

Oka's principle for holomorphic submersions with sprays

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Introduction

The *Oka principle* is a fundamental principle of complex analysis which says that on Stein manifolds (closed complex submanifolds of affine spaces), analytic problems of cohomological (or even homotopical) nature have only topological obstructions. The story began in 1939 when K. Oka proved that on a domain of holomorphy a second Cousin problem is solvable by holomorphic functions if it is solvable by continuous functions [Oka]. Oka's result has the following equivalent formulation: *If $E \rightarrow X$ is a holomorphic \mathbb{C}^* -bundle over a domain of holomorphy then every continuous section of E is homotopic to a holomorphic section.* In 1957 H. Grauert [Gra] extended Oka's result to sections of holomorphic fiber bundles with complex homogeneous fibers over Stein manifolds (see also the papers [Car, FR, HL]).

In 1989 M. Gromov [Gr1] announced a major generalization of Grauert's theorem, replacing complex Lie groups (and homogeneous spaces) by a larger class of complex manifolds with a spray. A *spray* on a complex manifold Y is a holomorphic vector bundle $p: E \rightarrow Y$ with a holomorphic map $s: E \rightarrow Y$ (the spray map) which is the identity on the zero section and whose restriction to each fibre of E is a submersion at zero. Such sprays exist on complex homogeneous spaces, on complements of affine algebraic subvarieties of codimension at least two and, more generally, on manifolds whose holomorphic tangent bundle is spanned by finitely many \mathbb{C} -complete holomorphic vector fields. The notion of spray extends in a natural way to submersions. The following is Theorem 4.5 in [Gr1]: *If $h: Z \rightarrow X$ is a holomorphic submersion onto a Stein manifold X such that every point $x \in X$ has an open neighborhood $U \subset X$ such that $Z|_U = h^{-1}(U) \rightarrow U$ admits a fiber-dominating spray then the inclusion of the*

space of holomorphic sections of h to the space of continuous sections of h is a weak homotopy equivalence. This will be referred to as the *parametric Oka principle* for sections of $Z \rightarrow X$.

In this paper we give a complete proof of this result and some extensions (Theorems 1.2 and 1.5). We also give examples (Theorem 1.6) and prove interpolation results (Theorems 1.7 and 1.9) which have been used in the construction of proper holomorphic embeddings of Stein manifolds into Euclidean spaces of minimal dimension [EGr, Sch, Pre1]. This paper is a sequel to [FP1] where we proved the same results for sections of fiber bundles whose fiber admits a dominating spray (thereby clarifying the results announced in Sect. 2.9 of [Gr1]). The relevant analytic tools were proved in [FP1] following [Gra] and [Gr1]; they include a homotopy version of the Oka-Weil theorem for sections of submersions with sprays [FP1, Sect. 4] and a gluing lemma for holomorphic sections over Cartan pairs [FP1, Sect. 5].

To prove the Oka principle we develop an inductive scheme for patching collections of local holomorphic sections into semi-global holomorphic sections. Such a scheme was proposed in [Gr1, Sect. 4], but we felt that an explanation would be welcome. This procedure requires that there exist holomorphic homotopies between the individual sections in the given family, two-parameter homotopies between the one-parameter homotopies, etc., in order to insure that all ‘triangles of homotopies’ which appear at various steps of the construction are ‘contractible’.

To keep track of the data we introduce in Sect. 3 the notion of holomorphic (resp. continuous) complexes and prisms. A *holomorphic complex* is a family of local holomorphic sections of $h: Z \rightarrow X$, parametrized by the *nerve* of an open covering $\mathcal{U} = \{U_j\}$ of X . A point in the parameter space determines an open set in X (the intersection of a certain collection of sets from the covering \mathcal{U}) on which the corresponding section of h is defined, and we have natural restriction conditions. A global section f is the same as a ‘constant complex’ in which any section is the restriction of f to the appropriate set. A *holomorphic k -prism* is a homotopy of holomorphic complexes with parameter in the k -dimensional cube. Similarly one defines continuous complexes.

The procedure runs as follows. We begin by deforming the initial continuous section $a = a_{*,0}$ by a homotopy of continuous complexes $a_{*,t}$ ($t \in [0, 1]$) to a holomorphic complex $a_{*,1}$ (Proposition 4.7). We then inductively construct a sequence of holomorphic complexes f_*^n ($n \in \mathbb{N}$), with $f_*^1 = a_{*,1}$, such that f_*^n is constant over the union of the first n sets in the given covering of X (i.e., it represents a holomorphic section there). When $n \rightarrow \infty$ the sequence f_*^n converges uniformly on compacts in X to a global holomorphic section $f: X \rightarrow Z$ of $h: Z \rightarrow X$. The heart of the proof is Proposition 5.1. This modification process requires special coverings of X , called *Cartan strings*, which were constructed by Henkin and Leiterer in [HL].

Grauert's Oka principle had numerous applications. Gromov's extensions have already been used in the embedding theorem for Stein manifolds into Euclidean spaces of minimal dimension [EGr, Sch]. Further references to recent applications are included at the end of the paper. Complete results on the Oka principle for maps of Riemann surfaces were obtained by Winkelmann [Win].

The Oka principle is a special case of the *homotopy principle* whose validity for an analytic problem means that *an analytic solution exists provided that there are no topological obstructions*. Classical examples include the Smale–Hirsch theory of immersions of real manifolds and the theory of totally real and lagrangian immersions due to Lees and Gromov. A good reference are the monographs [Gr2] and [Spr].

1 The results

Let $h: Z \rightarrow X$ be a holomorphic submersion onto a Stein manifold X . For each $x \in X$ we denote by $Z_x = h^{-1}(x)$ the fiber over x . At each point $z \in Z$ the tangent space $T_z Z$ contains the *vertical tangent space* $VT_z(Z) = \{e \in T_z Z: Dh(z)e = 0\} = T_z Z_{h(z)}$. If $p: E \rightarrow Z$ is a holomorphic vector bundle over Z , we denote by $E_z = p^{-1}(z) \subset E$ its fiber over $z \in Z$ and by $0_z \in E_z$ the zero element of E_z .

