldots, s\,.$$

If $\det A_j = 1$ *for each* j, *the above holds for a polynomial automorphism of* $\mathbf{C}^n$ *with Jacobian one.*

Before proving Proposition 2.1 we introduce the following terminology.

Definition 2.4. (i) An A-jet of order $m \ge 1$ (at $0 \in \mathbf{C}^n$) is a holomorphic polynomial $P\colon \mathbf{C}^n \to \mathbf{C}^n$ of degree $\le m$ with $P(0) = 0$ and $JP(0) = \det P'(0) \ne 0$.

(ii) An A_1-jet of order $m \ge 1$ is an A-jet P such that $JP(z) = 1 + O(|z|^m)$ as $z \to 0$.

If $F\colon U \subset \mathbf{C}^n \to \mathbf{C}^n$ is a holomorphic map, $a \in U$, and $m \ge 1$ an integer, we write

$$F(z) = F(a) + F_{m,a}(z - a) + O\left(|z - a|^{m+1}\right)\,. \tag{2.2}$$

Thus, $F_{m,a}$ is the Taylor polynomial of F of order m at a without the constant term. If F is nondegenerate at a, $F_{m,a}$ is an A-jet which we call *the A-jet of F of order m at a.* If $JF(z) = 1$ for all $z \in U$, then $F_{m,a}$ is an A_1-jet for each $a \in U$.

Conversely, given an A_1-jet P of order m, Andersén and Lempert proved [2, Proposition 6.3] that there is an $F \in \mathrm{Aut}_1\,\mathbf{C}^n$ such that $F(z) = P(z) + O(|z|^{m+1})$ as $z \to 0$, i.e., $P = F_{m,0}$.

The following lemma is evident by composing the power series.

Lemma 2.5. *If F and G are holomorphic maps defined on open subsets of* $\mathbf{C}^n$, *with values in* $\mathbf{C}^n$, *then for each integer* $m \ge 1$ *we have*

$$(G \circ F)_{m,a}(z) = G_{m,F(a)} \circ F_{m,a}(z) + O\left(|z|^{m+1}\right), \quad z \to 0 \tag{2.3}$$

at each point a where the composition $G \circ F$ is defined.

This generalizes to composition of several maps, and we also have

$$\left(F^{-1}\right)_{m,F(a)} \circ F_{m,a}(z) = z + O\left(|z|^{m+1}\right), \quad z \to 0 . \tag{2.4}$$

In the proof of Proposition 2.1 we shall use special automorphisms of $\mathbf{C}^n$, shears and generalized shears. These have been used extensively in the literature. Let $v \in \mathbf{C}^n$ be a vector of length one, let $\lambda: \mathbf{C}^n \to \mathbf{C}$ be a $\mathbf{C}$-linear form satisfying $\lambda(v) = 0$, and let $f: \mathbf{C} \to \mathbf{C}$ be a holomorphic function of one variable. Set $\langle z, v\rangle = \sum_{j=1}^{n} z_j \bar{v}_j$. The family of maps $F_t: \mathbf{C}^n \to \mathbf{C}^n$ ($t \in \mathbf{C}$),

$$F_t(z) = z + tf(\lambda(z))v, \qquad z \in \mathbf{C}^n , \tag{2.5}$$

is a complex one-parameter subgroup of $\mathrm{Aut}_1\, \mathbf{C}^n$ with infinitesimal generator

$$V(z) = \frac{\partial}{\partial t} F_t(z)|_{t=0} = f(\lambda(z))v . \tag{2.6}$$

Each automorphism of type (2.5) (for a fixed t) is called a *shear.* We have $JF_t \equiv 1$ for each t and $\operatorname{div} V = \sum_{j=1}^{n} \partial V_j / \partial z_j = 0$ (see [2] or the Appendix in [8]).

Observe that the shears F_t (2.5) are polynomial iff f is a polynomial, and every shear on $\mathbf{C}^n$ can be approximated by polynomial shears, uniformly on compacts in $\mathbf{C}^n$.

Similarly, the family of maps

$$G_t(z) = z + \left(e^{tf(\lambda(z))} - 1\right)\langle z, v\rangle v, \qquad z \in \mathbf{C}^n \tag{2.7}$$

for $t \in \mathbf{C}$ is a complex one-parameter subgroup of $\mathrm{Aut}\, \mathbf{C}^n$ with infinitesimal generator

$$W(z) = \frac{\partial}{\partial t} G_t(z)|_{t=0} = f(\lambda(z))\langle z, v\rangle v . \tag{2.8}$$

Every automorphism (2.7) is called a *generalized shear* [17] or an overshear [2]. In contrast to shears, generalized shears are never polynomial except when f is a constant. We have $\operatorname{div} W = f(\lambda(z))$ and $JG_t(z) = \exp(tf(\lambda z))$ (see [2] or [8]).

Conversely, every holomorphic vector field of the form (2.6) [resp. (2.8)] on $\mathbf{C}^n$ is complete, with the flow (2.5) [resp. (2.7)]. (Recall that the flow of a vector field V is the local solution $F_t(z)$ of $(\partial/\partial t)F_t(z)|_{t=0} = V(z)$, satisfying $F_0(z) = z$ for all z. V is complete on $\mathbf{C}^n$ if its flow $F_t(z)$ exists for all $t \in \mathbf{R}$ and $z \in \mathbf{C}^n$.)

We will need the following decomposition result which is due to Andersén [1] and Andersén and Lempert [2]. We denote by $(\mathbf{C}^n)^*$ the dual space of $\mathbf{C}^n$.

Lemma 2.6. *For each integer $k \geq 1$ there exist finitely many linear forms $\lambda_1, \lambda_2, \ldots, \lambda_r \in (\mathbf{C}^n)^*$ and vectors $v_1, v_2, \ldots, v_r \in \mathbf{C}^n$, with $\lambda_i(v_i) = 0$ and $|v_i| = 1$ for all i, such that every polynomial map $V: \mathbf{C}^n \to \mathbf{C}^n$ (which we think of as a vector field on $\mathbf{C}^n$) that is homogeneous of degree k is a finite sum*

$$V(z) = \sum_i c_i\, (\lambda_i(z))^k\, v_i + d_i\, (\lambda_i(z))^{k-1}\, \langle z, v_i\rangle\, v_i \tag{2.9}$$

for some $c_i, d_i \in \mathbf{C}$ (the λ_is and v_is may appear with repetition). If $\operatorname{div} V = 0$, *then (2.9) holds with $d_i = 0$ for all i. Moreover, we can choose the λ_is from any nonempty open subset of $(\mathbf{C}^n)^*$.*

Everything except the last statement is contained in [1] and [2, Proposition 3.9]. (See also the Appendix to [8].) To justify the last statement, we recall the main idea of the proof. By Lemma 5.7

in [1] we can choose for each $k_0 \in \mathbf{N}$ finitely many forms $\lambda_i \in (\mathbf{C}^n)^*$, $1 \le i \le r$, such that every homogeneous polynomial $P\colon \mathbf{C}^n \to \mathbf{C}$ of degree $k \le k_0$ has a decomposition $P(z) = \sum_{i=1}^r c_i(\lambda_i(z))^k$ for some $c_i \in \mathbf{C}$. From the proof given there it is clear that if we denote by r the number of all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbf{Z}_+)^n$ with total weight $|\alpha| = \alpha_1 + \cdots + \alpha_n = k$, the above holds for any set of r forms in "general position;" in particular, we can choose the λ_is from any nonempty subset of $(\mathbf{C}^n)^*$.

Applying this to div V (which is a homogeneous polynomial of degree $k-1$) we get $\operatorname{div} V(z) = \sum_i d_i(\lambda_i(z))^{k-1}$. Choose vectors $v_i \in \mathbf{C}^n$ with $|v_i| = 1$ and $\lambda_i(v_i) = 0$ for each i, and set $V_i(z) = d_i(\lambda_i(z))^{k-1}\langle z, v_i\rangle v_i$. Then $\operatorname{div} V_i = d_i(\lambda_i(z))^{k-1}$, so the field $\tilde V = V - \sum_i V_i$ has divergence zero. We then proceed as in [1] (see especially the remark on p. 232) to decompose $\tilde V$ into a finite sum $\tilde V(z) = \sum_i c_i(\lambda_i(z))^k v_i$ of divergence zero fields of type (2.6). [We may have to add more vectors v_i in the process, and this is why we must allow repetitions in (2.9).] For further details, we refer the reader to [1].

Lemma 2.6 can be applied to each homogeneous part of a polynomial vector field on $\mathbf{C}^n$ to decompose it into a finite sum of fields (2.6) and (2.8) whose flows consist of (generalized) shears. The following is an immediate consequence; see [13, Lemma 1.4].

Lemma 2.7. *Let V be an entire vector field on $\mathbf{C}^n$ ($n > 1$), let $K \subset \mathbf{C}^n$ be a compact set, and let $t > 0$. If the flow $F_\tau(z)$ of V exists for all $0 \le \tau \le t$ and for all points $z \in K$, then $F_t|_K$ is a uniform limit on K of compositions of shears and generalized shears. If* div $V = 0$, *then $F_t|_K$ is a limit of compositions of (polynomial) shears.*

Proof of Proposition 2.1. Set $K_0 = K$. Choose compact polynomially convex sets $K_1 \subset K_2 \subset K_3 \subset \mathbf{C}^n \backslash \{p, q\}$ and a number $\epsilon_0 > 0$ such that

$$K_0 \subset K_1, \quad \operatorname{dist}\big(K_j, \mathbf{C}^n \backslash K_{j+1}\big) > \epsilon_0, \quad j = 0, 1, 2\,.$$

Observe that condition (iii) in Proposition 2.1 will be satisfied if $\epsilon < \epsilon_0$ and

$$|F(z) - z| < \epsilon/2, \qquad z \in K_1\,. \tag{2.10}$$

Namely, if (2.10) holds, Rouché's theorem [6, p. 110] implies $F(K_1) \supset K$, and hence $F^{-1}(z) \in K_1$ for each $z \in K$. Setting $w = F^{-1}(z) \in K_1$ we have $|F^{-1}(z) - z| = |w - F(w)| < \epsilon/2$ by (2.10), and hence (iii) in Proposition 2.1 holds. Thus, it suffices to find an $F \in \operatorname{Aut}\mathbf{C}^n$ satisfying (i), (ii), and (2.10).

F will be constructed as a composition of four automorphisms:

$$F = H^{-1} \circ S \circ G \circ H \in \operatorname{Aut}\mathbf{C}^n\,. \tag{2.11}$$

Each of the automorphisms G, H, H^{-1}, and S will move points in K_2 for at most $\epsilon/8$, where $\epsilon < \epsilon_0$ is as above. Clearly this will imply (2.10).

The purpose of H is to move points p (resp. q) away from K to suitably chosen points p' (resp. q') such that there is an affine complex hyperplane $\Sigma \subset \mathbf{C}^n$ which contains p' and q' but does not intersect K_3. This step is not needed when K is convex (see [17, Section 1]).

The automorphism G is a shear which moves p' to q' within Σ, and it matches the identity to order N at each point $b_j = H(a_j)$, $j = 1, 2, \dots, s$.

The main step of the proof is construction of an automorphism S which fixes points q' and b_j for $1 \le j \le s$, it matches the identity to order N at each b_j, and its A-jet $S_{m,q'}$ at q' is chosen such that F (2.11) satisfies $F_{m,p} = P$. The construction of S is essentially that given (in the volume preserving case) by Andersén and Lempert in [2, Proposition 6.3], with necessary modifications to meet the additional requirements.

