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Solving the d- and $\bar\partial$-Equations in Thin Tubes and Applications to Mappings

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1. The Results

Let $\mathbf{C}^n$ denote the complex n-dimensional Euclidean space with complex coordinates $z = (z_1, \dots, z_n)$. A compact $\mathcal{C}^k$-submanifold $M \subset \mathbf{C}^n$ $(k \geq 1)$, with or without boundary, is *totally real* if for each $z \in M$ the tangent space $T_z M$ (which is a real subspace of $T_z\mathbf{C}^n$) contains no complex line; equivalently, the complex subspace $T_z^C M = T_z M + iT_z M$ of $T_z\mathbf{C}^n$ has complex dimension $m = \dim_{\mathbf{R}} M$ for each $z \in M$. We denote by $\mathcal{T}_\delta M = \{z \in \mathbf{C}^n : d_M(z) < \delta\}$ the tube of radius $\delta > 0$ around M; here $|z|$ is the Euclidean norm of $z \in \mathbf{C}^n$ and $d_M(z) = \inf\{|z-w| : w \in M\}$.

For any open set $U \subset \mathbf{C}^n$ and integers $p, q \in \mathbf{Z}_+$ we denote by $\mathcal{C}^l_{p,q}(U)$ the space of differential forms of class $\mathcal{C}^l$ and of bidegree (p,q) on U. For each multiindex $\alpha \in Z_+^{2n}$ we denote by ∂^α the corresponding partial derivative of order $|\alpha|$ with respect to the underlying real coordinates on $\mathbf{C}^n$.

The following is one of the main results of the paper; for additional estimates see Theorem 3.1.

1.1. Theorem. *Let $M \subset \mathbf{C}^n$ be a closed, totally real, $\mathcal{C}^1$-submanifold and let $0 < c < 1$. Denote by $\mathcal{T}_\delta$ the tube of radius $\delta > 0$ around M. There is a $\delta_0 > 0$ and for each integer $l \geq 0$ a constant $C_l > 0$ such that the following hold for all $0 < \delta \leq \delta_0$, $p \geq 0$, $q \geq 1$, and $l \geq 1$. For any $u \in \mathcal{C}^l_{p,q}(\mathcal{T}_\delta)$ with $\bar\partial u = 0$ there is a $v \in \mathcal{C}^l_{p,q-1}(\mathcal{T}_\delta)$ satisfying $\bar\partial v = u$ in $\mathcal{T}_{c\delta}$ and satisfying also the estimates*

$$\begin{aligned} \|v\|_{L^\infty(\mathcal{T}_{c\delta})} &\leq C_0 \delta \|u\|_{L^\infty(\mathcal{T}_\delta)}; \\ \|\partial^\alpha v\|_{L^\infty(\mathcal{T}_{c\delta})} &\leq C_l(\delta \|\partial^\alpha u\|_{L^\infty(\mathcal{T}_\delta)} + \delta^{1-|\alpha|}\|u\|_{L^\infty(\mathcal{T}_\delta)}), \quad |\alpha| \leq l. \end{aligned} \tag{1.1}$$

If $q = 1$ and the equation $\bar\partial v = u$ has a solution $v_0 \in \mathcal{C}^{l+1}_{(p,0)}(\mathcal{T}_\delta)$, then there is a solution $v \in \mathcal{C}^{l+1}_{(p,0)}(\mathcal{T}_\delta)$ of $\bar\partial v = u$ on $\mathcal{T}_{c\delta}$ satisfying

$$\|\partial_j \partial^\alpha v\|_{L^\infty(\mathcal{T}_{c\delta})} \leq C_{l+1}(\omega(\partial_j\partial^\alpha v_0, \delta) + \delta^{-l}\|u\|_{L^\infty(\mathcal{T}_\delta)})$$

for $1 \leq j \leq n$ and $|\alpha| = l$.

In the last estimate, $\omega(f,\delta) = \sup\{|f(x) - f(y)| : |x-y| \leq \delta\}$ is the *modulus of continuity* of a function; when f is a differential form on $\mathbf{C}^n$, $\omega(f,t)$ is defined as

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the sum of the moduli of continuity of its components (in the standard basis). The constants C_k appearing in the estimates are independent of u and δ (they depend only on M and c).

The solution in Theorem 1.1 is obtained by a family of integral kernels, depending on $\delta > 0$ and constructed specifically for thin tubes (and hence is given by a linear solution operator on each tube $\mathcal{T}_\delta$). Immediate examples show that the gain of δ in the estimate for v is the best possible. When u is a $(0,1)$-form (or a $(p,1)$-form), the estimates for the derivatives of v in (1.1) follow from the sup-norm estimate by shrinking the tube and applying the interior regularity for the $\bar\partial$-operator (Lemma 3.2). This is not the case in bidegrees (p,q) for $q > 1$. We refer to Section 3 for further details.

Another major result of the paper is Theorem 5.1 on solving the equation $dv = u$ for holomorphic forms in tubes $\mathcal{T}_\delta$ with precise estimates. Theorem 5.1 is obtained by using the solutions of the $\bar\partial$-equation (provided by Theorem 3.1) in the proof of Serre's theorem to the effect that, on pseudoconvex domains, the de Rham cohomology groups are given by holomorphic forms.

We now apply these results to the problem of approximating smooth diffeomorphisms between totally real submanifolds in $\mathbf{C}^n$ by biholomorphic maps in tubes $\mathcal{T}_\delta$ and by holomorphic automorphism of $\mathbf{C}^n$. The tools developed here give optimal results without any loss of derivatives in these approximation problems.

The *complex normal bundle* $\nu_M \to M$ of a totally real submanifold $M \subset \mathbf{C}^n$ is defined as the quotient bundle $\nu_M = T\mathbf{C}^n|_M / T^{\mathbf{C}}M$. It can be realized as a complex subbundle of $T\mathbf{C}^n|_M$ such that $T\mathbf{C}^n|_M = T^{\mathbf{C}}M \oplus \nu_M$. Given a diffeomorphisms $f\colon M_0 \to M_1$ between totally real submanifolds $M_0, M_1 \subset \mathbf{C}^n$, we say that the complex normal bundles $\pi_i\colon \nu_i \to M_i$ are isomorphic over f if there exists an isomorphism of $\mathbf{C}$-vector bundles $\phi\colon \nu_0 \to \nu_1$ satisfying $\pi_1 \circ \phi = f \circ \pi_0$.

1.2. THEOREM. *Let $f\colon M_0 \to M_1$ be a diffeomorphism of class $\mathcal{C}^k$ between compact totally real submanifolds $M_0, M_1 \subset \mathbf{C}^n$, with or without boundary* ($n \geq 1$, $k \geq 2$). *Assume that the complex normal bundles to M_0 and M_1 are isomorphic over f. Then there are numbers $\delta_0 > 0$ and $a > 0$ such that, for each $\delta \in (0, \delta_0)$, there exists an injective holomorphic map $F_\delta\colon \mathcal{T}_\delta M_0 \to \mathbf{C}^n$ such that $F_\delta(\mathcal{T}_\delta M_0) \supset \mathcal{T}_{a\delta} M_1$ and the following estimates hold for $0 \leq r \leq k$ as $\delta \to 0$:*

$$\|F_\delta|_{M_0} - f\|_{\mathcal{C}^r(M_0)} = o(\delta^{k-r}), \qquad \|F_\delta^{-1}|_{M_1} - f^{-1}\|_{\mathcal{C}^r(M_1)} = o(\delta^{k-r}). \tag{1.2}$$

The $\mathcal{C}^r(M)$-norm is defined as usual by using a finite open covering of M by coordinate charts and a corresponding partition of unity. An important aspect of Theorem 1.2 is the precise relationship between the rate of approximation on M_0 (resp., on M_1) and the radius δ of the tube on which the approximating biholomorphic map F_δ is defined. The condition that the complex normal bundles are isomorphic over f is necessary because the derivative of any biholomorphic map, defined near M_0 and sufficiently close to f in the $\mathcal{C}^1(M_0)$-norm, induces such an isomorphism. If M_0 and M_1 are contractible (such as arcs or totally real discs) or if they are of maximal real dimension n, then Theorem 1.2 applies to any $\mathcal{C}^k$-diffeomorphism $f\colon M_0 \to M_1$.

When all data in Theorem 1.2 are real-analytic, f extends to a biholomorphic map F from a neighborhood of M_0 onto a neighborhood of M_1 (see Remark (1) after the proof of Theorem 1.2 in Section 4). In such case we say that M_0 and M_1 are *biholomorphically equivalent*; such pairs of submanifolds have identical local analytic properties in $\mathbf{C}^n$. This is not so if f is smooth but non–real-analytic, for there exist smooth arcs in $\mathbf{C}^n$ that are complete pluripolar as well as arcs that are not pluripolar [DF], yet any diffeomorphism between smooth arcs can be approximated as in Theorem 1.2.

We don't know whether in general there exist biholomorphic maps F_δ in a *fixed* open neighborhood of M_0 and satisfying (1.2) as $\delta \to 0$. However, in certain situations we can approximate diffeomorphisms by global holomorphic automorphisms of $\mathbf{C}^n$. Recall that a compact set $K \subset \mathbf{C}^n$ is *polynomially convex* if for each $z \in \mathbf{C}^n \backslash K$ there is a holomorphic polynomial P on $\mathbf{C}^n$ such that $|P(z)| > \sup\{|P(x)| : x \in K\}$. We denote by $\mathrm{Aut}\,\mathbf{C}^n$ the group of all holomorphic automorphisms of $\mathbf{C}^n$.

DEFINITION 1.

(a) A $\mathcal{C}^k$-*isotopy* (or a $\mathcal{C}^k$-*flow*) in $\mathbf{C}^n$ is a family of $\mathcal{C}^k$-diffeomorphisms $f_t\colon M_0 \to M_t$ ($t \in [0,1]$) between $\mathcal{C}^k$-submanifolds $M_t \subset \mathbf{C}^n$ such that f_0 is the identity on M_0 and such that both $f_t(z)$ and $\frac{\partial}{\partial t} f_t(z)$ are continuous with respect to $(t,z) \in [0,1] \times M_0$ and of class $\mathcal{C}^k(M_0)$ in the second variable for each fixed $t \in [0,1]$.

(b) The isotopy in (a) is said to be *totally real* (resp., *polynomially convex*) if the submanifold $M_t \subset \mathbf{C}^n$ is totally real (resp., compact polynomially convex) for each $t \in [0,1]$.

(c) The *infinitesimal generator* of f_t as in (a) is the time-dependent vector field X_t on $\mathbf{C}^n$ that is uniquely defined along M_t by the equation $\frac{\partial}{\partial t} f_t(z) = X_t(f_t(z))$ ($z \in M_0$, $t \in [0,1]$).

(d) A *holomorphic isotopy* (or holomorphic *flow*) on a domain $D \subset \mathbf{C}^n$ is a family of injective holomorphic maps $F_t\colon D \to \mathbf{C}^n$ such that F_0 is the identity on D and such that the maps $F_t(z)$ and $\frac{\partial}{\partial t} F_t(z)$ are continuous with respect to $(t,z) \in [0,1] \times D$. Its infinitesimal generator X_t, defined as in (c), is a holomorphic vector field on the domain $D_t = F_t(D)$ for each $t \in [0,1]$.

1.3. THEOREM. *Let $M_0 \subset \mathbf{C}^n$ be a compact $\mathcal{C}^k$-submanifold of $\mathbf{C}^n$ ($n \ge 2$, $k \ge 2$). Assume that $f_t\colon M_0 \to M_t \subset \mathbf{C}^n$ ($t \in [0,1]$) is a $\mathcal{C}^k$-isotopy such that the submanifold $M_t = f_t(M_0) \subset \mathbf{C}^n$ is totally real and polynomially convex for each $t \in [0,1]$. Set $f = f_1\colon M_0 \to M_1$. Then there exists a sequence $F_j \in \mathrm{Aut}\,\mathbf{C}^n$ ($j = 1,2,3,\dots$) such that*

$$\lim_{j\to\infty} \|F_j|_{M_0} - f\|_{\mathcal{C}^k(M_0)} = 0, \qquad \lim_{j\to\infty} \|F_j^{-1}|_{M_1} - f^{-1}\|_{\mathcal{C}^k(M_1)} = 0. \tag{1.3}$$

Combining Theorem 1.3 with Corollary 4.2 from [FR] yields the following.

1.4. COROLLARY. *Let $f\colon M_0 \to M_1$ be a $\mathcal{C}^k$-diffeomorphism ($k \ge 2$) between compact, totally real, polynomially convex submanifolds of $\mathbf{C}^n$ of real dimension* *m. If* $1 \leq m \leq 2n/3$, *then there exists a sequence* $F_j \in \operatorname{Aut} \mathbf{C}^n$ $(j = 1, 2, 3, \ldots)$ *satisfying* (1.3).

Theorems 1.2 and 1.3 are proved in Section 4. A weaker version of Theorem 1.3 (with loss of derivatives) was obtained in [FL] by applying Hörmander's L^2-method for solving the $\bar{\partial}$-equations in tubes. For a converse to Theorem 1.3 see [FL, Rem. 2, p. 135]. When f is a real-analytic diffeomorphism as in Theorem 1.3, the approximating sequence $F_j \in \operatorname{Aut} \mathbf{C}^n$ can be chosen such that it converges to a biholomorphic map F in an open neighborhood of M_0 in $\mathbf{C}^n$ satisfying $F|_{M_0} = f$ [FR].

We now consider the approximation problem for maps preserving one of the forms

$$\omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n, \tag{1.4}$$

$$n = 2n', \quad \omega = \sum_{j=1}^{n'} dz_{2j-1} \wedge dz_{2j}. \tag{1.5}$$

A holomorphic map F between domains in $\mathbf{C}^n$ satisfying $F^*\omega = \omega$ will be called a *holomorphic* ω*-map.* The form (1.4) is the (standard) *complex volume form* on $\mathbf{C}^n$; in this case $F^*\omega = JF \cdot \omega$, where JF is the complex Jacobian determinant of F, and ω-maps are called *unimodular.* The form (1.5) is the *standard holomorphic symplectic form,* and holomorphic ω-maps are called *symplectic holomorphic.* We denote the corresponding automorphism group by

$$\operatorname{Aut}_\omega \mathbf{C}^n = \{F \in \operatorname{Aut} \mathbf{C}^n : F^*\omega = \omega\}.$$

For convenience we state the approximation results for ω-maps (Theorems 1.5 and 1.7 and Corollary 1.6) only for closed submanifolds; for an extension to manifolds with boundary, see the remark following Theorem 1.7.

1.5. THEOREM. *Let* ω *be any of the forms* (1.4), (1.5). *Let* $f: M_0 \to M_1$ *be a* $\mathcal{C}^k$*-diffeomorphism between closed totally real submanifolds in* $\mathbf{C}^n$ $(k, n \geq 2)$. *Assume that there is a* $\mathcal{C}^{k-1}$*-map* $L: M_0 \to \mathrm{GL}(n, \mathbf{C})$ *satisfying*

$$L_z|_{T_z M_0} = df_z, \quad L_z^*\omega = \omega \quad (z \in M_0). \tag{1.6}$$

Then for each sufficiently small $\delta > 0$ *there is an injective holomorphic map* $F_\delta: \mathcal{T}_\delta M_0 \to \mathbf{C}^n$ *such that* $F_\delta^*\omega = \omega$ *and* (1.2) *holds as* $\delta \to 0$. *If* M_0, M_1 *and* f *are real-analytic and if there exists a continuous* L *satisfying* (1.6), *then* f *extends to a biholomorphic map* F *on a neighborhood of* M_0 *satisfying* $F^*\omega = \omega$.

The notation $L_z^*\omega$ in (1.6) denotes the pull-back of the multi-covector $\omega_{f(z)}$ by the $\mathbf{C}$-linear map L_z (which we may interpret as a map $T_z\mathbf{C}^n \to T_{f(z)}\mathbf{C}^n$). Clearly (1.6) implies that the complex normal bundles $\nu_j \to M_j$ are isomorphic over f. Denoting by $\mathrm{SL}(n, \mathbf{C})$ the special linear group on $\mathbf{C}^n$ and by $\mathrm{Sp}(n, \mathbf{C})$ the linear symplectic group on $\mathbf{C}^{2n}$, we can express the condition in Theorem 1.5 as follows:

(*) *There exists a* $\mathcal{C}^{k-1}$*-map* $L: M_0 \to \mathrm{SL}(n, \mathbf{C})$ (*resp.,* $L: M_0 \to \mathrm{Sp}(n, \mathbf{C})$) *such that* $L_z = df_z$ *on* $T_z M_0$ *for each* $z \in M_0$.

The only obvious necessary conditions for the approximation of a $\mathcal{C}^k$-diffeomorphism $f: M_0 \to M_1$ by holomorphic ω-maps are that the complex normal bundles $\nu_j \to M_j$ be isomorphic over f and that $f^*(i_1^*\omega) = i_0^*\omega$, where $i_j: M_j \hookrightarrow \mathbf{C}^n$ is the inclusion. Theorem 1.5 reduces this analytic approximation problem to the geometric problem of finding an extension L of df satisfying (1.6). The regularity of L is not the key point; it would suffice to assume the existence of a *continuous* L satisfying (1.6), since an argument similar to the one in the proof of Theorem 1.5 for the real-analytic case then allows us to approximate L by a $\mathcal{C}^{k-1}$-map satisfying (1.6). We expect that such extension does not always exist, although we do not have specific examples. Here are some positive results.

1.6. Corollary. *Let ω be one of the forms* (1.4), (1.5), *and let $k, n \geq 2$. Let $f: M_0 \to M_1$ be a $\mathcal{C}^k$-diffeomorphism between closed totally real submanifolds such that the complex normal bundles to M_0* (*resp., M_1*) *in $\mathbf{C}^n$ are isomorphic over f and $f^*\omega = i_0^*\omega$. Then the conclusion of Theorem 1.5 holds in each of the following cases*:

(i) $\dim M_0 = \dim M_1 = n$;
(ii) $\omega = dz_1 \wedge \cdots \wedge dz_n$ *and M_0 is simply connected*;
(iii) $\omega = dz_1 \wedge \cdots \wedge dz_n$ *and ν_0 admits a complex line subbundle.*

In cases (ii) and (iii) we have $f^*\omega = i_0^*\omega = 0$ when $m < n$. Finally we present approximation results for ω-flows. We first introduce convenient terminology.

Definition 2. Let ω be a differential form on $\mathbf{C}^n$ and let $f_t: M_0 \to M_t \subset \mathbf{C}^n$ ($t \in [0,1]$) be a $\mathcal{C}^k$-isotopy with the infinitesimal generator X_t (see Definition 1).

(a) f_t is an *ω-flow* if the form $f_t^*\omega$ on M_0 is independent of $t \in [0,1]$.
(b) An ω-flow f_t is *closed* (resp., *exact*) if, for each $t \in [0,1]$, the pull-back to M_t of the form $\alpha_t = X_t \rfloor \omega$ (the contraction of ω by X_t) is closed (resp., exact).
(c) Let $U \subset \mathbf{C}^n$ be an open set and ω a holomorphic form on $\mathbf{C}^n$. A holomorphic flow $F_t: U \to \mathbf{C}^n$ ($t \in [0,1]$) satisfying $F_t^*\omega = \omega$ for all t is called a *holomorphic ω-flow.*

Remark. If $d\omega = 0$ (this holds for the forms (1.4), (1.5)) then a flow $f_t: M_0 \to M_t$ is an ω-flow if and only if the pull-back of $\alpha_t = X_t \rfloor \omega$ to M_t is a closed form on M_t for each $t \in [0,1]$. This can be seen from the following formula for the Lie derivative $L_{X_t}\omega$ [AMR, Thm. 5.4.1 and Thm. 6.4.8(iv)]:

$$\frac{d}{dt}(f_t^*\omega) = f_t^*(L_{X_t}\omega) = f_t^*(d(X_t \rfloor \omega) + X_t \rfloor d\omega) = f_t^*(d\alpha_t).$$

Hence $f_t^*\omega$ is independent of t if and only if $d(i_t^*\alpha_t) = 0$ on M_t for each $t \in [0,1]$.

1.7. Theorem. *Let ω be any of the forms* (1.4), (1.5). *Assume that $M_0 \subset \mathbf{C}^n$ is a closed totally real submanifold and that $f_t: M_0 \to M_t \subset \mathbf{C}^n$ ($t \in [0,1]$) is a totally real ω-flow of class $\mathcal{C}^k$ for some $k \geq 2$. Then for each sufficiently small $\delta > 0$ there is a holomorphic ω-flow $F_t^\delta: \mathcal{T}_\delta M_0 \to \mathbf{C}^n$ ($t \in [0,1]$) such that, for*

$0 \le r \le k$, *we have the following estimates as* $\delta > 0$ (*uniformly with respect to* $t \in [0,1]$):

$$\|F_t^\delta - f_t\|_{\mathcal{C}^r(M_0)} = o(\delta^{k-r}), \qquad \|(F_t^\delta)^{-1} - f_t^{-1}\|_{\mathcal{C}^r(M_t)} = o(\delta^{k-r}).$$

If in addition $n \ge 2$ *and* f_t *is an exact* ω*-flow that is totally real and polynomially convex, then for each* $\varepsilon > 0$ *there is a holomorphic* ω*-flow* $F_t \in \mathrm{Aut}_\omega\, \mathbf{C}^n$ *such that, for all* $t \in [0,1]$,

$$\|F_t - f_t\|_{\mathcal{C}^k(M_0)} < \varepsilon, \qquad \|F_t^{-1} - f_t^{-1}\|_{\mathcal{C}^k(M_t)} < \varepsilon.$$

Theorems 1.5 and 1.7 together with Corollary 1.6 extend to the following situation. Let M_0 be a compact domain in a totally real submanifold $M_0' \subset \mathbf{C}^n$, not necessarily closed or compact. In particular, M_0 may be a totally real submanifold with boundary ∂M_0 and M_0' a larger submanifold containing M_0. In the context of Theorem 1.5 or Corollary 1.6, assume that $f\colon M_0' \to M_1'$ is a $\mathcal{C}^k$-diffeomorphism between totally real submanifolds in $\mathbf{C}^n$ $(k \ge 2)$ and that $L\colon M_0' \to \mathrm{GL}(n,\mathbf{C})$ is a $\mathcal{C}^{k-1}$-map satisfying (1.6) on M_0'. Then the conclusion of Theorem 1.5 holds for M_0: There exist holomorphic ω-maps $F_\delta\colon \mathcal{T}_\delta M_0 \to \mathbf{C}^n$ for all sufficiently small $\delta > 0$ satisfying (1.2) as $\delta \to 0$. Likewise, if the flow f_t as in Theorem 1.7 is defined on M_0' then the conclusion of that theorem applies on the compact subdomain $M_0 \subset M_0'$.

In our last result we consider the problem of approximating a diffeomorphism $f\colon M_0 \to M_1$ by holomorphic ω-automorphisms of $\mathbf{C}^n$. Assuming that M_0 and M_1 are polynomially convex, we have two necessary conditions for such approximation:

(1) $f^*\omega = i_0^*\omega$; and
(2) there is a totally real, polynomially convex flow $f_t\colon M_0 \to M_t \subset \mathbf{C}^n$ $(t \in [0,1])$ with $f_0 = \mathrm{Id}_{M_0}$ and $f_1 = f$.

The second condition is necessary because the group $\mathrm{Aut}_\omega\, \mathbf{C}^n$ is connected (see [FR]). When $\dim M_0$ is smaller than the degree of ω, the first condition is trivial (both sides are zero). We summarize some of the situations when such an approximation is possible. Let β be a holomorphic form on $\mathbf{C}^n$ satisfying $d\beta = \omega$; when ω is given by (1.4) we may take $\beta = \frac{1}{n}\sum_{j=1}^n (-1)^{j-1} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n$, and when ω is the form (1.5) we may take $\beta = \sum_{j=1}^{n'} z_{2j-1}dz_{2j}$.

1.8. THEOREM. *Let* $n, k \ge 2$. *Let* $M_0 \subset \mathbf{C}^n$ *be a compact connected* $\mathcal{C}^k$*-submanifold of dimension* m *and let* $f_t\colon M_0 \to M_t$ $(t \in [0,1])$ *be a totally real, polynomially convex* $\mathcal{C}^k$*-flow. Assume either* (a) *that* ω *is the volume form* (1.4), $d\beta = \omega$, *and at least one of the following four conditions holds*:

(i) $m \le n-2$,
(ii) $m = n-1$ *and* $\mathrm{H}^{n-1}(M_0;\mathbf{R}) = 0$,
(iii) $m = n-1$, M_0 *is closed and orientable, and* $\int_{M_0}\beta = \int_{M_0} f_1^*\beta \ne 0$,
(iv) $m = n$, M_0 *is closed and satisfies* $\mathrm{H}^{n-1}(M_0;\mathbf{R}) = 0$, *and* $f_t^*\omega$ *is independent of* t;

or (b) *that* $n = 2n'$ ($n' \geq 2$), ω *is the form* (1.5), $d\beta = \omega$, *and at least one of the following three conditions holds*:

(v) M_0 *is an arc,*

(vi) M_0 *is a circle and* $\int_{M_0} \beta = \int_{M_0} f_1^*\beta$,

(vii) $m = 2$, M_0 *is closed and satisfies* $\mathrm{H}^1(M_0; \mathbf{R}) = 0$, *and* $f_t^*\omega$ *is independent of* $t \in [0, 1]$.

Set $f = f_1 \colon M_0 \to M_1$. *Then there is a sequence* $F_j \in \mathrm{Aut}_\omega \mathbf{C}^n$ *satisfying* (1.3).

For real-analytic data, Theorem 1.7 was proved in [F2] in the symplectic case and in [F3] in the unimodular case. In the latter situation the sequence $F_j \in \mathrm{Aut}_\omega \mathbf{C}^n$ can be chosen such that it converges to a holomorphic ω-map F in a neighborhood of M_0.

The paper is organized as follows. In Section 2 we collect some preliminary material, mostly extensions of certain well-known results. In Section 3 we construct a family of integral kernels for solving the $\bar\partial$-equation in tubes and prove the stated estimates; we conclude the section by historical remarks concerning such kernels. In Section 4 we apply Theorem 3.1 to prove Theorems 1.2 and 1.3. In Section 5 we solve the equation $dv = u$ in tubes, where u is an exact holomorphic form, and we find a holomorphic solution v satisfying good estimates. In Sections 6 and 7 we prove the results on approximating ω-diffeomorphisms by holomorphic ω-maps and ω-automorphisms. At the end of Section 4 we also include a correction to [FL].

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2. Geometric Preliminaries

We denote by $X \rfloor v$ the contraction of a form v by a vector field X. We shall use the following version of Poincaré's lemma [AMR, Deformation Lemma 6.4.17].

2.1. Lemma. *Let* M *be a* $\mathcal{C}^2$*-manifold and* w *a closed* $\mathcal{C}^1$ p*-form on* $I \times M$ *with* $I = [0, 1]$ *and* $p > 0$. *For* $t \in I$ *let* $i_t \colon M \to I \times M$ *be the injection* $x \to (t, x)$. *Then the* $(p-1)$*-form* $v = \int_0^1 i_t^*\big(\frac{\partial}{\partial t} \rfloor w\big)\, dt$ *on* M *satisfies* $dv = i_1^* w - i_0^* w$. *In particular, let* $F \colon I \times M \to N$ *be a* $\mathcal{C}^2$*-map and* u *a closed* $\mathcal{C}^1$ p*-form on* N *where* $p > 0$. *Setting* $f_t = F \circ i_t \colon M \to N$ *and* $w = F^* u$, *we have* $dv = f_1^* u - f_0^* u$.