1.1 Definition. (Gromov [Gr1]) *Let $h: Z \rightarrow X$ be a holomorphic submersion of a complex manifold Z onto a complex manifold X . A spray on Z associated to h (or a fiber-spray) is a triple (E, p, s) , where $p: E \rightarrow Z$ is a holomorphic vector bundle and $s: E \rightarrow Z$ is a holomorphic map such that for each $z \in Z$ we have*

- (i) $s(E_z) \subset Z_{h(z)}$ (equivalently, $h \circ p = h \circ s$),
- (ii) $s(0_z) = z$, and
- (iii) the derivative $Ds(0_z): T_{0_z}E \rightarrow T_z Z$ maps the subspace $E_z \subset T_{0_z}E$ surjectively onto the vertical tangent space $VT_z(Z)$.

The restriction $VDs(z) = Ds(0_z)|_{E_z}: E_z \rightarrow VT_z(Z)$ is called the *vertical derivative* of s at $z \in Z$. Gromov's definition of a spray only includes properties (i) and (ii), and a spray which also satisfies the domination property (iii) is called a (fiber-) *dominating spray*.

1.2 Theorem. (Gromov [Gr1], 4.5 Main Theorem) *Let $h: Z \rightarrow X$ be a holomorphic submersion of a complex manifold Z onto a Stein manifold X . Assume that each point in X has an open neighborhood $U \subset X$ such that the submersion $h^{-1}(U) \rightarrow U$ admits a (fiber dominating) spray. Then the inclusion $\iota: \text{Holo}(X, Z) \hookrightarrow \text{Cont}(X, Z)$ of the spaces of holomorphic sections into the space of continuous sections is a weak homotopy equivalence.*

This means that t induces an isomorphism of the respective homotopy groups of the two spaces (which are endowed with the compact–open topology). In particular their path connected components are in one-to-one correspondence, which means that

- (i) any continuous section of $Z \rightarrow X$ can be homotopically deformed to a holomorphic section, and
- (ii) any homotopy of sections $f_t: X \rightarrow Z$ ($0 \leq t \leq 1$) between two holomorphic sections f_0 and f_1 can be deformed into another homotopy consisting of holomorphic sections.

In the special case when $Z \rightarrow X$ is a fiber bundle whose fiber admits a spray, a complete Proof of Theorem 1.2 was given in [FP1] (thereby clarifying the results announced in Sect. 2.9 of [Gr1]). When the conclusion of Theorem 1.2 holds, we shall say that sections of h satisfy the *parametric Oka principle*. Likewise we say that the (parametric) Oka principle holds for maps $X \rightarrow Y$ of a Stein manifold X into a complex manifold Y if it holds for sections of the trivial fibration $Z = X \times Y \rightarrow X$.

Examples of spaces with sprays can be found in [Gr1, Sect.4.6.B] and in [FP1]. We recall the following example.

Example 1. Let $h: Z \rightarrow X$ be a holomorphic submersion. Suppose that Z admits finitely many \mathbb{C} -complete holomorphic vector fields V_1, V_2, \dots, V_N which are vertical (tangent to $VT(Z)$) and which span $VT_z(Z)$ at each point $z \in Z$. \mathbb{C} -completeness means that the flow ϕ_j^t of V_j is defined for all complex values of the time parameter t . The map $s: Z \times \mathbb{C}^N \rightarrow Z$, defined by

$$s(z; t_1, \dots, t_N) = \phi_1^{t_1} \circ \phi_2^{t_2} \circ \dots \circ \phi_N^{t_N}(z),$$

satisfies $s(z; 0, \dots, 0) = z$ and $\frac{\partial}{\partial t_j} s(z; 0, \dots, 0) = V_j(z)$ for $z \in Z$ and $1 \leq j \leq N$. Since these vectors span $VT_z(Z)$, s is a spray on Z . □

1.3 Corollary. *Let $h: Z \rightarrow X$ be a holomorphic submersion onto a Stein manifold X . Assume that each point in X has an open neighborhood $U \subset X$ such that there exist finitely many \mathbb{C} -complete holomorphic vector fields on $Z|U$ which are vertical with respect to h and which span the vertical tangent space $VT_z(Z)$ at each point $z \in Z|U$. Then the conclusion of Theorem 1.2 holds.*

We now describe a more general version of the Oka principle (compare with Theorem 1.4 and Corollary 1.5 in [FP1]). Let $h: Z \rightarrow X$ be a holomorphic submersion and let P be a compact Hausdorff space (the parameter space). Our basic objects now are continuous maps $f: X \times P \rightarrow Z$ such that $f(\cdot, p): X \rightarrow Z$ is a section of h for each fixed $p \in P$. A *homotopy* of such maps is a continuous map $H: X \times P \times [0, 1] \rightarrow Z$ such that $H_t(\cdot, p) = H(\cdot, p, t): X \rightarrow Z$ is a section of h for all $p \in P$ and $t \in [0, 1]$.

We denote by $\mathcal{H}(X)$ the algebra of all holomorphic functions on X . A compact set $K \subset X$ is called $\mathcal{H}(X)$ -convex (or *holomorphically convex*) if for each $x \in X \setminus K$ there is an $f \in \mathcal{H}(X)$ with $|f(x)| > \sup_K |f|$.

1.4 Definition. A subset P_0 in a topological space P is called nice if there exists an open set $U \subset P$ containing P_0 and a strong deformation retraction of U onto P_0 . The empty subset in P will be considered nice.