We now fill in the details of this scheme.

Step 2.8. Choose points $p_0,\ q_0 \in \mathbf{C}^n$ such that there exists an affine complex hyperplane in $\mathbf{C}^n$ which contains p_0 and q_0 and which does not intersect the set K_3. (If $p = q$ we choose $p_0 = q_0$.) Let $\eta > 0$ be sufficiently small so that the same property holds for any pair of points $p', q' \in \mathbf{C}^n$ with $|p' - p_0| < \eta$ and $|q' - q_0| < \eta$. We shall find an $H \in \mathrm{Aut}_1\, \mathbf{C}^n$ (a finite composition of shears) such that

(i) $|H(p) - p_0| < \epsilon$, $|H(q) - q_0| < \epsilon$, and
(ii) $|H(z) - z| + |H^{-1}(z) - z| < \epsilon/8$ for all $z \in K_2$.

It suffices to consider the case when the straight line segments $E_0 = \overline{pp_0}$ and $E_1 = \overline{qq_0}$ are disjoint and they do not intersect K_3. (If $p = q$, we only consider one segment.) The general case is then obtained by taking a finite composition of automorphisms obtained in special cases.

Let V be the vector field given by $V(z) = 0$ for z near K_3, $V(z) = p_0 - p$ for z near the segment E_0, and $V(z) = q_0 - q$ for z near the segment E_1. Since the set $L = K_3 \cup E_0 \cup E_1$ is polynomially convex [19], we can approximate V by polynomial holomorphic vector fields W on $\mathbf{C}^n$, uniformly in a neighborhood of L. Moreover, since V is a locally constant field and hence div $V = 0$, we claim that the approximating fields W can be chosen to have divergence zero. To see this we set

$$\Omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \ ,$$

and we associate to V the holomorphic $(n-1)$-form $\omega = V \rfloor \Omega$ (the contraction of Ω with V). Clearly this correspondence between holomorphic vector fields and holomorphic $(n-1)$-forms on $\mathbf{C}^n$ is an isomorphism. Then $\omega = 0$ near K_3 and $d\omega = (\mathrm{div}\, V)\Omega = 0$ near L. Since the sets E_0 and E_1 have small convex neighborhoods, it follows that $\omega = d\alpha$ for some holomorphic $(n-2)$-form α in a neighborhood of L such that $\alpha = 0$ near K_3.

We approximate α near L by a globally defined holomorphic $(n-2)$-form $\tilde{\alpha}$ with polynomial coefficients. Then $d\tilde{\alpha} = \tilde{\omega}$ is a closed holomorphic $(n-1)$-form on $\mathbf{C}^n$ which approximates ω near L. The polynomial vector field W defined by $\tilde{\omega} = W \rfloor \Omega$ has divergence zero and it approximates V near L.

The time-one map of W, which exists for each point z in a neighborhood of L, takes p approximately to p_0, q approximately to q_0, and it moves every point in a neighborhood of K_2 by as little as desired. By Lemma 2.7 this time-one map can be approximated by finite compositions of shears, uniformly in a neighborhood of L. This gives the desired automorphism H.

Step 2.9. Let H be as in Step 2.8, with $\eta > 0$ chosen sufficiently small such that the points $p' = H(p)$ and $q' = H(q)$ are contained in an affine complex hyperplane $\Sigma \subset \mathbf{C}^n \backslash K_3$. Set $b_j = H(a_j)$ for $j = 1, \ldots, s$. We shall find a polynomial shear G satisfying

(i) $G(p') = q'$,
(ii) $|G(z) - z| < \epsilon/8$ for all $z \in K_2$, and
(iii) $G(z) = z + O(|z - b_j|^{N+1})$ as $z \to b_j$ for $j = 1, \ldots, s$.

If $p = q$ (hence $p' = q'$), this is satisfied by the identity map and we go on to Step 2.10.

To obtain G, we first choose a linear form $\lambda: \mathbf{C}^n \to \mathbf{C}$ such that $\lambda(q'-p')=0$ and $\lambda(p') \notin \lambda(K_2)$. It suffices to choose λ such that $\ker\lambda$ is parallel to Σ. Choose a polynomial $f: \mathbf{C} \to \mathbf{C}$ such that $f(\lambda(p'))=1$, f vanishes to order N at points $\lambda(b_j)$ for $j=1,\dots,s$, and $|f(\zeta)(q'-p')| < \epsilon/8$ for each $\zeta \in \lambda(K_2)$. This is possible since the set $\lambda(K_2) \subset \mathbf{C}$ is polynomially convex. Then the shear

$$G(z) = z + f(\lambda(z))\left(q'-p'\right), \quad z \in \mathbf{C}^n$$

satisfies the required properties.

Step 2.10. We shall construct an $S \in \operatorname{Aut}\mathbf{C}^n$, a finite composition of shears and generalized shears, such that

(i) $S(q') = q'$,
(ii) $S(z) = z + O(|z-b_j|^{N+1})$ as $z \to b_j$ for $j = 1,\dots,s$,
(iii) $|S(z) - z| < \epsilon/8$ for all $z \in K_2$, and
(iv) the A-jet $S_{m,q'}$ of S at q' is determined by

$$S_{m,q'}(z) = H_{m,q} \circ P \circ \left(H^{-1}\right)_{m,p'} \circ \left(G^{-1}\right)_{m,q'}(z) + O\left(|z|^{m+1}\right), \quad z \to 0.$$

The automorphism F (2.11) then clearly satisfies conditions (ii) and (iii) in Proposition 2.1; condition (i) follows by Lemma 2.5 and the choice of $S_{m,q'}$.

We shall seek S as a composition $S = S_m \circ S_{m-1} \circ \cdots \circ S_1$, where each $S_k \in \operatorname{Aut}\mathbf{C}^n$ is a finite composition of (generalized) shears, and the S_ks will be determined by induction on k.

By a translation of coordinates we may assume that $q' = 0$. Fix a number $\eta > 0$ to be determined later. For every $k = 1, 2, \dots, m$ we require S_k to satisfy

$$\begin{aligned} &S_k(0) = 0\,, \\ &S_k(z) = z + O\left(|z-b_j|^{N+1}\right), \quad z \to b_j, \quad j = 1,\dots,s\,, \\ &|S_k(z) - z| < \eta, \quad z \in K_3\,. \end{aligned} \tag{2.12}$$

Let $Q = S_{m,q'}$ be the A-jet determined by condition (iv) above. Write $Q(z) = Q_1(z) + O(|z|^2)$ as $z \to 0$, so Q_1 is the linear part of Q. For $k = 1$ we shall find an automorphism S_1 satisfying (2.12) and such that $S_1(z) = Q_1(z) + O(|z|^2)$ as $z \to 0$. Then

$$Q \circ S_1^{-1}(z) = z + Q_2(z) + O\left(|z|^3\right), \quad z \to 0\,,$$

where $Q_2: \mathbf{C}^n \to \mathbf{C}^n$ is homogeneous of degree 2. Next we shall find $S_2 \in \operatorname{Aut}\mathbf{C}^n$, satisfying (2.12) and

$$S_2(z) = z + Q_2(z) + O\left(|z|^3\right), \quad z \to 0\,.$$

Then

$$Q \circ S_1^{-1} \circ S_2^{-1}(z) = z + Q_3(z) + O\left(|z|^4\right), \quad z \to 0\,,$$

where Q_3 is homogeneous of degree 3. Continuing this way we obtain after m steps a desired automorphism $S = S_m \circ S_{m-1} \circ \cdots \circ S_1$ satisfying conditions (i)–(iv) above.

Construction of S_1. Let $e_1, e_2, \dots, e_n$ be any complex basis of $\mathbf{C}^n$, with the dual basis $\lambda_1,\dots,\lambda_n$ of $(\mathbf{C}^n)^*$, so that $z = \sum_{j=1}^n \lambda_j(z)e_j$ for all $z \in \mathbf{C}^n$. Then the group $SL(n,\mathbf{C})$ is generated by linear shears

$$z \mapsto z + \alpha\lambda_j(z)\,e_k, \qquad 1 \le j \ne k \le n, \ \alpha \in \mathbf{C}\,. \tag{2.13}$$

(See [16, p. 357].) By Step 2.8 we can choose our basis $\{e_j\}$ such that $0 \notin \lambda_j(K_3)$ for $j = 1, 2, \dots, n$. In order to satisfy (2.12) we interpolate every shear (2.13) at the origin by a polynomial shear of the form

$$z \mapsto z + f\left(\lambda_j(z)\right) e_k ,$$

where f is a polynomial on $\mathbf{C}$ with $f(\zeta) = \alpha\zeta + O(|\zeta|^2)$ as $\zeta \to 0$, f vanishes to order N at all points $\lambda_j(b_l)$ for $l = 1, 2, \dots, s$, and $|f|$ is small on the set $\lambda_j(K_3) \subset \mathbf{C}\backslash\{0\}$. A suitable composition of such shears will give a desired S_1 in the case when $S_1'(0) = Q_1 \in SL(n, \mathbf{C})$.

In the general case when $Q_1 \in GL(n, \mathbf{C})$ we assume that $|e_2| = 1$, and we let S_0 be a generalized shear of the form (2.7):

$$S_0(z) = z + \left(e^{f(\lambda_1(z))} - 1\right) \langle z, e_2\rangle\, e_2 ,$$

where $f\colon \mathbf{C} \to \mathbf{C}$ is a polynomial which vanishes to order N at all points $\lambda_1(b_l)$ ($l = 1, 2, \dots, s$), $|f|$ is small on $\lambda_1(K_3)$, and $e^{f(0)} = JQ(0)$. Since $JS_0(0) = e^{f(0)}$ (see the Appendix in [8]), we have $JS_0(0) = JQ(0)$. The map $Q \circ S_0^{-1}$ has Jacobian one at the origin and hence we are back in the previous case. If we choose $\tilde{S}_1$ as in the special case above such that $\tilde{S}_1$ satisfies (2.12) and its derivative at $0 \in \mathbf{C}^n$ matches that of $Q \circ S_0^{-1}$, then $S_1 = \tilde{S}_1 \circ S_0$ satisfies all required conditions.

The inductive step. Suppose that $k \geq 2$ and we have constructed automorphisms $S_1, \dots, S_{k-1}$, satisfying (2.12), and such that

$$Q \circ S_1^{-1} \circ \cdots \circ S_{k-1}^{-1}(z) = z + V(z) + O\left(|z|^{k+1}\right), \quad z \to 0 \tag{2.14}$$

for some homogeneous polynomial map $V\colon \mathbf{C}^n \to \mathbf{C}^n$ of degree k. We will now construct the next automorphism S_k of $\mathbf{C}^n$, satisfying (2.12), such that

$$S_k(z) = z + V(z) + O\left(|z|^{k+1}\right), \quad z \to 0 . \tag{2.15}$$

From (2.14) and (2.15) it will follow that

$$Q \circ S_1^{-1} \circ \cdots \circ S_k^{-1}(z) = z + O\left(|z|^{k+1}\right), \quad z \to 0 .$$

This will finish the inductive step in the construction of S.

To construct S_k we apply Lemma 2.6 to decompose V in a finite sum

$$V(z) = \sum_j c_j \left(\lambda_j(z)\right)^k v_j + d_j \left(\lambda_j(z)\right)^{k-1} \langle z, v_j\rangle\, v_j \tag{2.16}$$

for some constants $c_j, d_j \in \mathbf{C}$, where the linear forms λ_j are chosen so that $0 \notin \lambda_j(K_3)$ for all j. This is possible since by Step 2.9 there is a complex hyperplane $\Sigma \subset \mathbf{C}^n\backslash K_3$ containing the point $q' = 0$; it suffices to choose λ_j such that $\ker \lambda_j$ is sufficiently close to Σ for each j. If $\operatorname{div} V = 0$, (2.16) holds with $d_j = 0$ for all j.