We shall apply this to the case when F is a deformation retraction of a tubular neighborhood $\mathcal{T}_\delta = \mathcal{T}_\delta M$ of a submanifold $M \subset \mathbf{C}^n$ onto M. This means that f_1

is the identity on $\mathcal{T}_\delta$, $f_t|_M$ is the identity for all t, and $f_0(\mathcal{T}_\delta) = M$. Set $\pi = f_0$. With u a closed $\mathcal{C}^1$ p-form on $\mathcal{T}_\delta$ and v as before, we obtain $dv = u - \pi^* u$ in $\mathcal{T}_\delta$.

In the situation that we shall consider, we have the following local description of the retraction F. Let M be a $\mathcal{C}^k$-submanifold in $\mathbf{C}^n$. For U a small open neighborhood in M of a point $z_0 \in M$ there is a $\mathcal{C}^k$-diffeomorphism $\phi\colon O \to \pi^{-1}(U)$, where O is open in $\mathbf{R}^m \times \mathbf{R}^{2n-m}$, such that:

(i) $F^{-1}(U) = O \cap (\mathbf{R}^m \times \{0\}^{2n-m}) = O' \times \{0\}^{2n-m}$;
(ii) for $x' \in O'$, the set $O_{x'} = \{y' \in \mathbf{R}^{2n-m} : (x', y') \in O\}$ is star-shaped with respect to 0; and
(iii) the map $f_t = F \circ i_t$ is ϕ-conjugate to $(x', y') \to (x', ty')$ for each $t \in I = [0, 1]$.

Let $u = \sum'_{|I|+|J|=p} u_{I,J}(x', y')dx'^I \wedge dy'^J$ in these coordinates. Then

$$w = \sum_{|I|+|J|=p}{}' u_{I,J}(x', ty')dx'^I \wedge d(ty')^J$$

and it is easy to check that

$$v = \sum_{|I|+|K|=p}{}' (-1)^{|K|} \sum_{j=1}^{n} \sum_{|J|=|K|+1} \varepsilon_J^{jK} y'_j \left(\int_0^1 u_{I,J}(x', ty')t^{|K|}\, dt \right) dx'^I \wedge dy'^K,$$

where ε_J^{jK} equals (if jK is a permutation of J) the signature of that permutation and equals zero otherwise.

The retraction F is constructed by retracting to M along the fibers of a vector bundle supplementary to the tangent bundle TM. The normal bundle to M in $\mathbf{C}^n$ is an obvious choice but is only of class $\mathcal{C}^{k-1}$ when M is a $\mathcal{C}^k$-submanifold. We shall show that there are $\mathcal{C}^k$-subbundles E of $M \times \mathbf{C}^n$ that are arbitrarily close to the normal bundle. When $k > 1$, it is easy to see that $(z + E_z) \cap \mathcal{T}_\delta$ is star-shaped with respect to z for all $z \in M$ when $\delta > 0$ is small enough and E is sufficiently close to the normal bundle. The map $G\colon E \to \mathbf{C}^n$, $G(z, v) = z + v$, maps the zero section 0_E diffeomorphically onto M, and its derivative dG is an isomorphism at each point of 0_E; hence G is a $\mathcal{C}^k$-diffeomorphism of a neighborhood $U_\delta \subset E$ of 0_E onto $\mathcal{T}_\delta$ for $\delta > 0$ small. We may assume that $U_\delta \cap E_z$ is star-shaped with respect to $(z, 0)$ for each $z \in M$. When f_t is G-conjugate to the map $(z, v) \to (z, tv)$ in U_δ for $t \in I$, the map F has the properties (i)–(iii) listed previously.

The local coordinates (x', y') are constructed as follows. Let $\varphi\colon O' \to U \subset M$ be a local $\mathcal{C}^k$-parametrization and let $s_1, \dots, s_{2n-m}$ be sections of $E \to M$ over U that form a $\mathcal{C}^k$-trivialization of $E|_U$. We set

$$\phi(x', y') = \varphi(x') + \sum_{j=1}^{2n-m} y'_j s_j(\varphi(x')) \quad \text{for } x' \in O' \text{ and } y' \in \mathbf{R}^{2n-m},$$

and we restrict it to $O = \phi^{-1}(\mathcal{T}_\delta)$. Then the fiber $O_{x'}$ is star-shaped for all $x' \in O'$ when $\delta > 0$ is small enough.

2.2. LEMMA (Approximation of subbundles). *Let M be a $\mathcal{C}^k$-submanifold of $\mathbf{C}^n$ and $E \to M$ a $\mathcal{C}^l$-subbundle* (*real or complex*) *of $M \times \mathbf{C}^n$ for some $0 \le l < k$.*

Then there is a $\mathcal{C}^k$-subbundle E' of $M \times \mathbf{C}^n$ arbitrarily close to E in the $\mathcal{C}^l$-topology. Moreover, if M is totally real in $\mathbf{C}^n$ and the bundle E is complex, then E' may be taken as the restriction to M of a holomorphic subbundle of $U \times \mathbf{C}^n$ for some open neighborhood U of M in $\mathbf{C}^n$.

Proof. A proof may be based on the following standard result. If $L: M \to \mathrm{Lin}_{\mathbf{C}}(\mathbf{C}^n, \mathbf{C}^n)$ is a $\mathcal{C}^l$-map such that L_z has constant rank r independent of $z \in M$ (abusing the language we shall say that L has rank r), then

$$E_L = \{(z, v) \in M \times \mathbf{C}^n : v \in L_z(\mathbf{C}^n)\} \tag{2.1}$$

is a complex $\mathcal{C}^l$-subbundle of rank r of the trivial bundle $M \times \mathbf{C}^n$, and every subbundle E of $M \times \mathbf{C}^n$ appears in this manner—for instance, by setting L_z to be the orthogonal projection of $\mathbf{C}^n$ onto the fiber E_z for $z \in M$. The analogous result holds for real vector bundles.

A more regular approximation to a subbundle E may then be obtained by approximating the corresponding map L defining E by a more regular map of rank r. The problem is that the rank of a generic perturbation of L may increase. To overcome this we use the following result (see [GLR]).

> *Let C be a positively oriented simple closed curve in $\mathbf{C}$, and let $L \in \mathrm{Lin}_{\mathbf{C}}(\mathbf{C}^n, \mathbf{C}^n)$ be a linear map with no eigenvalues on C. Then $\mathbf{C}^n = V_+ \oplus V_-$, where V_+ (resp., V_-) is a L-invariant subspace of $\mathbf{C}^n$ spanned by the generalized eigenvectors of L inside (resp., outside) of C. The map*
>
> $$P(L) = \frac{1}{2\pi i} \int_C (\zeta I - L)^{-1}\, d\zeta \tag{2.2}$$
>
> *is the projection onto V_+ with kernel V_-.*

Note that $P(L)$ depends holomorphically on L; thus, if L depends $\mathcal{C}^k$ or holomorphically on a parameter, so does $P(L)$.

We now take C to be a curve that encircles 1 but not 0; for instance,

$$C = \{\zeta \in \mathbf{C} : |\zeta - 1| = 1/2\}. \tag{2.3}$$

Let P be the associated projection operator (2.2). If L is a projection then $P(L) = L$. Moreover, for each L' sufficiently near a projection L, each eigenvalue of L' is either near 0 or near 1 and hence $P(L')$ is a projection with the same rank as L.

Thus, to smooth E, let L_z be the orthogonal projection onto E_z for $z \in M$; we approximate L by a $\mathcal{C}^k$-map $L': M \to \mathrm{Lin}_{\mathbf{C}}(\mathbf{C}^n, \mathbf{C}^n)$ and let E' be the bundle (2.1) associated to $P(L')$. By (2.2), the difference equals

$$P(L') - L = \frac{1}{2\pi i} \int_C ((\zeta I - L')^{-1} - (\zeta I - L)^{-1})\, d\zeta$$

and is $\mathcal{C}^l$-small when $L' - L$ is.

In the real case, we extend $L: \mathbf{R}^n \to \mathbf{R}^n$ to a complex linear map $L: \mathbf{C}^n \to \mathbf{C}^n$ and observe that $P(L)$ is also real (i.e., it maps $\mathbf{R}^n$ to itself) when C is the curve (2.3). Hence the restriction of $P(L)$ to $\mathbf{R}^n$ solves the problem.

Let now M be a totally real submanifold of $\mathbf{C}^n$ and $E \to M$ a $\mathcal{C}^l$ rank-r complex subbundle of $M \times \mathbf{C}^n$. For each $z \in M$ let $L_z : \mathbf{C}^n \to E_z$ be the orthogonal projection onto E_z. By [RS] we can approximate the $\mathcal{C}^l$-map $L : M \to \mathrm{Lin}_{\mathbf{C}}(\mathbf{C}^n, \mathbf{C}^n)$ as well as we like in the $\mathcal{C}^l$-topology on M by the restriction to M of a holomorphic map $L' : U \to \mathrm{Lin}_{\mathbf{C}}(\mathbf{C}^n, \mathbf{C}^n)$ defined on an open neighborhood $U \subset \mathbf{C}^n$ of M. By shrinking U we may assume that L'_z has exactly r eigenvalues inside C in (2.3) for each $z \in U$, so $P(L'_z)$ is a rank-r projection. The map $z \to P(L'_z)$ is holomorphic in U and determines a holomorphic rank-r vector bundle E' over U, with $E'|_M$ close to M. □

Let $d = \partial + \bar\partial$ be the splitting of the exterior derivative on a complex manifold.

DEFINITION 3 ($\bar\partial$-flat functions). If M is a closed subset in a complex manifold X and if u is a $\mathcal{C}^k$-function ($k \ge 1$) defined in a neighborhood of M in X, then we say that u is *$\bar\partial$-flat* (to order k) *on M* if $\partial^\alpha(\bar\partial u)(z) = 0$ for each $z \in M$ and for each derivative ∂^α of total order $|\alpha| \le k-1$ with respect to the underlying real local coordinates on X.

We shall commonly use the phrase "u is a $\bar\partial$-flat $\mathcal{C}^k$-function" when it is clear from the context which subset $M \subset X$ is meant.

2.3. LEMMA ($\bar\partial$-flat partitions of unity). *Let M be a totally real $\mathcal{C}^k$-submanifold of a complex manifold X where $k \ge 1$. For every open covering $\mathcal{U}$ of M in X there exists a $\mathcal{C}^k$-partition of unity on a neighborhood of M in X that is subordinate to the covering $\mathcal{U}$ and consists of functions that are $\bar\partial$-flat to order k on M.*

Proof. We may assume that $\mathcal{U}$ consists of coordinate neighborhoods. Let ϕ^0_ν be a $\mathcal{C}^k$-partition of unity subordinate to $\mathcal{U}|_M = \{U \cap M : U \in \mathcal{U}\}$. We may assume that the index sets agree, so $\operatorname{supp}\phi^0_\nu \subset U_\nu$ for each ν. By passing to local coordinates we may find a $\bar\partial$-flat $\mathcal{C}^k$-extension $\tilde\phi_\nu$ of ϕ^0_ν with $\operatorname{supp}\tilde\phi_\nu \subset U_\nu$. Since $\rho = \sum_\nu \tilde\phi_\nu = 1$ on M, it follows that $\rho \ne 0$ in a neighborhood V of M in X. It is immediate that $\phi_\nu = \tilde\phi_\nu/\rho$ is a $\mathcal{C}^k$-partition of unity on V that is $\bar\partial$-flat (to order k) on M. □

As a consequence of Lemma 2.3, we see that the usual results about $\bar\partial$-flat extensions of maps into $\mathbf{C}^N$ are also valid for totally real submanifolds in arbitrary complex manifolds.

2.4. LEMMA (Asymptotic complexifications). *Let M be a totally real $\mathcal{C}^k$-submanifold ($k \ge 1$) of $\mathbf{C}^n$ of real dimension $m \le n$. Then there exists a $\mathcal{C}^k$-submanifold $\tilde M \supset M$ in $\mathbf{C}^n$, of real dimension $2m$, with the following property: $\tilde M$ may be covered by $\mathcal{C}^k$ local parametrizations $Z : U \to Z(U) \subset \tilde M$, with $U \subset \mathbf{C}^m$ open subsets, such that $Z^{-1}(M) = U \cap \mathbf{R}^m$ and Z is $\bar\partial$-flat on $U \cap \mathbf{R}^m$. Moreover, there is a $\mathcal{C}^k$-retraction of a neighborhood of $\tilde M$ in $\mathbf{C}^n$ onto $\tilde M$ that is $\bar\partial$-flat on M.*

Proof. By a theorem of Whitney [W2, Thm. 1], there exists a $\mathcal{C}^\omega$-manifold M_0 and a $\mathcal{C}^k$-diffeomorphism $G^0\colon M_0 \to M$. The manifold M_0 has a complexification $\tilde M_0$ that is a complex manifold containing M_0 as a maximal real submanifold. The map G^0 has a $\bar\partial$-flat extension $G\colon \tilde M_0 \to \mathbf{C}^n$ that is an injective immersion at M. (To obtain G it suffices to patch local $\bar\partial$-flat extensions of G_0 by a $\bar\partial$-flat partition of unity provided by Lemma 2.3.) Hence G maps a neighborhood of M_0 in $\tilde M_0$ diffeomorphically onto its image $\tilde M \subset \mathbf{C}^n$. When $Z^0\colon U \to \tilde M_0$ (U open in $\mathbf{C}^m$) is a local holomorphic parametrization with $(Z^0)^{-1}(M_0) = U \cap \mathbf{R}^m$, the map $Z = G \circ Z^0\colon U \to \tilde M$ is a local parametrization of the type described in Lemma 2.4. Note that $T_z\tilde M = T_z^{C} M$ for each $z \in M$.

Next we prove the existence of a retraction onto $\tilde M$ that is $\bar\partial$-flat on M. Let $\nu \to M$ be the complex normal bundle of M in $\mathbf{C}^n$. By Lemma 2.2 there is an open neighborhood O of M in $\mathbf{C}^n$ and a holomorphic rank-$(n-m)$ subbundle $N \subset O \times \mathbf{C}^n$ such that $N|_M$ approximates ν well. By shrinking O we may assume that N is transversal to $\tilde M$ in O. This means that the map $\phi\colon N|_{\tilde M} \to \mathbf{C}^n$, $\phi(z,v) = z+v$, is a $\mathcal{C}^k$-diffeomorphism from a neighborhood W of the zero section in $N|_{\tilde M}$ onto its image $O_0 \subset O \subset \mathbf{C}^n$. We may assume that $W \cap N_z$ is star-shaped with respect to $0_z \in N_z$ for each $z \in \tilde M$. Now the deformation retraction $(z,v) \to (z,tv)$ ($t \in [0,1]$) of W onto the zero section in $N|_{\tilde M}$ may be transported by ϕ to a retraction $F\colon [0,1] \times O_0 \to O_0$ of O_0 onto the submanifold $\tilde M \cap O_0$. Set $\pi = F_0\colon O_0 \to \tilde M \cap O_0$. Let $U \subset \mathbf{C}^m$ and let $Z\colon U \to \tilde M$ be a local $\mathcal{C}^k$-parametrization such that $Z(U \cap \mathbf{R}^m) \subset M$ and Z is $\bar\partial$-flat on $U \cap \mathbf{R}^m$. Choose holomorphic sections $s_1, \dots, s_{n-m}$ of N that provide a trivialization of N near $Z(U)$. Then

$$(z', w') \to Z(z') + \sum_{j=1}^{n-m} w'_j s_j(Z(z'))$$

is a $\mathcal{C}^k$-diffeomorphism of a neighborhood W of $U \times \{0\}^{n-m}$ in $\mathbf{C}^n$ onto $\pi^{-1}(Z(U))$, and it is $\bar\partial$-flat on $(U \cap \mathbf{R}^m) \times \{0\}^{n-m}$. In these coordinates the maps F_t are given by $(z', w') \to (z', tw')$; hence F_t is $\bar\partial$-flat on $\pi^{-1}(M)$. □

2.5. Lemma (Rough multiplication). *Let U be an open set in $\mathbf{R}^N$, $f \in \mathcal{C}^k(U)$ and $g \in \mathcal{C}^{k-1}(U)$, where $k \ge 1$. Let E be a closed subset of U such that $f(x) = 0$ for all $x \in E$. Then there exists a function $h \in \mathcal{C}^k(U)$ such that*:

(i) $|\partial^\alpha(h - fg)| = o(d_E^{k-|\alpha|})$ *for $|\alpha| < k$, uniformly on compacts in U;*
(ii) *at points of E we have $\partial^\alpha h = \sum_{0 \ne \beta \le \alpha} \binom{\alpha}{\beta} \partial^\beta f\, \partial^{\alpha-\beta} g$ for $|\alpha| \le k$; and*
(iii) *if $U \subset \mathbf{C}^N$ and if f and g as before are $\bar\partial$-flat on $E \subset f^{-1}(0)$, then so is h.*

Proof. The proof is similar to the better known "Glaeser–Kneser rough composition theorem"; the main point is to verify that the collection of functions $(\partial^\alpha h)_{|\alpha| \le k}$ on E, defined by (ii), are a Whitney system (i.e., that they satisfy the assumptions of the Whitney extension theorem; see [W1] or [T]). We shall leave out the details of this verification. Let h be a $\mathcal{C}^k$-function, provided by Whitney's theorem, whose partial derivatives are given by (ii) at points of E. Then (i) follows easily by comparing the Taylor expansions of $\partial^\alpha h$, $\partial^\beta f$, and $\partial^{\alpha-\beta} g$ about the nearest point in E. Case (iii) follows from (ii), which is seen as follows. From (ii) we get, at points of E,

$$\partial^\alpha \bar\partial h = \partial^\alpha(\bar\partial f \cdot g) + \sum_{0\neq\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta f\, \partial^{\alpha-\beta}\bar\partial g$$

for $|\alpha| \leq k-1$. If f and g are $\bar\partial$-flat on E, then this expression vanishes when $|\alpha| \leq k-1$ and so we get (iii). □

The following lemma is needed in the proof of Theorem 1.4 and its corollaries.

2.6. LEMMA. *Let M be a totally real, m-dimensional $\mathcal{C}^k$-submanifold of $\mathbf{C}^n$, $f\colon M \to \mathbf{C}^p$ a $\mathcal{C}^k$-map, and $l\colon M \to \mathrm{Lin}_{\mathbf{C}}(\mathbf{C}^n, \mathbf{C}^p)$ a $\mathcal{C}^{k-1}$-map such that, for each $z \in M$, l_z agrees with df_z on T_zM. Then there is a neighborhood $U \subset \mathbf{C}^n$ of M and a $\mathcal{C}^k$-map $F\colon U \to \mathbf{C}^p$ that is $\bar\partial$-flat on M and satisfies $F(z) = f(z)$ and $dF_z = l_z$ for all $z \in M$.*

Proof. It suffices to prove the result for functions ($p = 1$); the general case then follows by applying it componentwise. So we shall assume $p = 1$.

We first consider the local case. Fix a point $z_0 \in M$. Choose $e_1, \dots, e_{n-m} \in \mathbf{C}^n$ such that these vectors, together with the tangent space $T_{z_0}M$, span a totally real subspace of $T_{z_0}\mathbf{C}^n$ of maximal dimension n. If $\kappa\colon U \to M$ is a $\mathcal{C}^k$-parametrization of a small neighborhood of z_0 in M with $\kappa(0) = z_0$ and if V is a sufficiently small neighborhood of 0 in $\mathbf{R}^{n-m}$, then the map $\phi(x, y) = \kappa(x) + \sum_{j=1}^{n-m} y_j e_j$ ($x \in U$, $y \in V$) is a $\mathcal{C}^k$-diffeomorphism onto an n-dimensional totally real submanifold in $\mathbf{C}^n$. Observe that, for $x \in U$ and $(u, v) \in \mathbf{R}^m \times \mathbf{R}^{n-m}$, we have

$$l_{\kappa(x)} \circ d\phi_{(x,0)}(u, v) = df_{\kappa(x)} \circ d\kappa_x(u) + \sum_{j=1}^{n-m} v_j l_{\kappa(x)}(e_j).$$

Since $l_{\kappa(x)}(e_j)$ is only of class $\mathcal{C}^{k-1}$ in x, we apply the rough multiplication lemma to the pairs $y_j, l_{\kappa(x)}(e_j)$ for $1 \leq j \leq n-m$ to get a $\mathcal{C}^k$ function h on $U \times V$ satisfying $\frac{\partial h}{\partial x_i}(x, 0) = 0$ and $\frac{\partial h}{\partial y_j}(x, 0) = l_{\kappa(x)}(e_j)$ for $1 \leq i \leq m$ and $1 \leq j \leq n-m$. With $F^0(x, y) = f(\kappa(x)) + h(x, y)$ it follows that $dF^0_{(x,0)} = l_{\kappa(x)} \circ d\phi_{(x,0)}$. When $\tilde F^0$ (resp., $\tilde\phi$) is a $\mathcal{C}^k$-extension of F^0 (resp., ϕ) that is $\bar\partial$-flat on $\mathbf{R}^n$, we see that $\tilde\phi$ is a $\mathcal{C}^k$-diffeomorphism of a neighborhood of $0 \in \mathbf{C}^n$ onto a neighborhood of $z_0 \in \mathbf{C}^n$. Thus, near z_0, $F = \tilde F^0 \circ \tilde\phi^{-1}$ is a $\mathcal{C}^k$ $\bar\partial$-flat extension of f. When $z \in M$ we have $dF_z = l_z$ on a maximal totally real subspace, so these two linear maps are equal on $T_z\mathbf{C}^n$. This establishes the local case.

For the global case let $\mathcal{U} = \{U_i\}$ be an open covering of M and let $F^{(i)}$ be a $\bar\partial$-flat extension of f in U_i, with $dF^{(i)}_z = l_z$ for $z \in U_i \cap M$. By Lemma 2.3 there is a partition of unity $\{\phi_i\}$ by $\bar\partial$-flat $\mathcal{C}^k$-functions on a neighborhood of M subordinate to $\mathcal{U}$. We set $F = \sum_i \phi_i F^{(i)}$, where the term with index i is zero outside U_i. When $z \in M$, $dF_z = \sum_i \phi_i(z) dF^{(i)}_z + \sum_i f(z) d(\phi_i)_z$. Since $\sum_i \phi_i = 1$, it follows that $\sum_i d\phi_i = 0$ and we have $dF_z = l_z$. □

3. Solving the $\bar\partial$-Equation in Tubes around Totally Real Manifolds

In this section we construct a family of integral kernels, depending on a parameter $\delta > 0$, for solving the $\bar\partial$-equation in tubes $\mathcal{T}_\delta M$ around compact totally real submanifolds $M \subset \mathbf{C}^n$ of class $\mathcal{C}^1$. The main result is Theorem 3.1, which is identical to Theorem 1.1 except that it contains additional Hölder estimates (3.3) and (3.4).

We denote by d_M the Euclidean distance to M. If M is of class $\mathcal{C}^k$, it is well known that $\rho = d_M^2$ is a $\mathcal{C}^k$ strictly plurisubharmonic function in a neighborhood of M when $k > 1$; when $k = 1$, there is a strictly plurisubharmonic $\mathcal{C}^2$-function ρ such that $\rho = d_M^2 + o(d_M^2)$. As in Section 1, let $\mathcal{T}_\delta$ denote the tubular neighborhood of M of radius δ, that is, the set of points whose distance to M is less than δ.

For a domain D in $\mathbf{R}^n$ (or in $\mathbf{C}^n$), a bounded function u in D belongs to the Hölder class $\Lambda^s(D)$ for some $0 < s < 1$ if $|u|_{s,D} := \sup\{|u(z+h) - u(z)||h|^{-s} : h \neq 0,\ z, z+h \in D\} < \infty$; in this case the Hölder s-norm of u is defined by $\|u\|_{\Lambda^s(D)} = \|u\|_{L^\infty(D)} + |u|_{s,D}$. When $s = 1$ we set

$$|u|_{1,D} := \sup\{|u(z+h) + u(z-h) - 2u(z)||h|^{-1} : h \neq 0,\ z, z-h, z+h \in D\};$$

$\Lambda^1(D)$ is called the *Zygmund class* on D. When D is a tubular neighborhood $\mathcal{T}_\delta M$ of a submanifold M, we write $|u|_{s,\delta}$ for $|u|_{s,\mathcal{T}_\delta M}$. When $s = k + \alpha$ with $k \in \mathbf{Z}_+$ and $0 < \alpha \leq 1$, we take $\|u\|_{\Lambda^s(D)} = \|u\|_{\mathcal{C}^k(D)} + |D^k u|_{\alpha,D}$. We sometimes write $\mathcal{C}^{k+\alpha}(D)$ for $\Lambda^{k+\alpha}(D)$ when $0 < \alpha < 1$.

We extend function space norms to vector fields or differential forms on open sets in $\mathbf{R}^n$ as the sum of the norms of the components. When M is a compact $\mathcal{C}^k$-manifold, we define the norms on functions or forms on M as follows. Let $\Phi_j \colon U_j \to V_j \subset M$, $j = 1, \ldots, p$, be a covering of M by local parametrizations, and let $\{\phi_1, \ldots, \phi_p\}$ be a $\mathcal{C}^k$-partition of unity subordinate to the covering $\{V_1, \ldots, V_p\}$ of M. Then we set $\|u\| = \sum_{j=1}^p \|\Phi_j^*(\phi_j u)\|$, where $\|\cdot\|$ is a Hölder or some other function space norm. Different choices of $\{\Phi_j\}$ and $\{\phi_j\}$ give rise to equivalent norms on the same space.

Let $z = (z_1, \ldots, z_n)$ be the complex coordinates and let $(x_1, y_1, \ldots, x_n, y_n)$ ($z_j = x_j + iy_j$) be the underlying real coordinates on $\mathbf{C}^n = \mathbf{R}^{2n}$. For $1 \leq j \leq 2n$, ∂_j denotes the partial derivative with respect to the jth variable. If $\alpha = (\alpha_1, \ldots, \alpha_{2n})$ is a multi-index of length $2n$, then ∂^α denotes the corresponding partial derivative of order $|\alpha| = \alpha_1 + \cdots + \alpha_{2n}$ with respect to the real variables on $\mathbf{C}^n = \mathbf{R}^{2n}$.

If f is a function or a form near M, we shall say that f *vanishes to order l on M* if $|f(z)| = o(d_M(z)^l)$ and $\partial^\alpha f = 0$ on M when $|\alpha| \leq l$. Recall that any $\mathcal{C}^k$-function f on M can be extended to a $\mathcal{C}^k$-function on $\mathbf{C}^n$ such that $\bar\partial f$ vanishes to order $k-1$ on M [HöW, Lemma 4.3].