1.5 Theorem. Let X be a Stein manifold, $K \subset X$ a compact $\mathcal{H}(X)$ -convex subset and $h : Z \rightarrow X$ a holomorphic submersion onto X . Assume that for each point $x \in X \setminus K$ there is a neighborhood $U_x \subset X$ such that the submersion $h : h^{-1}(U_x) \rightarrow U_x$ admits a spray (def. 1.1). Let P be a compact Hausdorff space and $P_0 \subset P$ a nice compact subset (def. 1.4). Assume that $a : X \times P \rightarrow Z$ is a continuous map such that for each $p \in P$, $a(\cdot, p) : X \rightarrow Z$ is a section of $h : Z \rightarrow X$ which is holomorphic in an open set $U_0 \supset K$, and $a(\cdot, p)$ is holomorphic on X for each $p \in P_0$. Let d be a metric on Z compatible with the manifold topology. Then for each $\epsilon > 0$ there exists a homotopy $H_t : X \times P \rightarrow Z$ ($t \in [0, 1]$) satisfying:

- (i) $H_0 = a$,
- (ii) the section $H_1(\cdot, p) : X \rightarrow Z$ is holomorphic for each $p \in P$,
- (iii) the homotopy is fixed on P_0 , i.e., $H_t(\cdot, p)$ is independent of t for $p \in P_0$, and
- (iv) $d(H_t(x, p), a(x, p)) < \epsilon$ for all $x \in K, p \in P$ and $0 \leq t \leq 1$.

Theorem 1.5 implies Theorem 1.2. If we take P to be the n -sphere S^n and $P_0 = \emptyset$, Theorem 1.5 shows that each continuous map $S^n \rightarrow \text{Cont}(X, Z)$ can be homotopically deformed to a map $S^n \rightarrow \text{Holo}(X, Z)$. Similarly, if P is the closed $(n + 1)$ -ball $B^{n+1} \subset \mathbb{R}^{n+1}$ and $P_0 = \partial B^{n+1} = S^n$, Theorem 1.5 shows that each map $S^n \rightarrow \text{Holo}(X, Z)$ which extends to a map $B^{n+1} \rightarrow \text{Cont}(X, Z)$ also extends to a map $B^{n+1} \rightarrow \text{Holo}(X, Z)$. This means that the inclusion $\text{Holo}(X, Z) \hookrightarrow \text{Cont}(X, Z)$ induces an isomorphism of the respective homotopy groups of the two spaces as claimed by Theorem 1.2. □

Next we consider the validity of the Oka principle for maps of Stein manifolds into $\mathbb{C}^q \setminus \Sigma$, where Σ is a closed complex subvariety of \mathbb{C}^q . Immediate examples show that the Oka principle fails in general when Σ is a complex hypersurface, except in special cases such as when a complex Lie group acts transitively on $\mathbb{C}^q \setminus \Sigma$. The situation is fairly complicated even for subsets of higher codimension as the following result shows:

1.6 Theorem. (a) If $q \geq 2$ and $\Sigma \subset \mathbb{C}^q$ is a closed analytic subset of complex codimension ≥ 2 in \mathbb{C}^q such that, with respect to some holomorphic coordinates $z = (z', z_q) \in \mathbb{C}^q$ and some constant $C > 0$ we have

$$\Sigma \subset \Gamma = \{z \in \mathbb{C}^q : |z_q| \leq C(1 + |z'|)\}, \tag{1.1}$$

then the Oka principle holds for maps from any Stein manifold into $\mathbb{C}^q \setminus \Sigma$. This is the case in particular if Σ is an algebraic subset of codimension at least two in \mathbb{C}^q .

- (b) For each $q \geq 1$ there exist discrete sets $\Sigma \subset \mathbb{C}^q$ such that the Oka principle fails for maps of $X = (\mathbb{C}^*)^{2q-1}$ into $\mathbb{C}^q \setminus \Sigma$.
- (c) For each $1 \leq k < q$ there exist proper holomorphic embeddings $g: \mathbb{C}^k \hookrightarrow \mathbb{C}^q$ such that the Oka principle fails for maps of $X = (\mathbb{C}^*)^{2(q-k)-1}$ into $\mathbb{C}^q \setminus g(\mathbb{C}^k)$.

Remark. In the forthcoming paper [Fo5] we extend the Oka principle to maps from Stein manifolds into complements $\mathbb{C}P^n \setminus \Sigma$ of projective-algebraic subvarieties of codimension at least two. □

We now consider the problem of avoiding analytic subsets in holomorphic vector bundles by graphs of holomorphic sections. Theorems 1.7 and 1.9 below have been used in [EGr], [Sch] and [Pre1, Pre2].

1.7 Theorem. *Let $\pi: V \rightarrow X$ be a holomorphic vector bundle of rank $q \geq 2$ over a Stein manifold X , let $X_0 \subset X$ be a closed complex subvariety in X , and let $K \subset X$ be a compact $\mathcal{H}(X)$ -convex subset. Set $X' = X \setminus (X_0 \cup K)$ and $V' = \pi^{-1}(X') \subset V$. Suppose that $\Sigma \subset V'$ is a closed analytic subset in V' such that for each $x \in X'$, the fiber $\Sigma_x = \Sigma \cap V_x$ has complex codimension at least two in V_x , and there are an open neighborhood $U \subset X'$ of x , a set $\Gamma \subset \mathbb{C}^q$ as in (1.1), and a biholomorphic map $\Phi: U' = \pi^{-1}(U) \rightarrow U \times \mathbb{C}^q$ of the form $\Phi(x, v) = (x, \phi(x, v))$ ($x \in U, v \in V_x$), satisfying*

$$\Phi(U' \cap \Sigma) \subset U \times \Gamma. \tag{1.2}$$

Given a continuous section $f_0: X \rightarrow V \setminus \overline{\Sigma}$ that is holomorphic in an open set $U_0 \supset X_0 \cup K$, there is for each $k \in \mathbf{Z}_+$ a homotopy of continuous sections $f_t: X \rightarrow V \setminus \overline{\Sigma}$ ($0 \leq t \leq 1$) which are holomorphic in a neighborhood of $X_0 \cup K$, which match f_0 to order k along X_0 and approximate f_0 uniformly on K , and the section f_1 is holomorphic on X .