For each j we choose holomorphic polynomials $f_j, g_j\colon \mathbf{C} \to \mathbf{C}$ which are close to 0 on $\lambda_j(K_3)$, which vanish to order N at points $\lambda_j(b_l)$ for $l = 1, 2, \dots, s$, and which satisfy

$$f_j(\zeta) = c_j\zeta^k + O\left(|\zeta|^{k+1}\right), \quad g_j(\zeta) = d_j\zeta^{k-1} + O\left(|\zeta|^k\right), \quad \zeta \to 0 .$$

We then set

$$\Phi_j(z) \quad = \quad z + f_j\left(\lambda_j(z)\right) v_j$$

$$
\begin{aligned}
&= z + c_j\left(\lambda_j(z)\right)^k v_j + O\left(|z|^{k+1}\right), \quad z \to 0\,, \\
\Psi_j(z) &= z + \left(e^{g_j(\lambda_j(z))} - 1\right)\langle z, v_j\rangle v_j \\
&= z + d_j\left(\lambda_j(z)\right)^{k-1}\langle z, v_j\rangle v_j + O\left(|z|^{k+1}\right), \quad z \to 0\,.
\end{aligned}
$$

These are the time-one maps of vector fields $f_j(\lambda_j z)v_j$ (resp. $g_j(\lambda_j(z))\langle z, v_j\rangle v_j$) which appear in (2.16). Take S_k to be the composition of all maps Φ_j and Ψ_j (in any order). Note that the Taylor expansion of each of these maps begins with z+homogeneous terms of order k. It is immediate that, when composing such maps, their homogeneous parts of degree k add up, and we get

$$
S_k(z) = z + V(z) + O\left(|z|^{k+1}\right), \quad z \to 0\,.
$$

This finishes the inductive step and concludes the construction of S.

Since each of the m automorphism S_k moves points of K_3 by less than η, we can achieve (by choosing $\eta > 0$ sufficiently small) that their composition S moves points of K_2 by less than $\epsilon/8$ as required by (iii). This completes the proof of Proposition 2.1 under assumption (e).

Suppose now that (e') holds. Recall that H and G are finite compositions of polynomial shears. Since the A-jet $Q = S_{m,q'}$ is conjugate to the A_1-jet P by a volume preserving automorphism, it is itself an A_1-jet of order m. Thus, $JQ(0) = 1$ and hence S_1 constructed above is volume preserving. Assuming inductively that map S_l is volume preserving for $1 \le l \le k-1$, it follows from chain rule that the Jacobian of the map in (2.14) agrees with 1 to order m at the origin. This implies

$$
1 + O\left(|z|^m\right) = \det\left(I + V'(z) + O\left(|z|^k\right)\right) = 1 + \operatorname{div} V(z) + O\left(|z|^k\right), \quad z \to 0\,.
$$

Since div V is homogeneous of degree $k-1$, we get div $V = 0$. Consequently, in the decomposition (2.16) of V, we only need vector fields of the first type whose time-one maps Φ_j are shears. This shows that each S_k for $k = 1, 2, \ldots, m$ is a finite composition of shears, and hence, the same is true for the maps S and F. This completes the proof of Proposition 2.1. □

3. Jet interpolation by nowhere degenerate holomorphic maps on $\mathbf{C}^n$

To motivate the discussion we recall the following classical interpolation theorem for entire functions: *Given a discrete set $\{a_j\} \subset \mathbf{C}^n$ and for each j a holomorphic polynomial P_j of degree at most m_j, there exists an entire function $F\colon \mathbf{C}^n \to \mathbf{C}$ satisfying*

$$
F(z) = P_j\left(z - a_j\right) + O\left(\left|z - a_j\right|^{m_j+1}\right), \quad z \to a_j, \quad j = 1, 2, 3, \ldots
$$

(See [15, Corollary 1.5.4] for $n = 1$.) In particular, one can prescribe values of an entire function at any discrete set of points.

This result extends to holomorphic mappings $F\colon \mathbf{C}^n \to \mathbf{C}^n$ by applying it to each component F_j of F. However, the problem becomes much harder if we require in addition that $JF(z) \ne 0$ for all $z \in \mathbf{C}^n$. The following interpolation theorem is due to Rosay and Rudin.

Theorem (Rosay and Rudin [17, Theorem 1.1]). *Let $n > 1$. Given a discrete sequence $\{a_j\} \subset \mathbf{C}^n$ (without repetition) and an arbitrary sequence $\{b_j\} \subset \mathbf{C}^n$, there is a holomorphic map $F\colon \mathbf{C}^n \to \mathbf{C}^n$ such that $F(a_j) = b_j$ for each j and $JF(z) = 1$ for each $z \in \mathbf{C}^n$.*

Our main result in this section is the following.

Theorem 3.1. *Let $n > 1$. Assume that $\{a_j\}$ is a discrete sequence in $\mathbf{C}^n$ (without repetition), that $\{b_j\}$ is an arbitrary sequence in $\mathbf{C}^n$, and that $P_j: \mathbf{C}^n \to \mathbf{C}^n$ is a holomorphic polynomial map of degree at most $m_j \geq 1$ with $P_j(0) = 0$ and $JP_j(0) \neq 0$. Then there exists a nowhere degenerate holomorphic mapping $F: \mathbf{C}^n \to \mathbf{C}^n$ such that for every $j = 1, 2, 3, \dots$ we have $F(a_j) = b_j$ and*

$$F(z) = b_j + P_j\left(z - a_j\right) + O\left(\left|z - a_j\right|^{m_j+1}\right), \quad z \to a_j . \tag{3.1}$$

If in addition the polynomials P_j satisfy

$$JP_j(z) = 1 + O\left(|z|^{m_j}\right), \quad z \to 0, \qquad j = 1, 2, 3, \dots , \tag{3.2}$$

there is an F as above satisfying $JF \equiv 1$.

Corollary 3.2. *Let $n > 1$. Assume that $\{a_j\}$ is a discrete sequence in $\mathbf{C}^n$ (without repetition), that $\{b_j\}$ is an arbitrary sequence in $\mathbf{C}^n$, and that $A_j \in GL(n, \mathbf{C})$ for each $j \in \mathbf{N}$. Then there exists a nowhere degenerate holomorphic map $F: \mathbf{C}^n \to \mathbf{C}^n$ satisfying*

$$F\left(a_j\right) = b_j, \quad F'\left(a_j\right) = A_j, \qquad j = 1, 2, 3, \dots$$

If $\det A_j = 1$ *for each j, there is an F as above such that $JF(z) = 1$ for all $z \in \mathbf{C}^n$.*

It is easy to see that condition (3.2) on P_j is necessary for the existence of F with Jacobian one satisfying (3.1).

Proof of Theorem 3.1. We follow the scheme of the proof of [17, Theorem 1.1], replacing in [17, Corollary 1.3] by our Proposition 2.1. We first choose the origin of $\mathbf{C}^n$ so that $0 < |a_1| < |a_2| < |a_3| < \dots$ and then choose coordinate axes so that the hyperplane $\{z_1 = 0\}$ contains none of the points b_j. Denote by $\pi_j: \mathbf{C}^n \to \mathbf{C}$ the jth coordinate projection. Let $E: \mathbf{C}^n \to \mathbf{C}^* \times \mathbf{C}^{n-1}$ be the entire map

$$E\left(z_1, z_2, \dots, z_n\right) = \left(e^{z_1}, z_2 e^{-z_1}, z_3, \dots, z_n\right) .$$

Clearly E is a holomorphic covering projection onto $\mathbf{C}^* \times \mathbf{C}^{n-1}$ with $JE \equiv 1$. In particular, every $z \in \mathbf{C}^n$ with $\pi_1(z) \neq 0$ belongs to the range of E, and one can find points $w \in \mathbf{C}^n$ such that $E(w) = z$ and $|\pi_1(w)|$ is as large as desired.

We will find F as a composition $F = E \circ G$, where E is as above and

$$G = \lim_{k \to \infty} G_k, \qquad G_k = \Psi_1 \circ \Psi_2 \circ \cdots \circ \Psi_k$$

is a limit of certain sequences of compositions of (generalized) shears on $\mathbf{C}^n$. Since $JE \equiv 1$, we have $JF(z) = JG(z)$ for all $z \in \mathbf{C}^n$. The automorphisms G_k will be constructed inductively by choosing at each step a suitable automorphism $\Psi_k \in \operatorname{Aut} \mathbf{C}^n$ and setting $G_k = G_{k-1} \circ \Psi_k$. In the case when all P_js are A_1-jets, all maps Ψ_k and G_k will be compositions of shears (with Jacobian one).

We begin by setting $G_0(z) = z$. Suppose that $k \geq 1$ and $G_{k-1} \in \operatorname{Aut} \mathbf{C}^n$ has been chosen in such a way that the map $F_{k-1} = E \circ G_{k-1}$ satisfies (3.1) for $j = 1, 2, \dots, k-1$. Moreover, if all P_js are A_1-jets, we may assume in addition that $JG_{k-1} \equiv 1$ and hence $JF_{k-1} \equiv 1$.

Choose $v_k \in \mathbf{C}^n$ such that $E(v_k) = b_k$ and $|\pi_1(v_k)|$ is so large that v_k lies outside the compact set $G_{k-1}(r_k \overline{B})$, where $r_k = |a_k|$ and $B \subset \mathbf{C}^n$ is the unit ball centered at 0. Thus, there is a (unique) $q_k \in \mathbf{C}^n$ such that $G_{k-1}(q_k) = v_k$ and $|q_k| > r_k$. Then $F_{k-1}(q_k) = b_k$.

Since the map $F_{k-1} = E \circ G_{k-1}$ is locally invertible, Lemma 2.5 provides a unique A-jet $\tilde{P}_k$ of degree m_k such that

$$(F_{k-1})_{m_k,q_k} \circ \tilde{P}_k(z) = P_k(z) + O\left(|z|^{m_k+1}\right), \quad z \to 0,$$

where P_k is as in Theorem 3.1. If P_k is an A_1-jet and $JF_{k-1} \equiv 1$, then $\tilde{P}_k$ is also an A_1-jet. Choose a δ_k, $0 < \delta_k < r_k - r_{k-1}$, such that

$$|G_{k-1}(z) - G_{k-1}(w)| < 2^{-k} \tag{3.3}$$

for all $z, w \in r_k\overline{B}$ with $|z - w| < \delta_k$. Proposition 2.1 gives us a $\Psi_k \in \operatorname{Aut}\mathbf{C}^n$, a finite composition of (generalized) shears, such that

$$\begin{aligned} &\Psi_k(z) = z + O\left(|z-a_j|^{m_j+1}\right), \quad z \to a_j, \quad 1 \le j \le k-1, \\ &\Psi_k(z) = q_k + \tilde{P}_k(z - a_k) + O\left(|z-a_k|^{m_k+1}\right), \quad z \to a_k, \\ &|\Psi_k(z) - z| < \delta_k, \qquad z \in r_{k-1}\overline{B}. \end{aligned} \tag{3.4}$$

Set $G_k = G_{k-1} \circ \Psi_k$ and $F_k = E \circ G_k$. Then F_k satisfies (3.1) for $j = 1, 2, \ldots, k$. Moreover, (3.3) and the last line in (3.4) imply

$$|G_k(z) - G_{k-1}(z)| < 2^{-k}, \qquad |z| \le r_{k-1}.$$

It follows that $G = \lim_{k\to\infty} G_k$ exists uniformly on compacts in $\mathbf{C}^n$. By construction we have $JG(a_j) \ne 0$ for each $j = 1, 2, 3, \ldots$. Hence, $JG(z) \ne 0$ for all $z \in \mathbf{C}^n$ and G is one-to-one. The map $F = E \circ G\colon \mathbf{C}^n \to \mathbf{C}^n$ then satisfies Theorem 3.1. $\square$

4. Jet interpolation by injective maps and automorphisms

The main result in this section is the following jet-interpolation theorem for automorphisms of $\mathbf{C}^n$.