We call a continuous function $\omega \colon \mathbf{R}_+ \to \mathbf{R}_+$ a *modulus of continuity* if it is nondecreasing and subadditive and if $\omega(0) = 0$. If $f \colon A \to \mathbf{C}$ ($A \subset \mathbf{R}^n$) is uniformly continuous, we define the modulus of continuity of f by $\omega(f, t) = \sup\{|f(x) - f(y)| \colon |x - y| \leq t\}$, $t \geq 0$, where $\omega(f, \cdot)$ is clearly a modulus of continuity as defined previously. We say that a modulus of continuity ω is a modulus of continuity for a function f if $\omega(f,t) \le \omega(t)$ for all $t \ge 0$. If f is a form on A, then $\omega(f,t)$ is defined as the sum of the moduli of continuity of its components.

We denote by $\mathcal{C}^l_{p,q}(U)$ the space of (p,q)-forms of class $\mathcal{C}^l$ on an open set $U \subset \mathbf{C}^n$.

3.1. THEOREM. *Let $M \subset \mathbf{C}^n$ be a closed totally real $\mathcal{C}^1$-submanifold and let $0 < c < 1$. Denote by $\mathcal{T}_\delta$ the tube of radius $\delta > 0$ around M. Then there is a $\delta_0 > 0$ and, for each integer $l \ge 1$, a constant $C_l > 0$ such that the following hold for $0 < \delta \le \delta_0$ with $p \ge 0$ and $q \ge 1$: For each $u \in \mathcal{C}^l_{p,q}(\mathcal{T}_\delta)$ with $\bar\partial u = 0$, there is a $v \in \mathcal{C}^l_{(p,q-1)}(\mathcal{T}_\delta)$ satisfying $\bar\partial v = u$ in $\mathcal{T}_{c\delta}$ and*

$$\|\partial^\alpha v\|_{L^\infty(\mathcal{T}_{c\delta})} \le C_l(\delta\|\partial^\alpha u\|_{L^\infty(\mathcal{T}_\delta)} + \delta^{1-|\alpha|}\|u\|_{L^\infty(\mathcal{T}_\delta)}), \quad |\alpha| \le l. \tag{3.1}$$

In particular we have $\|v\|_{L^\infty(\mathcal{T}_{c\delta})} \le C\delta\|u\|_{L^\infty(\mathcal{T}_\delta)}$. If $q = 1$ and the equation $\bar\partial v = u$ has a solution $v_0 \in \mathcal{C}^{l+1}_{(p,0)}(\mathcal{T}_\delta)$, then there is a solution $v \in \mathcal{C}^{l+1}_{(p,0)}(\mathcal{T}_\delta)$ of $\bar\partial v = u$ satisfying, for $1 \le j \le n$,

$$\|\partial_j\partial^\alpha v\|_{L^\infty(\mathcal{T}_{c\delta})} \le C_{l+1}(\omega(\partial_j\partial^\alpha v_0, \delta) + \delta^{-l}\|u\|_{L^\infty(\mathcal{T}_\delta)}), \quad |\alpha| = l. \tag{3.2}$$

If we assume in addition that $\partial^\alpha u \in \Lambda^s(\mathcal{T}_\delta)$ for some $|\alpha| \le l$ and $0 < s \le 1$, we may choose v as above satisfying also the following estimates (with constants $C_{l,s}$ independent of u and δ):

$$\|\partial_j\partial^\alpha v\|_{L^\infty(\mathcal{T}_{c\delta})} \le C_{l,s}(\delta^s\|\partial^\alpha u\|_{\Lambda^s(\mathcal{T}_\delta)} + \delta^{-|\alpha|}\|u\|_{L^\infty(\mathcal{T}_\delta)}), \tag{3.3}$$

$$\|\partial_j\partial^\alpha v\|_{\Lambda^s(\mathcal{T}_{c\delta})} \le C_{l,s}(\|\partial^\alpha u\|_{\Lambda^s(\mathcal{T}_\delta)} + \delta^{-|\alpha|-s}\|u\|_{L^\infty(\mathcal{T}_\delta)}) \tag{3.4}$$

REMARKS. (1) If u is of class $\mathcal{C}^l$ then there is, in general, no $\mathcal{C}^{l+1}$ solution v to $\bar\partial v = u$.

(2) In (3.3) one may be tempted to delete δ^s and use instead the $L^\infty(\mathcal{T}_\delta)$-norm of $\partial^\alpha u$ in the first term on the right-hand side. Yet that this proves false even when $n = 1$ is a well-known phenomenon. Since the Bochner–Martinelli operator used in the proof is a homogeneous convolution operator, it gains one derivative in norms such as Hölder, Zygmund, and Sobolev, but not in the sup-norm or the $\mathcal{C}^l$-norm.

(3) Theorem 3.1 has the following extension to nonclosed totally real $\mathcal{C}^1$-submanifolds M' in $\mathbf{C}^n$. Let K be a compact subset of M' and let $K' \subset M'$ be a compact neighborhood of K in M'. For $\delta > 0$ we set

$$U_\delta = \{z \in \mathbf{C}^n : d_K(z) < \delta\}, \qquad U'_\delta = \{z \in \mathbf{C}^n : d_{K'}(z) < \delta\}.$$

Choose $c \in (0,1)$. Given a form $u \in \mathcal{C}^l_{p,q}(U'_\delta)$ with $\bar\partial u = 0$, we can solve $\bar\partial v = u$ in $U_{c\delta}$; the estimates in Theorem 3.1 remain valid when the tube $\mathcal{T}_{c\delta}$ is replaced by $U_{c\delta}$ on the left-hand side and $\mathcal{T}_\delta$ is replaced by U'_δ on the right-hand side of each estimate. The proof can be obtained by simple modifications of the kernel construction that follows. This applies to compact totally real submanifolds with boundary in $\mathbf{C}^n$ since any such is a compact domain in a larger totally real submanifold.

In this section, C denotes some constant whose value may change every time it occurs but does not depend on quantities such as u, δ, etc.

For $(0,1)$-forms u (and hence for $(p,1)$-forms, $0 \le p \le n$), a large part of the result comes from the interior elliptic regularity of the $\bar\partial$-operator and has nothing to do with the particular solution v.

3.2. LEMMA. *Let $0 < c < 1$. There exist constants $C_l > 0$ ($l \in \mathbf{N}$) satisfying the following. If $K \subset \mathbf{C}^n$ is a compact subset with $\mathcal{T}_\delta = \{z : d(z,K) < \delta\}$ and if v is a continuous function in $\mathcal{T}_\delta$ such that $\bar\partial v \in \mathcal{C}^l_{(0,1)}(\mathcal{T}_\delta)$ for some $l \in \mathbf{N}$, then $v \in \mathcal{C}^l(\mathcal{T}_\delta)$ and*

$$\|\partial^\alpha v\|_{L^\infty(\mathcal{T}_{c\delta})} \le C_l(\delta\|\partial^\alpha\bar\partial v\|_{L^\infty(\mathcal{T}_\delta)} + \delta^{-|\alpha|}\|v\|_{L^\infty(\mathcal{T}_\delta)}); \quad |\alpha| \le l.$$

If $\bar\partial v = \bar\partial f$ for some $f \in \mathcal{C}^{l+1}(\mathcal{T}_\delta)$, then v is also of class $\mathcal{C}^{l+1}$ and satisfies

$$\|\partial_j\partial^\alpha v\|_{L^\infty(\mathcal{T}_{c\delta})} \le C_{l+1}(\omega(\partial_j\partial^\alpha f,\delta) + \delta^{-l-1}\|v\|_{L^\infty(\mathcal{T}_\delta)}); \quad |\alpha| = l.$$

Proof. We apply the Bochner–Martinelli formula

$$g(z) = \int_{\partial\mathcal{T}_\delta} g(\zeta)B(\zeta,z) - \int_{\mathcal{T}_\delta} \bar\partial g(\zeta)\wedge B(\zeta,z),$$

valid for $g \in \mathcal{C}^1(\bar{\mathcal{T}}_\delta)$, where $B(\zeta,z)$ is the Bochner–Martinelli (B-M) kernel

$$B(\zeta,z) = c_n\sum_{j=1}^n(-1)^{j-1}\frac{\overline{\zeta_j - z_j}}{\|\zeta - z\|^{2n}}d\bar\zeta[j]\wedge d\zeta,$$

which is a closed integrable $(n,n-1)$-form. Let $z \in \mathcal{T}_{c\delta}$ and let $\chi : \mathbf{R} \to [0,1]$ be a cut-off function with $\chi(t) = 1$ when $|t| \le \frac{1}{2}$ and $\chi(t) = 0$ when $|t| \ge 1$. For $w \in \mathbf{C}^n$, set $\chi_\delta(w) = \chi\left(\frac{2|w|}{(1-c)\delta}\right)$. It follows that the partial derivatives satisfy $|\partial^\alpha\chi_\delta| \le C_\alpha\delta^{-|\alpha|}$ for all α. Applying the Bochner–Martinelli formula to $g(\zeta) = \chi_\delta(\zeta - z)v(\zeta)$, we obtain

$$\begin{aligned} v(z) &= -\int_{\mathcal{T}_\delta}\bar\partial_\zeta(\chi_\delta(\zeta - z)v(\zeta))\wedge B(\zeta,z) \\ &= -\int_{\mathcal{T}_\delta}\bar\partial v(\zeta)\wedge\chi_\delta(\zeta - z)B(\zeta,z) - \int_{\mathcal{T}_\delta} v(\zeta)\,\bar\partial_\zeta\chi_\delta(\zeta - z)\wedge B(\zeta,z) \\ &= I_1(z) + I_2(z). \end{aligned} \tag{3.5}$$

These are convolution operators and we may differentiate on either integrand. This gives, for $|\alpha| \le l$,

$$\begin{aligned} \partial^\alpha v(z) &= \partial^\alpha I_1(z) + \partial^\alpha I_2(z) \\ &= -\int_{\mathcal{T}_\delta}\partial^\alpha\bar\partial v(\zeta)\wedge\chi_\delta(\zeta - z)B(\zeta,z) \\ &\quad - \int_{\mathcal{T}_\delta} v(\zeta)\,\partial_z^\alpha(\bar\partial\chi_\delta(\zeta - z)\wedge B(\zeta,z)). \end{aligned}$$

Setting $c' = \frac{1-c}{2}$ and $c'' = \frac{1}{2}c'$ and using $|B(\zeta, z)| \le C|\zeta - z|^{1-2n}$, we can estimate the integrals for $|\alpha| \le l$ as follows:

$$\begin{aligned}|\partial^\alpha I_1(z)| &\le C \int_{|\zeta - z| \le c'\delta} |\partial^\alpha \bar\partial v(\zeta)| \cdot |\zeta - z|^{1-2n}\, dV \\ &\le C\|\partial^\alpha \bar\partial v\|_{L^\infty(\mathcal{T}_\delta)} \int_{|\zeta - z| \le c'\delta} |\zeta - z|^{1-2n}\, dV \\ &\le C\|\partial^\alpha \bar\partial v\|_{L^\infty(\mathcal{T}_\delta)} \int_0^{c'\delta} \frac{r^{2n-1}dr}{r^{2n-1}} \le C\delta\|\partial^\alpha \bar\partial v\|_{L^\infty(\mathcal{T}_\delta)}, \\ |\partial^\alpha I_2(z)| &\le C \int_{c''\delta \le |\zeta - z| \le c'\delta} |v(\zeta)| \cdot \delta^{-2n-|\alpha|}\, dV \le C\|v\|_{L^\infty(\mathcal{T}_\delta)} \delta^{-|\alpha|}.\end{aligned}$$

This proves the first estimate in Lemma 3.2. The estimate for $|\partial^\alpha I_2|$ also holds for derivatives of order $|\alpha| = l + 1$.

We now assume that $\bar\partial v = \bar\partial f$ for some $f \in \mathcal{C}^{l+1}(\mathcal{T}_\delta)$; then $v - f$ is holomorphic and hence v is also $\mathcal{C}^{l+1}$. We wish to estimate the derivatives of order $l+1$ of $I_1(z)$. For $|\alpha| = l$ we have

$$\partial_j \partial^\alpha I_1(z) = -\partial_j \int_{\mathcal{T}_\delta} \partial^\alpha \bar\partial v(\zeta) \wedge \chi_\delta(\zeta - z) B(\zeta, z).$$

We now apply (3.5) to f, replacing $\bar\partial f$ by $\bar\partial v (= \bar\partial f)$ in the first term on the right-hand side and differentiating under the integral, to get

$$\begin{aligned}\partial_j \partial^\alpha f(z) = -\partial_j &\int_{\mathcal{T}_\delta} \partial^\alpha \bar\partial v(\zeta) \wedge \chi_\delta(\zeta - z) B(\zeta, z) \\ &- \int_{\mathcal{T}_\delta} \partial_j \partial^\alpha f(\zeta) \wedge \bar\partial \chi_\delta(\zeta - z) \wedge B(\zeta, z).\end{aligned}$$

Observe that the first term on the right-hand side equals $\partial_j \partial^\alpha I_1(z)$ from the previous display. For a fixed $z \in \mathbf{C}^n$ we also apply (3.5) to the constant function $\partial_j \partial^\alpha f(z)$:

$$\partial_j \partial^\alpha f(z) = -\int_{\mathcal{T}_\delta} \partial_j \partial^\alpha f(z)\, \bar\partial \chi_\delta(\zeta - z) \wedge B(\zeta, z).$$

Combining the three preceding formulas yields

$$\partial_j \partial^\alpha I_1(z) = \int_{\mathcal{T}_\delta} (\partial_j \partial^\alpha f(\zeta) - \partial_j \partial^\alpha f(z))\, \bar\partial \chi_\delta(\zeta - z) \wedge B(\zeta, z)$$

and hence $|\partial_j \partial^\alpha I_1(z)| \le C\omega(\partial_j \partial^\alpha f, \delta)$. □

From Lemma 3.2 it follows that the estimates (3.1) and (3.2) in Theorem 3.1 will be proved for $(p, 1)$-forms u if we can find a solution v that satisfies a sup-norm estimate $\|v\|_{L^\infty(\mathcal{T}_{c\delta})} \le C\delta\|u\|_{L^\infty(\mathcal{T}_\delta)}$. Such a solution is obtained by a linear operator given by an integral kernel that we now construct.

Construction of the Kernel for $(0,1)$*-Forms*

We shall use Koppelman's formula, which we now recall. For $v, w \in \mathbf{C}^n$, let $\langle v, w\rangle = \sum_{i=1}^n v_i w_i$. Let $V \subset \mathbf{C}^{2n}$ and $\Omega' \subset \Omega \subset\subset \mathbf{C}^n$ be open subsets such that Ω has piecewise $\mathcal{C}^1$-boundary and $\bar\Omega \times \Omega \subset V$. Let $P = P(\zeta, z) = (P_1, \dots, P_n)\colon V \to \mathbf{C}^n$ be a $\mathcal{C}^1$-map satisfying

(i) $P(\zeta, z) = \overline{\zeta - z}$ in a neighborhood of the diagonal of $\Omega' \times \Omega'$ and
(ii) the function $\Phi\colon V \to \mathbf{C}$, $\Phi(\zeta, z) = \langle P(\zeta, z), \zeta - z\rangle$, satisfies $\Phi(\zeta, z) \neq 0$ when $z \in \Omega'$ and $\zeta \in \bar\Omega \backslash \{z\}$.

Any such map P is called a *Leray map* for the pair $\Omega' \subset \Omega$, and Φ is the corresponding *support function*. We shall use the notation

$$d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_n,$$

$$\bar\partial_\zeta P[j] = \bar\partial_\zeta P_1 \wedge \cdots \wedge \widehat{\bar\partial_\zeta P_j} \wedge \cdots \wedge \bar\partial_\zeta P_n,$$

$$\bar\partial_\zeta P[i,j] = \bar\partial_\zeta P_1 \wedge \cdots \wedge \widehat{\bar\partial_\zeta P_i} \wedge \cdots \wedge \widehat{\bar\partial_\zeta P_j} \wedge \cdots \wedge \bar\partial_\zeta P_n.$$

Define the integral kernels

$$K(\zeta, z) = c_n \Phi(\zeta, z)^{-n} \sum_{j=1}^n (-1)^{j-1} P_j\, \bar\partial_\zeta P[j] \wedge d\zeta,$$

$$L(\zeta, z) = c_n \Phi(\zeta, z)^{-n} \sum_{i \neq j} (-1)^{i+j} P_j\, \bar\partial_z P_i \wedge \bar\partial_\zeta P[i,j] \wedge d\zeta.$$

Note that the kernel $K(\zeta, z)$ is locally integrable when $z \in \Omega'$. It is also important to observe that, if $a(\zeta, z)$ is a $\mathcal{C}^1$ function, then the kernels generated by P (resp., aP) are identical outside the zero set $a = 0$. For a suitable choice of the constant $c_n \in \mathbf{R}$, we then have the following Koppelman–Leray representation formula for $\bar\partial$-closed $(0,1)$-forms $u \in \mathcal{C}^1_{0,1}(\bar\Omega)$:

$$u(z) = \int_{\partial\Omega} L(\zeta, z) \wedge u(\zeta) + \bar\partial_z \int_\Omega K(\zeta, z) \wedge u(\zeta), \quad z \in \Omega'. \tag{3.6}$$

This follows by applying the Stokes formula to the first integral on the right-hand side to transfer the integration to an ε-sphere around z and using $\bar\partial_z K = -\bar\partial_\zeta L$; in the limit as $\varepsilon \to 0$ we obtain (3.6) by a usual residue calculation. For ζ near z, the kernel L coincides with the B-M kernel for $(0,1)$-forms; in fact, for the Leray map $P(\zeta, z) = \overline{\zeta - z}$, (3.6) is the classical Bochner–Martinelli–Koppelman formula.

We have a lot of freedom in the choice of the map P that determines K and L. If we choose it such that $P(\zeta, \cdot)$ is holomorphic in Ω' when $\zeta \in \partial\Omega$, then $L(\zeta, z) = 0$ for such ζ and z (since each term in L contains a derivative $\bar\partial_z P_i$) and hence the function

$$v(z) = \int_\Omega K(\zeta, z) \wedge u(\zeta) \tag{3.7}$$

solves the equation $\bar\partial v = u$ in Ω'.

We shall construct the integral kernel of our solution operator on $\mathcal{T}_\delta$ by combining the B-M kernel near the diagonal $\zeta = z$ of the smaller tube $\mathcal{T}_{c\delta}$ with the Henkin kernel when ζ is near the boundary of $\mathcal{T}_\delta$ and $z \in \mathcal{T}_{c\delta}$. This will give a family of linear solution operators of the form (3.7) depending on δ for small $\delta > 0$.

Let ρ be the strongly plurisubharmonic function mentioned in the beginning of this section. Since $\{\rho < (1-\varepsilon)\delta^2\} \subset \mathcal{T}_\delta \subset \{\rho < (1+\varepsilon)\delta^2\}$ for sufficiently small $\delta > 0$, we may replace the tube $\mathcal{T}_\delta$ with the sublevel sets $\{\rho < \delta^2\}$, which we still denote by $\mathcal{T}_\delta$.

The construction of the kernel will proceed through several lemmas. First we recall from [HL] the following well-known result about the existence of the Henkin support function Φ and the corresponding Leray map P on a fixed strongly pseudoconvex domain, which in our case is a tube $\mathcal{T}_{\delta_0}$.

3.3. LEMMA. *There exist constants C, $R > 0$ such that, for $\delta_0 > 0$ sufficiently small, there are functions $\Phi(\zeta, z)$ and $A(\zeta, z)$ in $\mathcal{C}^1(\mathcal{T}_{\delta_0} \times \mathcal{T}_{\delta_0})$, with Φ holomorphic in z, and there is a $\mathcal{C}^1$-function $B(\zeta, z)$ defined for $\zeta, z \in \mathcal{T}_{\delta_0}$ and $|\zeta - z| \le R$ that satisfies the following*:

(i) $\Phi(\zeta, z) = A(\zeta, z)B(\zeta, z)$,
(ii) $|B(\zeta, z)| \ge C$ *and* $\operatorname{Re} A(\zeta, z) \ge \rho(\zeta) - \rho(z) + C|\zeta - z|^2$ *when* $|\zeta - z| \le R$,
(iii) $|\Phi(\zeta, z)| \ge C$ *when* $|\zeta - z| \ge \frac{1}{2}R$, *and*
(iv) *with Φ as described here, there exists a map $P = P(\zeta, z) = (P_1, \dots, P_n)$ such that, for all j, $P_j \in \mathcal{C}^1(\mathcal{T}_{\delta_0} \times \mathcal{T}_{\delta_0})$, P_j is holomorphic in z, and $\Phi(\zeta, z) = \langle P(\zeta, z), \zeta - z\rangle$.*

Proof. This follows from the proof of Theorems 2.4.3 and 2.5.5. in [HL]. Here $A(\zeta, z)$ is an approximate Levi polynomial in $z \in \mathbf{C}^n$ of the form

$$A(\zeta, z) = 2\sum_{j=1}^{n} \frac{\partial\rho(\zeta)}{\partial\zeta_j}(\zeta_j - z_j) - \sum_{j,k=1}^{n} a_{jk}(\zeta)(\zeta_j - z_j)(\zeta_k - z_k),$$

where the a_{jk} are $\mathcal{C}^1$ functions that approximate the partial derivatives $\partial^2\rho/\partial\zeta_j\partial\zeta_k$ sufficiently well on $\mathcal{T}_{\delta_0}$ [HL, Lemma 2.4.2]. In fact, when ρ is of class $\mathcal{C}^3$ or better, we might simply take $a_{jk} = \partial^2\rho/\partial\zeta_j\partial\zeta_k$.

The only small change from [HL] is that, in our situation, the maps Φ and P may be defined globally for $\zeta \in \mathcal{T}_{\delta_0}$, and not only for ζ near the boundary of $\mathcal{T}_{\delta_0}$, provided that $\delta_0 > 0$ is sufficiently small. This follows from the thinness of the tube $\mathcal{T}_{\delta_0}$ and can be seen as follows. Observe that, for $\zeta \in M$, the linear terms in $A(\zeta, \cdot)$ vanish and we have $\Re A(\zeta, z) < 0$ for all points $z \in M\backslash\{\zeta\}$ sufficiently close to ζ. Hence for $\delta_0 > 0$ small we can choose $\varepsilon > 0$ (depending on δ_0) such that $\Re A(\zeta, z) < 0$ whenever $z, \zeta \in \mathcal{T}_{\delta_0}$ and $\varepsilon \le |\zeta - z| \le 2\varepsilon$. The proof of Theorem 2.4.3 in [HL] (which proceeds by cutting of $\log A$ on $B(\zeta, 2\varepsilon) \cap \mathcal{T}_{\delta_0}$ and solving a $\bar\partial$-equation on $\mathcal{T}_{\delta_0}$) then gives a globally defined Φ (and hence P). □

Let Φ, P, A, and B be as in Lemma 3.3, constructed on a fixed tube $\mathcal{T}_{\delta_0}$; P is not quite a Leray map because it does not equal $\overline{\zeta - z}$ near the diagonal, and we shall now modify it suitably on tubes $\mathcal{T}_\delta$ for $0 < \delta \le \delta_0$. Let $0 < c < c' < 1$. Choose a cut-off function λ_δ such that $\lambda_\delta = 1$ in $\mathcal{T}_{c'\delta}$ and $\lambda_\delta = 0$ near $\partial\mathcal{T}_\delta$. We may assume that its (real) gradient satisfies $\|\nabla\lambda_\delta\| \le C\delta^{-1}$ for some $C > 0$ independent of δ. We will show that, for a suitably chosen function $\phi(\zeta, z)$ on $\bar{\mathcal{T}}_\delta \times \bar{\mathcal{T}}_\delta$, the conditions in Koppelman's formula (3.6) are satisfied for the pair of domains $\Omega = \mathcal{T}_\delta$ and $\Omega' = \mathcal{T}_{c\delta}$ if we define the Leray map $\tilde{P}$ by

$$\tilde{P}(\zeta, z) = (1 - \lambda_\delta(\zeta))\phi(\zeta, z)P(\zeta, z) + \lambda_\delta(\zeta)\overline{\zeta - z},$$

with the corresponding support function given by

$$\tilde{\Phi} = \langle \tilde{P}, \zeta - z\rangle = (1 - \lambda_\delta)\phi\Phi + \lambda_\delta|\zeta - z|^2.$$

We need to find ϕ such that $\tilde{\Phi}(\zeta, z) \neq 0$ when $z \in \mathcal{T}_{c\delta}$ and $\zeta \in \bar{\mathcal{T}}_\delta\backslash\{z\}$. When $\zeta \in \bar{\mathcal{T}}_{c\delta}$, we have $\tilde{\Phi}(\zeta, z) = |\zeta - z|^2$, so the condition is satisfied for any choice of ϕ. Hence it suffices to consider the points ζ where $\rho(\zeta) > \rho(z)$. Let $\psi\colon \mathbf{R} \to [0, 1]$ be a cut-off function such that $\psi(t) = 1$ for $|t| \le \frac{1}{2}R$ and $\psi(t) = 0$ for $|t| \ge \frac{2}{3}R$. Set

$$\phi(\zeta, z) = \psi(|\zeta - z|)B(\zeta, z)^{-1} + (1 - \psi(|\zeta - z|))\overline{\Phi(\zeta, z)},$$

where B is as in Lemma 3.3. Then $\phi\Phi = \psi A + (1 - \psi)|\Phi|^2$ (since $B^{-1}\Phi = A$), and we have the following estimates for the real part $\theta(\zeta, z) := \operatorname{Re}\phi\Phi(\zeta, z)$ when $\rho(\zeta) > \rho(z)$.

(a) When $|\zeta - z| \le \frac{1}{2}R$: $\theta = \operatorname{Re} A \ge C|\zeta - z|^2$.

(b) When $\frac{1}{2}R \le |\zeta - z| \le R$:

$$\theta = \psi \operatorname{Re} A + (1 - \psi)|\Phi|^2 \ge \psi C|\zeta - z|^2 + (1 - \psi)C^2 > 0.$$

(c) When $|\zeta - z| > R$: $\theta = |\Phi|^2 \ge C^2$.

This verifies the required properties, and hence (3.6) is valid when $K(\zeta, z)$ and $L(\zeta, z)$ are the kernels generated by the Leray map $\tilde{P}$. For ζ near $\partial\mathcal{T}_\delta$ we have $\tilde{P} = \phi P$; since $\phi \neq 0$ there, the kernel L is identical to the one generated by the holomorphic Leray map P, and hence the first term in (3.6) is zero. This gives us the solution formula (3.7) for the equation $\bar\partial v = u$ in $\mathcal{T}_{c\delta}$. This completes the construction of the kernel for $(0, 1)$-forms.

Proof of the sup-norm Estimates

It suffices to show that the sup-norm estimate holds in our situation when $n \ge 3$. In case $n < 3$ we simply identify $\mathbf{C}^n$ with $\mathbf{C}^n \times \{0\} \subset \mathbf{C}^3$ and extend f independently of the additional variables; the solution to the extended problem will satisfy the estimates, and its restriction to $\mathbf{C}^n$ will be a solution to the original $\bar\partial$-problem.