We emphasize that Σ is not assumed to be closed in V , and its closure $\overline{\Sigma}$ need not be an analytic subset of V . We do not assume anything about the projection $\pi(\Sigma)$, and the sets $V_x \setminus \Sigma_x$ need not be biholomorphically equivalent for different base points x . The condition (1.2) will insure the existence of a spray on $U' \setminus \Sigma$.

We state an important special case of Theorem 1.7. To each holomorphic vector bundle $\pi: V \rightarrow X$ of rank q we can associate a holomorphic fiber bundle $\bar{\pi}: \bar{V} \rightarrow X$ whose fiber $\bar{V}_x \cong \mathbb{C}P^q$ is obtained by compactifying $V_x \cong \mathbb{C}^q$ with the hyperplane at infinity. Since every complex linear automorphism of \mathbb{C}^q extends to a unique projective automorphism of $\mathbb{C}P^q$, the fibers \bar{V}_x patch together into a holomorphic bundle over X .

1.8 Corollary. *Let $\pi : V \rightarrow X$ be a holomorphic vector bundle of rank q over a Stein manifold X , and let $\bar{\pi} : \bar{V} \rightarrow X$ be the associated bundle with fiber $\mathbb{C}\mathbb{P}^q$ as above. If $\Sigma \subset \bar{V}$ is a closed analytic subset such that for each $x \in X$ the fiber $\Sigma_x = \Sigma \cap \bar{V}_x$ is of complex codimension at least two in \bar{V}_x then the Oka principle holds for sections $f : X \rightarrow V \setminus \Sigma$ (in the strong form described by Theorem 1.7).*

We will show that, if Σ is as in the corollary, the condition (1.2) holds with respect to some local vector bundle charts Φ on V and hence we get sprays. By Chow’s theorem [Chi] each fiber Σ_x is an algebraic subset of \bar{V}_x , and the codimension condition insures that Σ_x does not contain the hyperplane at infinity. This last condition can be equivalently described as follows. In a local bundle chart on $\pi^{-1}(U) \cong U \times \mathbb{C}^q$ over a small subset $U \subset X$, the set $\Sigma \cap \pi^{-1}(U)$ is given by finitely many equations

$$g_j(x, w) = \sum_{|I| \leq d_j} g_{j,I}(x)w^I = 0 \quad (1 \leq j \leq m)$$

which are polynomial in $w \in \mathbb{C}^q$ and where the coefficients $g_{j,I}$ are holomorphic in U . The closure $\bar{\Sigma} \subset U \times \mathbb{C}\mathbb{P}^q$ is defined by the corresponding homogenized equations, obtained by adding a suitable power of an additional variable w_0 to each term (so $w_0 = 0$ is the hyperplane at infinity). Let P_j be the top order homogeneous part of g_j (with respect to w); this is the part of g_j which gets no w_0 terms after homogenization. We see that $\bar{\Sigma}_x \subset \mathbb{C}\mathbb{P}^q$ contains the hyperplane at infinity $\{w_0 = 0\}$ if and only if $P_j(x, w) = 0$ for all $w \in \mathbb{C}^q$ and $j = 1, \dots, m$. The hypothesis precludes this from happening and hence Lemma 7.1 applies.

Example 2. If $D = \{z \in \mathbb{C} : |z| < 1\}$ and $\Sigma \subset D \times \mathbb{C}\mathbb{P}^1$ is a closed analytic subset of dimension one (a complex curve), it is easy to show that Σ can be avoided by graphs of continuous (or smooth) functions $f : D \rightarrow \mathbb{C}$, but in general there is no holomorphic map $f : D \rightarrow \mathbb{C}\mathbb{P}^1$ (i.e., meromorphic function on D) whose graph avoids Σ , due to hyperbolicity. In fact, let Σ_n be the union of graphs of the following functions on D : $z \rightarrow 0$, $z \rightarrow 1$, $z \rightarrow \infty$, and $z \rightarrow kz$ for $k = 1, 2, \dots, n$. Then for sufficiently large n there is no holomorphic (or meromorphic) function on D whose graphs avoids Σ_n . Proof: if the graph of f_n avoids Σ_n for each n , the sequence $\{f_n\}$ is a normal family, and hence a subsequence f_{n_k} converges to a holomorphic function $f : D \rightarrow \mathbb{C}$. For sufficiently large n the complex line $w = nz$ intersects the graph of f transversely at some point; the same line then intersects the graph of f_l for all sufficiently large l , a contradiction.

In our last theorem the hypotheses are similar as in Theorem 1.7, except that we replace the codimension condition on the fibers Σ_x by a homogeneity condition (compare with Sect. 1 in [EGr]). We state it without approximation

on a $\mathcal{H}(X)$ -convex set, although the result actually holds in the same form as Theorem 1.7.

1.9 Theorem. *Let $\pi : V \rightarrow X$ be a holomorphic vector bundle over a Stein manifold X , let $X_0 \subset X_1 \subset X$ be closed complex subvarieties of X , and let $\Sigma \subset V$ be a closed complex subvariety such that $\pi(\Sigma) = X_1 \setminus X_0$. Assume that for each point $x_0 \in X_1 \setminus X_0$ there are an open neighborhood $U \subset X$ and a holomorphic action of a complex Lie group G on $U' = \pi^{-1}(U) \subset V$, satisfying:*

- (i) $\pi(g \cdot v) = \pi(v)$ for each $v \in U'$ and $g \in G$, and
- (ii) for each $x \in U \cap X_1$, G preserves $V_x \setminus \Sigma_x$ and acts transitively there.

Then for any continuous section $f_0 : X \rightarrow V \setminus \Sigma$ that is holomorphic in a neighborhood of X_0 and for any integer $k \in \mathbf{Z}_+$ there is a homotopy $f_t : X \rightarrow V \setminus \Sigma$ ($0 \leq t \leq 1$) satisfying the conclusion of Theorem 1.7.