Theorem 4.1. *Assume that $n > 1$, $m \ge 1$, and that for each $j = 1, 2, 3, \ldots$, $P_j\colon \mathbf{C}^n \to \mathbf{C}^n$ is a holomorphic polynomial map of degree at most m with $P_j(0) = 0$ and $JP_j(0) \ne 0$. Let $e_1 = (1, 0, \ldots, 0) \in \mathbf{C}^n$. Then there exists an $F \in \operatorname{Aut}\mathbf{C}^n$, a finite composition of shears and generalized shears, such that for each $j \in \mathbf{N}$ we have $F(je_1) = je_1$ and*

$$F(z) = je_1 + P_j(z - je_1) + O\left(|z - je_1|^{m+1}\right), \quad z \to je_1. \tag{4.1}$$

If in addition the polynomial maps P_j satisfy

$$JP_j(z) = 1 + O\left(|z|^m\right), \quad z \to 0, \qquad j = 1, 2, 3, \ldots, \tag{4.2}$$

then (4.1) holds for an $F \in \operatorname{Aut}_1\mathbf{C}^n$ which is a finite composition of shears. Moreover, if $R > 0$ and $P_j(z) = z$ for $j \le R$, we can choose F such that

$$|F(z) - z| + \left|F^{-1}(z) - z\right| < \epsilon, \qquad |z| \le R. \tag{4.3}$$

Remark. In a subsequent paper with Buzzard [5] we proved that Theorem 4.1 remains valid even if the degrees $m_j = \deg P_j$ form an unbounded sequence, i.e., (4.1) and (4.2) hold with m replaced by m_j for each $j = 1, 2, 3, \ldots$.

We now consider the jet-interpolation problem for more general discrete sequences in $\mathbf{C}^n$. In doing so we must remember that, in general, one cannot map an infinite discrete set in $\mathbf{C}^n$ onto another discrete set by an automorphism of $\mathbf{C}^n$ [17]. In fact, the infinite discrete sets in $\mathbf{C}^n$ form uncountably many different equivalence classes under this relation [17, Section 5]. Rosay and Rudin introduced the following notion:

Definition. A discrete infinite sequence $\{a_j\} \subset \mathbf{C}^n$ (without repetition) is *tame* if there exists a holomorphic automorphism $\Phi \in \operatorname{Aut} \mathbf{C}^n$ such that

$$\Phi\left(a_j\right) = je_1 = (j, 0, \ldots, 0), \qquad j = 1, 2, 3, \ldots$$

The sequence $\{a_j\}$ is *very tame* if the above holds for a $\Phi \in \operatorname{Aut}_1 \mathbf{C}^n$.

Rosay and Rudin [17] found several geometric conditions which imply that a given sequence is tame (resp. very tame). They showed that members of every tame sequence are permutable by automorphisms of $\mathbf{C}^n$, and hence one can speak about tameness of infinite discrete sets. On the other hand, there exist discrete sets $E \subset \mathbf{C}^n$ such that the only nondegenerate holomorphic map $F: \mathbf{C}^n \to \mathbf{C}^n$ satisfying $F(\mathbf{C}^n \backslash E) \subset \mathbf{C}^n \backslash E$ is the identity map [17, Theorem 5.1].

From the definition of (very) tame sequences and Theorem 4.1 we get the following corollaries.

Corollary 4.2. *Assume that $n > 1$ and $m \geq 1$, that $\{a_j\}$ and $\{b_j\}$ are tame discrete sequences in $\mathbf{C}^n$ (without repetition), and that for each $j = 1, 2, 3, \ldots$, $P_j: \mathbf{C}^n \to \mathbf{C}^n$ is a holomorphic polynomial map of degree at most m with $P_j(0) = 0$ and $JP_j(0) \neq 0$. Then there exists an $F \in \operatorname{Aut} \mathbf{C}^n$ such that for each $j \in \mathbf{N}$, $F(a_j) = b_j$ and*

$$F(z) = b_j + P_j\left(z - a_j\right) + O\left(\left|z - a_j\right|^{m+1}\right), \quad z \to a_j .$$

If the sequences $\{a_j\}$, $\{b_j\}$ are very tame and all polynomials P_j satisfy (4.2), we can choose $F \in \operatorname{Aut}_1 \mathbf{C}^n$.

Corollary 4.3. *Assume that $n > 1$, that $\{a_j\}$ and $\{b_j\}$ are tame discrete sequences in $\mathbf{C}^n$ (without repetition), and that $A_j \in GL(n, \mathbf{C})$ for each $j \in \mathbf{N}$. Then there exists an $F \in \operatorname{Aut} \mathbf{C}^n$ satisfying*

$$F\left(a_j\right) = b_j, \quad F'\left(a_j\right) = A_j, \qquad j = 1, 2, 3, \ldots$$

If the sequences $\{a_j\}$, $\{b_j\}$ are very tame and $\det A_j = 1$ for each j, the above is satisfied by an $F \in \operatorname{Aut}_1 \mathbf{C}^n$.

Rosay and Rudin proved in [17, Theorem 3.7] that for every infinite discrete sequence $\{a_j\} \subset \mathbf{C}^n$ (not necessarily tame), there is an injective holomorphic map $F: \mathbf{C}^n \to \mathbf{C}^n$ with $JF \equiv 1$, satisfying $F(a_j) = je_1$ for $j = 1, 2, 3, \ldots$. Together with Theorem 4.1 we get the following:

Corollary 4.4. *Let $n > 1$ and $m \geq 1$. Assume that $\{a_j\}$ is a discrete sequence in $\mathbf{C}^n$ (without repetition), and for each $j \in \mathbf{N}$, P_j is a holomorphic polynomial of degree at most m with $P_j(0) = 0$ and $JP_j(0) \neq 0$. Then there exists an injective holomorphic map $F: \mathbf{C}^n \to \mathbf{C}^n$ such that for each $j = 1, 2, 3, \ldots$, $F(a_j) = je_1 = (j, 0, \ldots, 0)$ and*

$$F(z) = je_1 + P\left(z - a_j\right) + O\left(\left|z - a_j\right|^{m+1}\right), \quad z \to a_j .$$

Proof of Theorem 4.1. The proof is similar to Step 2.10 in the proof of Proposition 2.1, except that we now perform the same operation simultaneously at all points $je_1 \in \mathbf{C}^n$ $(j = 1, 2, 3, \ldots)$ at every step of the process. We will find F as a finite composition

$$F = G_m \circ G_{m-1} \circ \cdots \circ G_1 \circ G_0 ,$$

where each G_k will be a finite composition of shears and generalized shears which will be constructed inductively. The number of composition factors in each G_k will depend only on n and k. Hence, to satisfy (4.3), it suffices to choose a slightly larger number $R' > R$ and ensure that each composition factor is sufficiently close to the identity on the ball $R'B$. We choose R' such that $j > R$ implies $j > R'$ for every integer j.

We begin the construction by setting

$$G_0(z) = \left(z_1, e^{f(z_1)} z_2, z_3, \ldots, z_n\right) ,$$

where $f\colon \mathbf{C} \to \mathbf{C}$ is an entire function satisfying

$$e^{f(j)} = JP_j(0), \qquad j = 1, 2, 3, \ldots ,$$

and such that $|f(\zeta)|$ is small for $|\zeta| \le R'$. Then $G_0(je_1) = je_1$ and $JG_0(je_1) = JP_j(0)$ for all j, and G_0 is close to the identity on $R'B$. Replacing F by $F \circ G_0^{-1}$ we reduce our problem to the case when $JP_j(0) = 1$ for all j.

Next we find an automorphism G_1 which fixes all points je_1 ($j \in \mathbf{N}$) and satisfies

$$G_1'(je_1) = P_j'(0) \in SL(n, \mathbf{C}), \qquad j = 1, 2, 3, \ldots .$$

Choose a basis $\lambda_1, \ldots, \lambda_n$ of the dual space $(\mathbf{C}^n)^*$ such that for each i, $\lambda_i(e_1) = 1$ and $\ker \lambda_i$ is almost orthogonal to the first coordinate axis so that $\lambda_i(je_1) = j \notin \lambda_i(R'\overline{B})$ whenever $j > R$. Let $v_1, \ldots, v_n$ be the dual basis of $\mathbf{C}^n$ satisfying $\lambda_i(v_k) = \delta_{i,k}$. Recall [16, p. 357] that the group $SL(n, \mathbf{C})$ is generated by linear shears

$$z \mapsto z + \alpha \lambda_i(z) v_k, \qquad i \ne k,\ \alpha \in \mathbf{C} . \tag{4.4}$$

Fix $i \ne k$, $1 \le i, k \le n$. Suppose that we are given for each $j = 1, 2, 3, \ldots$ a constant $\alpha_j \in \mathbf{C}$, with $\alpha_j = 0$ for $1 \le j \le R$. By the choice of λ_i we can find an entire function f on $\mathbf{C}$ satisfying $f(j) = 0$ and $f'(j) = \alpha_j$ for each j, and such that $|f(\zeta)|$ is small for $\zeta \in \lambda_i(R'B)$. The shear automorphism

$$z \mapsto z + f\left(\lambda_i(z)\right) v_k, \qquad z \in \mathbf{C}^n$$

then fixes all points je_1, its derivative at je_1 is the linear shear (4.4) with $\alpha = \alpha_j$, and it is close to the identity on $R'B$. Since $P_j'(0) \in SL(n, \mathbf{C})$ is a finite composition of linear shears (4.4) for each j, with the number of factors depending only on n, a finite composition of such shears gives G_1.

Suppose inductively that $1 < k \le m$ and that we have already constructed automorphisms $G_0, G_1, \ldots, G_{k-1}$ whose composition $F_{k-1} = G_{k-1} \circ \cdots \circ G_1 \circ G_0$ fixes all points je_1, it is close to the identity near $R\overline{B}$, and it satisfies for each $j = 1, 2, 3, \ldots$

$$P_j \circ \left(F_{k-1}^{-1}\right)_{k, je_1}(z) = z + V_j(z) + O\left(|z|^{k+1}\right), \quad z \to 0 , \tag{4.5}$$

where $V_j\colon \mathbf{C}^n \to \mathbf{C}^n$ is a homogeneous polynomial map of degree k for each $j \in \mathbf{N}$. (In (4.5) we are using the A-jet notation introduced in Section 2.) If we find G_k, a composition of finitely many (generalized) shears, which is close to the identity near $R\overline{B}$, which fixes all points je_1 ($j \in \mathbf{N}$) and satisfies for each $j = 1, 2, 3, \ldots$

$$G_k(z) = z + V_j\left(z - je_1\right) + O\left(|z - je_1|^{k+1}\right), \quad z \to je_1 , \tag{4.6}$$

then the map $F_k = G_k \circ F_{k-1}$ satisfies the same requirements, with $k-1$ replaced by k. After m steps we thus obtain the desired automorphism $F = F_m$.