3.4. Lemma. *The solution $v(z) = \int_{\mathcal{T}_\delta} K(\zeta, z) \wedge u(\zeta)$ defined previously satisfies the estimate $\|v\|_{L^\infty(\mathcal{T}_{c\delta})} \le C\delta\|u\|_{L^\infty(\mathcal{T}_\delta)}$ when $n \ge 3$.*

Proof. Let $P^0 = \phi P$; P^0 is independent of δ and $\tilde{P} = (1 - \lambda)P^0 + \lambda(\overline{\zeta - z})$. This gives

$$\begin{aligned}\bar\partial\tilde P_j &= \bar\partial\lambda(\overline{\zeta_j - z_j} - P_j^0) + (1-\lambda)\bar\partial P_j^0 + \lambda d\bar\zeta_j \\ &=: \bar\partial\lambda(\overline{\zeta_j - z_j} - P_j^0) + \eta_j. \end{aligned}\tag{3.8}$$

The terms in $K(\zeta, z)$ are of the form $\tilde\Phi^{-n}\tilde P_j \bar\partial\tilde P[j] \wedge d\zeta$. Since $\bar\partial\lambda \wedge \bar\partial\lambda = 0$, this is a sum of terms of the following two types:

$$\tilde\Phi^{-n}\tilde P_j \eta[j] \wedge d\zeta \quad \text{and} \quad \tilde\Phi^{-n}\tilde P_j(\overline{\zeta_k - z_k} - P_k^0)\bar\partial\lambda \wedge \eta[j,k] \wedge d\zeta.$$

We shall estimate the integrals of these over $\mathcal{T}_\delta$ when $z \in \mathcal{T}_{c\delta}$. We have already shown that $\operatorname{Re}\tilde\Phi(\zeta, z) \ge C|\zeta - z|^2$. For $|\zeta - z| \le \frac{1}{2}R$ we have

$$\begin{aligned}\sum_{j=1}^n P_j^0(\zeta, z)(\zeta_j - z_j) &= \langle P^0(\zeta, z), \zeta - z\rangle = A(\zeta, z) \\ &= 2\sum_{j=1}^n \frac{\partial\rho(\zeta)}{\partial\zeta_j}(\zeta_j - z_j) + \mathcal{O}(|\zeta - z|^2).\end{aligned}$$

This implies $P_j^0(\zeta, z) = 2\frac{\partial\rho(\zeta)}{\partial\zeta_j} + \mathcal{O}(|\zeta - z|) = \mathcal{O}(\delta + |\zeta - z|)$. By choice of λ this gives $(1-\lambda(\zeta, z))P_j^0(\zeta, z) = \mathcal{O}(|\zeta - z|)$ and therefore $\tilde P_j(\zeta, z) = \mathcal{O}(|\zeta - z|)$. Since $|\eta_j| \le C$, we have

$$\begin{aligned}\tilde\Phi^{-n}\tilde P_j\eta[j] \wedge d\zeta &= \mathcal{O}(|\zeta - z|^{1-2n}), \\ \tilde\Phi^{-n}\tilde P_j(\overline{\zeta_k - z_k} - P_k^0)\bar\partial\lambda \wedge \eta[j,k] \wedge d\zeta &= \mathcal{O}(|\zeta - z|^{1-2n} + \delta^{-1}|\zeta - z|^{2-2n}),\end{aligned}$$

which shows that the kernel $K(\zeta, z)$ has a singularity of the same type as the Bochner–Martinelli kernel on the diagonal.

Locally we may straighten M; that is, for each $p \in M$ there is a neighborhood V_p of p and a $\mathcal{C}^1$-diffeomorphism $\Psi\colon U \to V_p$, where U is a neighborhood of the origin in $\mathbf{R}^{2n}$, such that Ψ is nearly volume- and distance-preserving and $\Psi(U \cap \mathbf{R}^m) = V_p \cap M$, where m is the dimension of M. We denote the points in $\mathbf{R}^{2n}$ by $(u', u'') \in \mathbf{R}^m \times \mathbf{R}^{2n-m}$. By compactness we may assume that $\mathcal{T}_\delta$ is covered by a finite number (independent of δ) of sets

$$K_j^\delta = \Psi_j(\{(u', u'');\ |u'| \le a,\ |u''| \le \delta\})$$

for some constant a. We keep the notation ζ and z for the points in the new coordinates also. We then have the estimate ($\zeta = (u', u'')$):

$$\begin{aligned}&\left|\int_{K_j} K(\zeta, z) \wedge u(\zeta)\right| \\ &\qquad \le C\|u\|_{L^\infty(\mathcal{T}_\delta)} \int_{|u'| \le a, |u''| \le \delta} (|\zeta - z|^{1-2n} + \delta^{-1}|\zeta - z|^{2-2n})\, dV(\zeta) \\ &\qquad \le C\|u\|_{L^\infty(\mathcal{T}_\delta)} \int_{|u'| \le a, |u''| \le \delta} (|\zeta|^{1-2n} + \delta^{-1}|\zeta|^{2-2n})\, dV(\zeta).\end{aligned}$$

For $m < t < 2n$ we estimate these integrals as follows:

$$\int_{|u'|\le a,|u''|\le\delta} \frac{1}{|\zeta|^t} \le C\left(\int_0^{\sqrt{2}\delta} \frac{r^{2n-1}}{r^t}\,dr + \int_\delta^a \frac{\delta^{2n-m} r^{m-1}}{r^t}\,dr\right) \le C\delta^{2n-t}. \quad (3.9)$$

Hence

$$\left|\int_{K_j} K(\zeta,z)\wedge u(\zeta)\right| \le C\|u\|_{L^\infty(\mathcal{T}_\delta)}(\delta+\delta^{-1}\delta^2) = 2C\|u\|_{L^\infty(\mathcal{T}_\delta)}$$

when $2n-2>m$. Since $m\le n$, this holds for $n>2$. □

Construction of the Kernel for Forms of Higher Degree

We consider the form

$$K(\zeta,z) = c_n\tilde\Phi(\zeta,z)^{-n}\sum_{j=1}^{n}(-1)^{j-1}\tilde P_j\bar\partial\tilde P[j]\wedge d(\zeta-z)$$

on $\mathcal{T}_\delta\times\mathcal{T}_{c'\delta}$, where $\bar\partial$ is now taken with respect to both ζ and z. We decompose

$$K(\zeta,z) = \sum_{p\le n}\sum_{q\le n-1} K_{p,q}(\zeta,z),$$

where $K_{p,q}$ has bidegree (p,q) with respect to z and $(n-p,n-q-1)$ with respect to ζ. If $q>0$ then $K_{p,q}(\zeta,z)=0$ when $z\in\mathcal{T}_{c'\delta}$ and ζ is near $\partial\mathcal{T}_\delta$. (Recall that $K(\zeta,z)=K^\delta(\zeta,z)$ depends on δ via the cut-off function λ_δ.) It follows that the $(p,q-1)$-form

$$v(z) = \int_{\mathcal{T}_\delta} K_{p,q-1}(\zeta,z)\wedge u(\zeta) = (-1)^{p+q}\int_{\mathcal{T}_\delta} u(\zeta)\wedge K_{p,q-1}(\zeta,z)$$

solves $\bar\partial v=u$ in $\mathcal{T}_{c'\delta}$ for each $\bar\partial$-closed (p,q)-form u in $\mathcal{T}_\delta$, $q>0$. The precise meaning of the integral is as follows. Write

$$K_{p,q-1}(\zeta,z) = \sum_{|I|=p}\sum_{|J|=q-1} k_{I,J}(\zeta,z)dz^I\wedge d\bar z^J,$$

where $k_{I,J}(\zeta,z)$ is an $(n-p,n-q)$-form in $\zeta\in\mathcal{T}_\delta$ depending smoothly on $z\in\mathcal{T}_{c'\delta}$. Then

$$v(z) = \sum_{|I|=p}\sum_{|J|=q-1}(-1)^{p+q}\left(\int_{\mathcal{T}_\delta} u(\zeta)\wedge k_{I,J}(\zeta,z)\right)dz^I\wedge d\bar z^J.$$

This completes the construction of the kernel. The reader may find some additional references and historical remarks about the solution formula at the end of this section.

Before proceeding we make the following elementary comments.

Geometric observations. Let M be a compact m-dimensional $\mathcal{C}^1$-submanifold of $\mathbf{R}^N$. There exists a constant $B>0$ such that, if $z_0,z_1\in\mathcal{T}_\delta(M)$ for sufficiently small δ, then z_0 and z_1 may be joined by a path in $\mathcal{T}_\delta(M)$ of length no more than $B|z_1-z_0|$. This is due to the fact that the tubes may be locally straightened, in a uniform way, to tubes around $\mathbf{R}^m\times\{0\}$ in $\mathbf{R}^N$.

From this we obtain the following: If $u \in \mathcal{C}^1(\mathcal{T}_\delta M)$, $\|u\|_{L^\infty(\mathcal{T}_\delta)} \le A$, and $\|u\|_{\mathcal{C}^1(\mathcal{T}_\delta)} \le At^{-1}$ for $t \le 1$ and $0 < s < 1$, then $|u|_{s,\delta} \le \max(2, B)At^{-s}$. We see this as follows. If $|h| \le t$, we can integrate Du from z to $z+h$ to get $|u(z+h) - u(z)||h|^{-s} \le BAt^{-1}|h|^{1-s} \le BAt^{-s}$. If $|h| \ge t$, the triangle inequality gives $|u(z+h) - u(z)||h|^{-s} \le 2At^{-s}$.

We also have a corresponding result for compact manifolds M: If $\|u\|_{\mathcal{C}^r(M)} \le A$ and $\|u\|_{\mathcal{C}^{r+1}(M)} \le At^{-1}$ for $t \ge 0$, then $\|u\|_{\mathcal{C}^{r+s}(M)} \le CAt^{-s}$, where C is a constant independent of u.

Proof of the Estimates for Forms of Higher Degree

The proof of the sup-norm estimate, which we gave for $(0,1)$-forms, carries over almost verbatim to the general case. However, Lemma 3.2 fails and we must proceed differently to estimate the derivatives.

With $c_0 = c' - c$ we introduce smooth cut-off functions $\chi_\delta \in \mathcal{C}_0^\infty(B(0, c_0\delta))$ with $\chi_\delta(w) = 1$ when $|w| < c_0\delta/2$ and $|\partial^\alpha \chi_\delta| \le C_\alpha \delta^{-|\alpha|}$. Then we decompose v as $v' + v''$, with

$$v'(z) = \int_{\mathcal{T}_\delta} \chi_\delta(\zeta - z) K_{p,q-1}(\zeta, z) \wedge u(\zeta),$$

$$v''(z) = \int_{\mathcal{T}_\delta} (1 - \chi_\delta(\zeta - z)) K_{p,q-1}(\zeta, z) \wedge u(\zeta);$$

we then estimate each summand separately.

Recall that if $z \in \mathcal{T}_{c\delta}$ and $|\zeta - z| \le c_0\delta$ then $K(\zeta, z)$ equals the Bochner–Martinelli kernel. Thus $v'(z)$ is obtained for $z \in \mathcal{T}_{c\delta}$ by applying a convolution operator to u; hence

$$\partial^\alpha v'(z) = \int_{\mathcal{T}_\delta} \chi_\delta(\zeta - z) K_{p,q-1}(\zeta, z) \wedge \partial^\alpha u(\zeta).$$

Thus the components of $\partial^\alpha v'(z)$ are linear combinations of terms $h(z) = (k * g)(z)$, where $k(w) = \chi_\delta(w)\bar{w}_j|w|^{-2n}$ and g is a component of $\partial^\alpha u$. Since $|k(w)| \le |w|^{1-2n}$ and k is supported by $B(0, c_0\delta)$, an obvious estimate gives $|h(z)| \le C\delta\|g\|_\infty$, so

$$\|\partial^a v'\|_{L^\infty(\mathcal{T}_{c\delta})} \le C\delta\|\partial^\alpha u\|_{L^\infty(\mathcal{T}_\delta)}.$$

In order to estimate the finer norms of h, we introduce the auxiliary kernels

$$k_t(w) = \chi_\delta(w)\bar{w}_j(t^2 + |w|^2)^{-n}, \quad t > 0.$$

This is a smooth function of (t, z) satisfying $|k_t(z)| \le |k(z)|$ and $\lim_{t\to 0} k_t(z) = k(z)$. Since each k_t has compact support, it follows that $\int \partial_j D^\beta k_t(w)\, dV(w) = 0$ for every (t, z)-derivative D^β. Thus, setting $h_t(z) = (k_t * g)(z)$, we see that

$$D^\beta \partial_j h_t(z) = \int_{\mathcal{T}_\delta} \partial_j D^\beta k_t(w)(g(z - w) - g(z))\, dV(w).$$

Observing that $|\partial^\gamma \chi_\delta(w)| \le C_\gamma |w|^{-|\gamma|}$ on $\operatorname{supp} \chi_\delta$, a simple calculation gives

$$|D^\beta \partial_j k_t(w)| \le C_\beta |w|^{-|\beta|}(t + |w|)^{-2n}.$$

Assume that $\partial^\alpha u \in \Lambda^s(\mathcal{T}_\delta)$ for some $s \in (0,1)$. We have $g \in \Lambda^s(\mathcal{T}_\delta)$, and for $t > 0$ we can estimate in polar coordinates:

$$|\partial_j h_t(z)| \le C|g|_{s,\delta} \int_{|w|<c_0\delta} (t+|w|)^{s-2n}\, dV(w) \le C|g|_{s,\delta} \int_0^{c_0\delta} r^{s-1}\, dr = C_s \delta^s |g|_{s,\delta}.$$

For the first-order derivatives with respect to (t,z), in the same way we derive

$$|D\partial_j h_t(z)| \le C|g|_{s,\delta} \int_{|w|<c_0\delta} |w|^{s-1}(t+|w|)^{-2n}\, dV(w) \le C|g|_{s,\delta} \int_0^{c_0\delta} (t+r)^{s-2}\, dr \le C'_s t^{s-1} |g|_{s,\delta}. \tag{3.10}$$

By the dominated convergence theorem we have $h_t(z) \to h(z)$ and

$$\partial_j h_t(z) \to h_{(j)}(z) = \int_{\mathcal{T}_\delta} \partial_j k(w)(g(z-w) - g(z))\, dV(w)$$

as $t \to 0$. We also have

$$|\partial_j h_t(z) - h_{(j)}(z)| \le \int_0^t \left| \frac{\partial}{\partial \tau} \partial_j h_\tau(z) \right| d\tau \le C|g|_{s,\delta} t^s;$$

hence the convergence of the derivatives is uniform and therefore $h_{(j)}(z) = \partial_j h(z)$. Thus $|\partial_j h_t(z)| \le C_s \delta^s |g|_{s,\delta}$, and we conclude that

$$\|\partial_j \partial^\alpha v'\|_{L^\infty(\mathcal{T}_{c\delta})} \le C_s \delta^s |\partial^\alpha u|_{s,\delta}.$$

We have also shown that

$$\partial_j \partial^\alpha v'(z) = \int_{\mathcal{T}_\delta} \partial_j(\chi_\delta(\zeta - z) K_{p,q-1}(\zeta, z)) \wedge (\partial^\alpha u(\zeta) - (\partial^a u)_z).$$

In order to estimate the Λ^s-norm of $\partial_j \partial^\alpha v'$, we need the following standard.

Lemma. *Let $\phi \in \mathcal{C}(\mathcal{T}_\delta)$ have an extension $\tilde{\phi} \in \mathcal{C}^1(\mathbf{R}^+ \times \mathcal{T}_\delta)$ satisfying $|D\tilde{\phi}(t,z)| \le A t^{s-1}$ for some $0 < s < 1$. Then $\phi \in \Lambda^s(\mathcal{T}_\delta)$ and $|\phi|_{s,\delta} \le (B + 2/s)A$.*

This is just a slight modification of [HL, Apx. 1, Prop. 2]. Applying this to $\phi = \partial_j h$ and $\tilde{\phi}(t,z) = \partial_j h_t(z)$, (3.10) gives $|\partial_j h|_{s,c\delta} \le C'_s(B + 2/s)|g|_{s,\delta}$. Thus

$$|\partial_j \partial^\alpha v'|_{s,c\delta} \le C_s |\partial^\alpha u|_{s,\delta}.$$

In order to study v'', we set $K''_{p,q-1}(\zeta,z) = (1 - \chi_\delta(\zeta - z))K_{p,q-1}(\zeta,z)$ on $\mathcal{T}_\delta \times \mathcal{T}_{c\delta}$. This kernel has continuous z-derivatives of all orders, and it equals zero when $|\zeta - z| \le c_0\delta/2$. It follows that v'' is a smooth form with

$$\partial^\alpha v''(z) = \int_{\mathcal{T}_\delta} \partial_z^\alpha K''_{p,q-1}(\zeta, z) \wedge u(\zeta).$$

We recall formula (3.8) and point out that $|\bar{\partial}\lambda_\delta| = \mathcal{O}(\delta^{-1})$ and $|\tilde{\phi}(\zeta,z)| \ge C|\zeta - z|^2$ on $\mathcal{T}_\delta \times \mathcal{T}_{c\delta}$; moreover, the quantities $|D_z\tilde{\phi}(\zeta,z)|$, $|\eta_j|$, and $|P_j^0|$ are all bounded by $C|\zeta - z|$, while their derivatives with respect to z are bounded independently of δ (since λ_δ is independent of z).

An induction on $|\alpha|$ shows that the components of $\partial_z^\alpha K''_{p,q-1}(\zeta, z)$ are linear combinations of terms of the type

$$\tilde{\phi}^{-n-k}\partial_z^\beta(1-\chi_d(\zeta - z))a_0(\zeta, z)\cdots a_t(\zeta, z)$$

(with $\beta \le \alpha$, $k \le |\alpha - \beta|$, and $t \ge 2k+1-|\alpha-\beta|$) and terms of the type

$$\tilde{\phi}^{-n-k}\frac{\partial\lambda_\delta(\zeta)}{\partial\bar{\zeta}_i}a_0(\zeta, z)\cdots a_t(\zeta, z)$$

(with $k \le |\alpha|$ and $t \ge 2k+2-|\alpha|$), where the $a_j(\zeta, z)$ have continuous z-derivatives of all orders that have upper bounds independent of δ and where $|a_j(\zeta, z)| \le C|\zeta - z|$ when $t > 0$ and $1 \le j \le t$. Because $|\zeta - z| > c_0\delta/2$ when $K''_{p,q-1} \ne 0$, it follows easily that

$$|\partial_z^\alpha K''_{p,q-1}(\zeta, z)| \le C_\alpha \delta^{-1}|\zeta - z|^{2-2n-|\alpha|}.$$

Thus

$$\begin{aligned}|\partial^\alpha v''(z)| &\le C_\alpha\delta^{-1}\|u\|_{L^\infty(\mathcal{T}_\delta)}\int_{\mathcal{T}_\delta\setminus B(z,c_0\delta/2)}|\zeta - z|^{2-2n-|\alpha|}\,dV(\zeta)\\ &\le C_\alpha\delta^{1-|\alpha|}\|u\|_{L^\infty(\mathcal{T}_\delta)}\end{aligned}$$

for $z \in \mathcal{T}_{c\delta}$ and $\alpha \in \mathbf{Z}^n_+$. The last estimate follows for $|\alpha| > 2$, $|\alpha| = 2$, and $|\alpha| = 1$, respectively, from the following three integral estimates:

$$\int_{|\zeta - z|>\delta}|\zeta - z|^{-2n-t}\,dV(\zeta) = C_t\delta^{-t}, \quad t > 0; \tag{3.11}$$

$$\int_{\mathcal{T}_\delta\setminus B(z,c_1\delta)}|\zeta - z|^{-2n}\,dV(\zeta) \le C(c_1), \quad z \in \mathcal{T}_\delta, \tag{3.12}$$

$$\int_{\mathcal{T}_\delta}|\zeta - z|^{s-2n}\,dV(\zeta) \le C_s\delta^s, \quad 0 < s < 2n - m. \tag{3.13}$$

Equation (3.11) follows immediately by a change of variable. Inequality (3.12) is proved exactly like (3.9); in the sum in the middle of (3.9), the first integral has lower limit $c_1\delta$ instead of 0. Finally, (3.13) follows by setting $t = 2n - s$ in (3.9).

Using the geometric observation following the construction of the kernel, we have

$$\|\partial^\alpha v''\|_{\Lambda^s(\mathcal{T}_{c\delta})} \le C_{\alpha,s}\delta^{1-|\alpha|-s}\|u\|_{L^\infty(\mathcal{T}_{c\delta})}.$$

This completes the proof of the Hölder estimates in Theorem 3.1 for the case $0 < s < 1$. The proof for $s = 1$ follows the same lines, with certain small modifications; since that case will not be used in this paper, we omit the details.

Remarks on Constructions of Kernels

The first integral kernel operators with holomorphic kernels, those solving the $\bar{\partial}$-equation on strongly pseudoconvex domains in $\mathbf{C}^n$, have been constructed by

Henkin and independently by R. de Arellano (see the references in [HL]). Henkin's approach is to patch the Bochner–Martinelli and Leray kernels on the boundary $\partial\Omega$. Our patching of the two kernels (by first multiplying by ϕ) is the same as in Øvrelid [Ø1; Ø2]. The whole construction is similar to the one by Harvey and Wells [HW].

It seems that the first really precise L^∞ and $\mathcal{C}^k$-estimates for the $\bar\partial$-equation in thin tubes around a totally real submanifold $M \subset \mathbf{C}^n$, proved by means of integral solution operators, are due to Harvey and Wells [HW] in 1972. In 1974, Range and Siu [RS] used a more refined kernel construction to prove estimates for the highest-order derivatives of their solution on M and deduced $\mathcal{C}^k$-approximation of $\mathcal{C}^k$-functions on a $\mathcal{C}^k$-submanifold $M \subset \mathbf{C}^n$ by holomorphic functions, a case left open in [HW]. In fact, this approximation problem has been one of the original motivations in proving such estimates. This approximation was later accomplished more efficiently (and in greater generality) by Baouendi and Treves [BT1; BT2], who used the convolution with the complex Gaussian kernel. This latter method does not seem to give the approximation of diffeomorphisms obtained in this paper because we must work in tubular neighborhoods and not solely on the submanifold.

As remarked previously, our construction of the kernel in this paper is close to [HW] and our main contribution is the way in which we estimate the solutions. We find it quite striking that this simple and seemingly crude construction of the kernel gives rise to results that are essentially optimal for the applications to mappings presented in this paper. For the benefit of the reader we have given a fairly self-contained presentation based on the text [HL]. Another closely related paper is [BB], where Bruna and Burgués approximate $\bar\partial$-closed jets on a totally real set X in Hölder norms by functions holomorphic in a neighborhood of X. It seems likely that their method—making use of weighted integral kernels of Anderson and Berndtsson type [AB]—may also be used to prove our results. However, we believe that our approach is simpler and more elementary. Our results, suitably reformulated, may also be proved for neighborhoods of totally real sets.

4. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2. We consider first the case $\dim M_0 = \dim M_1 = n$. Let $d(z)$ denote the Euclidean distance of z to M_0, and let $\mathcal{T}_\delta$ (resp., $\mathcal{T}_\delta{}'$) denote the open tube of radius δ around M_0 (resp., around M_1). The $\mathcal{C}^k$-diffeomorphism $f\colon M_0 \to M_1$ can be extended to a $\mathcal{C}^k$-map on $\mathbf{C}^n$, still denoted f, that is $\bar\partial$-flat to order k at M_0:

$$|\partial^\alpha(\bar\partial f)(z)| = o(d(z)^{k-1-|\alpha|}), \quad 0 \le |\alpha| \le k-1.$$

In particular, the derivative $Df(z)$ is a nondegenerate $\mathbf{C}$-linear map at each point $z \in M_0$ (the complexification of $df_z\colon T_z M_0 \to T_{f(z)} M_1$) and hence f is a $\mathcal{C}^k$ diffeomorphism in some neighborhood of M_0 in $\mathbf{C}^n$. The $(0,1)$-form $u = \bar\partial f$ of class $\mathcal{C}^{k-1}$ satisfies $\bar\partial u = 0$ and

$$\|\partial^\alpha u\|_{L^\infty(\mathcal{T}_\delta)} = o(\delta^{k-1-|\alpha|}), \quad 0 \le |\alpha| \le k-1,$$

as $\delta \to 0$. Applying Theorem 3.1 (specifically, the estimates (3.1) with $l = k-1 \geq 0$ and a fixed constant $0 < c < 1$), for each sufficiently small $\delta > 0$ we obtain a solution v_δ to $\bar\partial v_\delta = u$ in $\mathcal{T}_\delta$ satisfying the following estimates:

$$\begin{aligned}\|\partial^\alpha v_\delta\|_{L^\infty(\mathcal{T}_{c\delta})} &\leq C(\delta\|\partial^\alpha u\|_{L^\infty(\mathcal{T}_\delta)} + \delta^{1-|\alpha|}\|u\|_{L^\infty(\mathcal{T}_\delta)}) \\ &\leq C(\delta o(\delta^{k-1-|\alpha|}) + \delta^{1-|\alpha|} o(\delta^{k-1})) \\ &= o(\delta^{k-|\alpha|}), \quad |\alpha| \leq k-1.\end{aligned}$$

Moreover, since $\bar\partial v = u$ has a solution of class $\mathcal{C}^{l+1} = \mathcal{C}^k$ (namely, f), we can choose v_δ to satisfy in addition the estimates (3.2) for the derivatives of top-order k:

$$\|\partial^\alpha v_\delta\|_{L^\infty(\mathcal{T}_{c\delta})} \leq C(\omega_k(f;\delta) + \delta^{-k+1}\|\bar\partial f\|_{L^\infty(\mathcal{T}_{c\delta})}) = o(1), \quad |\alpha| = k.$$

Here $\omega_k(f;\delta)$ denotes the modulus of continuity of the kth-order derivatives of f. Set $F_\delta = f - v_\delta$ in $\mathcal{T}_\delta$. Then $\bar\partial F_\delta = 0$ and the estimates on v_δ imply

$$\|F_\delta - f\|_{\mathcal{C}^r(\mathcal{T}_{c\delta})} = \|v_\delta\|_{\mathcal{C}^r(\mathcal{T}_{c\delta})} = o(\delta^{k-r}), \quad 0 \leq r \leq k,$$

which gives the first estimate in (1.2). It remains to prove that F_δ is biholomorphic and satisfies the inverse estimates in (1.2) for all sufficiently small $\delta > 0$. To simplify the notation we replace δ by $c\delta/2$, so that F_δ is holomorphic in the tube $\mathcal{T}_{2\delta}$ and satisfies

$$\|F_\delta - f\|_{\mathcal{C}^r(\mathcal{T}_{2\delta})} = o(\delta^{k-r}), \quad 0 \leq r \leq k, \tag{4.1}$$

as $\delta \to 0$. Since f is a diffeomorphism near M_0, so is any sufficiently close $\mathcal{C}^1$ approximation of f; hence (4.1) with $r = 1$ implies that, for $\delta > 0$ sufficiently small (say, $0 < \delta \leq \delta_0 \leq 1$), the map F_δ is diffeomorphic (and hence biholomorphic) in $\mathcal{T}_{2\delta}$. Decreasing δ_0 if necessary, there is a number $a > 0$ such that

$$|f(z) - f(z')| \geq 2a|z - z'|, \quad z, z' \in \mathcal{T}_{\delta_0}.$$

Since $f(M_0) = M_1$, this implies that $f(\mathcal{T}_\delta)$ contains the tube $\mathcal{T}'_{2a\delta}$.