The paper is organized as follows. In Sect. 2 we explain the idea of the Proof of Theorems 1.2 and 1.5. In Sect. 3 we introduce the concept of a holomorphic (resp. continuous) complex and prism, and in Sect. 4 we introduce the notion of a Cartan string. The heart of the proof is Proposition 5.1 and the subsequent results of Sect. 5. We conclude the Proof of Theorem 1.5 by an inductive construction in Sect. 6. In Sect. 7 we prove Theorem 1.6, and in Sect. 8 we prove Theorems 1.7 and 1.9.

2 Outline of the proof of the main theorem

In this section we explain the scheme of Proof of Theorems 1.2 and 1.5. We concentrate on the case without parameters (i.e., when P is a single point). Thus, given a compact $\mathcal{H}(X)$ -convex set $K \subset X$ and a continuous section $a : X \rightarrow Z$ which is holomorphic in an open set $U_0 \supset K$, we shall construct a homotopy of sections $H^s : X \rightarrow Z$ ($0 \leq s \leq 1$) such that $H^0 = a$, the section $f = H^1$ is holomorphic on X , and every section H^s is holomorphic near K and it approximates a uniformly on K .

We begin by constructing a collection of holomorphic sections $a_{(j)} : U_j \rightarrow Z$ ($j = 0, 1, 2, \dots$) on a suitably chosen open covering $\mathcal{U} = \{U_0, U_1, U_2, \dots\}$ of X , where $U_0 \supset K$ is the given set on which a is already holomorphic, such that for each $j \in \mathbf{Z}_+$ there is a homotopy of continuous sections $a_{(j),s} : U_j \rightarrow Z$ ($0 \leq s \leq 1$) satisfying $a_{(j),0} = a|_{U_j}$ and $a_{(j),1} = a_{(j)}$. For $j = 0$ we take $a_{(0),s} = a|_{U_0}$ for all s . (See Proposition 4.7.)

We think of the collection $\{a_{(j)} : j \in \mathbf{Z}_+\}$ as a puzzle whose pieces should be rearranged into a holomorphic section $f : X \rightarrow Z$ which is homotopic to the initial section a .

Let us first consider what is involved in patching a pair of local holomorphic sections into a single holomorphic section over the union of the two sets. To

simplify the notation we assume that $A, B \subset X$ are compact subsets, $U \supset A$ and $V \supset B$ are open sets, and $a: U \rightarrow Z, b: V \rightarrow Z$ are holomorphic sections of $Z \rightarrow X$ over U resp. V . We wish to replace the pair (a, b) by a section \tilde{a} which is holomorphic in a neighborhood of $A \cup B$ and which approximates a on A . (The problem is nontrivial only when $C = A \cap B \neq \emptyset$.) We proceed in two steps. First we replace b by another holomorphic section $b_1: V \rightarrow Z$ (perhaps shrinking V around B if necessary) which approximates a very closely in some neighborhood W of $A \cap B$. In the second step we ‘glue’ a with b_1 into a new holomorphic section \tilde{a} over $A \cup B$. The first step can be achieved if the following hold:

- the set $C = A \cap B$ is Runge in B , i.e., we can approximate holomorphic functions in small neighborhoods of C by functions holomorphic in a neighborhood of B ,
- the submersion $Z \rightarrow X$ admits a spray over a neighborhood of B , and
- there is a neighborhood $W \supset A \cap B$ and a homotopy of holomorphic sections $b_t: W \rightarrow Z$ connecting $b_0 = b|_W$ and $b_1 = a|_W$.

Granted these conditions, the h-Runge theorem (see Theorems 4.1 and 4.2 in [FP1]) shows that we can approximate the homotopy $\{b_t\}$ uniformly in a neighborhood of $C = A \cap B$ by a homotopy of sections \tilde{b}_t ($0 \leq t \leq 1$) which are holomorphic in a neighborhood of B . In particular the section \tilde{b}_1 approximates a as well as desired in some fixed neighborhood of C . Replacing b by \tilde{b}_1 we may thus assume that b approximates a near C .

To glue a and b we linearize the problem and solve a certain $\bar{\partial}$ -equation in a neighborhood of $A \cup B$. This is accomplished by Theorems 5.1 and 5.5 in [FP1], provided that (A, B) is a *Cartan pair* in X (def. 4.1 below). This means that each of the sets A, B and $A \cup B$ has a basis of Stein neighborhoods, C is Runge in B , and $(A \setminus B) \cap (B \setminus A) = \emptyset$.

In order to glue the local holomorphic sections $a_{(j)}: U_j \rightarrow Z$ into a single holomorphic section $f: X \rightarrow Z$ we perform the above steps inductively. At each step the sets U_j on which our sections are defined may shrink, and we must control the shrinking so that in the end we still have a covering of X . For this reason we initially choose a special locally finite covering $\mathcal{A} = \{A_0, A_1, A_2, \dots\}$ of X by *compact* sets satisfying

- (i) $K \subset A_0 \subset U_0$;
- (ii) for each $n \geq 1$, the ordered collection of sets (A_0, A_1, \dots, A_n) is a *Cartan string* (def. 4.2 below). This property enables us to carry out the gluing procedure by induction on n . In particular this means that for each $n \geq 1$, $A^{n-1} = A_0 \cup A_1 \cup \dots \cup A_{n-1}$ and A_n form a Cartan pair;
- (iii) for each $j \in \mathbf{Z}_+$ there is a holomorphic section $a_{(j)}$ in an open neighborhood $U_j \supset A_j$ which is homotopic to $a|_{U_j}$, and for $j \geq 1$ the restriction $Z|_{U_j}$ admits a spray;

- (iv) for each pair $i \neq j$ such that $A_i \cap A_j \neq \emptyset$, there is a holomorphic homotopy between $a_{(i)}$ and $a_{(j)}$ in $U_{(i,j)} = U_i \cap U_j$; etc.