The construction of G_k is similar to the construction of S_k in Section 2 above. We begin by choosing finitely many linear forms $\lambda_i \in (\mathbf{C}^n)^*$ and vectors $v_i \in \ker \lambda_i$ of length one $(1 \le i \le r)$, possibly with repetition, such that for each i we have $\lambda_i(e_1) = 1$, $j \notin \lambda_i(R'\overline{B})$ for $j > R$ (this holds if $\ker \lambda_i$ is nearly orthogonal to the first coordinate axis), and such that Lemma 2.6 holds. Applying Lemma 2.6 to V_j for $j = 1, 2, 3, \ldots$ we get

$$V_j(z) = \sum_{i=1}^{r} c_{i,j}\,(\lambda_i(z))^k\ v_i + d_{i,j}\,(\lambda_i(z))^{k-1}\,\langle z, v_i\rangle\, v_i$$

for some constants $c_{i,j}, d_{i,j} \in \mathbf{C}$. Note that $V_j = 0$ for $j \le R$ and hence $c_{i,j} = d_{i,j} = 0$ for $j \le R$. Choose entire functions $f_i, g_i \colon \mathbf{C} \to \mathbf{C}$ $(1 \le i \le r)$ satisfying

$$\begin{aligned} f_i(\zeta) &= c_{i,j}(\zeta - j)^k + O\left(|\zeta - j|^{k+1}\right), \quad \zeta \to j\,, \\ g_i(\zeta) &= d_{i,j}(\zeta - j)^{k-1} + O\left(|\zeta - j|^{k}\right), \quad \zeta \to j \end{aligned}$$

for each $i = 1, \ldots, r$ and $j = 1, 2, 3, \ldots$, and such that $|f_i|$ and $|g_i|$ are small on $\lambda_i(R'B)$. As in Section 2 we set

$$\begin{aligned} \Phi_i(z) &= z + f_i\,(\lambda_i(z))\, v_i\,, \\ \Psi_i(z) &= z + \left(e^{g_i(\lambda_i(z))} - 1\right)\langle z, v_i\rangle\, v_i\,, \end{aligned}$$

and we let G_k be the composition of all Φ_is and Ψ_is for $1 \le i \le r$. Then G_k satisfies (4.6). This completes the proof of Theorem 4.1. $\square$

5. Convergence of certain compositions of automorphisms

The situation described in the next proposition arises naturally in the construction of proper holomorphic embeddings in Section 6 below. This construction has appeared in several recent papers, most notably in the paper [14] by Globevnik and Stensønes where they constructed proper holomorphic embeddings of certain finitely connected planar domains into $\mathbf{C}^2$, and more recently in the paper [4] by Buzzard and Fornaess.

Proposition 5.1. *Let D be a connected open set in $\mathbf{C}^n$ which is exhausted by compact sets $K_0 \subset K_1 \subset K_2 \subset \cdots \subset \cup_{j=0}^{\infty} K_j = D$ such that $K_{j-1} \subset \mathrm{Int} K_j$ for each $j \in \mathbf{N}$. Choose numbers ϵ_j $(j = 1, 2, 3, \ldots)$ such that*

$$0 < \epsilon_j < \mathrm{dist}\left(K_{j-1}, \mathbf{C}^n \backslash K_j\right) \ \ (j \in \mathbf{N}), \qquad \sum_{j=1}^{\infty} \epsilon_j < \infty\,. \tag{5.1}$$

Suppose that for each $j = 1, 2, 3, \ldots$, Ψ_j is a holomorphic automorphism of $\mathbf{C}^n$ satisfying

$$\left|\Psi_j(z) - z\right| < \epsilon_j, \quad z \in K_j\,. \tag{5.2}$$

Set $\Phi_m = \Psi_m \circ \Psi_{m-1} \circ \cdots \circ \Psi_1$. Then there is an open set $\Omega \subset \mathbf{C}^n$ such that $\lim_{m\to\infty} \Phi_m = \Phi$ exists on Ω (uniformly on compacts), and Φ is a biholomorphic map of Ω onto D. In fact, $\Omega = \cup_{m=1}^{\infty} \Phi_m^{-1}(K_m)$.

The following special case deserves to be stated separately.

Proposition 5.2. *Let $K_0 \subset K_1 \subset K_2 \subset \cdots \subset \cup_{j=0}^{\infty} K_j = \mathbf{C}^n$ be compact sets such that $K_{j-1} \subset \mathrm{Int} K_j$ for each $j \in \mathbf{N}$. Let $\epsilon_j > 0$, Ψ_j and Φ_j be as in Proposition 5.1, satisfying (5.1) and (5.2). Let $\Omega \subset \mathbf{C}^n$ consist of all points $z \in \mathbf{C}^n$ such that the sequence $\{\Phi_m(z) : m \in \mathbf{N}\} \subset \mathbf{C}^n$ is bounded. Then $\lim_{m\to\infty} \Phi_m = \Phi$ exists on Ω (uniformly on compacts), and Φ is a biholomorphic map of Ω onto $\mathbf{C}^n$.*

Before proving Proposition 5.1, we give an application to the problem considered in the recent paper [12] by Forstneric et al. Since the additional ingredient developed here is rather minor in comparison with the construction in [12], the following should be considered a joint result with Globevnik and Stensønes.

Theorem 5.3. *For each discrete set E in a connected pseudoconvex Runge domain $D \subset \mathbf{C}^n$ ($n > 1$) there exist a pseudoconvex Runge domain $\Omega \subset \mathbf{C}^n$, a biholomorphic map $F : \Omega \to D$ onto D, and a connected component A of $\Omega \cap (\mathbf{C} \times \{0\}^{n-1})$ such that $F(A) \supset E$.*

If Ω is Runge in $\mathbf{C}^n$ and Λ is an affine complex line in $\mathbf{C}^n$, then every connected component of $\Lambda \cap \Omega$ is Runge in Λ and hence conformally a disc or $\mathbf{C}$. Thus, Theorem 5.3 implies the following:

Corollary 5.4. *Every discrete set in a connected pseudoconvex Runge domain $D \subset \mathbf{C}^n$ ($n > 1$) is contained in one leaf of a nonsingular holomorphic foliation of D by complex discs and lines.*

To see this, take the image by F (from Theorem 5.3) of the foliation of Ω by translates of the first coordinate axis.

We indicate how Theorem 5.3 follows from the construction in [12] and from Proposition 5.1 above. Given a discrete set E in a connected pseudoconvex Runge domain $\Omega \subset \mathbf{C}^n$ ($n > 1$), the authors constructed in [12] a sequence of compact, polynomially convex sets $K_0 \subset K_1 \subset K_2 \subset \cdots \subset \cup_{j=0}^{\infty} K_j = D$, and a sequence of holomorphic automorphisms $\Psi_j \in \mathrm{Aut}\, \mathbf{C}^n$ satisfying (5.2), where $\epsilon_j > 0$ can be chosen arbitrarily small in each step, such that the sequence of compositions $F_m = \Psi_m \circ \Psi_{m-1} \circ \cdots \circ \Psi_1$ converges to a holomorphic map F on certain simply connected domains $A \subset \mathbf{C} \times \{0\}^{n-1}$ (contained in the first coordinate axis), and such that $F : A \to D$ is a proper holomorphic embedding of A into D whose image $F(A)$ contains the given discrete set E. (In [12] we used the notation $\triangle$ instead of A.)

If we choose in each step the number $\epsilon_j > 0$ sufficiently small such that (5.1) holds, Proposition 5.1 shows that the sequence F_m converges to a biholomorphic map F from a domain $\Omega \subset \mathbf{C}^n$ onto D, and by construction A is a connected component of $\Omega \cap (\mathbf{C} \times \{0\}^{n-1})$. Clearly Ω is pseudoconvex, and it is Runge in $\mathbf{C}^n$ since D is Runge and $F : \Omega \to D$ is a limit of automorphisms of $\mathbf{C}^n$. We refer the reader to [12] for further details.

Proof of Proposition 5.1. Set $\Phi_0(z) = z$ for $z \in \mathbf{C}^n$. Let

$$L_m = \Phi_m^{-1}(K_m) \quad (m \in \mathbf{N}), \qquad \Omega = \cup_{m=1}^{\infty} L_m \subset \mathbf{C}^n . \tag{5.3}$$

From (5.1) and (5.2) we get $\Psi_{m+1}(K_m) \subset \mathrm{Int} K_{m+1}$ ($m \geq 0$), and hence,

$$\Phi_{m+1}(L_m) = \Psi_{m+1}(\Phi_m(L_m)) = \Psi_{m+1}(K_m) \subset \mathrm{Int} K_{m+1} .$$

Thus, $L_m \subset \mathrm{Int} L_{m+1}$ for $m = 1, 2, 3, \ldots$, and hence Ω is open in $\mathbf{C}^n$. By induction we get

$$\Phi_j(L_m) \subset K_j, \qquad j \geq m .$$

Hence, (5.2) gives for $l > m \geq 1$ and $z \in L_m$:

$$\begin{aligned} |\Phi_l(z) - \Phi_m(z)| &\leq \sum_{j=m+1}^{l} \left|\Phi_j(z) - \Phi_{j-1}(z)\right| \\ &= \sum_{j=m+1}^{l} \left|\Psi_j\left(\Phi_{j-1}(z)\right) - \Phi_{j-1}(z)\right| \\ &< \sum_{j=m+1}^{l} \epsilon_j \,. \end{aligned} \tag{5.4}$$

This shows that $\lim_{l\to\infty} \Phi_l = \Phi$ exists on L_m and it satisfies

$$|\Phi(z) - \Phi_m(z)| \leq \sum_{j=m+1}^{\infty} \epsilon_j < \operatorname{dist}\left(K_m, \mathbf{C}^n \backslash D\right), \quad z \in L_m \,.$$

The last inequality follows from (5.1). Thus, $\Phi(L_m) \subset D$ for each $m \in \mathbf{N}$, and hence $\Phi(\Omega) \subset D$.

We claim that Φ is injective on Ω. Fix $m \in \mathbf{N}$. For $z \in L_m$ we write $\Phi_m(z) = w \in K_m$. If $l > m$, then (5.4) implies

$$\left|\Phi_l \circ \Phi_m^{-1}(w) - w\right| = |\Phi_l(z) - \Phi_m(z)| < \sum_{j=m+1}^{\infty} \epsilon_j \,.$$

Letting $l \to \infty$ we get

$$\left|\Phi \circ \Phi_m^{-1}(w) - w\right| \leq \sum_{j=m+1}^{\infty} \epsilon_j, \quad w \in K_m \,. \tag{5.5}$$

This shows that for each fixed compact set $K \subset D$, $\Phi \circ \Phi_m^{-1}$ is biholomorphic on K for all sufficiently large m, and hence Φ is biholomorphic on the subset $\Phi_m^{-1}(K) \subset \Omega$. Since Φ is a limit of injective holomorphic maps on Ω, it follows that Φ is injective on Ω.