Fix an $\varepsilon > 0$. By (4.1) applied with $r = 0$, we obtain a constant $\delta_1 = \delta_1(\varepsilon)$ with $0 < \delta_1 \leq \delta_0$ such that $\|F_\delta - f\|_{L^\infty(\mathcal{T}_{2\delta})} < a\varepsilon\delta^k$ for $0 < \delta \leq \delta_1$. Fix a point $z \in \mathcal{T}_\delta$ and let $w = f(z)$. For each z' with $|z' - z| = \varepsilon\delta^k$, we have

$$\begin{aligned}|F_\delta(z') - w| &= |(F_\delta(z') - f(z')) + (f(z') - f(z))| \\ &\geq |f(z') - f(z)| - |F_\delta(z') - f(z')| \\ &\geq 2a\varepsilon\delta^k - a\varepsilon\delta^k = a\varepsilon\delta^k.\end{aligned}$$

This means that the image by F_δ of the sphere $S = \{z' : |z' - z| = \varepsilon\delta^k\}$ is a hypersurface containing the ball $B(w; a\varepsilon\delta^k) = \{w' : |w' - w| < a\varepsilon\delta^k\}$ in the bounded component of its complement. By degree theory, the F_δ-image of the ball $B(z; \varepsilon\delta^k)$ contains the ball $B(w; a\varepsilon\delta^k)$. Hence there is a point $\zeta \in B(z; \varepsilon\delta^k)$ such that $F_\delta(\zeta) = w = f(z)$, and we have $|F_\delta^{-1}(w) - f^{-1}(w)| = |\zeta - z| < \varepsilon\delta^k$. Since this applies to any point $w \in \mathcal{T}'_{2a\delta}$, we conclude that $F_\delta(\mathcal{T}_{2\delta}) \supset \mathcal{T}'_{2a\delta}$ and

$$\|F_\delta^{-1} - f^{-1}\|_{L^\infty(\mathcal{T}'_{2a\delta})} \le \varepsilon\delta^k, \quad 0 < \delta \le \delta_1(\varepsilon). \tag{4.2}$$

Since $\varepsilon > 0$ was arbitrary, this gives the inverse estimate in (1.2) for $r = 0$.

We proceed to estimate the derivatives of the inverse maps. Denote by $\|A\|$ the spectral norm of a linear map $A \in \mathrm{GL}(\mathbf{R}, 2n)$. Note that $Df^{-1}(w) = Df(z)^{-1}$, where $w = f(z)$. Fix a point $w \in \mathcal{T}'_{2a\delta}$ and let $z = f^{-1}(w)$ and $z_\delta = F_\delta^{-1}(w)$ (these are points in $\mathcal{T}_{2\delta}$). By (4.2) we have $|z - z_\delta| \le \varepsilon\delta^k$. Writing $A = Df(z)$ and $B = DF_\delta(z_\delta)$, we get

$$\begin{aligned}\|DF_\delta^{-1}(w) - Df^{-1}(w)\| &= \|A^{-1} - B^{-1}\| \\ &= \|A^{-1}(B - A)B^{-1}\| \\ &\le \|A^{-1}\| \cdot \|A - B\| \cdot \|B^{-1}\|.\end{aligned}$$

Since f is a diffeomorphism and F_δ is $\mathcal{C}^1$-close to f, the eigenvalues of A and B are uniformly bounded away from zero, and this gives a uniform estimate on $\|A^{-1}\|$ and $\|B^{-1}\|$ (independent of δ). The middle term is

$$\|A - B\| = \|Df(z) - DF_\delta(z_\delta)\| \le \|Df(z) - Df(z_\delta)\| + \|Df(z_\delta) - DF_\delta(z_\delta)\|.$$

The second term on the right-hand side is of size $o(\delta^{k-1})$, according to (4.1). As $\delta \to 0$, we have $z_\delta \to z$, and hence the first term on the right-hand side goes to zero (by continuity of Df). Hence $\sup\{\|DF_\delta^{-1}(w) - Df^{-1}(w)\| : w \in \mathcal{T}'_{2a\delta}\}$ goes to zero as $\delta \to 0$. This completes the proof when $k = 1$. If $k > 1$, we can further estimate $\|Df(z) - Df(z_\delta)\| \le C|z - z_\delta| \le C\varepsilon\delta^k$, where C is an upper bound for the second derivatives of f. This gives

$$\sup\{\|DF_\delta^{-1}(w) - Df^{-1}(w)\| : w \in \mathcal{T}'_{2a\delta}\} = o(\delta^{k-1}),$$

as required by (1.2) for derivatives of order $r = 1$. To obtain the estimates (1.2) for the higher derivatives of $F_\delta^{-1} - f^{-1}$, we may apply the same method to the tangent map—that is, the induced map on tangent bundles over the tubes that equals the derivative of the given map on each tangent space. We leave out the details. This proves Theorem 1.2 when $\dim M_0 = n$.

Suppose now that $m = \dim M_0 < n$. We are assuming that there is an isomorphism $\phi\colon \nu_0 \to \nu_1$ of the complex normal bundles $\nu_0 \to M_0$ (resp., $\nu_1 \to M_1$) over f; by approximation we may assume that ϕ is of class $\mathcal{C}^{k-1}$. For each $z \in M_0$ we have $T_z\mathbf{C}^n = T_z^{\mathbf{C}}M_0 \oplus \nu_{0,z}$. Let l_z be the $\mathbf{C}$-linear map on $\mathbf{C}^n$ that is uniquely defined by taking $l_z = df_z$ on $T_z^{\mathbf{C}}M_0$ and $l_z = \phi_z$ on $\nu_{0,z}$. Clearly $l_z \in \mathrm{GL}(n, \mathbf{C})$ for each $z \in M$. Applying Lemma 2.6, we obtain a $\mathcal{C}^k$-extension $\tilde f$ of f that is $\bar\partial$-flat on M_0. Now the proof may proceed exactly as before. This proves Theorem 1.2. □

Remarks. (1) If $f_0\colon M_0 \to M_1$ is a *real-analytic diffeomorphism* and if the complex normal bundles to M_0 (resp., M_1) are isomorphic over f, then f *extends* to a biholomorphic map F from neighborhood of M_0 onto a neighborhood of M_1. We see this as follows. Let $\phi\colon \nu_0 \to \nu_1$ be the continuous isomorphism (over f) of the complex normal bundles to M_0 (resp., M_1). There exist complexifications

$\tilde{M}_i \subset \mathbf{C}^n$ of M_i $(i = 0, 1)$ such that f extends to a biholomorphic map $\tilde{f}\colon \tilde{M}_0 \to \tilde{M}_1$ and such that the complex normal bundles $\nu_i \to M_i$ extend to holomorphic vector bundles $\tilde{\nu}_i \to \tilde{M}_i$. We define a continuous map $\psi\colon M_0 \to \mathrm{GL}(n, \mathbf{C})$ by $\psi(z) = d\tilde{f}_z \oplus \phi_z$. Since $M_0 \subset \tilde{M}_0$ is totally real, ψ may be approximated by a holomorphic map $\tilde{\psi}\colon \tilde{M}_0 \to \mathrm{GL}(n, \mathbf{C})$. We now define $\tilde{\phi}\colon \tilde{\nu}_0 \to \tilde{M}_1 \times \mathbf{C}^n$ by $\tilde{\phi}_z(v) = (\tilde{f}(z), \tilde{\psi}_z(v))$. Clearly $\tilde{\phi}$ is a holomorphic vector bundle isomorphism between $\tilde{\nu}_0$ and a holomorphic subbundle $\tilde{\nu}_2 \subset \tilde{M}_1 \times \mathbf{C}^n$ that is an approximation of $\tilde{\nu}_1$. In particular, $\tilde{\phi}$ is a biholomorphic map between neighborhoods V_i of the zero sections of $\tilde{\nu}_i$. These neighborhoods map biholomorphically onto neighborhoods of M_0 (resp., M_1) under the projection maps (the Docquier–Grauert theorem). This gives the desired biholomorphic extension of f.

(2) If instead of Theorem 3.1 we use Hörmander's L^2-estimates when solving $\bar{\partial} v_\delta = u$ $(= \bar{\partial} f)$ in $\mathcal{T}_\delta$, the resulting holomorphic maps $F_\delta = f - v_\delta$ can be shown to satisfy the weaker estimate $\|F_\delta|_{M_0} - f\|_{\mathcal{C}^r(M_0)} = o(\delta^{k-r-l})$ for $0 \le r \le l$, where l is the smallest integer larger than $\frac{1}{2}\dim M_0$. This approach had been used in [FL].

Proof of Theorem 1.3. The proof can be obtained by repeating the proof of Theorem 1.1 in [FL] (or of its more technical version, [FL, Thm. 2.1]), except that one applies our Theorem 3.1 whenever solving a $\bar{\partial}$-equation. This gives the improved estimates in (1.3) with no loss of derivatives. We leave out the details. □

A correction to [FL]. We take this opportunity to correct an error in the proof of Lemma 4.1 in [FL]. (Equation numbers in the balance of this section refer to that paper.) The lemma is correct as stated, but the proof of the estimate (4.5) is not correct. Using the notation of that proof, we have the higher variational equations

$$\frac{\partial}{\partial t} D^p \phi_t(x) = DX_t(\phi_t(x)) \circ D^p \phi_t(x) + H_X^p(t, x)$$

for $p \le k$, where $D^p f$ denotes the pth-order derivative of a map $f\colon \Omega \subset \mathbf{R}^n \to \mathbf{R}^n$, so $D^p f \in L^p(\mathbf{R}^n, \mathbf{R}^n)$. Here $H_X^p(t, x)$ is a sum of terms involving derivatives of the vector field X_t and derivatives of order less than p of the flow ϕ_t, and $H_X^1 = 0$. We use the same notation for Y_t^ε and its flow ψ_t^ε.

Choose unit vectors $v_1, \dots, v_p \in \mathbf{R}^n$ and set

$$y(t) = [D^p \phi_t(x) - D^p \psi_t^\varepsilon(x)](v_1, \dots, v_p).$$

It will be sufficient to show that $\|y(t)\| = o(\varepsilon^{k-p})$ uniformly for $0 \le t \le t_0$, $x \in K(\varepsilon)$, and unit vectors $v_1, \dots, v_p$. Now $y(t)$ satisfies the differential equation

$$\begin{aligned} y'(t) = DY_t^\varepsilon(\psi_t^\varepsilon(x)) \cdot y(t) + (DX_t(\phi_t(x)) - DY_t^\varepsilon(\psi_t^\varepsilon(x)) \circ D^p \phi_t(x)(v_1, \dots, v_p) \\ + (H_X^p(t, x) - H_{Y^\varepsilon}^p(t, x))(v_1, \dots, v_p). \end{aligned}$$

This is a linear system $y' = A(t) \circ y + b(t)$, $y \in \mathbf{R}^n$. Suppose the matrix norms satisfy $\|A(t)\| \le A$ and $\|b(t)\| \le b$ for $t \in [0, t_0]$. The function $u(t) = \|y(t)\|$ is differentiable outside the zeroes of u, with $u'(t) = y'(t) \cdot y(t)/\|y(t)\| \le \|y'(t)\|$, so $u'(t) \le Au(t) + b$ outside the zeroes of u. Since $\phi_0 = \psi_0^\varepsilon = \mathrm{Id}$, we have

$y(0) = 0$. We shall first show that $u(t) \le \frac{b}{A}(e^{At} - 1)$ for $t \in [0, t_0]$. If $u(t) = 0$ then there is nothing to prove. If not, let t_1 be the largest zero of u on $[0, t]$. Thus $u'(s) \le Au(s) + b$ for $s \in (t_1, t]$. Setting $v(s) = u(s)e^{-As}$ yields $v'(s) \le be^{-As}$ for $s \in (t_1, t]$. Integration from t_1 to t gives $v(s) \le \frac{b}{A}(e^{-At_1} - e^{-At})$. Thus $u(t) \le \frac{b}{A}(e^{A(t-t_1)} - 1) \le \frac{b}{A}(e^{At} - 1)$.

In our situation, by (4.4) the matrix norm of $A(t) = DY_t^\varepsilon(\psi_t^\varepsilon(x))$ is bounded independently of $\varepsilon > 0$, $x \in K(\varepsilon)$, and t. It is therefore sufficient to prove that $b = o(\varepsilon^{k-p})$ uniformly in x, t, and unit vectors $v_1, \dots, v_n$. It is shown in [FL] that the matrix norm $\|DX_t(\phi_t(x)) - DY_t^\varepsilon(\psi_t^\varepsilon(x))\|_{L^\infty(K(\varepsilon))} = o(\varepsilon^{k-1})$. Since the flow $\phi_t(x)$ is of class $\mathcal{C}^k$, it follows that the matrix norm $\|D^p\phi_t(x)\|$ is uniformly bounded for $x \in K(\varepsilon)$ and $t \in [0, t_0]$. Applying (4.4) and (4.5) inductively as in [FL], we obtain $\|H_X^p - H_{Y^\varepsilon}^p\|_{L^\infty(K(\varepsilon))} = o(\varepsilon^{k-p})$ uniformly in t, which proves the claim. □

5. Solving the Equation $dv = u$ for Holomorphic Forms in Tubes

Let d denote the exterior derivative. In this section we solve the equation $dv = u$ with sup-norm estimates for holomorphic forms in tubes $\mathcal{T}_\delta = \mathcal{T}_\delta M$ around totally real submanifolds $M \subset \mathbf{C}^n$. We denote by Λ^s the Hölder spaces as in Section 3. We first state our main result for closed submanifolds; for an extension to compact submanifolds with boundary, see Remark (3) following Theorem 5.1.

5.1. Theorem. *Let $i\colon M \hookrightarrow \mathbf{C}^n$ denote the inclusion of a closed, m-dimensional, totally real submanifold of class $\mathcal{C}^2$ in $\mathbf{C}^n$. Let a positive constant $c < 1$ be given. Then there exist positive constants C, δ_0, and C_s for all $s \in (0,1)$ such that, if u is a d-closed holomorphic p-form in the tube $\mathcal{T}_\delta = \mathcal{T}_\delta M$ for some $0 < \delta \le \delta_0$ and $1 \le p \le n$, then*:

(a) *if $p > m$, the equation $dv = u$ has a holomorphic solution v in $\mathcal{T}_\delta$ satisfying*

$$\|v\|_{L^\infty(\mathcal{T}_{c\delta})} \le C\delta\|u\|_{L^\infty(\mathcal{T}_\delta)}; \tag{5.1}$$

(b) *if $p \le m$ and the form i^*u is exact on M, then for any solution of $dv_0 = i^*u$ of class $\Lambda^s(M)$ $(0 < s < 1)$ there is a holomorphic solution v of $dv = u$ in $\mathcal{T}_\delta$ satisfying*

$$\|v\|_{L^\infty(\mathcal{T}_{c\delta})} \le C_s(\delta\|u\|_{L^\infty(\mathcal{T}_\delta)} + \|v_0\|_{L^\infty(M)} + \delta^s\|v_0\|_{\Lambda^s(M)}); \tag{5.2}$$

(c) *if $p \le m$ and i^*u is exact on M, then there is a holomorphic solution of $dv = u$ with*

$$\|v\|_{L^\infty(\mathcal{T}_{c\delta})} \le C\|u\|_{L^\infty(\mathcal{T}_\delta)}. \tag{5.3}$$

Remarks. (1) If Ω is a Stein manifold, then the Rham cohomology groups $H^p(\Omega; \mathbf{C})$ can be calculated by holomorphic forms in the following sense: Each closed form is cohomologous to a closed holomorphic p-form, and if a holomorphic form u is exact (i.e., if $u = dv_0$ for some not necessarily holomorphic $(p-1)$-form v_0) then also $u = dv$ for a holomorphic $(p-1)$-form v on Ω. (See [Hö, Thm. 2.7.10].)

(2) On a $\mathcal{C}^2$-manifold M, the $\mathcal{C}^1$-forms and the d-operator $d\colon \mathcal{C}^1_{p-1}(M) \to \mathcal{C}^0_p(M)$ are intrinsically defined. By duality, the notion $dv = u$ (weakly) is well-defined on M. The condition in Theorem 5.1 that i^*u be exact on M need only hold in the weak sense.

(3) Theorem 5.1 has an extension to nonclosed totally real $\mathcal{C}^1$-submanifolds M' in $\mathbf{C}^n$. Let K be a compact subset of M' and let $K' \subset M'$ be a compact neighborhood of K in M'. (For instance, $K = M$ may be a compact totally real submanifold with boundary in $\mathbf{C}^n$.) For $\delta > 0$ we set

$$U_\delta = \{z \in \mathbf{C}^n : d_K(z) < \delta\}, \quad U'_\delta = \{z \in \mathbf{C}^n : d_{K'}(z) < \delta\}.$$

Choose $c \in (0,1)$. Assume that u is a d-closed holomorphic p-form in U'_δ with i^*u exact on $U'_\delta \cap M'$ (where $i\colon M' \hookrightarrow \mathbf{C}^n$ is the inclusion map). Then there is a holomorphic solution of $dv = u$ in $U_{c\delta}$ such that the estimates (5.1)–(5.3) are valid when $\mathcal{T}_{c\delta}$ is replaced by $U_{c\delta}$ and $\mathcal{T}_\delta$ is replaced by U'_δ.

Proof of Theorem 5.1

We give the details in the case when M is closed (compact and without boundary); for the nonclosed case, see Remark (4) following the proof.

Since M is a strong deformation retraction of the tube $\mathcal{T}_\delta$, the equation $dv = u$ has a differentiable solution on $\mathcal{T}_\delta$ under the previous assumptions. The strategy is to first find a good differentiable solution v_1 and then successively get rid of its $(p-q-1, q)$-components for $q > 0$. The second part, Lemma 5.2, follows the proof of Serre's theorem [Hö, Thm. 2.7.10], which amounts to solving a $\bar\partial$-equation at each step. We use the solution provided by Theorem 3.1; it is here that we need the sharp estimates (3.3) and (3.4) for the Hölder norms.

5.2. LEMMA. *Let $0 < c < c_1 < 1$. Let u be a closed holomorphic p-form on $\mathcal{T}_\delta$ for $0 < \delta \le \delta_0$, as in Theorem 5.1. Suppose that there exists a differentiable $(p-1)$-form v_1 on $\mathcal{T}_{c_1\delta}$ satisfying $dv_1 = u$ and*

$$\|v_1\|_{L^\infty(\mathcal{T}_{c_1\delta})} \le A_\delta, \quad \|v_1\|_{\Lambda^s(\mathcal{T}_{c_1\delta})} \le A_\delta \delta^{-s}, \tag{5.4}$$

where A_δ depends on δ and u. Then there exists a holomorphic $(p-1)$-form v in $\mathcal{T}_{c\delta}$ satisfying $dv = u$ and $\|v\|_{L^\infty(\mathcal{T}_{c\delta})} \le C_0 A_\delta$ for $0 < \delta \le \delta_0$, where C_0 is an absolute constant.

Proof. Let $v_1 = \sum_{q \le q_0} v_{(q)}$, where $v_{(q)}$ is of bidegree $(p-1-q, q)$. By comparing the terms of bidegree $(p-1-q_0, q_0+1)$ in the equation $dv_1 = \partial v_1 + \bar\partial v_1 = u$ and taking into account that u is holomorphic, we see that $\bar\partial v_{(q_0)} = 0$. If $q_0 > 0$ then we have (by Theorem 3.1) a form w on $\mathcal{T}_\delta$ solving $\bar\partial w = v_{(q_0)}$ and satisfying the following estimates for some fixed $c < c_2 < 1$ and for all $1 \le j \le 2n$:

$$\|\partial_j w\|_{L^\infty(\mathcal{T}_{c_2\delta})} \le C_1(\|v_{(q_0)}\|_{L^\infty(\mathcal{T}_{c_1\delta})} + \delta^s \|v_{(q_0)}\|_{\Lambda^s(\mathcal{T}_{c_1\delta})}) \le 2C_1 A_\delta,$$

$$\|\partial_j w\|_{\Lambda^s(\mathcal{T}_{c_2\delta})} \le C_1(\|v_{(q_0)}\|_{\Lambda^s(\mathcal{T}_{c_1\delta})} + \delta^{-s} \|v_{(q_0)}\|_{L^\infty(\mathcal{T}_{c_1\delta})}) \le 2C_1 A_\delta \delta^{-s}.$$

Thus the form $v_2 = v_1 - w$ solves $dv_2 = dv_1 = u$, has only components of bidegree $(p-q-1, q)$ for $q < q_0$, and it satisfies

$$\|v_2\|_{L^\infty(\mathcal{T}_{c_2\delta})} \le C' A_\delta, \qquad \|v_2\|_{\Lambda^s(\mathcal{T}_{c_2\delta})} \le C' A_\delta \delta^{-s}.$$

Repeated use of this argument gives a holomorphic solution of the equation $dv = u$ satisfying $\|v\|_{L^\infty(\mathcal{T}_{c\delta})} \le C_0 A_\delta$. □

To prove Theorem 5.1 it thus suffices to construct a good differentiable solution satisfying Lemma 5.2, with A_δ as small as possible. Let $F = \{F_t\}: [0,1] \times \mathcal{T}_{\delta_0} \to \mathcal{T}_{\delta_0}$ be a $\mathcal{C}^2$ deformation retraction of a tube $\mathcal{T}_{\delta_0}$ onto M, with F_1 the identity map and $\pi = F_0: \mathcal{T}_{\delta_0} \to M$ a retraction onto M. Let $i_t: M \to [0,1] \times M$ be the map $x \to (t,x)$. By Lemma 2.1, the form $\tilde v = \int_0^1 i_t^*\big(\frac{\partial}{\partial t} \rfloor F^* u\big)\, dt$ solves $d\tilde v = u - \pi^* u$ in $\mathcal{T}_{\delta_0}$. In the special local coordinates provided by Lemma 2.4, if $u = \sum_{|I|+|J|=p} u_{I,J} dx^I \wedge dy^J$ then the components of $\tilde v$ are linear combinations of terms $y_j \int_0^1 t^{|J|-1} u_{I,J}(x,ty)\, dt$ for $j \in J$. Since the variables y_j are transverse to M, we have $|y_j| = O(\delta)$ on $\mathcal{T}_\delta$ and hence $\|\tilde v\|_{L^\infty(\mathcal{T}_\delta)} \le C\delta \|u\|_{L^\infty(\mathcal{T}_\delta)}$, with C independent of δ. Replacing δ by $c_1\delta$ and changing C in each step below if necessary (but keeping it independent of δ), it follows from Cauchy's inequalities that $\|Du\|_{L^\infty(\mathcal{T}_{c_1\delta})} \le C\delta^{-1} \|u\|_{L^\infty(\mathcal{T}_\delta)}$ and thus

$$\|D\tilde v\|_{L^\infty(\mathcal{T}_{c_1\delta})} \le C\|u\|_{L^\infty(\mathcal{T}_\delta)}, \qquad \|\tilde v\|_{\Lambda^s(\mathcal{T}_{c_1\delta})} \le C\delta^{1-s} \|u\|_{L^\infty(\mathcal{T}_\delta)}.$$

In part (a) of Theorem 5.1 we have $p > m$ and so $\pi^* u = \pi^*(i^* u) = 0$ by degree reasons; hence the form $v_1 = \tilde v$ satisfies $dv_1 = u$ and the estimate (5.4) with $A_\delta = C\delta \|u\|_{L^\infty(\mathcal{T}_\delta)}$. Lemma 5.2 now completes the proof in this case.

To prove case (b) we set $v_1 = \tilde v + \pi^* v_0$, where $v_0 \in \Lambda^s(M)$ solves $dv_0 = i^* u$. We obtain

$$\|v_1\|_{L^\infty(\mathcal{T}_{c_1\delta})} \le C(\delta \|u\|_{L^\infty(\mathcal{T}_\delta)} + \|v_0\|_{L^\infty(M)}),$$

$$\|v_1\|_{\Lambda^s(\mathcal{T}_{c_1\delta})} \le C(\delta^{1-s} \|u\|_{L^\infty(\mathcal{T}_\delta)} + \|v_0\|_{\Lambda^s(M)}).$$

Lemma 5.2 then provides a holomorphic solution of $dv = u$ satisfying the estimates (5.2).

Finally, to prove part (c) in Theorem 5.1 we shall construct a good solution of $dv_0 = i^* u$ on M belonging to $\Lambda^s(M)$ and then apply (b). In order to circumvent problems caused by low differentiability of M, we use the following result of Whitney [W2]: If M is a compact $\mathcal{C}^k$ manifold ($k \ge 1$), possibly with boundary, then the underlying topological manifold may be given a structure of a $\mathcal{C}^\infty$ manifold, denoted M_0, such that the set-theoretical identity map $i_0: M_0 \to M$ is a $\mathcal{C}^k$-diffeomorphism.

Let $i_0: M_0 \to M$ be as before. We choose a smooth Riemann metric on M_0 and refer to Wells [We] for what follows. Let d^* denote the Hilbert space adjoint of the exterior derivative d with respect to the corresponding inner product on forms. The *Laplace operator* $\triangle = d^*d + dd^*$ has a corresponding *Green operator* $G: L^2_{(p)}(M_0) \to H^2_{(p)}(M_0))$ with the property that $\beta = d^* G(\alpha)$ is the solution of $d\beta = \alpha$ with minimal L^2-norm (orthogonal to the null-space of $\triangle$), provided that the equation $d\beta = \alpha$ is (weakly) solvable. For further details, see [We, Sec. 4.5].

The Green operator is a classical pseudodifferential operator of order -2, so it induces bounded operators $L^\infty \to \Lambda^2$ and $\Lambda^s \to \Lambda^{s+2}$ for $s > 0$. (See [S, Sec. VI, 5.3.) Now $i_0^* u$ is a $\mathcal{C}^1$-form on M_0, and $v_0 = (i_0^{-1})^*(d^* G i_0^* u)$ is a $\mathcal{C}^1$-form on M with $dv_0 = i^* u$, satisfying

$$\|v_0\|_{\Lambda^s(M)} \le C_s' \|i^* u\|_{L^\infty(M)} \le C_s \|u\|_{L^\infty(\mathcal{T}_\delta)}.$$

Substituting this into (5.2) gives (5.3). □

REMARKS. (1) In general the constant $C = C_\delta$ in the estimate (5.3) cannot be chosen so that $\lim_{\delta\to 0} C_\delta = 0$. To see this, let $i^* u \ne 0$ and choose a form $\phi \in \mathcal{C}^1_{(m-p)}(M)$ with $\int_M u \wedge \phi \ne 0$. If v_δ solves $dv_\delta = u$ in $\mathcal{T}_{c\delta}$ and satisfies $\lim_{\delta\to 0} \|v_\delta\|_{L^\infty(\mathcal{T}_\delta)} = 0$, we get

$$\int_M u \wedge \phi = \int_M dv_\delta \wedge \phi = \pm \int_M v_\delta \wedge d\phi \to 0$$

as $\delta \to 0$, a contradiction.

(2) If M is only of class $\mathcal{C}^1$, the operator d is not well-defined on M. Instead, we call a p-form α on M *exact* if there exists an integrable $(p-1)$-form β on M such that, for each smooth $(m-p)$-form ϕ on a neighborhood of M, we have $\int_M \beta \wedge i^*(d\phi) = (-1)^p \int_M \alpha \wedge i^*\phi$. Then it is not hard to verify that $d(i_0^*\beta) = i_0^*\alpha$ (weakly) on M_0 and also that $d(\pi^*\beta) = \pi^*\alpha$ on $\mathcal{T}_{\delta_0}$. Using this, the proof carries over with only minor changes to the case where M is of class $\mathcal{C}^{1+\varepsilon}$ for some $\varepsilon > 0$, when $dv_0 = i^*u$ is interpreted as above.