In general, for each multiindex $J = (j_0, j_1, \dots, j_n)$ such that $A_J = A_{j_0} \cap \dots \cap A_{j_n} \neq \emptyset$, there is an n -dimensional holomorphic homotopy $a_J(t)$ in $U_J = U_{j_0} \cap \dots \cap U_{j_n}$, with the parameter t belonging to the standard n -simplex $\Delta^n \subset \mathbb{R}^n$, such that for t belonging to a boundary face of Δ^n determined by a shorter multiindex $I \subset J$ we have $a_J(t) = a_I(t)|_{U_J}$. The parameter space of our entire collection of holomorphic sections and homotopies between them is the geometric realization of the simplicial complex called the *nerve* of the covering \mathcal{A} (see Sect. 3). The sets U_j will shrink but the A_j 's will stay the same during the entire construction.

Suppose inductively that we have already joined the sections $a_{(0)}, \dots, a_{(n-1)}$ into a holomorphic section f^{n-1} in a neighborhood of $A^{n-1} = A_0 \cup A_1 \cup \dots \cup A_{n-1}$, using the homotopies between them and the gluing procedure. We emphasize that all modifications are done by holomorphic homotopies. In the next step we must glue f^{n-1} with the section $a_{(n)}$. For this we need a holomorphic homotopy between the two sections in a neighborhood of $A^{n-1} \cap A_n$. In the special case when $h: Z \rightarrow X$ is a fiber bundle whose fiber admits a spray such a homotopy can be constructed from a continuous homotopy between the two sections, provided that our sets are chosen sufficiently carefully (i.e., A_n must be a pseudoconvex bump on A^{n-1}). This was explained in [FP1]; see especially Proposition 6.1 there. The argument in [FP1] does not seem to carry over to the present situation and we need an alternative systematic way of insuring the existence of such homotopies.

Our inductive construction is such that for each $j = 0, 1, \dots, n - 1$ we have a holomorphic homotopy between f^{n-1} and $a_{(n)}$ in a neighborhood of $A_j \cap A_n$, inherited from the initial homotopy between $a_{(j)}$ and $a_{(n)}$. We now patch these n partial homotopies into a single homotopy defined in a neighborhood of $A^{n-1} \cap A_n = \cup_{j=0}^{n-1} (A_j \cap A_n)$. This can be done in the same way as above by induction on n , provided that the ordered collection $(A_0 \cap A_n, A_1 \cap A_n, \dots, A_{n-1} \cap A_n)$ is also a Cartan string. Finally, since (A^{n-1}, A_n) is a Cartan pair, we can glue f^{n-1} and $a_{(n)}$ into a section f^n in a neighborhood of A^n which approximates f^{n-1} on A^{n-1} . This completes the induction step. (The details are given in Sect. 5.) The sequence of sections f^n obtained in this way converges uniformly on compacts in X to a holomorphic section $f: X \rightarrow Z$ which solves the problem (Sect. 6).

3 Holomorphic complexes and prisms

3.1 Definition. Let $\mathcal{A} = \{A_0, A_1, A_2, \dots\}$ be any finite or countable family of nonempty subsets of X which is locally finite. The nerve of \mathcal{A} is the (combinatorial) simplicial complex $\mathcal{K}(\mathcal{A})$ consisting of precisely those multiindices

$J = (j_0, j_1, \dots, j_k) \in \mathbf{Z}_+^{k+1}$ ($k \in \mathbf{Z}_+$) with increasing entries $0 \leq j_0 < j_1 < \dots < j_k$ for which

$$A_J = A_{j_0} \cap A_{j_1} \cap \dots \cap A_{j_k} \neq \emptyset.$$

We denote the geometric realization of $\mathcal{K}(\mathcal{A})$ by $K(\mathcal{A})$.

For simplicial complexes and their realization we refer the reader to [HW]; here we only recall a few basic ideas. The geometric realization of a simplicial complex is a topological space which is a union of topological simplexes of various dimensions such that any two of them either do not intersect, they intersect along a simplex of lower dimension, or one is contained in the other.

In the nerve complex, each multiindex $J = (j_0, j_1, \dots, j_k) \in \mathcal{K}(\mathcal{A})$ of length $k + 1$ determines a closed k -dimensional face $|J| \subset K(\mathcal{A})$, called the *body* (or *carrier*) of J , and J is called the *vertex scheme* of $|J|$. $|J|$ is homeomorphic to the standard k -simplex $\Delta^k \subset \mathbb{R}^k$ (the closed convex hull of the set $\{0, e_1, e_2, \dots, e_k\} \subset \mathbb{R}^k$ containing the origin and the standard basis vectors), and any k -dimensional face of $K(\mathcal{A})$ is of this form. Thus the topological space $K(\mathcal{A})$ is the body of the combinatorial object $\mathcal{K}(\mathcal{A})$. The vertices of $K(\mathcal{A})$ correspond to the individual sets $A_j \in \mathcal{A}$, i.e., to singletons $\{j\} \in \mathcal{K}(\mathcal{A})$. Given $I, J \in \mathcal{K}(\mathcal{A})$ we have $|I| \cap |J| = |I \cap J|$. Thus for any two (bodies of) simplexes in $K(\mathcal{A})$, either one is a subset of the other or else their intersection is a simplex of lower dimension, possibly empty.