It remains to show that $\Phi(\Omega) = D$. Fix an integer $m \geq 1$. Choose $l > m$ such that

$$\sum_{j=l+1}^{\infty} \epsilon_j < \operatorname{dist}\left(K_m, \mathbf{C}^n \backslash K_{m+1}\right) \,.$$

Set $\tilde{\Phi} = \Phi \circ \Phi_l^{-1}$; this is a holomorphic map in a neighborhood of K_l. From (5.5) we get

$$\left|\tilde{\Phi}(w) - w\right| < \operatorname{dist}\left(K_m, \mathbf{C}^n \backslash K_{m+1}\right), \quad w \in K_l \,.$$

Rouché's theorem [6, p. 110] implies $\tilde{\Phi}(K_l) \supset K_m$. Since $\tilde{\Phi}(K_l) = \Phi(\Phi_l^{-1}(K_l)) = \Phi(L_l)$, we get

$$K_m \subset \Phi\left(L_l\right) \subset \Phi(\Omega) \,.$$

Since this holds for every m, we conclude that $D \subset \Phi(\Omega)$ and hence $D = \Phi(\Omega)$.

Our proof shows that Ω (5.3) consists of all points $z \in \mathbf{C}^n$ for which the sequence

$$\{\Phi_m(z) \colon m \in \mathbf{N}\} \tag{5.6}$$

is relatively compact in D. Indeed, if $z \in \Omega$, then $\lim_{m\to\infty} \Phi_m(z) = \Phi(z) \in D$, and hence the sequence (5.6) is contained in some compact subset of D. Conversely, if (5.6) is relatively compact in D, it is contained in the set K_s for all sufficiently large s, so $\Phi_s(z) \in K_s$ and hence $z \in L_s \subset \Omega$. If $D = \mathbf{C}^n$, Ω contains exactly those points $z \in \mathbf{C}^n$ for which (5.6) is a bounded sequence. This completes the proof of Propositions 5.1 and 5.2. □

6. An interpolation theorem for proper holomorphic embeddings

We are interested in the following problem: *If M is a Stein manifold which admits a proper holomorphic embedding into $\mathbf{C}^n$ for some $n > 1$, what other properties of the embedding can one prescribe?* The following is our main result in this section.

Theorem 6.1. *Let M be a closed complex subvariety of $\mathbf{C}^n$, with $1 \le \dim M < n$, and set $d = n - \dim M$. Given a discrete set $\{p_j\} \subset \mathbf{C}^n$ and a $\mathbf{C}$-linear subspace $\Lambda_j \subset \mathbf{C}^n$ of dimension $\dim M$ for each $j = 1, 2, 3, \ldots$, there exist*

- *(i) a domain $\Omega \subset \mathbf{C}^n$ containing M, and*
- *(ii) a biholomorphic mapping $F: \Omega \to \mathbf{C}^n$ onto $\mathbf{C}^n$ satisfying $JF(z) = 1$ for each $z \in \Omega$, such that the image variety $M' = F(M) \subset \mathbf{C}^n$ satisfies*

$$p_j \in \operatorname{Reg} M', \quad T_{p_j} M' = \Lambda_j, \qquad j = 1, 2, 3, \ldots , \tag{6.1}$$

and such that any entire map $G: \mathbf{C}^d \to \mathbf{C}^n$ of rank d intersects M' at infinitely many points. If $d = 1$, there is an F as above such that $\mathbf{C}^n \backslash M'$ is Kobayashi hyperbolic.

Recall that the rank of G is the maximal (generic) rank of $G'(\zeta)$ for $\zeta \in \mathbf{C}^d$.

The main idea of this construction was developed in the papers [11, 12], and [4]. The last part concerning intersections and hyperbolicity was proved for one-dimensional complex varieties M in $\mathbf{C}^2$ by Buzzard and Fornæss [4].

Since the map $F: \Omega \to \mathbf{C}^n$ in Theorem 6.1 is biholomorphic onto $\mathbf{C}^n$ (a Fatou–Bieberbach map), Ω is pseudoconvex and the restriction $F|_M: M \hookrightarrow \mathbf{C}^n$ is a proper holomorphic embedding of M onto the closed subvariety $M' = F(M)$ of $\mathbf{C}^n$. Moreover, since F will be constructed as a locally uniform limit on Ω of a sequence of automorphisms, the domain Ω is Runge in $\mathbf{C}^n$ (see [2]).

Remark. Our proof of Theorem 6.1 will show that one can also prescribe any finite order jet of $F(M) = M'$ at each point p_j. To make this precise, we choose for each $j = 1, 2, 3, \ldots$ a complex subspace $L_j \subset \mathbf{C}^n$ so that $\Lambda_j \oplus L_j = \mathbf{C}^n$. Fix a j and write $z = (z', z'')$, with $z' \in \Lambda_j$ and $z'' \in L_j$. There is a neighborhood $U_j \subset \Lambda_j$ of 0 and a holomorphic map $g_j: U_j \to L_j$ satisfying $g_j(0) = 0$ and $Dg_j(0) = 0$, such that near the point p_j, M' is given by

$$\left\{p_j + \left(z', g_j\left(z'\right)\right) : z' \in U_j\right\} . \tag{6.2}$$

Suppose now that we are given for each $j \in \mathbf{N}$ a jet h_j of a holomorphic map $\Lambda_j \to L_j$ at 0, of order $m_j \ge 1$, such that $h_j(0) = 0$ and $Dh_j(0) = 0$. We can construct an F as in Theorem 6.1 such that $F(M)$ is locally near each p_j of the form (6.2), with the m_j-jet of g_j at 0 equal to the given jet h_j.

Corollary 6.2. *Let $\{p_j\}_{j=1}^{\infty}$ be a discrete subset of $\mathbf{C}^n$ for $n > 1$. If a Stein manifold M admits a proper holomorphic embedding $F_0: M \hookrightarrow \mathbf{C}^n$, then M also admits an embedding $F: M \hookrightarrow \mathbf{C}^n$*

whose image $F(M)$ contains the set $\{p_j\}$, and such that every entire map $G: \mathbf{C}^d \to \mathbf{C}^n$ of rank $d = n - \dim M$ intersects $F(M)$ at infinitely many points. If $d = 1$, there is an embedding F as above such that $\mathbf{C}^n \backslash F(M)$ is Kobayashi hyperbolic.

Corollary 6.3. *Let $m, d \geq 1$ and $n = m + d$. There exists a proper holomorphic embedding $F: \mathbf{C}^m \hookrightarrow \mathbf{C}^n$ such that every entire map $G: \mathbf{C}^d \to \mathbf{C}^n$ of rank d intersects $F(\mathbf{C}^m)$ at infinitely many points. For each $n \geq 1$ there exist proper holomorphic embeddings $F: \mathbf{C}^n \hookrightarrow \mathbf{C}^{n+1}$ such that $\mathbf{C}^{n+1} \backslash F(\mathbf{C}^n)$ is Kobayashi hyperbolic.*

Corollary 6.3 implies the following result. We identify $\mathbf{C}^m$ with $\mathbf{C}^m \times \{0\} \subset \mathbf{C}^{m+1}$.

Corollary 6.4. *For each pair of integers (m, n) such that $1 \leq m < n \leq 2m + 1$, there exists a proper holomorphic embedding $F: \mathbf{C}^m \hookrightarrow \mathbf{C}^n$ which does not extend to an injective holomorphic map of $\mathbf{C}^{m+1}$ into $\mathbf{C}^n$.*

Remark. Observe that, on the other hand, every injective entire map $F: \mathbf{C}^m \to \mathbf{C}^n$ for $n > 2m+1$ extends to an injective entire map $\tilde{F}: \mathbf{C}^{m+1} \to \mathbf{C}^n$ by taking $\tilde{F}(z, z_{m+1}) = F(z) + z_{m+1}v$ for a suitably chosen vector $v \in \mathbf{C}^n$; if F is an immersion, we can choose $\tilde{F}$ to be an immersion as well. (Choose $v \in \mathbf{C}^n$ which is not be contained in the image of $DF(z)$ for any $z \in \mathbf{C}^m$, and which is not a complex multiple of any vector $F(z) - F(z')$ for $z, z' \in \mathbf{C}^m$. Clearly, such v exists if $2m + 1 < n$.) Hence the codimension assumption in Corollary 6.4 is sharp.

Proof of Corollary 6.4. Choose $F: \mathbf{C}^m \hookrightarrow \mathbf{C}^n$ as in Corollary 6.3, and suppose that F extends to an injective map $\tilde{F}: \mathbf{C}^{m+1} \to \mathbf{C}^n$. Then for each $t \neq 0$, $\tilde{F}(\cdot, t): \mathbf{C}^m \to \mathbf{C}^n$ is an injective map whose image misses $F(\mathbf{C}^m)$. If $n \leq 2m$, we have $d = n - m \leq m$, and hence this is impossible by our choice of F. In the remaining case $n = 2m+1$ we have $d = m+1$. There exists a biholomorphic (Fatou–Bieberbach) map Φ of $\mathbf{C}^{m+1}$ onto a subset of $\mathbf{C}^{m+1} \backslash \mathbf{C}^m \times \{0\}$ (see [17]). The composition $G = \tilde{F} \circ \Phi: \mathbf{C}^{m+1} \to \mathbf{C}^n$ is injective and its image avoids $F(\mathbf{C}^m)$ in contradiction with the choice of F. $\square$

Corollary 6.2 does not give any new results concerning the *existence* of embeddings of a given Stein manifold into $\mathbf{C}^n$; however, once we know that one such embedding exists, it provides embeddings that are more "twisted" than the original one. In this context, we recall the embedding theorem of Eliashberg and Gromov [7]: *Every Stein manifold of dimension m admits a proper holomorphic embedding into $\mathbf{C}^N$ for the minimal integer $N > (3m + 1)/2$.* It is not known whether every open Riemann surface admits an embedding into $\mathbf{C}^2$. Recently it became clear that finitely connected planar domains admit such embeddings; see the paper [14] by Globevnik and Stensønes.

We will need the following lemma.

Lemma 6.5. *Let K be a compact polynomially convex subset of $\mathbf{C}^n$, and let $A \subset \mathbf{C}^n$ be a closed complex subvariety of $\mathbf{C}^n$. Suppose that A_0 is a compact, holomorphically convex subset of A such that $K \cap A$ is contained in the (relative) interior of A_0. Then the set $K \cup A_0$ is polynomially convex.*

Proof. Let $K_1 = K \cup A_0$. We first show that $\hat{K}_1 \subset K \cup A$. For every point $z \in \mathbf{C}^n \backslash A$, Cartan's [15, Theorem A] provides a holomorphic function f on $\mathbf{C}^n$ satisfying $f(z) = 1$ and $f|_A = 0$. If also $z \notin K$, there is a holomorphic function g on $\mathbf{C}^n$ such that $g(z) = 1$ and $|g| < 1$ on K. The function $F = fg^N$ for sufficiently large integer N satisfies $F(z) = 1$ and $|F| < 1$ on K_1. Thus, $z \notin \hat{K}_1$ and our claim is established.

Suppose now that $\hat{K}_1 \neq K_1$, and choose a point $z \in \hat{K}_1 \backslash K_1 \subset A \backslash A_0$. Since A_0 is holomorphically convex in A, there is a holomorphic function f on A satisfying $|f(z)| > \sup\{|f(w)| : w \in A_0\}$. By Cartan's [15, Theorem B] we can extend f to a holomorphic function on $\mathbf{C}^n$. Since $U = \hat{K}_1 \backslash K_1 \subset A \backslash A_0$ is a relative neighborhood of z in $\hat{K}_1$ whose relative boundary (in $\hat{K}_1$) is contained in A_0, f contradicts Rossi's local maximum modulus principle [18]. This proves that K_1 is polynomially convex. $\square$

Proof of Theorem 6.1. To make the proof more accessible, we first prove the interpolation result (6.1) in Theorem 6.1, following the idea of [11]. We will then explain the modifications in the construction which ensure that every map $G : \mathbf{C}^d \to \mathbf{C}^n$ of rank d intersects $F(M)$ at infinitely many points. This part is similar to the paper [4] by Buzzard and Fornæss who proved the results for the case $n = 2$ and $d = 1$. The main idea goes back to the construction by Rosay and Rudin [17, Section 4] of discrete sets in $\mathbf{C}^n$ which are unavoidable by nondegenerate holomorphic maps $G : \mathbf{C}^n \to \mathbf{C}^n$.