(3) If M is of class $\mathcal{C}^{2+\varepsilon}$ for some $\varepsilon > 0$, a more refined argument gives a holomorphic solution of $dv = u$ that also satisfies $\|v\|_{\mathcal{C}^1(\mathcal{T}_{c\delta})} \le C \log(1/\delta)\|u\|_{L^\infty(\mathcal{T}_\delta)}$ whenever i^*u is exact. This reflects the fact that one expects to "gain almost a derivative" in the interior estimates for the d-equation. We cannot establish such estimates with a constant independent of δ. In fact, when $M = \{z \in \mathbf{C}^n : |z_j| = 1, 1 \le j \le n\}$, this would lead to the estimate $\|\beta\|_{\mathcal{C}^1(M)} \le \text{const}\|\alpha\|_{L^\infty(M)}$ for a solution of $d\beta = \alpha$, a contradiction.

(4) Small changes are needed to prove Theorem 5.1 when $M = K$ is a compact subset of a larger totally real $\mathcal{C}^2$-submanifold $M' \subset \mathbf{C}^n$ (see Remark (3) following the statement of Theorem 5.1). We follow the same proof as before, using the appropriate version of $\bar\partial$-results given by Remark (3) following Theorem 3.1. During the proof we shrink $K' \supset K$ and $\delta > 0$ several times. In the proof of (5.2), we observe that the L^2-minimal solution of $dv_0 = i^*u$ in $U_\delta' \cap M'$ also satisfies $d^*v_0 = 0$ when $p > 1$, and we may apply the interior elliptic estimates to obtain Hölder estimates for v_0 in a neighborhood of K. There are also arguments to secure the necessary control of $\|v_0\|_{L^2}$, for instance, the Hodge decomposition in a manifold with boundary.

6. Proof of Theorem 1.5

In this section we prove Theorem 1.5. We shall adapt a method of J. Moser [M] to the holomorphic setting.

Let ω be either the holomorphic volume form $dz_1 \wedge \cdots \wedge dz_n$ or the holomorphic symplectic form $\sum_{j=1}^{n'} dz_{2j-1} \wedge dz_{2j}$, $n = 2n'$. Write $M = M_0$ and let

$f: M = M_0 \to M_1$ be a $\mathcal{C}^k$-diffeomorphism as in Theorem 1.5 ($k \ge 2$), satisfying condition (1.6) for some $\mathcal{C}^{k-1}$-map $L: M \to GL(n, \mathbf{C})$. Let $i: M \hookrightarrow \mathbf{C}^n$ denote the inclusion. We assume in the proofs that M is compact and without boundary. As usual, we denote by $\mathcal{T}_\delta = \mathcal{T}_\delta M$ the tube of radius δ around M.

By Lemma 2.6 there is a neighborhood $U \subset \mathbf{C}^n$ of M and a $\mathcal{C}^k$-diffeomorphism $\tilde f: U \to \tilde f(U) \subset \mathbf{C}^n$ extending f such that $\tilde f$ is $\bar\partial$-flat on M and satisfies $(\tilde f^*\omega)_z = \omega_z$ at all points $z \in M$. The proof of Theorem 1.2 then gives, for each small $\delta > 0$, a holomorphic map $F'_\delta: \mathcal{T}_\delta \to \mathbf{C}^n$ of the form

$$F'_\delta = \tilde f + R_\delta, \quad \|R_\delta\|_{\mathcal{C}^j(\mathcal{T}_\delta M)} = o(\delta^{k-j}); \quad 0 \le j \le k. \tag{6.1}$$

In order to prove Theorem 1.5, we must construct biholomorphic maps F_δ as before but which in addition satisfy $F_\delta^*\omega = \omega$. We need the following two lemmas.

6.1. Lemma (Existence of a good $\bar\partial$-flat extension). *If $\tilde f$ is any $\bar\partial$-flat $\mathcal{C}^k$-extension of f satisfying $(\tilde f^*\omega)_z = \omega_z$ for all $z \in M$, then there exists another $\bar\partial$-flat $\mathcal{C}^k$-extension $\hat f$ of f satisfying $|\hat f^*\omega - \omega| = o(d_M^{k-1})$ near M and $d\tilde f_z = d\hat f_z$ for all $z \in M$.*

6.2. Lemma (Approximation of a good $\bar\partial$-flat extension). *Assume that $\tilde f$ is any $\bar\partial$-flat $\mathcal{C}^k$-extension of f satisfying $|\tilde f^*\omega - \omega| = o(d_M^{k-1})$. Then, for all sufficiently small $\delta > 0$, there exist biholomorphic maps $F_\delta: \mathcal{T}_\delta \to \mathbf{C}^n$ with $F_\delta^*\omega = \omega$ and $\|F_\delta - \tilde f\|_{\mathcal{C}^j(\mathcal{T}_\delta M)} = o(\delta^{k-j})$ for $0 \le j \le k$.*

We postpone the proof of Lemmas 6.1 and 6.2 for a moment.

Proof of Theorem 1.5 in the Smooth Case. Let $f: M = M_0 \to M_1$ be a $\mathcal{C}^k$-diffeomorphism as in Theorem 1.5. By Lemma 2.6, there is a $\bar\partial$-flat extension $\tilde f$ of f satisfying $\tilde f^*\omega = \omega$ at points of M. By Lemma 6.1 we can modify this extension, still denoting it $\tilde f$, such that $|\tilde f^*\omega - \omega| = o(d_M^{k-1})$. Finally we apply Lemma 6.2 to derive biholomorphic maps F_δ in tubes $\mathcal{T}_\delta$ around M satisfying $F_\delta^*\omega = \omega$ and the estimates (1.2). This proves Theorem 1.5 in the smooth case, granted that Lemmas 6.1 and 6.2 hold. We postpone the proof in the real-analytic case to the end of this section. □

Proof of Lemma 6.2. Let $\tilde f$ be as described in the lemma and let $F'_\delta: \mathcal{T}_\delta \to \mathbf{C}^n$ (for small $\delta > 0$) be holomorphic maps of the form (6.1) obtained as in the proof of Theorem 1.2. From the estimates on R_δ in (6.1) and the assumption $|\tilde f^*\omega - \omega| = o(d_M^{k-1})$, it follows that

$$\|(F'_\delta)^*\omega - \omega\|_{\mathcal{C}^j(\mathcal{T}_\delta)} = o(\delta^{k-j-1}), \quad 0 \le j \le k-1.$$

Set $\omega^\delta = (F'_\delta)^*\omega$; this is a holomorphic p-form on $\mathcal{T}_\delta$ that is close to ω. Choose constants $0 < a < c < 1$. Using Moser's method [M] we shall construct a holomorphic map $G_\delta: \mathcal{T}_{a\delta} \to \mathcal{T}_\delta$ that is very close to the identity map and satisfies $G_\delta^*\omega^\delta = \omega$ on $\mathcal{T}_{a\delta}$. The holomorphic map $F_\delta = F'_\delta \circ G_\delta: \mathcal{T}_{a\delta} \to \mathbf{C}^n$ is then close to F'_δ (and hence to $\tilde f$), and it satisfies $F_\delta^*\omega = G_\delta^*(\omega^\delta) = \omega$.

We first outline Moser's method, postponing the estimates for a moment. Set $\omega_1^\delta = (F_\delta')^*\omega$ and $\omega_t^\delta = (1-t)\omega + t\omega_1^\delta$ for $t \in [0,1]$. Then $d\omega_t^\delta = 0$, and ω_t^δ is close to ω for each t and δ. Our goal is to construct a $\mathcal{C}^1$-family of holomorphic maps $G_t = G_{\delta,t}\colon \mathcal{T}_{a\delta} \to \mathcal{T}_\delta$ satisfying $G_0 = \mathrm{Id}$ and $G_t^*\omega_t^\delta = \omega$ for all $t \in [0,1]$; the time-1 map $G_\delta = G_{\delta,1}$ will then solve the problem.

To simplify the notation we suppress δ for the moment, writing $\omega_t^\delta = \omega_t$ and $G_{\delta,t} = G_t$. Suppose that such a flow G_t exists. Denote by Z_t its infinitesimal generator; this is a holomorphic time-dependent vector field on the image of G_t that satisfies $\frac{d}{dt}G_t(z) = Z_t(G_t(z))$ for each $t \in [0,1]$ and each z in the domain of G_t. Differentiating the equation $G_t^*\omega_t = \omega$ on t and applying the time-dependent Lie derivative theorem [AMR, Thm. 5.4.5], we have

$$0 = \frac{d}{dt}(G_t^*\omega_t) = G_t^*\left(L_{Z_t}\omega_t + \frac{d}{dt}\omega_t\right) = G_t^*(d(Z_t \rfloor \omega_t) + \omega_1 - \omega). \tag{6.2}$$

We have also used the Cartan formula for the Lie derivative $L_{Z_t}\omega_t$, as well as $d\omega_t = 0$. This shows that $G_t^*\omega_t = \omega$ holds for all $t \in [0,1]$ if and only if the generator Z_t satisfies the equation $d(Z_t\rfloor\omega_t) + \omega_1 - \omega = 0$ for all $t \in [0,1]$.

At this point we observe that ω is exact holomorphic on $\mathbf{C}^n$, $\omega = d\beta$; in fact, when ω is the volume form (1.4) we may take $\beta = \frac{1}{n}\sum_{j=1}^n (-1)^{j+1} dz[j]$, and when ω is the symplectic form (1.5) we may take $\beta = \sum_{j=1}^{n'} z_{2j-1}dz_{2j}$. Hence the difference $\omega_1 - \omega = F_\delta'^* d\beta - d\beta = d(F_\delta'^*\beta - \beta)$ is exact holomorphic on $\mathcal{T}_\delta$. By Theorem 5.1 we can solve the equation $dv = \omega_1 - \omega$ to get a small holomorphic $(p-1)$-form $v = v_\delta$ in $\mathcal{T}_{c\delta}$. Let Z_t be the unique holomorphic vector field on $\mathcal{T}_{c\delta}$ solving the (algebraic!) equation $Z_t\rfloor\omega_t + v = 0$. Integrating Z_t yields a flow G_t that satisfies $G_t^*\omega_t = \omega$ on its domain of definition.

For this approach to work we must choose v_δ on $\mathcal{T}_{c\delta}$ to have as small sup-norm as possible; this will imply that $|Z_t|$ is small and hence its flow $G_t(z)$ will not escape the tube $\mathcal{T}_{c\delta}$ (on which Z_t is defined) before time $t = 1$, provided that the initial point $G_0(z) = z$ belongs to the smaller tube $\mathcal{T}_{a\delta}$. (In particular, the solution $v_\delta = (F_\delta')^*\beta - \beta$ may not work because F_δ' is not close to the identity map.)

In order to apply Theorem 5.1 efficiently we must first show that $dv_0 = i^*(\omega_1 - \omega)$ has a solution on M with small norm. Consider the map $h\colon [0,1]\times M \to \mathbf{C}^n$, $h(t,z) = \tilde f(z) + tR_\delta(z)$, and set $w = h^*\omega$. Also let $i_t\colon M \to [0,1]\times M$ denote the injection $i_t(z) = (t,z)$ ($z \in M$, $t \in [0,1]$). It follows from Lemma 2.1 that $v_0 = \int_0^1 i_t^*\big(\frac{\partial}{\partial t}\rfloor w\big)\,dt$ solves $dv_0 = i_1^*w - i_0^*w$. We have $i_1^*w = i^*\omega_1$ and $i_0^*w = i^*\tilde f^*\omega = i^*\omega$, so $dv_0 = i^*(\omega_1 - \omega)$. It follows from the preceding formula that $v_0 = \sum_{j=1}^n r_j^\delta v_j$, where $r_1^\delta, \dots, r_n^\delta$ are the components of R_δ and $v_1, \dots, v_n$ are $(p-1)$-forms on M with $\|v_j\|_{\mathcal{C}^{k-1}(M)}$ bounded independently of δ. This gives $\|v_0\|_{\mathcal{C}^l(M)} = o(\delta^{k-l})$ for $0 \le l \le k-1$. It follows that $\|v_0\|_{\Lambda^s(M)} = o(\delta^{k-s})$ for a given $s \in (0,1)$. Since $\|\omega_1 - \omega\|_{L^\infty(\mathcal{T}_\delta)} = o(\delta^{k-1})$, it follows from Theorem 5.1 that, for all sufficiently small $\delta > 0$, we have a holomorphic solution of $dv_\delta = \omega_1 - \omega$ in $\mathcal{T}_{c\delta}$ that satisfies $\|v_\delta\|_{L^\infty(\mathcal{T}_{c\delta})} = o(\delta^k)$.

Let Z_t^δ be the holomorphic vector field in $\mathcal{T}_{c\delta}$ satisfying $Z_t^\delta\rfloor\omega_t^\delta = v_\delta$. The previous estimate on v_δ implies $\|Z_t^\delta\|_{L^\infty(\mathcal{T}_{c\delta})} = o(\delta^k)$ uniformly in $t \in [0,1]$. The standard formula for the rate of escape of the flow shows that we can choose $\delta_0 > 0$ sufficiently small such that, for all $\delta \in (0, \delta_0)$ and all initial points $z \in \mathcal{T}_{a\delta}$, the flow $G_{\delta,t}(z)$ of Z_t^δ remains in $\mathcal{T}_{c\delta}$ for all $t \in [0,1]$. At $t = 1$ we have a map $G_\delta = G_{\delta,1}\colon \mathcal{T}_{a\delta} \to \mathcal{T}_{c\delta}$ satisfying $G_\delta^* \omega_1^\delta = \omega$ and $|G_\delta(z) - z| = o(\delta^k)$ for $z \in \mathcal{T}_{a\delta}$.

Set $F_\delta = G_\delta \circ F'_\delta$. Since the maps F'_δ have uniformly bounded $\mathcal{C}^1$-norms on $\mathcal{T}_\delta$, we see that $\|F_\delta - F'_\delta\|_{L^\infty(\mathcal{T}_{a\delta})} = o(\delta^k)$. Replacing a by a smaller constant and applying the Cauchy inequalities, we also get

$$\|F_\delta - \tilde f\|_{\mathcal{C}^j(\mathcal{T}_{a\delta})} \le \|F_\delta - F'_\delta\|_{\mathcal{C}^j(\mathcal{T}_{a\delta})} + \|F'_\delta - \tilde f\|_{\mathcal{C}^j(\mathcal{T}_\delta)} = o(\delta^{k-j}), \quad j \le k.$$

By construction we have $F_\delta^* \omega = \omega$, so F_δ solves the problem. □

REMARK. This method applies on any domain $D \subset\subset \mathbf{C}^n$ on which we can solve the $\bar\partial$-equations with estimates (e.g., on pseudoconvex domains); it shows that, for any holomorphic map $F'\colon D \to \mathbf{C}^n$ for which $|F'^*\omega - \omega|$ is sufficiently uniformly small on D, there exists a holomorphic map $F\colon D' \to \mathbf{C}^n$ on a slightly smaller domain $D' \subset\subset D$ such that $F^*\omega = \omega$ and F is uniformly close to F' on D'. We obtain F in the form $F = F' \circ G$, where $G\colon D' \to D$ is a holomorphic map close to the identity and chosen such that $G^*(F'^*\omega) = \omega$. The precise amount of shrinking of the domain depends on $\|F'^*\omega - \omega\|_{L^\infty(D)}$ and on the constants in the solutions of the $\bar\partial$-equations; we do not know if there is a solution to this problem on all of D.

We now turn to the proof of Lemma 6.1. We shall need the following.

6.3. LEMMA. *Let u be a d-closed p-form of class $\mathcal{C}^{k-1}$ in a neighborhood of M, with $p \ge 1$, such that the $(p,0)$-component u' of u is $\bar\partial$-flat on M and $u'' = u - u'$ is $(k-1)$-flat on M. Assume $i^*u = 0$, where $i\colon M \hookrightarrow \mathbf{C}^n$ is the inclusion. Then there exists a $(p-1,0)$-form v in a neighborhood of M such that $v = \sum_{j=1}^N \zeta_j v_j$, where each ζ_j is a $\bar\partial$-flat $\mathcal{C}^k$-function vanishing on M, each v_j is a $\bar\partial$-flat $\mathcal{C}^{k-1}$-form, and $|u - dv| = o(d_M^{k-1})$. If $u = 0$ on M, we may take $v_j = 0$ on M for all j.*

REMARK. Using rough multiplication (Lemma 2.5), we see that there is a $\bar\partial$-flat $(p,0)$-form v of class $\mathcal{C}^k$ that also satisfies $|dv - u| = o(d_M^{k-1})$. However, the version stated here is often technically more convenient, since we may wish to postpone the use of rough multiplication.

Proof of Lemma 6.3. In the case $m = n$ we may take $v = 0$, which can be seen as follows. We have $u' = \sum_{|I|=p} u_I dz^I$, where the coefficients u_I are $\mathcal{C}^k$-functions that are $\bar\partial$-flat on M; hence $i^*u = 0$ means that $u_I = 0$ on M for all I (since the coefficients of u'' vanish on M). It follows from the Cauchy–Riemann equations that each u_I is flat on M, so we may choose $v = 0$.

When $m < n$, we use the asymptotically holomorphic extension $\tilde M$ of M (Lemma 2.4) and the $\bar\partial$-flat retraction F to $\tilde M$. Recall that a neighborhood of M may be covered by $\mathcal{C}^k$-charts $G_i\colon U_i \to V_i$ ($G_i(z) = (z'_{(i)}(z), w'_{(i)}(z)) \in \mathbf{C}^m \times \mathbf{C}^{n-m}$, $1 \le i \le r$) satisfying:

(a) G_i is $\bar\partial$-flat on M, $G_i(M \cap U_i) = V_i \cap (\mathbf{R}^m \times \{0\})$, and $G_i(\tilde M \cap U_i) = V_i \cap (\mathbf{C}^m \times \{0\})$;
(b) the retraction F is given in these local coordinates by $(t, (z', w')) \to (z', tw')$.

Let $\tilde i \colon \tilde M \hookrightarrow \mathbf{C}^n$ be the inclusion. Arguing as in the case $m = n$ and making use of the $\bar\partial$-flat local parametrizations of $\tilde M$, we see that $\tilde i^* u$ is flat on M and so is $\tilde\pi^* u = \tilde\pi^* \tilde i^* u$, where $\tilde\pi = F_0$. When $F \colon [0,1] \times W \to W$ is the retraction to $\tilde M$, the form

$$\hat v = \int_0^1 i_t^* \left(\frac{\partial}{\partial t} \rfloor F^* u \right) dt \tag{6.3}$$

solves $d\hat v = u - \tilde\pi^* u$ on a neighborhood of M, according to Lemma 2.1. Expressing u in the G_i-coordinates $(z'_{(i)}(z), w'_{(i)}(z))$ (which are $\bar\partial$-flat on M) yields

$$u = \sum_{|I|+|J|=p} a_{I,J}(z'_{(i)}, w'_{(i)}) dz'^{I}_{(i)} \wedge dw'^{J}_{(i)} + r'_{(i)}$$

on U_i, where the $a_{I,J}$ are $\mathcal{C}^{k-1}$-functions that are $\bar\partial$-flat on $\mathbf{R}^m \times \{0\}$ and where $r'_{(i)}$ is a $\mathcal{C}^{k-1}$-form that is flat on M. Using the formula following Lemma 2.1, we see that $\hat v$ in (6.3) is a linear combination of terms

$$w'_{(i),j} \left(\int_0^1 a_{I,J}(z'_{(i)}, tw'_{(i)}) t^{|K|}\, dt \right) dz'^{I}_{(i)} \wedge dw'^{K}_{(i)},$$

where $|I| + |K| = p - 1$ and $1 \le j \le n - m$, plus a remainder term $r''_{(i)}$ satisfying $|\partial^\alpha r''_{(i)}| = o(d_M^{k-|\alpha|})$ on U_i for $|\alpha| \le k - 1$. Here $w'_{(i),j}$ denotes the jth component of $w'_{(i)}$. Since $G_{(i)}$ is $\bar\partial$-flat, it follows that

$$\hat v = \sum_{j=1}^{n-m} \sum_{|L|=p-1} w'_{(i),j} g^{(i)}_{j,L} dz^L + r_{(i)}$$

in U_i, where each $g^{(i)}_{j,L}$ is a $\bar\partial$-flat $\mathcal{C}^{k-1}$-function and where $r_{(i)}$ behaves like $r''_{(i)}$.

Choose a $\bar\partial$-flat partition of unity $\{\psi_i\}_{i=1}^r$ subordinate to the covering $\{U_i\}_{i=1}^r$, and choose $\bar\partial$-flat cut-off functions $\chi_i \in \mathcal{C}_0^\infty(U_i)$ with $\chi_i = 1$ near $\operatorname{supp} \psi_i \cap M$ for $i = 1, \dots, r$. Let $\zeta_1, \dots, \zeta_N$ (with $N = r(n-m)$) be some enumeration of the collection of functions $\{\psi_i w'_{(i),j} \colon i \le r,\ j \le n - m\}$. Furthermore, let $v_1, \dots, v_N$ be the corresponding enumeration of the forms $\chi_i \sum_{|L|=p-1} g^{(i)}_{j,L} dz^L$, prolonged by zero outside U_i. Set $v = \sum_{l=1}^N \zeta_l v_l$. Clearly $|dv - u| = o(d_M^{k-1})$. Furthermore, if $u = 0$ on M, we also see that $\int_0^1 a_{I,J}(z', tw') t^{|K|}\, dt = 0$ on $V_i \cap (\mathbf{R}^m \times \{0\})$ and hence $v_1 = \cdots = v_N = 0$ on M. □

Proof of Lemma 6.1. In the unimodular case, $\omega = dz_1 \wedge \cdots \wedge dz_n$, we could successively increase the order of vanishing of $\tilde f \omega - \omega$ on M by adding certain correction terms to $\tilde f$. This seems harder to do in the symplectic case, so we shall instead present an argument that works uniformly in both cases. It is a modification of Moser's method: with $\omega_t = (1-t)\omega + t\tilde f^* \omega$, we shall construct a $\mathcal{C}^1$-family of $\bar\partial$-flat $\mathcal{C}^k$-maps g_t on a neighborhood of M, with $g_0 = \mathrm{id}$ and $\left| \frac{d}{dt} g_t^* \omega_t \right| = o(d_M^{k-1})$

uniformly in t. Given such a family, integration in t gives $\|g_1^*\omega_1-\omega\| = o(d_M^{k-1})$. We will also show that g_1 is $\bar\partial$-flat on M. Hence the map $\hat f = \tilde f \circ g_1$ will satisfy Lemma 6.1. Furthermore, we shall see that $|g_1(z)-z| = O(d_M(z)^2)$, so $Dg_1 =$ Id on M and hence $\hat f$ and $\tilde f$ have the same differential on M.

We shall obtain g_t by integrating a certain real time-dependent vector field X_t of class $\mathcal{C}^k$. Differentiating $\frac{d}{dt} g_t^*\omega_t$ as in (6.2), we see that X_t must satisfy $|d(X_t\rfloor\omega_t)+\omega_1-\omega| = o(d_M^{k-1})$. We shall now construct such a vector field. More precisely, we shall construct a continuous family of $\mathcal{C}^k$ real vector fields X_t, on a tube $\mathcal{T}_0 = \mathcal{T}_{\delta_0}$, satisfying the following properties for each $t \in [0,1]$.

(1) X_t, considered as a map $\mathcal{T}_0 \to \mathbf{C}^n$, is $\bar\partial$-flat on M. (Here we identify a real tangent vector $X = \sum_{j=1}^n a_j \frac{\partial}{\partial x_j} + b_j \frac{\partial}{\partial y_j} \in T_z\mathbf{C}^n$ with the corresponding complex vector $(a_1+ib_1,\ldots,a_n+ib_n) \in \mathbf{C}^n$.)
(2) $|X_t(z)| \le C d_M(z)^2$ for some $C>0$ independent of $t\in[0,1]$.
(3) $|d(X_t\rfloor\omega_t)+\omega_1-\omega| = o(d_M^{k-1})$ uniformly in $t\in[0,1]$.

Let us first show that this solves the problem. We must show that X_t can be integrated from $t=0$ to $t=1$ for all initial values in a smaller tube. Recall that, after shrinking δ_0 if necessary, the function d_M is differentiable in $\mathcal{T}_0\backslash M$, with a gradient of length 1. Let $z(t)$ be an integral curve of X_t in $\mathcal{T}_0\backslash M$, $t\in[0,t_0]$, and set $u(t) = d_M(z(t))$. Then

$$u'(t) = \nabla d_M(z(t))\cdot X_t(z(t)) \le |X_t(z(t))| \le Cu(t)^2.$$

Here we denote by $v\cdot w$ the real inner product of the vectors $v, w\in\mathbf{C}^n$. Integrating the inequality $u'(t)/u(t)^2 \le C$ from 0 to t gives $1/u(0)-1/u(t)\le Ct$ and thus $u(t)(1-Ctu(0)) \le u(0)$ for $0\le t\le t_0$. Let the initial value $z(0)\in\mathcal{T}_{\delta_1}\backslash M$, where $\delta_1 \le \min(\delta_0/2, 1/2C)$. It follows that $u(t)\le u(0)/(1-Ctu(0)) \le 2u(0)$ and hence the integral curve extends to all values $t\in[0,1]$. Since $|X_t(z(t))|\le Cu(t)^2$, we see that $|z(t)-z(0)|\le 4Cu(0)^2 t$. In other words, the time-t diffeomorphisms g_t are well-defined on $\mathcal{T}_{\delta_1}$ for all $t\in[0,1]$ and satisfy $|g_t(z)-z|\le 4Ctd_M(z)^2$. In particular, $g_t(z) = z$ and $Dg_t(z) =$ Id for $z\in M$ and $t\in[0,1]$.

To show that the $\mathcal{C}^k$-maps g_t are $\bar\partial$-flat on M, we consider the variational equation $\frac{\partial}{\partial t} D_z g_t(z) = D_z X_t(g_t(z)) \circ D_z g_t(z)$ with the initial condition $D_z g_0 =$ Id. Decomposing the differential $D\phi$ as the sum of a $\mathbf{C}$-linear part $D'\phi$ and a $\mathbf{C}$-conjugate part $D''\phi$, we get

$$\begin{aligned}\frac{\partial}{\partial t} D_z'' g_t(z) &= D_z''\left(\frac{\partial}{\partial t} g_t(z)\right) = D_z''(X_t(g_t(z)))\\ &= (D_z' X_t)(g_t(z))\circ D_z'' g_t(z) + (D_z'' X_t)(g_t(z))\circ D_z' g_t(z).\end{aligned}$$

We apply both sides to a unit vector $v\in\mathbf{C}^n$ and set $y(t) = D_z'' g_t(z)v \in \mathbf{C}^n$. We obtain a linear differential equation $y'(t) = A(t)y(t)+b(t)$ with the initial condition $y(0) = D_z'' g_0(z)v = 0$. The function $u(t) = |y(t)|$ is differentiable when $u(t)\ne 0$ and $u'(t) = y'(t)\cdot y(t)/|y(t)| \le |y'(t)|$. Thus, if $|A(t)|\le A$ and $|b(t)|\le b$ then $u'(t)\le Au(t)+b$, where $u(t)\ne 0$. We shall prove that $u(t)\le \frac{b}{A}(e^{At}-1)$, $t\in[0,1]$. If $u(t)=0$ then there is nothing to prove. If not, let t_0 be the largest zero of

u on the interval $[0, t]$. Then $v(s) = u(s)a^{-As}$ satisfies the differential inequality $v'(s) \le be^{-As}$ for $s \in (t_0, t]$. Integration from t_0 to t gives $v(t) \le \frac{b}{A}(e^{-At_0} - e^{-At})$ and $u(t) \le \frac{b}{A}(e^{A(t-t_0)} - 1) \le \frac{b}{A}(e^{At} - 1)$.