For each $n \in \mathbf{Z}_+$ we denote by

$$\mathcal{K}^n(\mathcal{A}) = \mathcal{K}(A_0, A_1, \dots, A_n) \subset \mathcal{K}(\mathcal{A})$$

the nerve of the finite subcollection $\mathcal{A}_n = \{A_0, \dots, A_n\}$ and by $K^n(\mathcal{A})$ its body. Occasionally we delete \mathcal{A} in the notation. Clearly $\mathcal{K}^n \subset \mathcal{K}^{n+1}$ for each n , and $\mathcal{K}(\mathcal{A}) = \bigcup_{n=0}^\infty \mathcal{K}^n$. More generally, for any multiindex $J = (j_0, j_1, \dots, j_k) \in \mathbf{Z}_+^{k+1}$ (not necessarily belonging to $\mathcal{K}(\mathcal{A})$) we denote by

$$\mathcal{K}_J(\mathcal{A}) = \mathcal{K}(A_{j_0}, A_{j_1}, \dots, A_{j_k})$$

the nerve of the indicated collection of sets and by $K_J(\mathcal{A})$ its body. Note that $\mathcal{K}_J(\mathcal{A})$ is a finite subcomplex of $\mathcal{K}(\mathcal{A})$ whose body is a k -dimensional simplex if and only if $J \in \mathcal{K}(\mathcal{A})$; otherwise it is a union of simplexes of lower dimension. Occasionally we shall write simply \mathcal{K} instead of $\mathcal{K}(\mathcal{A})$ when it is clear from the context which collection \mathcal{A} is meant.

From now on we assume that $\mathcal{A} = \{A_0, A_1, A_2, \dots\}$ is a (finite or countable) locally finite family of compact subsets of X . Later on, \mathcal{A} will be a covering of X with additional properties, but this is not important for the moment. An open neighborhood of \mathcal{A} is a collection $\mathcal{U} = \{U_0, U_1, U_2, \dots\}$ of open subsets of X (with the same index set) such that $A_i \subset U_i$ for each i . Such an neighborhood is called *faithful* if $\mathcal{K}(\mathcal{U}) = \mathcal{K}(\mathcal{A})$, that is, the two families have the same nerve

complex. Clearly any locally finite family \mathcal{A} as has an open faithful neighborhood \mathcal{U} . As before we write for each $J = (j_0, j_1, \dots, j_k)$

$$U^J = U_{j_0} \cup U_{j_1} \cup \dots \cup U_{j_k}, \quad U_J = U_{j_0} \cap U_{j_1} \cap \dots \cap U_{j_k}.$$

If $h: Z \rightarrow X$ is a holomorphic submersion onto X and $U \subset X$ is an open subset, we denote by $\mathcal{O}_h(U, Z)$ (resp. $\mathcal{C}_h(U, Z)$) the set of all holomorphic (resp. continuous) sections $f: U \rightarrow Z$ of $h: Z \rightarrow X$ over U .

3.2 Definition. *Let $h: Z \rightarrow X$ be a holomorphic map of a complex manifold Z onto a complex manifold X and let $\mathcal{A} = \{A_0, A_1, A_2, \dots\}$ be a locally finite, at most countable family of compact sets in X .*

(i) *A holomorphic $\mathcal{K}(\mathcal{A})$ -complex with values in Z is a collection*

$$f_* = \{f_J: |J| \rightarrow \mathcal{O}_h(U_J, Z), \quad J \in \mathcal{K}(\mathcal{A})\},$$

where $\mathcal{U} = \{U_0, U_1, U_2, \dots\}$ is a faithful neighborhood of \mathcal{A} , satisfying the following compatibility conditions:

$$I, J \in \mathcal{K}(\mathcal{A}), \quad I \subset J \implies f_J(t) = f_I(t)|_{U_J}, \quad t \in |I|.$$

A continuous $\mathcal{K}(\mathcal{A})$ -complex with values in Z is a collection

$$f_* = \{f_J: |J| \rightarrow \mathcal{C}_h(U_J, Z), \quad J \in \mathcal{K}(\mathcal{A})\}$$

satisfying the same compatibility conditions.

(ii) *If f_* is a (holomorphic or continuous) $\mathcal{K}(\mathcal{A})$ -complex and $\mathcal{K}' \subset \mathcal{K}(\mathcal{A})$ is a subcomplex of $\mathcal{K}(\mathcal{A})$, we denote by $f_*|_{\mathcal{K}'}$ the restriction of f_* to \mathcal{K}' .*

(iii) *A $\mathcal{K}(\mathcal{A})$ -complex f_* , defined on a faithful neighborhood \mathcal{U} of \mathcal{A} , is constant if there is a section $g: \cup \{U: U \in \mathcal{U}\} \rightarrow Z$ such that $f_J(t)|_{U_J} = g|_{U_J}$ for each $J \in \mathcal{K}(\mathcal{A})$ and each $t \in |J|$.*

Remark. We shall consider $\mathcal{K}(\mathcal{A})$ -complexes in the sense of their germs on the sets in \mathcal{A} . Thus, two $\mathcal{K}(\mathcal{A})$ -complexes f_* and g_* are considered equivalent if there is an open faithful neighborhood $\mathcal{U} = \{U_i\}$ of $\mathcal{A} = \{A_i\}$ such that for each $J \in \mathcal{K}(\mathcal{A})$ and $t \in |J|$, the sections $f_J(t)$ and $g_J(t)$ are defined and equal in U_J . In practice we shall not distinguish between equivalent complexes. □

A holomorphic complex is a bookkeeping tool that we shall need to keep track of the sections and homotopies between them alluded to in Sect.2. For each $J = (j_0, \dots, j_k) \in \mathcal{K}(\mathcal{A})$ we have a family of holomorphic sections

$$f_J(t): U_J \rightarrow Z, \quad t \in |J|,$$

depending continuously on the parameter $t \in |J|$, which we may think of as a homotopy of holomorphic sections over the set $U_J = U_{j_0} \cap U_{j_1} \cap \dots \cap U_{j_k}$, with the parameter t belonging to the simplex $|J| \subset K(\mathcal{A})$. For each face $I \subset J$ of J

and for $t \in |I| \subset |J|$ the section $f_J(t)$ agrees with the section $f_I(t)$, restricted from its original domain $U_I \subset X$ to the subdomain U_J . It is worthwhile to consider separately the one dimensional case.