Choose the origin of $\mathbf{C}^n$ such that $0 < |p_1| < |p_2| < |p_3| < \cdots$, and then choose numbers $0 < r_1 < r_2 < r_3 < \cdots$ such that $r_j < |p_j| < r_{j+1}$ for $j = 1, 2, 3, \ldots$. Recall that B is the unit ball in $\mathbf{C}^n$.

To start the induction we set $F_0(z) = z - a$, where $a \in \mathbf{C}^n$ is chosen such that $p_1 \notin M_0 = F_0(M)$. Suppose that $k \geq 1$ and we have already constructed a volume preserving automorphism F_{k-1} of $\mathbf{C}^n$ such that

(i) the subvariety $M_{k-1} = F_{k-1}(M) \subset \mathbf{C}^n$ contains the points $p_1, p_2, \ldots, p_{k-1}$ in its regular locus of top dimension $\dim M$,

(ii) $T_{p_j} M_{k-1} = \Lambda_j$ and the jet of M_{k-1} of order m_j at p_j (see the remark following Theorem 6.1) matches the prescribed jet of order m_j for $1 \leq j \leq k-1$,

(iii) $p_k \notin M_{k-1}$.

We now construct the next automorphism $\Psi_k \in \mathrm{Aut}_1 \mathbf{C}^n$ such that $F_k = \Psi_k \circ F_{k-1}$ satisfies the requirements (i)–(iii) for $1 \leq j \leq k$. We also want to ensure the convergence of the sequence F_k on M. To this end we first choose a number $\rho_k \geq k$ such that

$$|F_{k-1}(z)| \geq r_{k+1}, \qquad z \in M,\ |z| \geq \rho_k . \tag{6.3}$$

Set

$$K_k = F_{k-1}\left(M \cap \rho_k \overline{B}\right) \cup \left(r_k \overline{B}\right) \subset M_{k-1} \cup \left(r_k \overline{B}\right) . \tag{6.4}$$

K_k is polynomially convex by Lemma 6.5. Now choose a point $q_k \in (\mathrm{Reg} M_{k-1}) \backslash K_k$ such that $\dim_{q_k} M_{k-1} = \dim M$. Given a number $\epsilon_k > 0$, Proposition 2.1 furnishes an automorphism $\Psi_k \in \mathrm{Aut}_1 \mathbf{C}^n$ satisfying the following:

(a) $\Psi_k(q_k) = p_k$ and the m_k-jet (6.2) of the subvariety $M_k = \Psi_k(M_{k-1})$ at p_k matches the prescribed m_k-jet; in particular, $T_{p_k} M_k = \Lambda_k$,

(b) $\Psi_k(z) = z + O(|z - p_j|^{m_j+1})$ as $z \to p_j$ for $1 \leq j \leq k-1$;

(c) $|\Psi(z) - z| < \epsilon_k$ for all $z \in K_k$; and

(d) $p_{k+1} \notin M_k$.

Regarding condition (a), the reader should observe that any two local submanifolds (6.2) at p_j can be mapped one onto the other by a finite composition of shear automorphisms $z \mapsto z + f(\lambda(z))v$ ($z \in \mathbf{C}^n$), where $\lambda : \mathbf{C}^n \to \Lambda_j$ is a linear projection onto Λ_j with kernel L_j, $v \in L_j$, and f is an entire function on Λ_j. Since these transformations are volume preserving, we can choose Ψ_k to be volume preserving.

The map $F_k = \Psi_k \circ F_{k-1}$ then satisfies conditions (i)–(iii) with $k-1$ replaced by k. If we choose in each step the number $\epsilon_k > 0$ so that

$$\epsilon_k < \min\left(r_{k+1} - r_k, 2^{-k}\right) ,$$

then Proposition 5.1 (applied to the compacts sets $r_k\overline{B}$ which exhaust $\mathbf{C}^n$) ensures that $F = \lim_{k\to\infty} F_k$ exists on a pseudoconvex Runge domain $\Omega \subset \mathbf{C}^n$ and $F\colon \Omega \to \mathbf{C}^n$ is biholomorphic onto $\mathbf{C}^n$. Moreover, since $\rho_k \to \infty$ as $k \to \infty$, condition (c) on Ψ_k implies that the sequence F_k converges on M, so $M \subset \Omega$. This proves that F satisfies (6.1). Since $JF_k \equiv 1$ for each k, we have $JF \equiv 1$ on Ω. □

Proof of the last part of Theorem 6.1. We now explain the necessary modifications in the construction of Ψ_k in kth step which ensure that F also satisfies the last requirement in Theorem 6.1. As mentioned earlier, this part is similar in spirit to [4].

For any subset $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \subset \{1, 2, \ldots, n\}$ with $m = \dim M$ elements, we denote by $\mathbf{C}_\alpha$ the coordinate plane in $\mathbf{C}^n$ of dimension m in coordinate directions $z_{\alpha_1}, \ldots, z_{\alpha_m}$. For each such α, let $\alpha' \subset \{1, \ldots, n\}$ be the complementary subset with $d = n - m$ elements, and let $\pi_\alpha\colon \mathbf{C}^n \to \mathbf{C}_{\alpha'}$ be the linear projection with $\ker \pi_\alpha = \mathbf{C}_\alpha$.

We shall describe the induction step. Fix $k \geq 1$ and suppose that F_{k-1} has already been constructed. For every α as above we choose a nonempty open spherical shell

$$S_\alpha \subset (r_{k+1}B) \setminus \left(r_k\overline{B}\right)$$

such that $\overline{S}_\alpha \cap \overline{S}_\beta = \emptyset$ for $\alpha \neq \beta$, and such that $p_k \notin \cup_\alpha \overline{S}_\alpha$. In each shell S_α we choose a countable dense set $\{a_{\alpha,l}\colon l \in \mathbf{N}\}$ such that the projections $\pi_\alpha(a_{\alpha,l}) \in \mathbf{C}_{\alpha'}$ for distinct l are distinct. Let $B_{\alpha,l}$ be the largest open ball in $\mathbf{C}_\alpha$, centered at 0, such that

$$a_{\alpha,l} + 2B_{\alpha,l} \subset S_\alpha \setminus K_k ,$$

where K_k is given by (6.4). Choose an integer l_0 (to be specified later) and set

$$D_{k,\alpha} = \cup_{l=1}^{l_0} \left(a_{\alpha,l} + \overline{B}_{\alpha,l}\right), \quad D_k = \cup_\alpha D_{k,\alpha} . \tag{6.5}$$

We denote by $\triangle$ the unit ball in $\mathbf{C}^d$. If $\delta > 0$ and if $A, B \subset \mathbf{C}^n$ are analytic sets of dimension d (not necessarily closed), we say that B is a δ-perturbation of A if the Hausdorff distance between A and B is less than δ. Recall that $M_{k-1} = F_{k-1}(M)$.

Lemma 6.6. *We can choose an integer l_0 sufficiently large and a number $\delta_k > 0$ sufficiently small such that the set D_k (6.5) satisfies the following property: If $G\colon \triangle \to r_{k+2}B$ is any holomorphic map satisfying*

(i) $|G(0)| \leq r_k$,
(ii) $\operatorname{dist}(G(0), M_{k-1}) \geq 1/k$,
(iii) $\max_\alpha |J(\pi_a \circ G)(0)| \geq 1/k$, *and*
(iv) $G(\triangle)$ *avoids a* δ_k*-perturbation* $D'_k \subset \mathbf{C}^n$ *of* D_k,

then G maps the ball $(1-2^{-k})\triangle \subset \mathbf{C}^d$ into the ball $r_{k+1}B \subset \mathbf{C}^n$.

In practice we will consider perturbations D'_k which are graphs of holomorphic mappings over the balls in D_k, in a direction complementary to $\mathbf{C}_\alpha$, with small uniform norm (see Lemma 6.8 below).

The proof of Lemma 6.6 is almost the same as in [4, Lemma 2.2]. We shall include it for the sake of completeness. If Lemma 6.6 does not hold, there is a sequence of holomorphic maps $G_j\colon \triangle \to r_{k+2}B$ $(j=1,2,3,\ldots)$, satisfying conditions (i)–(iii) in the lemma, such that

$$G_j(\triangle) \cap D_{k,j} = \emptyset \tag{6.6}$$

for some d-dimensional analytic set $D_{k,j} \subset \mathbf{C}^n$ which is a $1/j$-perturbation of the set D_k (6.5) in which we take $l_0 = j$, and such that $G_j\left((1-2^{-k})\triangle\right) \not\subset r_{k+1}B$ for all j. Passing to a subsequence we may assume that $\lim_{j\to\infty} G_j = G$ exists uniformly on compacts in $\triangle$. Hence, the convergence is uniform on $(1-2^{-k})\overline{\triangle}$, and G satisfies:

(α) $G(0) \in (r_k\overline{B})\backslash M_{k-1}$,
(β) $J(\pi_\alpha \circ G)(0) \neq 0$ for some α, and
(γ) $G\left((1-2^{-k})\overline{\triangle}\right) \not\subset r_{k+1}B$.

From (α) and (γ) it follows that $G(\triangle)$ intersects the set $S_\alpha\backslash M_{k-1}$, and ($\beta$) implies that the map $G_\alpha = \pi_\alpha \circ G$ has generic rank d. Thus, there is a point $\zeta_0 \in \triangle$ such that

$$G(\zeta_0) \in S_\alpha\backslash M_{k-1}, \qquad J(G_\alpha)(\zeta_0) \neq 0\,.$$

Hence, G is transverse to the affine plane $G(\zeta) + \mathbf{C}_\alpha$ for all $\zeta \in \triangle$ sufficiently close to ζ_0. Since the sequence $\{a_{\alpha,l}\colon l = 1,2,3,\ldots\}$ is dense in $S_\alpha\backslash M_{k-1}$, we can choose a point $a_{\alpha,l}$ sufficiently close to $G(\zeta_0)$ such that $G(\triangle)$ intersects the ball $B' = a_{\alpha,l} + B_{\alpha,l}$ transversely at a point near $G(\zeta_0)$. Since $d+m=n$, such intersections are stable under small deformations of both G and B'. Hence, for sufficiently large j, $G_j(\triangle)$ intersects any $1/j$-perturbation of the ball B'. Since $B' \subset D_k$ if $l_0 = j$ is chosen sufficiently large, we get a contradiction to (6.6). This proves Lemma 6.6. □

We now fix $l_0 = l_0(k)$ and $\delta_k > 0$ so that Lemma 6.6 holds, and we denote by $D_{k,\alpha}$ (resp. D_k) the sets (6.5) for this l_0. We will show that D_k can be approximated by (parts of) the image $\Psi_k(M_{k-1})$ for a suitably chosen automorphism $\Psi_k \in \mathrm{Aut}_1\,\mathbf{C}^n$. We will need the following lemma (notation as above).