We know that $|D_z X_t(z)|$ and $|D_z g_t(z)|$ are bounded uniformly in $z \in \mathcal{T}_{\delta_1}$ and $t \in [0, 1]$, while $|D_z'' X_t(z)| = o(d_M(z)^{k-1})$. Thus we may choose the upper bound A for $|A(t)|$ independently of $z \in \mathcal{T}_{\delta_1}$ and the unit vector v, and we may choose the upper bound b of $|b(t)|$ to be of size $b = o(d_M(z)^{k-1})$ uniformly in v. Since $u(t) = |D_z'' G_t(z)v|$, it follows that $|D_z'' g_t(z)| = o(d_M(z)^{k-1})$, so each g_t is $\bar\partial$-flat on M.

By assumption we have $|d(X_t \rfloor \omega_t)_z + (\omega_1 - \omega_0)_z| = o(d_M(z)^{k-1})$. Since $d_M(g_t(z)) \le 2d_M(z)$ and the norms $|D_z g_t(z)|$ are bounded uniformly in $z \in \mathcal{T}_{\delta_1}$ and $t \in [0, 1]$, we have $\left|\frac{\partial}{\partial t}(g_t^* \omega_t)_z\right| = o(d_M(z)^{k-1})$ uniformly in t. By integration in t we obtain $|(g_1^* \omega_1 - \omega)_z| = o(d_M(z)^{k-1})$. Setting $\hat f = \tilde f \circ g_1$, we see that $\hat f$ is a $\bar\partial$-flat $\mathcal{C}^k$-extension of $\tilde f$, $D\hat f = D\tilde f$ on M, and $|(\hat f^* \omega - \omega)_z| = o(d_M(z)^{k-1})$. Thus $\hat f$ satisfies Lemma 6.1.

It remains to construct the vector field X_t. Applying Lemma 6.3 to $\omega - \omega_1$ yields a $(p-1, 0)$-form v near M with $|dv - (\omega - \omega_1)| = o(d_M^{k-1})$ and $v = 0$ on M. We decompose ω_t as $\omega_t' + \omega_t''$, where ω_t' is the $(p, 0)$-component of ω_t. Then $\omega_t' = \omega + t(\omega_1' - \omega)$, and $\omega_t' = \omega$ on M for each t. Hence the map $\phi \colon Z \to Z \rfloor \omega_t'$, taking the $(1, 0)$-vectors $Z \in T_z^{(1,0)}\mathbf{C}^n$ to $\Lambda^{(p-1,0)} T_z^* \mathbf{C}^n$, is an isomorphism for z near M and $t \in [0, 1]$. Hence the equation $Z_t' \rfloor \omega_t = v$ uniquely defines a time-dependent $(1, 0)$-vector field Z_t' on $\mathbf{C}^n$ near M.

With respect to the basis $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$ for $(1, 0)$-vectors and the basis $dz[1], \dots, dz[n]$ (resp., $dz_1, \dots, dz_n$) for the $(p-1, 0)$-covectors, the map ϕ is represented by an $(n \times n)$ matrix–valued function $A(t, z) = A_0 + tB(z)$, where A_0 is constant and invertible and where the entries of $B(z)$ are $\bar\partial$-flat $\mathcal{C}^{k-1}$-functions that vanish on M. It follows that the entries of $A(t, z)^{-1}$ are rational functions $b(t, z)$ in t with coefficients that are $\bar\partial$-flat $\mathcal{C}^{k-1}$-functions. From the properties of v as given by Lemma 6.3, it follows that $Z_t' = \sum_{j=1}^N \zeta_j \sum_{k=1}^n r_{jk}(t, z) \frac{\partial}{\partial z_k}$, where $\zeta_1, \dots, \zeta_N$ are $\mathcal{C}^k$-functions that vanish on M and are $\bar\partial$-flat on M and where each r_{jk} is a rational function in t with coefficients that are $\bar\partial$-flat $\mathcal{C}^{k-1}$-functions with $r_{jk}(t, z) = 0$ for $z \in M$.

We next apply the rough multiplication lemma to the pairs $(\zeta_j(z), r_{jk}(t, z))$ with respect to the compact subset $M \times [0, 1]$ in $\mathbf{C}^n \times \mathbf{R}$. We thus obtain $\mathcal{C}^k$-functions $a_l(z, t)$ $(1 \le l \le n)$, $\bar\partial$-flat on M with respect to z, such that

$$\left|\sum_{j=1}^N \zeta_j r_{jl}(t, \cdot) - a_l(t, \cdot)\right| = o(d_M^k)$$

uniformly in t. (*Note:* Use of the parametrized version of rough multiplication gives a smooth family of $\mathcal{C}^k$-functions, but we do not need this.)

We set $Z_t = \sum_{l=1}^n a_l(t, z) \frac{\partial}{\partial z_l}$ and $X_t = Z_t + \bar Z_t$. Writing $a_l = u_l + iv_l$, with u_l and v_l real, we have $X_t = \sum_{l=1}^n u_l(z, t) \frac{\partial}{\partial x_l} + v_l(z, t) \frac{\partial}{\partial y_l}$. If we consider X_t as a map $\mathcal{T}_0 \to \mathbf{C}^n$, this means that $X_t = (a_1(t, z), \dots, a_n(t, z))$ and is $\bar\partial$-flat on M.

Furthermore, since $\zeta_j(z)$ and $r_{jl}(t,z)$ both vanish when $z \in M$, we see that the X_t vanish to the second order on M.

Finally, we must show that (3) is satisfied. Writing $X_t = \bar{Z}_t + (Z_t - Z'_t) + Z'_t$, we see that

$$d(X_t \rfloor \omega_t) + \omega_1 - \omega = d(\bar{Z}_t \rfloor \omega''_t) + d((Z_t - Z'_t)\rfloor \omega'_t) + (dv + \omega_1 - \omega).$$

The first term on the right-hand side is $o(d_M^k)$, since ω''_t vanishes to order $k-1$ and Z_t vanishes to the second order on M. Furthermore, $Z_t - Z'_t$ vanishes to the kth order, so the second term is $o(d_M^{k-1})$ and the third term is $|dv + \omega_1 - \omega| = o(d_M^{k-1})$. Thus, (3) holds uniformly in t, since the derivatives are continuous in (z,t). □

Proof of Theorem 1.5 in the Real-Analytic Case. By assumption, there is a continuous map $\psi_0\colon M_0 \to \mathrm{SL}(n,\mathbf{C})$ (resp., $\psi_0\colon M_0 \to \mathrm{Sp}(n,\mathbf{C})$) such that $\psi_{0,z}$ agrees with $d_z f$ on $T_z M_0$ for each $z \in M_0$. By Remark (1) following the proof of Theorem 1.2 in Section 4, ψ_0 may be approximated by a holomorphic map ψ_1 from a neighborhood of M_0 to $\mathrm{GL}(n,\mathbf{C})$ with $\psi_{1,z} = d_z\tilde{f}$ on $T_z\tilde{M}_0$ for each $z \in \tilde{M}_0$. Since $\psi_{0,z}^*\omega = \omega$ for $z \in M_0$ and since ψ_1 approximates ψ_0 on M_0, it follows that the form $\psi_{1,z}^*\omega = (\det\psi_{1,z})\omega$ is close to ω for all $z \in \tilde{M}_0$ sufficiently near M_0.

We may think of ψ_1 as a holomorphic automorphism of the trivial bundle $\tilde{M}_0 \times \mathbf{C}^n \to \tilde{M}_0$. We claim that there is another holomorphic automorphism g of $\tilde{M}_0 \times \mathbf{C}^n$ such that $g|_{T\tilde{M}_0} = \mathrm{Id}$ and $g^*\psi_1^*\omega = \omega$. In the unimodular case, we let g act as the identity on $T\tilde{M}_0$ and as multiplication by $(\det\psi_1)^{-1/(n-m)}$ on $\tilde{\nu}_0$ (the holomorphic extension of the complex normal bundle ν_0 to $\tilde{M}_0$); the root is well-defined because the function $\det\psi_{1,z}$ is close to 1. In the symplectic case, g is a reduction to symplectic normal form with holomorphic dependence on $z \in \tilde{M}_0$. In both cases the map $\psi = \psi_1 \circ g$ is an automorphism of the trivial bundle $\tilde{M}_0 \times \mathbf{C}^n$ satisfying $\psi^*\omega = \omega$.

Let F_1 be a biholomorphic extension of $\tilde{f}$, constructed from $\psi = \psi_1 \circ g$ as in Remark (1) (Section 4), that satisfies $d_z F_1 = \psi_z$ at points $z \in \tilde{M}_0$. Thus $F_1^*\omega = \omega$ at points of $\tilde{M}_0$. Applying Moser's method as before, we can construct a biholomorphism G in a tubular neighborhood of $\tilde{M}_0$ that equals the identity on $\tilde{M}_0$ and satisfies $G^*(F_1^*\omega) = \omega$. Then $F = F_1 \circ G$ is a biholomorphic map near M_0 that extends f and satisfies $F^*\omega = \omega$. □

7. Proof of Theorems 1.7 and 1.8

We need to consider maps that have different degrees of smoothness with respect to the time variable and the space variable. We use the following terminology.

DEFINITION 4. Let U be an open subset of $[0,1] \times \mathbf{R}^m$. A mapping $f\colon U \to \mathbf{R}^n$ is called a *$\mathcal{C}^l$-family of $\mathcal{C}^k$-maps* if $\partial_t^j(\partial_x^\alpha f)$ is continuous in U for $0 \le j \le l$ and $|\alpha| \le k$. There is an obvious extension of this notion to maps $f\colon [0,1] \times M \to N$ where M and N are $\mathcal{C}^k$ manifolds. If in addition $f_t = f(t,\cdot)$ is a diffeomorphism (of its domain onto its image) for each $t \in [0,1]$, we call $f = \{f_t\}$ a $\mathcal{C}^l$*-family of* $\mathcal{C}^k$*-diffeomorphisms.*

Thus, a $\mathcal{C}^1$-family of $\mathcal{C}^k$-diffeomorphisms is the same as a $\mathcal{C}^k$-isotopy (or a $\mathcal{C}^k$-flow) in the sense of Definition 1 in Section 1. We remark that if f_t is a $\mathcal{C}^l$-family of diffeomorphisms on domains $U_t \subset \mathbf{R}^n$ for $t \in [0,1]$, then the family of inverses f_t^{-1} are not necessarily a $\mathcal{C}^l$-family if $l > 0$; the reason is that the t-derivatives of the (derivatives of the) inverse map will involve higher order x-derivatives of the original map.

In the situation of Theorem 1.7 we shall say that a time-dependent family of $\mathcal{C}^k$-forms on submanifolds $M_t \subset \mathbf{C}^n$, $\alpha_t = \sum_{|I|=p} \alpha_{I,t}\, dz^I$ with $\alpha_{I,t} \in \mathcal{C}^k(M_t)$, is a *continuous family of* $\mathcal{C}^k$*-forms* if $\alpha_{I,t} \circ f_t$ is a continuous family of $\mathcal{C}^k$-functions on M for all multi-indices I. Recall that $\mathcal{T}_\delta = \mathcal{T}_\delta M$ is the open tube of radius δ around a submanifold $M \subset \mathbf{C}^n$.

The main step in the proof of Theorem 1.7 is the following result.

7.1. THEOREM. *Let $f_t\colon M = M_0 \to M_t \subset \mathbf{C}^n$ ($t \in [0,1]$) be a $\mathcal{C}^1$-family of $\mathcal{C}^k$-diffeomorphisms between compact, totally real $\mathcal{C}^k$-submanifolds of $\mathbf{C}^n$, with f_0 the identity on M. By $i_t\colon M_t \hookrightarrow \mathbf{C}^n$ we denote the inclusion map. Let α_t ($t \in [0,1]$) be a continuous family of $(p,0)$-forms of class $\mathcal{C}^k$ on M_t such that $i_t^*\alpha_t$ is closed on M_t for each t. Then there exists an extension of α_t to a continuous family $\hat{\alpha}_t$ of $(p,0)$-forms of class $\mathcal{C}^k$ on a neighborhood of $\tilde{M} = \bigcup_{t\in[0,1]}\{t\} \times M_t$ in $[0,1] \times \mathbf{C}^n$ such that, for all sufficiently small $\delta > 0$, there exists a continuous family of closed holomorphic p-forms u_t^δ on $U_\delta = \bigcup_{t\in[0,1]}\{t\} \times \mathcal{T}_\delta M_t$ satisfying*

$$\|u_t^\delta - \hat{\alpha}_t\|_{\mathcal{C}^r(\mathcal{T}_\delta M_t)} = o(\delta^{k-r}), \quad 0 \le r \le k,$$

uniformly in $t \in [0,1]$. If $i_t^*\alpha_t$ is exact on M_t for each $t \in [0,1]$ then we may choose u_t^δ exact for every t; in this case, u_t^δ can be chosen to be entire if each M_t is polynomially convex.

In the simplest case, when $M_t = M$ and $\alpha_t = \alpha$ for all $t \in [0,1]$, the main steps in the proof of Theorem 7.1 are as follows (we write $\mathcal{T}_\delta = \mathcal{T}_\delta M$).

(i) We construct a $(p,0)$-form $\hat{\alpha}$ on a neighborhood of M such that $d\hat{\alpha}$ is flat on M. In particular, $\hat{\alpha}$ is $\bar{\partial}$-flat on M.

(ii) We approximate the coefficients of $\hat{\alpha}$ by holomorphic functions to obtain a holomorphic p-form u' in $\mathcal{T}_\delta$ with $\|du'\|_{L^\infty(\mathcal{T}_\delta)} = o(\delta^{k-1})$.

(iii) We solve $dv = du'$, with v holomorphic and $\|v\|_{L^\infty(\mathcal{T}_\delta} = o(\delta^k)$, and set $u = u' - v$.

(iv) If $i^*\alpha$ is exact, the norm of the de Rham cohomology class of i^*u is $o(\delta^k)$, and this class may be represented by a holomorphic p-form u_0 on $\mathcal{T}_\delta$ of size $o(\delta^k)$. Then $u_1 = u - u_0$ is exact and approximates α to the correct order on M.

In the parametric case we perform these steps such that the solutions are continuous with respect to the parameter t. Before giving the proof of Theorem 7.1, we summarize (slight extensions of) certain well-known results that we shall need.

We begin by considering the *parameter dependence in Whitney's extension theorem*. Instead of a general compact subset $K \subset \mathbf{R}^n$ (or $K \subset \mathbf{C}^n$), we consider the case when K is a compact $\mathcal{C}^1$-submanifold, with or without boundary. This is a so-called 1-regular set, so we have the following more precise results (see [T, Chap. IV, Secs. 1 and 2, esp. p. 76).

(i) Let $A = \{\alpha \in \mathbf{Z}^n_+ : |\alpha| \le k\}$. The collections $F = (f_\alpha)_{\alpha\in A} \in \mathcal{C}(K)^A$ satisfying the Whitney condition form a closed subspace $\mathcal{E}^k(K)$ of $\mathcal{C}(K)^A$ with respect to the sup-norm; we shall call such collections *Whitney functions.*

(ii) The Whitney extension operator $\mathcal{W}: \mathcal{E}^k(K) \to \mathcal{C}^k_0(K')$, where $K' \subset \mathbf{R}^n$ is a closed neighborhood of K, is linear and norm-continuous. Thus $\partial^\alpha \mathcal{W}(F) = f_\alpha$ on K for each $\alpha \in A$ and

$$\|\mathcal{W}(F)\|_{\mathcal{C}^k(K')} \le C \sup\{\|f_\alpha\|_{L^\infty(K)}: |\alpha| \le k\}.$$

(iii) There exists a constant $C > 0$ such that $C\omega$ is a modulus of continuity for $\partial^\alpha\mathcal{W}(F)$, $|\alpha| = k$, whenever ω is a modulus of continuity for all f_α, $|\alpha| = k$.

From this it follows immediately that if $f_{\alpha,t}$ $(\alpha \in A)$ are $\mathcal{C}^l$-families of continuous functions on K and if $F_t = (f_{\alpha,t})_{\alpha\in A}$ is a Whitney function for each $t \in [0, 1]$, then their Whitney extensions $\mathcal{W}(F_t)$ are a $\mathcal{C}^l$-family of $\mathcal{C}^k$-functions and we may bound the t and x derivatives of $\mathcal{W}(F_t)$ in terms of F_t.

Using these results, the proof of Lemma 2.5 gives the following lemma.

7.2. Lemma (Parameter-dependent rough multiplication). *Let $K \subset \mathbf{R}^n$ be a compact $\mathcal{C}^1$-submanifold, with or without boundary. Let f_t be a $\mathcal{C}^l$-family of $\mathcal{C}^k$-functions, and let g_t be a $\mathcal{C}^l$-family of $\mathcal{C}^{k-1}$-functions on a neighborhood of K in $\mathbf{R}^n$ such that $f_t = 0$ on K for each $t \in [0, 1]$. Then there exists a $\mathcal{C}^l$-family of $\mathcal{C}^k$-functions h_t on a neighborhood of K such that $|h_t - f_t g_t| = o(d^k_K)$ uniformly in $t \in [0, 1]$. If $K \subset \mathbf{C}^n$ and if f_t and g_t are $\bar\partial$-flat on K, then so is h_t.*

We next prove an extension lemma.

7.3. Lemma. *Let $M \subset \mathbf{C}^n$ be a compact, totally real $\mathcal{C}^k$-submanifold. For any $\mathcal{C}^l$-family of $\mathcal{C}^k$-maps $f_t: M \to \mathbf{C}^N$ $(t \in [0, 1])$, there exist an open set $U \subset \mathbf{C}^n$ containing M and a $\mathcal{C}^l$-family of $\mathcal{C}^k$-maps $\tilde f_t: U \to \mathbf{C}^N$ such that each $\tilde f_t$ is $\bar\partial$-flat on M and restricts to f_t on M. If $N = n$ and $f_t: M \to M_t = f_t(M) \subset \mathbf{C}^n$ is a diffeomorphism for each $t \in [0, 1]$, we can choose $\tilde f_t$ as before to be a $\mathcal{C}^l$-family of $\mathcal{C}^k$-diffeomorphisms on U.*

Proof. Let $m = \dim_{\mathbf{R}} M \le n$. We consider first the case when $M = \bar V$ is a smoothly bounded compact domain in $\mathbf{R}^m \subset \mathbf{C}^m \subset \mathbf{C}^n$. Write $z_j = x_j + iy_j$ with $x_j, y_j \in \mathbf{R}$. Given $f \in \mathcal{C}^k(\bar V)$, we consider the following Whitney function on $\bar V$ for the real coordinates $x_1, \dots, x_m, y_1, \dots, y_m$ in $\mathbf{C}^m$:

$$F: f_{(\alpha',\alpha'')} = i^{|\alpha''|}\, \partial_x^{\alpha'+\alpha''}(f), \quad \alpha', \alpha'' \in \mathbf{Z}^m_+,\ |\alpha'| + |\alpha''| \le k.$$

From the Cauchy–Riemann equations $\frac{\partial g}{\partial y_j} = i\,\frac{\partial g}{\partial x_j}$ $(1 \le j \le m)$ for a function g in a neighborhood of $\bar V$ in $\mathbf{C}^m$, it follows that the Whitney extension $\tilde f = \mathcal{W}(F)$ of

F to $\mathbf{C}^m$ is $\bar\partial$-flat on $\bar V$. If $m < n$ then we extend $\mathcal{W}(F)$ trivially in the variables $z_{m+1}, \dots, z_n$ to get a Whitney extension on $\mathbf{C}^n$. Moreover, if $\{f_t : t \in [0,1]\}$ is a $\mathcal{C}^l$-family of $\mathcal{C}^k$-functions on $\bar V$ and if F_t is defined as before, then the Whitney extensions $\mathcal{W}(F_t)$ are a $\mathcal{C}^l$-family of $\mathcal{C}^k$-functions that are $\bar\partial$-flat on $\bar V$.

Next we consider a local $\mathcal{C}^k$-parametrization $\phi: U \to M$ around a point $w_0 \in M$, where U is an open set in $\mathbf{R}^m$. Let $z_0 = \phi^{-1}(w_0) \in U$. Choose a smoothly bounded domain $V \subset\subset U$ containing z_0 and set $W = \phi(\bar V) \subset M$. Let $\tilde\phi$ be an extension of ϕ to $\mathbf{C}^n$ (as constructed previously) that is $\bar\partial$-flat on $\bar V$. If $m < n$, we also choose a basis $v_1, \dots, v_{n-m}$ of the complex normal space $(T^{\mathbf{C}}_{w_0} M)^\perp$ to M at w_0. The map $\Phi(z) = \tilde\phi(z) + \sum_{j=1}^{n-m} z_{m+j} v_j$ is then a $\mathcal{C}^k$-diffeomorphism in a neighborhood of z_0 that is $\bar\partial$-flat on $\bar V$; hence its inverse Φ^{-1} is well-defined in a neighborhood $\tilde W \subset \mathbf{C}^n$ of w_0 and is $\bar\partial$-flat on $W \cap \tilde W \subset M$.

The first part of the proof also provides an extension ψ_t of the map $f_t \circ \phi: \bar V \to \mathbf{C}^n$ to a neighborhood of $\bar V$ in $\mathbf{C}^n$ such that ψ_t is $\bar\partial$-flat on $\bar V$. The composition $\psi_t \circ \Phi^{-1}: \tilde W \to \mathbf{C}^n$ is a $\mathcal{C}^k$-extension of the map f_t that is $\bar\partial$-flat on $W \cap \tilde W \subset M$.

This gives us a local $\bar\partial$-flat $\mathcal{C}^k$-extension of f_t in a neighborhood of each point $w_0 \in M$. We can patch these local extensions by a $\bar\partial$-flat partition of unity along M (as in Lemma 2.6) to obtain a desired $\mathcal{C}^l$-family $\tilde f_t$ satisfying Lemma 7.3.

It remains to consider the case when $f_t: M \to M_t$ is a diffeomorphism for each $t \in [0,1]$. Let $\tilde M = \bigcup_{t\in[0,1]} \{t\} \times M_t \subset [0,1] \times \mathbf{C}^n$, and let $\tilde f: [0,1] \times M \to \tilde M$ be the map $\tilde f(t,z) \to (t, f_t(z))$. Let ν denote the complex normal bundle of M and ν^t the complex normal bundle of M_t in $\mathbf{C}^n$. Then $\tilde\nu = \bigcup_{t\in[0,1]} \{t\} \times \nu^t$ is, in an obvious way, a vector bundle over $\tilde M$, and $[0,1] \times \nu$ is a vector bundle over $[0,1] \times M$. By standard bundle theory (see [Ati, Lemma 1.4.5]) there exists a bundle equivalence $\psi: [0,1] \times \nu \to \tilde\nu$ over $\tilde f$. Thus we have continuously varying isomorphisms $\nu_z \to \tilde\nu^t_{f_t(z)}$ $(z \in M,\ t \in [0,1])$ that we extend to a continuous map $A': [0,1] \times M \to \mathrm{End}_{\mathbf{C}}(\mathbf{C}^n)$. Then we approximate A' by a $\mathcal{C}^l$-family of $\mathcal{C}^k$-maps $A: [0,1] \times M \to \mathrm{End}_{\mathbf{C}}(\mathbf{C}^n)$ so that $A(t,z)\nu_z$ is a supplementary subspace to $(Df_t)_z(T^C_z M)$ for each $(t,z) \in [0,1] \times M$. Let $L(t,z)$ equal $(Df_t)^C_z$ on $T^C_z M$ and $A(t,z)$ on ν_z. Since $T_z\mathbf{C}^n = T^C_z M \oplus \nu_z$, it follows that $L(t,z)$ belongs to $GL(n,\mathbf{C})$; it is not hard to check that $L_t = L(t,\cdot): M \to GL(n,\mathbf{C})$ is a $\mathcal{C}^l$-family of $\mathcal{C}^{k-1}$-maps extending Df_t. Using Lemma 7.2, it is easy to see that Lemma 2.6 has a parameter-dependent version that gives the desired conclusion. □

Proof of Theorem 7.1. Set $M = M_0$ and $i = i_0: M \hookrightarrow \mathbf{C}^n$. We first apply Lemma 7.3 to find a neighborhood $U \subset \mathbf{C}^n$ of M and a continuous family of $\mathcal{C}^k$-diffeomorphisms $\hat f_t: U \to U_t \subset \mathbf{C}^n$ that are $\bar\partial$-flat on M. The family of inverses $(\hat f_t)^{-1}: U_t \to U$ is then a continuous family of $\mathcal{C}^k$-diffeomorphisms on $\tilde U = \bigcup_{t\in[0,1]} \{t\} \times U_t$ which are $\bar\partial$-flat on M_t and which extend $f_t^{-1}: M_t \to M$.

Let $\alpha_t = \sum_{|I|=p} \alpha_{I,t}\, dz^I$ be as in Theorem 7.1, with $\alpha_{I,t} \in \mathcal{C}^k(M_t)$. Our assumption is that $\alpha_{I,t} \circ f_t$ $(t \in [0,1])$ is a continuous family of $\mathcal{C}^k$-functions for each I. Applying Lemma 7.3, we can extend it to a continuous family $\alpha'_{I,t}$ of $\mathcal{C}^k$-functions on $[0,1] \times U$ that are $\bar\partial$-flat on M. Set $\tilde\alpha_{I,t} = \alpha'_{I,t} \circ (\hat f_t)^{-1}$ and $\tilde\alpha_t = \sum_{|I|=p} \tilde\alpha_{I,t}\, dz^I$; this is a continuous family of $\mathcal{C}^k$ $(p,0)$-forms on $\tilde U$, and $\tilde\alpha_t$ is $\bar\partial$-flat on M_t.

The next step is to modify $\tilde\alpha_t$ so as to make its differential flat on M_t. We observe that both $\hat f_t^*\tilde\alpha_t$ and $\beta_t := d\hat f_t^*\tilde\alpha_t = \hat f_t^*(d\hat\alpha_t)$ are continuous families of $\mathcal{C}^{k-1}$-forms on U. By assumption, $di_t^*\alpha_t = 0$ and hence $i^*\beta_t = 0$.

It is clear that the proof of Lemma 6.3 produces a $\mathcal{C}^l$-family of solutions v_t for any $\mathcal{C}^l$-family u_t satisfying the assumptions in that lemma. Applying this to the forms $u_t = \beta_t$ constructed here, we obtain a continuous family of $(p, 0)$-forms $\gamma_t' = \sum_{j=1}^N \zeta_j\gamma_{j,t}'$ $(t \in [0, 1])$ such that $d\gamma_t' - \beta_t$ is $(k-1)$-flat on M, where the $\zeta_1, \dots, \zeta_N$ are $\bar\partial$-flat $\mathcal{C}^k$-functions vanishing on M and where the $\gamma_{j,t}'$ are continuous families of $\mathcal{C}^{k-1}$ $(p, 0)$-forms that are $\bar\partial$-flat on M.

Then $(\hat f_t^{-1})^*\gamma_{j,t}' = \sum_{|I|=p} a_{j,I,t}dz^I + \lambda_{j,t}$ where $a_{j,I,t}$ are continuous families of $\mathcal{C}^{k-1}$-functions that are $\bar\partial$-flat on M_t and $\lambda_{j,t} = o(d_{M_t}^{k-1})$ uniformly in t. Applying parameter-dependent rough multiplication (Lemma 7.2) to ζ_j and $a_{j,I,t}\circ\hat f_t$ gives continuous families $b_{j,I,t}$ of $\mathcal{C}^k$-functions near M that are $\bar\partial$-flat on M. Setting $\gamma_t = \sum_{|I|=p}\sum_{j=1}^N(b_{j,I,t}\circ\hat f_t^{-1})dz^I$ and $\hat\alpha_t = \tilde\alpha_t - \gamma_t$, we have $\hat\alpha_t|_{M_t} = \alpha_t$ $(t \in [0, 1])$ and $|d\hat\alpha_t| = o(d_{M_t}^{k-1})$ uniformly in t.

The next step is to approximate $\hat f_t$ well by biholomorphic maps in tubular neighborhoods $\mathcal{T}_\delta$ of M. Note that $\hat f_t$ maps M onto M_t and is a diffeomorphism from a neighborhood U of M on a neighborhood U_t of M_t, with estimates on derivatives that are valid for all $t \in [0, 1]$. It follows that, for some $\bar a > 0$ and all sufficiently small $\delta > 0$, we have $\hat f_t(\mathcal{T}_{\bar a\delta}M) \subset \mathcal{T}_\delta M_t$ and $\hat f_t^{-1}(\mathcal{T}_{\bar a\delta}M_t) \subset \mathcal{T}_\delta M$ for all $t \in [0, 1]$.

If we apply the solution operator of Theorem 3.1 to the equation $\bar\partial R_t^\delta = \bar\partial\hat f_t$ in $\mathcal{T}_\delta = \mathcal{T}_\delta M$ and set $h_t^\delta = \hat f_t - R_t^\delta$, we obtain a continuous family of holomorphic maps h_t^δ on $\mathcal{T}_\delta$ satisfying $\|h_t^\delta - \hat f_t\|_{\mathcal{C}^j(\mathcal{T}_\delta M)} = o(\delta^{k-j})$ for $j \le k$, where $k \ge 2$. It follows that, for small $\delta > 0$, the map h_t^δ is a biholomorphism of $\mathcal{T}_\delta$ onto its image and $g_t^\delta := \hat f_t^{-1}\circ h_t^\delta$ is a $\mathcal{C}^k$-diffeomorphism of the tube $\mathcal{T}_\delta$ onto a small perturbation of $\mathcal{T}_\delta$.

Since h_t^δ is close to $\hat f_t$, it is not hard to see (using the argument in the proof of Theorem 1.2) that, if $0 < a < \bar a$ and $\varepsilon > 0$ are given, then for $\delta > 0$ sufficiently small (depending on a and ε) we have the inclusions $h_t^\delta(\mathcal{T}_{\delta'}M) \supset \mathcal{T}_{a\delta'}M_t$ for $\varepsilon\delta \le \delta' \le \delta$ and $(h_t^\delta)^{-1}(\mathcal{T}_{\delta'}M_t) \supset \mathcal{T}_{a\delta'}M$ for $\varepsilon\delta \le \delta' \le a\delta$. For all t, we also have $\mathcal{T}_{d'/2}M \subset g_t^\delta(T_{d'}M) \subset \mathcal{T}_{2d'}M$ for $\varepsilon\delta \le \delta' \le \delta$.

The next step is to approximate $\hat\alpha_t$ by a continuous family of holomorphic p-forms u_t' $(= u_t'^\delta)$ on tubes $\mathcal{T}_\delta M_t$. Suppose that $\hat\alpha_t = \sum_{|I|=p}\hat\alpha_{t,I}dz^I$. For small $\delta > 0$, we have $h_t^{\delta/a}(\mathcal{T}_{\delta/a}M) \supset \mathcal{T}_\delta M_t$ for $t \in [0, 1]$. Let $u_{t,I}''$ be holomorphic approximations to $\hat\alpha_{t,I}\circ h_t^{\delta/a}$, constructed as F_δ in Section 4. Set $u_{t,I}' = u_{t,I}''\circ(h_t^{\delta/a})^{-1}$. Then the p-form $u_t' = \sum_{|I|=p}u_{t,I}'dz^I$ is holomorphic in $\mathcal{T}_\delta M_t$ and satisfies $\|u_t' - \hat\alpha_t\|_{\mathcal{C}^j(\mathcal{T}_\delta M_t)} = o(\delta^{k-j})$ uniformly in t. We also see that $\|du_t'\|_{L^\infty(\mathcal{T}_\delta M_t)} = o(\delta^{k-1})$, and if we set $v_{0,t} = i_t^*(u_t' - \hat\alpha_t)$ then $dv_{0,t} = i_t^*du_t'$.

We wish to prove the existence of a continuous family of holomorphic $(p-1)$-forms v_t $(= v_t^\delta)$ on $\mathcal{T}_{b\delta}M_t$ for some $b > 0$, with $\|v_t\|_{L^\infty(\mathcal{T}_{b\delta}M_t)} = o(\delta^k)$, uniformly in t, and solving $dv_t = du_t'$. Then $u_t^\delta = u_t' - v_t$ would be a continuous family of closed holomorphic p-forms with $\|u_t|_{M_t} - \alpha_t\|_{\mathcal{C}^j(M_t)} = o(\delta^{k-j})$, uniformly in t, as required.

A parameter-dependent version of Theorem 5.1 for the family M_t would yield that result. The following argument will give this for a small $b > 0$, but we shall restrict ourselves to the special case we need. Choose $a < \bar{a}$ and $\varepsilon = a/2$. For $\delta > 0$ small, $w'_t = \hat{f}_t^*(du_t'^\delta)$ are $\mathcal{C}^{k-1}$-forms on $\mathcal{T}_{\bar{a}\delta}M$ with $\|w'_t\|_{L^\infty(\mathcal{T}_{\bar{a}\delta}M)} = o(\delta^{k-1})$ and $\|w'_t\|_{\mathcal{C}^s(\mathcal{T}_{\bar{a}\delta}M)} = o(\delta^{k-1-s})$, uniformly in t.

Furthermore, with $v'_{0,t} = f_t^* v_{0,t}$, we have $dv'_{0,t} = i^* w'_t$ on M, with $\|v'_{0,t}\|_{L^\infty} = o(\delta^k)$ and $\|v'_{0,t}\|_{\mathcal{C}^s} = o(\delta^{k-s})$, uniformly in t. Then the first part of the proof of Theorem 5.1 and the remarks on continuous t-dependence give a continuous family of $\mathcal{C}^{k-1}$-forms ω'_t on $\mathcal{T}_{\bar{a}\delta}M$ solving $d\omega'_t = w'_t$, with $\|\omega'_t\|_{L^\infty} = o(\delta^k)$ and $\|\omega'_t\|_{\mathcal{C}^s} = o(\delta^{k-s})$, uniformly in t. Then $\omega_t = (g_t^\delta)^*\omega'_t$ are defined on $\mathcal{T}_{\bar{a}\delta/2}M$ and satisfy the same kind of estimates, and $d\omega_t = (h_t^\delta)^*(\hat{f}_t^{-1})^* w'_t = (h_t^\delta)^* du'_t$ is holomorphic. Since $a < \bar{a}$, the second part of the proof of Theorem 5.1 gives the existence of a continuous family of holomorphic p-forms v'_t on $\mathcal{T}_{a\delta/2}M$ satisfying $dv'_t = (h_t^\delta)^* du'_t$ and $\|v'_t\|_{L^\infty(\mathcal{T}_{a\delta/2}M)} = o(\delta^k)$ uniformly in t. By assumption, $h_t^\delta(\mathcal{T}_{a\delta/2}M) \supset \mathcal{T}_{a^2\delta/2}M_t$ for each t and so $v_t^\delta = (h_t^\delta)^{-1*} v'_t$ is a continuous family of holomorphic p-forms on $\mathcal{T}_{a^2\delta/2}M_t$ with $dv_t^\delta = du'_t$ on $\mathcal{T}_{a^2\delta/2}M_t$ and $\|v_t^\delta\|_{L^\infty} = o(\delta^k)$ uniformly in t.

We now show that if $i_t^*\alpha_t$ is exact for every t then the holomorphic forms u_t^δ described here may be chosen to be exact. We recall that the de Rham cohomology group $H^p(M, \mathbf{C})$ is finite dimensional and $f_t^*\colon H^p(M_t, \mathbf{C}) \to H^p(M, \mathbf{C})$ is an isomorphism for every t. We have that $H^p(M, \mathbf{C}) \approx \{\alpha \in \mathcal{C}_{(p)}(M) : d\alpha = 0\}/(\text{exact forms})$, where derivatives are taken in the weak sense, and we may equip $H^p(M, \mathbf{C})$ with the quotient norm.

For each $t_0 \in [0,1]$, there exist closed holomorphic p-forms $\hat{u}_1, \dots, \hat{u}_N$ on an open neighborhood U of M_{t_0} such that $[i_{t_0}^*\hat{u}_j]$, $1 \le j \le N$, is a basis for $H^p(M_{t_0}, \mathbf{C})$. Then $t \to [f_t^* u_t^\delta]$ is a continuous map $[0,1] \to H^p(M, \mathbf{C})$, and $t \to [f_t^*\hat{u}_j]$ $(1 \le j \le N)$ is continuous for t near t_0. It follows that $\{[f_t^*\hat{u}_j] : j \le N\}$ is a basis for $H^p(M, \mathbf{C})$ for t in a neighborhood $J \subset [0,1]$ of t_0, and we may write $[f_t^* u_t^\delta] = \sum_{j=1}^N c_j^\delta(t)[f_t^*\hat{u}_j]$ with c_j^δ continuous on J. Each form $f_t^*\alpha_t$ is exact on M, so

$$\|[f_t^* u_t^\delta]\| \le \|f_t^*(u_t^\delta) - \hat{\alpha}_t\|_{L^\infty(M)} = o(\delta^k).$$

This means that, for $J_1 \subset\subset J$, we have $\max_{t\in J_1}|c_j^\delta(t)| = o(\delta^k)$ for all $j \le N$. For $\delta > 0$ small and $t \in J_1$ we have $\mathcal{T}_\delta M_t \subset U$; $u_t^{0\delta} = u_t^\delta - \sum_{j=1}^N c_j^\delta(t)\hat{u}_j$ is exact on $\mathcal{T}_\delta M_t$ (since $[i_t^* u_t^{0\delta}] = 0$) and approximates α_t well enough. We can now patch these together with a partition of unity in t to obtain a solution u_t^δ for $t \in [0,1]$ that satisfies Theorem 7.1.

Finally, assume M_t is polynomially convex for all $t \in [0,1]$ and let u_t^δ be the exact solution on U_δ. For $\delta > 0$ sufficiently small we may also assume that $\mathcal{T}_\delta M_t$ is Runge in $\mathbf{C}^n$ for all t. Given $a < a' < 1$ and $\varepsilon > 0$, there exist $t_j \in [0,1]$, $j = 1, \dots, N$, and (relatively) open intervals $I_j \subset [0,1]$, $t_j \in I_j$, such that $U_{a\delta} \subset \bigcup_{j=1}^N I_j \times \mathcal{T}_{a'\delta}M_{t_j} \subset U_\delta$; for all $t \in I_j$ we have $\|u_t^\delta - u_{t_j}^\delta\|_{\mathcal{C}^k(\mathcal{T}_{a'\delta}M_{t_j})} < \varepsilon$ and $\|\hat{\alpha}_t^\delta - \hat{\alpha}_{t_j}^\delta\|_{\mathcal{C}^k(\mathcal{T}_{a'\delta}M_{t_j})} < \varepsilon$. Let β_j be a holomorphic $(p-1)$-form on $\mathcal{T}_\delta M_{t_j}$

such that $d\beta_j = u^\delta_{t_j}$. By Oka's theorem there is an entire $(p-1)$-form v_j such that $\|\beta_j - v_j\|_{L^\infty(\mathcal{T}_\delta M_{t_j})} < \varepsilon$. The Cauchy estimates imply $\|\beta_j - v_j\|_{\mathcal{C}^r(\mathcal{T}_{a'\delta} M_{t_j})} = \varepsilon o(\delta^{-r})$ and hence $\|u^\delta_{t_j} - dv_j\|_{\mathcal{C}^r(\mathcal{T}_{a'\delta} M_{t_j})} = \varepsilon o(\delta^{-(r+1)})$. Choosing $\varepsilon = o(\delta^{k+1})$, we obtain $\|dv_j - \hat{\alpha}^\delta_t\|_{\mathcal{C}^r(\mathcal{T}_{a\delta} M_t)} = o(\delta^{k-r})$ whenever $t \in I_j$. If $\chi_j(t)$ is a partition of unity on $[0, 1]$ subordinate to the covering $\{I_j\}$ and if we define $v_t = \sum_{j=1}^N \chi_j(t) v_j(z)$, then $u_t = dv_t$ is an entire form for each t that satisfies Theorem 7.1. □

Proof of Theorem 1.7. By assumption, $f_t: M \to M_t$ is $\mathcal{C}^1$-family of $\mathcal{C}^k$-diffeomorphisms and ω is one of the forms (1.4), (1.5). Let X_t be the infinitesimal generator of f_t, that is, $\partial_t f_t(z) = X_t(f_t(z))$ for $z \in M$ and $t \in [0, 1]$. Then $\alpha_t = X_t \rfloor \omega$ is a continuous family of $(p, 0)$-forms on M_t, with $p = n - 1$ when ω is the volume form (1.4) and $p = 1$ when ω is the symplectic form (1.5). Since f_t is an ω-flow, $i^*_t \alpha_t$ is closed on M_t for each t (by the remark after Definition 2).

By Theorem 7.1 there exists an extension of α_t to a continuous family $\hat{\alpha}_t$ of $(p, 0)$-forms of class $\mathcal{C}^k$ on a neighborhood of $\tilde{M} = \bigcup_{t\in[0,1]}\{t\} \times M_t$ such that, for all sufficiently small $\delta > 0$, there exists a continuous family of closed holomorphic p-forms u^δ_t on $U_\delta = \bigcup_{t\in[0,1]}\{t\} \times \mathcal{T}_\delta M_t$ with $\|u^\delta_t - \hat{a}_t\|_{\mathcal{C}^r(\mathcal{T}_\delta M_t)} = o(\delta^{k-r})$, uniformly in t, for $0 \le r \le k$.

The equation $u^\delta_t = Y^\delta_t \rfloor \omega$ uniquely defines a time-dependent holomorphic vector field Y^δ_t on U_δ. Since u^δ_t is closed, the flow F^δ_t of Y^δ_t is a holomorphic ω-flow wherever it is defined (see Definition 2). If we let X_t denote the extension of X_t to U_δ defined by $\hat{\alpha}_t = X_t \rfloor \omega$, then $\|Y^\delta_t - X_t\|_{\mathcal{C}^r(\mathcal{T}_\delta M_t)} = o(\delta^{k-r})$ uniformly in t. We may apply [FL, Lemma 4.1] to see that, for small $\delta > 0$, the flow $F^\delta_t(z)$ exists for all $t \in [0, 1]$ and $z \in \mathcal{T}_\delta M_0$ and that $\|F^\delta_t - f_t\|_{\mathcal{C}^r(\mathcal{T}_\delta M_0)} = o(\delta^{k-r})$ uniformly in t. In fact, it follows from the proof of this lemma (see Section 4 and [FL]) that the same approximation also holds for the flow from time t to time s; if we let $f_{t,s} = f_s \circ f_t^{-1}: \mathcal{T}_\delta M_t \to \mathbf{C}^n$ denote the flow of X_t from t to s and let $F^\delta_{t,s} = F^\delta_s \circ (F^\delta_t)^{-1}$ denote the flow of Y^δ_t from t to s, then for small $\delta > 0$ the flow $F^\delta_{t,s}$ exists for all $s, t \in [0, 1]$ and we have $\|F^\delta_{t,s} - f_{t,s}\|_{\mathcal{C}^r(\mathcal{T}_\delta M_t)} = o(\delta^{k-r})$ uniformly in s and t. Since $f_t^{-1} = f_{t,0}$, the second estimate in Theorem 1.7 follows.

Finally, if f_t is an exact ω-flow (i.e., $i^*_t \alpha_t$ is exact on M_t for each t) and if each M_t is also polynomially convex, then by (the proof of) Theorem 7.1 above we may choose $u^\delta_t(z) = \sum_{j=1}^N \chi^\delta_j(t) dv_j(z)$, where $v_j(z)$ are entire $(p-1)$-forms on $\mathbf{C}^n$ and χ^δ_j $(1 \le j \le N)$ are $\mathcal{C}^\infty$ functions with compact support in $\mathbf{R}$ that form a partition of unity on $[0, 1]$. We may even assume that v_j are $(p-1)$-forms with polynomial coefficients. This means that the polynomial vector fields X_j on $\mathbf{C}^n$, uniquely defined by the equation $dv_j = X_j \rfloor \omega$, are divergence-free (resp., Hamiltonian). By [F4, Prop. 4.1] these can be written as finite sums $X_j(z) = \sum_{k=1}^{N_j} X_{j,k}(z)$, where $X_{j,k}$ are complete divergence-free (resp., Hamiltonian) polynomial vector fields on $\mathbf{C}^n$ (in fact they are shear fields). "Completeness" means that the fields X_{jk} may be integrated in time for all $t \in \mathbf{C}$ (and initial points $z \in \mathbf{C}^n$). Then $Y_{jk}(t, z) := \chi^\delta_j(t) X_{jk}(z)$ is also a complete vector field whose integral curves are reparametrizations of the integral curves of X_{jk}. Hence we may write $Y^\delta_t = \sum_{j,k} Y_{jk}(t, z)$; that is, Y^δ_t is the sum of complete, divergence-free (resp., Hamiltonian), time-dependent, polynomial (in $z \in \mathbf{C}^n$) vector fields. For the rest of this proof it is more convenient to write this sum as $\sum_{l=1}^{N} Y_l(t,z)$, where each Y_l is one of the Y_{jk} above.

Let $G^l_{t,t+s}$ be the flow of $Y_l(t,z)$ from time t to time $t+s$. This means that $G^l_{t,t}(z) = z$ and $\frac{d}{ds}G^l_{t,t+s}(z) = Y_l(t+s, G^l_{t,t+s}(z))$. Define $G_{t,t+s}(z) = (G^N_{t,t+s} \circ \cdots \circ G^1_{t,t+s})(z)$. We can regard this as the flow of a time-dependent vector field $X^{t,t+s}_{t'}$ defined for times t' between t and $t+s$; for $t + \frac{j-1}{N}s \le t' \le t + \frac{j}{N}s$ we define $X^{t,t+s}_{t'}(z) = \frac{1}{N}Y_j\big(t + N\big(t' - \frac{j-1}{N}s\big), z\big)$. If we reparametrize time such that the joints are passed at zero speed, we may even assume that $X^{t,t+s}_{t'}$ is smooth and vanishes near the endpoints. We denote this smooth flow by $G^{t,t+s}_{t'}(z)$. By definition, $G_{t,t+s}(z) = G^{t,t+s}_{t+s}(z)$. Since the vector fields Y_j are complete divergence-free (resp., Hamiltonian) entire vector fields, it follows that $G^{t,t+s}_{t'}$ is a holomorphic ω-flow; that is, $(G^{t,t+s}_{t'})^*\omega = \omega$ when $t \le t' \le t+s$.

For each $m \in \mathbf{N}$ we define the concatenations $F^m_1(z) = (G_{1-\frac{1}{m},1} \circ \cdots \circ G_{0,\frac{1}{m}})(z)$. Then (by [AMR, Suppl. 4.1.A]) we have $\lim_{m\to\infty} F^m_1(z) = F^\delta_1(z)$ uniformly for $z \in \mathcal{T}_\delta M_0$. As before, we can view $F^m_1(z)$ as the time-1 map of the flow of the vector field X_t defined by $X_t = X_t^{\frac{j-1}{m},\frac{j}{m}}$ for $t \in \big[\frac{j-1}{m}, \frac{j}{m}\big]$, $1 \le j \le m$. Let $F^m_t(z)$ be the flow of this vector field. It is easy to see that we can arrange for $\lim_{m\to\infty} F^m_t = F^\delta_t$ uniformly in $[0,1] \times \mathcal{T}_\delta M_0$ and that the Cauchy estimates imply $\|F^m_t - F^\delta_t\|_{\mathcal{C}^k(M_0)} < \varepsilon$ for all $t \in [0,1]$ and all sufficiently large $m \in \mathbf{N}$. Similarly, $(F^m_1)^{-1}$ is a concatenation and hence $\lim_{m\to\infty}(F^m_1)^{-1} = (F^\delta_1)^{-1}$ uniformly on $\mathcal{T}_\delta M_1$; it follows that $\lim_{m\to\infty}(F^m_t)^{-1} = (F^\delta_t)^{-1}$ on M_t and thus the result follows by the Cauchy estimates. □

Proof of Theorem 1.8. We shall see that, in all cases except (iii) and (vi), the pull-back $i_t^*\alpha_t$ of the form $\alpha_t = X_t \rfloor \omega$ to M_t is exact for each t; hence f_t is an exact ω-flow and the result follows from the second part of Theorem 1.7.

In case (i) we have $i_t^*\alpha_t = 0$ by degree reason. In cases (ii), (iv), (v), and (vii), we first see that the form $i_t^*\alpha_t$ is closed on M_t, either by degree reasons or by the comment after Definition 2 in Section 1; hence the cohomological assumptions imply, in each of these cases, that $i_t^*\alpha_t$ is exact on M_t.

For the two remaining cases (iii) and (vi) it is shown in [F3, pp. 439, 441] that the initial family f_t may be altered to an exact, totally real, and polynomially convex ω-flow without changing the maps $f_0 = \mathrm{Id}$ and f_1; hence the result again follows from Theorem 1.7. □

References

[AMR] R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, tensor analysis, and applications,* 2nd ed., Springer, New York, 1988.

[AB] M. Andersson and B. Berndtsson, *Henkin–Ramirez formulas with weight factors,* Ann. Inst. Fourier (Grenoble) 32 (1982), 91–110.

[Ati] M. F. Atiyah, *K-Theory,* Benjamin, New York, 1967.

[BT1] M. S. Baouendi and F. Trèves, *A property of the functions and distributions annihilated by a locally integrable system of complex vector fields,* Ann. of Math. (2) 113 (1981), 387–421.

[BT2] ———, *Approximation of solutions of linear PDE with analytic coefficients,* Duke Math. J. 50 (1983), 285–301.

[BB] J. Bruna and J. Burgués, *Holomorphic approximation in $\mathcal{C}^m$-norms on totally real compact sets in* $\mathbf{C}^n$, Math. Ann. 269 (1984), 103–117.

[C] H. Cartan, *Espaces fibrés analytiques,* Symposium Internat. de topologia algebraica (Mexico City, 1958) pp. 97–121, Universidad Nacional Autonamde Mexico and UNESCO, Mexico City, 1958.

[DF] K. Diederich and J. E. Fornæss, *A smooth curve in* $\mathbf{C}^2$ *which is not a pluripolar set,* Duke Math. J. 49 (1982), 931–936.

[F1] F. Forstnerič, *Approximation by automorphisms on smooth submanifolds of* $\mathbf{C}^n$, Math. Ann. 300 (1994), 719–738.

[F2] ———, *A theorem in complex symplectic geometry,* J. Geom. Anal. 5 (1995), 379–393.

[F3] ———, *Equivalence of real submanifolds under volume preserving holomorphic automorphisms of* $\mathbf{C}^n$, Duke Math. J. 77 (1995), 431–445.

[F4] ———, *Holomorphic automorphisms of* $\mathbf{C}^n$: *A survey,* Complex analysis and geometry (Trento, 1993), Lecture Notes in Pure and Appl. Math., 173, pp. 173–200, Dekker, New York, 1996.

[FL] F. Forstnerič and E. Løw, *Global holomorphic equivalence of smooth submanifolds in* $\mathbf{C}^n$, Indiana Univ. Math. J. 46 (1997), 133–153.

[FR] F. Forstnerič and J.-P. Rosay, *Approximation of biholomorphic mappings by automorphisms of* $\mathbf{C}^n$, Invent. Math. 112 (1993), 323–349.

[GLR] I. Gohberg, P. Lancaster, and L. Rodman, *Invariant subspaces of matrices with applications,* Wiley, New York, 1986.

[HW] F. R. Harvey and R. O. Wells, Jr., *Holomorphic approximation and hyperfunction theory on a* $\mathcal{C}^1$ *totally real submanifold of a complex manifold,* Math. Ann. 197 (1972), 287–318.

[HL] G. M. Henkin and J. Leiterer, *Theory of functions on complex manifolds,* Monogr. Math., 79, Birkhäuser, Boston, 1984.

[Hö] L. Hörmander, *An introduction to complex analysis in several variables,* 3rd ed., North-Holland, Amsterdam, 1990.

[HöW] L. Hörmander and J. Wermer, *Uniform approximations on compact sets in* $\mathbf{C}^n$, Math. Scand. 23 (1968), 5–21.

[M] J. Moser, *On the volume elements on a manifold,* Trans. Amer. Math. Soc. 120 (1965), 286–294.

[Ø1] N. Øvrelid, *Integral representation formulas and* L^p*-estimates for the* $\bar\partial$*-equation,* Math. Scand. 29 (1971), 137–160.

[Ø2] ———, *Integral representation formulae for differential forms, and solutions of the* $\bar\partial$*-equation,* Fonctions analytiques de plusieurs variables et analyse complexe (Colloq. Internat. CNRS, no. 208, Paris, 1972), pp. 180–198, Gauthier-Villars, Paris, 1974.

[RS] R. M. Range and Y. T. Siu, C^k *approximation by holomorphic functions and* $\bar\partial$*-closed forms on* C^k *submanifolds of a complex manifold,* Math. Ann. 210 (1974), 105–122.

[S] E. M. Stein, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals* (Princeton Math. Ser., 43), Princeton Univ. Press, Princeton, NJ, 1993.

[T] J. C. Tougeron, *Idéaux de fonctions différentiables,* Ergeb. Math. Grenzgeb. (3), 71, Springer-Verlag, Berlin, 1972.

[We] R. O. Wells, *Differential analysis on complex manifolds,* 2nd ed., Grad. Texts in Math., 65, Springer, New York, 1980.

[W1] H. Whitney, *Analytic extensions of differentiable functions defined on closed sets,* Trans. Amer. Math. Soc. 36 (1934), 69–89.

[W2] ———, *Differentiable manifolds,* Ann. of Math. (2) 37 (1936), 645–680.

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