Lemma 6.7. *The set $E_k = K_k \cup D_k$ is polynomially convex.*

Proof. Fix $\alpha = \alpha_1$ such that S_{α_1} is the inner-most of all shells S_α chosen in step k. We shall first prove that the set $R = K_k \cup D_{k,\alpha_1}$ is polynomially convex.

Since K_k is polynomially convex, the first step in the proof of Lemma 6.5 shows that the polynomial hull of R is contained in the union of K_k with the affine subspaces $\Sigma_l = a_{\alpha_1,l} + \mathbf{C}_{\alpha_1}$, $1 \le l \le l_0(k)$.

It suffices to show that $R \cap \Sigma_l$ is polynomially convex for each l. Namely, if this holds but R is not polynomially convex, there is a point $p \in (\hat{R}\backslash R) \cap \Sigma_l$ for some l. The set $U = \hat{R} \cap \Sigma_l$ is a relative neighborhood of p in $\hat{R}$, with the relative boundary of U (in $\hat{R}$) contained in $R \cap \Sigma_l$. Since $R \cap \Sigma_l$ is polynomially convex, there is a holomorphic function f satisfying $f(p) = 1$ and $|f| < 1$ on $R \cap \Sigma_l$, and hence $|f| < 1$ on bU. This contradicts Rossi's local maximum modulus theorem [18], thereby establishing our claim.

To prove polynomial convexity of $R \cap \Sigma_l$ we observe that

$$R \cap \Sigma_l = B' \cup B'' \cup A_0 ,$$

where $B' = \overline{B}_{\alpha_1,l}$ and $B'' = \Sigma_l \cap (r_k \overline{B})$ are disjoint closed balls in Σ_l (the second one may be empty or a point), and

$$A_0 = \Sigma_l \cap F_{k-1}\left(M \cap \rho_k \overline{B}\right)$$

is a holomorphically convex subset of the complex subvariety $A = \Sigma_l \cap F_{k-1}(M)$. Observe that $A \cap B' = \emptyset$, and $A \cap B''$ is contained in the relative interior of A_0 (with respect to A). Since the union $B' \cup B''$ of two disjoint closed balls in Σ_l is polynomially convex, Lemma 6.5 implies that $R \cap \Sigma_l$ is polynomially convex for each l. This proves that the set $R = K_k \cup D_{k,\alpha_1}$ is polynomially convex.

Let S_{α_2} be the next concentric shell. We claim that the set

$$R_2 = K_k \cup D_{k,\alpha_1} \cup D_{k,\alpha_2} = R \cup D_{k,\alpha_2}$$

is polynomially convex. To see this, choose a ball $r\overline{B} \subset \mathbf{C}^n$ which contains the shell S_{α_1} but does not intersect S_{α_2}. Repeating the same proof as above, with K_k replaced by $K_k \cup r\overline{B}$, we see that $K_k \cup (r\overline{B}) \cup D_{k,\alpha_2}$ is polynomially convex. So it remains to show that if $p \in r\overline{B} \backslash R$, then p is not in the hull of R_2. Fix such a p. Since R is polynomially convex, there is a polynomial f on $\mathbf{C}^n$ with $f(p) = 1$ and $|f| < 1$ on R. Let g equal f in a neighborhood of $K_k \cup r\overline{B}$ and $g = 0$ in a neighborhood of D_{k,α_2}. By Oka–Weil theorem [15] we can approximate g by a polynomial h, uniformly on $K_k \cup (r\overline{B}) \cup D_{k,\alpha_2}$. If the approximation is sufficiently close, then $|h(p)| > \sup\{|h(z)| : z \in R_2\}$, thereby establishing our claim.

Continuing inductively we see after finitely many steps that the set E_k is polynomially convex. This proves Lemma 6.7. $\square$

Let $\delta_k > 0$ and $l_0(k)$ be such that Lemma 6.6 holds. Recall that $\alpha' \subset \{1, \ldots, n\}$ is the complement of α. Let $B'_{\alpha',l} \subset \mathbf{C}_{\alpha'}$ be the ball of radius δ_k, centered at the origin, and set

$$P_{\alpha,l} = a_{\alpha,l} + \left(B_{\alpha,l} \times B'_{\alpha',l}\right) \subset \mathbf{C}^n . \tag{6.7}$$

If $U_k \supset K_k$ is a small compact, polynomially convex neighborhood of K_k in $\mathbf{C}^n$, and if $\delta_k > 0$ is chosen sufficiently small, then the closures $\overline{P}_{\alpha,l}$ are pairwise disjoint for all α and $1 \leq l \leq l_0$, none of them intersects U_k, and the set

$$\tilde{E}_k = U_k \cup \left(\cup_{\alpha,l} \overline{P}_{\alpha,l}\right) \tag{6.8}$$

(where the union is over all α and $1 \leq l \leq l_0(k)$) is polynomially convex. The last assertion follows from Lemma 6.7.

Lemma 6.8. *There exists a* $\Psi_k \in \mathrm{Aut}_1 \mathbf{C}^n$ *which satisfies the properties (a)–(d) above and such that for each* (α, l) $(1 \leq l \leq l_0(k))$ *there is a holomorphic map* $g_{\alpha,l} \colon B_{\alpha,l} \to \mathbf{C}_{\alpha'} = \mathbf{C}^d$*, with* $||g_{\alpha,l}||_\infty < \delta_k/2$*, such that*

$$\left\{a_{\alpha,l} + \left(w, g_{\alpha,l}(w)\right) : w \in B_{\alpha,l}\right\} \subset \Psi_k(M_{k-1}) \cap P_{\alpha,l} .$$

On the left-hand side above we use the splitting of coordinates $z = (w, \zeta) \in \mathbf{C}^n$ such that $w \in \mathbf{C}_\alpha$ and $\zeta \in \mathbf{C}_{\alpha'}$.

Proof. Proposition 2.1 furnishes a preliminary automorphism $\Theta_k \in \mathrm{Aut}_1 \mathbf{C}^n$ satisfying conditions (a)–(d) in the first part of the proof of Theorem 6.1, such that the variety $\Theta_k(M_{k-1}) = M'_k$ satisfies

$$a_{\alpha,l} \in \mathrm{Reg} M'_k, \qquad T_{a_{\alpha,l}} M'_k = \mathbf{C}_\alpha \tag{6.9}$$

for all α and all l with $1 \le l \le l_0(k)$. We may assume that $\Theta_k(K_k) \subset U_k$.

To explain the second step we fix a pair (α, l), and we choose coordinates $w \in \mathbf{C}^n$ such that, in these coordinates, $\mathbf{C}_\alpha = \{0\}^d \times \mathbf{C}^m$, $\mathbf{C}_{\alpha'} = \mathbf{C}^d \times \{0\}^m$, and $w(a_{\alpha,l}) = 0$. We write $w = (w', w'')$ accordingly, with $w' \in \mathbf{C}^d$ and $w'' \in \mathbf{C}^m$. Consider the linear vector field $V(w) = (-\lambda w', \mu w'')$, where $\lambda > 0$ and $\mu > 0$ satisfy $d\lambda - m\mu = 0$. The last condition is equivalent to $\mathrm{div}\, V = 0$. The flow of V, given by

$$\phi_t\left(w', w''\right) = \left(e^{-\lambda t} w', e^{\mu t} w''\right) ,$$

is volume preserving; it is contracting on $\mathbf{C}_{\alpha'}$ and expanding on $\mathbf{C}_\alpha$. Hence, (6.9) implies that for sufficiently large $\lambda > 0$ and $\mu > 0$, the time-one map ϕ_1 of V stretches a suitably chosen neighborhood of the point $a_{\alpha,l} \in M'_k$ in M'_k to a graph as in Lemma 6.8 (the left-hand side of the display).

We do this simultaneously on every set $P_{\alpha,l}$. Let V be the divergence zero vector field in a neighborhood of the polynomially convex set $\tilde{E}_k$ (6.8) which is defined as above near each set $\overline{P}_{\alpha,l}$ and which is zero near U_k. We can approximate V, uniformly on $\tilde{E}_k$, by a divergence zero polynomial vector field on $\mathbf{C}^n$. The argument is the same as in Step 2.8 in the proof of Proposition 2.1. Lemma 2.7 implies that the time-one map of V is a (locally uniform) limit of compositions of shears.

This gives an automorphism Φ_k, a composition of finitely many shears, which is close to the identity on U_k, which fixes the points $p_1, p_2, \ldots, p_k$ and it matches the identity map to order $\max\{m_j + 1 \colon 1 \le j \le k\}$ at these points, and such that $\Psi_k = \Phi_k \circ \Theta_k$ satisfies Lemma 6.8. □

Conclusion of the proof of Theorem 6.1. Let $P_{k,\alpha,l}$ be the set (6.7) defined in step k. At each step k we choose Ψ_k such that Lemma 6.8 holds, and we make sure that the total deformation of $r_{k+1}B$ at all later steps of the construction is less than $\delta_k/2$. This implies that the intersection of the final subvariety $F(M) \subset \mathbf{C}^n$ with $P_{k,\alpha,l}$ contains a δ_k-perturbation of the ball $a_{\alpha,l} + B_{\alpha,l}$ constructed in step k. (In fact we can ensure that $F(M) \cap P_{k,\alpha,l}$ contains a graph over $a_{\alpha,l} + B_{\alpha,l}$ as in Lemma 6.8.) Let

$$D'_k = \cup_{\alpha,l}\left(F(M) \cap P_{k,\alpha,l}\right) ,$$

where the union is over all α and l, $1 \le l \le l_0(k)$.

In order to get a contradiction we suppose that $G \colon \mathbf{C}^d \to \mathbf{C}^n$ is a holomorphic map of rank d which intersects $F(M)$ in at most finitely many points. We may assume that G has rank d at the origin and $G(0) \notin F(M)$. Choose α and $k_0 \ge 2$ such that

(i) $|G(0)| \le r_{k_0}$,
(ii) $\mathrm{dist}(G(0), M_{k-1}) \ge 1/k$ for all $k \ge k_0$,
(iii) $|J(\pi_a \circ G)(0)| \ge 1/k_0$, and
(iv) $G(\mathbf{C}^d) \cap D'_k = \emptyset$ for $k \ge k_0$.

Condition (ii) can be achieved since M_{k-1} is close to $F(M)$ on any fixed compact set for large k, and $G(0) \notin F(M)$; (iv) follows from the assumption on G since $D'_k \subset F(M)$.

Recall that $\triangle$ is the unit ball in $\mathbf{C}^d$. Fix an $R > 1$ and choose $k = k(R)$ such that $G(R\triangle)$ is contained in $r_{k+2}B$ but not in $r_{k+1}B$. If $k \ge k_0$, we apply Lemma 6.6 to the map $\tilde{G}(\zeta) = G(R\zeta)$

from $\triangle \subset \mathbf{C}^d$ to $r_{k+2}B \subset \mathbf{C}^n$ to conclude that G maps the ball in $\mathbf{C}^d$ of radius $(1-2^{-k})R$ into $r_{k+1}B$. If $k-1 \geq k_0$, we apply Lemma 6.6 again to conclude that G maps the ball of radius $(1-2^{-k})(1-2^{-k+1})R$ into r_kB. We keep repeating the argument until we reach k_0. Since $c = \prod_{l=2}^{\infty}(1-2^{-l}) > 0$, we conclude that G maps the ball $(cR)\triangle$ into $r_{k_0+1}B$. Since this is true for all $R > 1$, G is bounded on $\mathbf{C}^d$ and hence constant, a contradiction. If $d = 1$, the same argument proves that $\mathbf{C}^n \backslash F(M)$ is hyperbolic. This completes the proof of Theorem 6.1. □

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