Example 3. Let $J = (j_0, j_1) \in \mathcal{K}(\mathcal{A})$ be a simplex of length two. Its body $|J| \subset K(\mathcal{A})$ is a segment which we can represent by $[0, 1] \subset \mathbb{R}$. J contains two zero dimensional faces, namely the vertices (j_0) and (j_1) (corresponding to the sets A_{j_0} resp. A_{j_1}), and the bodies of these faces are identified with the endpoints $\{0\}$ resp. $\{1\}$ of $[0, 1]$. The restriction of a $\mathcal{K}(\mathcal{A})$ -complex f_* to the subcomplex $\mathcal{K}_J(\mathcal{A}) \subset \mathcal{K}(\mathcal{A})$, determined by $J = (j_0, j_1)$, consists of a one parameter family (a homotopy) $f_J(t)$ ($t \in |J| = [0, 1]$) of sections over $U_J = U_{j_0} \cap U_{j_1}$ such that $f_J(0)$ is the restriction to U_J of a section $f_{j_0}: U_{j_0} \rightarrow Z$, and likewise $f_J(1)$ is the restriction to U_J of a section $f_{j_1}: U_{j_1} \rightarrow Z$. □

We shall also need the notion of a multiparameter homotopy of $\mathcal{K}(\mathcal{A})$ -complexes. A suitable concept for this is the following.

3.3 Definition. Let $h: Z \rightarrow X$ and \mathcal{A} be as in def. 3.2, and let $k \in \mathbf{Z}_+$.

(i) A holomorphic $(\mathcal{K}(\mathcal{A}), k)$ -prism (or a k -prism over $\mathcal{K}(\mathcal{A})$) with values in Z is a collection

$$f_* = \{f_J: |J| \times [0, 1]^k \rightarrow \mathcal{O}_h(U_J, Z), J \in \mathcal{K}(\mathcal{A})\},$$

where \mathcal{U} is a faithful open neighborhood of \mathcal{A} , such that for each fixed $y \in [0, 1]^k$ the associated family

$$f_{*,y} = \{f_{J,y} = f_J(\cdot, y): |J| \rightarrow \mathcal{O}_h(U_J, Z), J \in \mathcal{K}(\mathcal{A})\},$$

is a holomorphic $\mathcal{K}(\mathcal{A})$ -complex. A continuous $(\mathcal{K}(\mathcal{A}), k)$ -prism with values in Z is a collection $f_* = \{f_J: |J| \times [0, 1]^k \rightarrow \mathcal{C}_h(U_J, Z): J \in \mathcal{K}(\mathcal{A})\}$ such that $f_{*,y}$ is a continuous $\mathcal{K}(\mathcal{A})$ -complex for each $y \in [0, 1]^k$.

(ii) A prism f_* is sectionally constant if there is an open set $U \supset \cup_{i \geq 0} A_i$ such that the complex $f_{*,y}$ is represented by a section $f_y: U \rightarrow Z$ for each fixed $y \in [0, 1]^k$. If this holds for all y in a subset $Y \subset [0, 1]^k$, we say that f_* is sectionally constant on Y .

Thus a $(\mathcal{K}(\mathcal{A}), 0)$ -prism is a $\mathcal{K}(\mathcal{A})$ -complex, a $(\mathcal{K}(\mathcal{A}), 1)$ -prism is the same thing as a homotopy of $\mathcal{K}(\mathcal{A})$ -complexes, a 2-prism is a homotopy of 1-prisms, etc. A sectionally constant $(\mathcal{K}(\mathcal{A}), k)$ -prism is a k -parameter homotopy of sections over an open neighborhood of the union of sets in \mathcal{A} .

4 Cartan strings and the initial holomorphic complex

Let X be a complex manifold. A compact set $C \subset X$, contained in another compact set $B \subset X$, is said to be *Runge* in B if C has a basis of open neighborhoods

each of which is Runge in a fixed neighborhood of B . A compact set $K \subset X$ is said to be a *Stein compactum* if it has a basis of open neighborhoods which are Stein.

4.1 Definition. *An ordered pair of compact sets (A, B) in a complex manifold X is said to be a Cartan pair (or a Cartan string of length 2) if*

- (i) *each of the sets A , B , and $A \cup B$ has a basis of Stein neighborhoods,*
- (ii) *$A \setminus B \cap B \setminus A = \emptyset$ (the separation condition), and*
- (iii) *the set $C = A \cap B$ is Runge in B . (C may be empty.)*

Our definition of a Cartan pair in [FP1] was similar, except that it did not include the property (iii). We now recall Gromov's definition of a *Cartan string*. It is by induction on $n \in \mathbb{N}$, a Cartan string of length 2 being just a Cartan pair.

4.2 Definition. ([Gr1], 4.2.D'.) *Let X be a complex manifold and $A_0, A_1, \dots, A_n \subset X$ compact subsets ($n \geq 1$). The sequence (A_0, A_1, \dots, A_n) is a Cartan string of length $n + 1$ if*

- (i) *$(A_0 \cup \dots \cup A_{n-1}, A_n)$ is a Cartan pair, and*
- (ii) *if $n \geq 2$, the sequences (A_0, \dots, A_{n-1}) and $(A_0 \cap A_n, \dots, A_{n-1} \cap A_n)$ are Cartan strings of length n .*

A locally finite covering $\mathcal{A} = \{A_0, A_1, A_2, \dots\}$ of X by compact sets is a Cartan covering if (A_0, A_1, \dots, A_n) is a Cartan string for each $n \in \mathbb{N}$.

Note that each set A_k in a Cartan string, and also each finite union $A^k = \bigcup_{i=0}^k A_i$, is a Stein compactum in X . The order of sets in a Cartan string is important because of the property (iii) in Definition 4.1; hence a Cartan string is a *sequences* of sets and not merely a family. Cartan strings have the following *hereditary property*:

4.3 Proposition. *If (A_0, A_1, \dots, A_n) is a Cartan string in a complex manifold X , and if $B \subset X$ is a Stein compactum in X , then $(A_0 \cap B, \dots, A_n \cap B)$ is also a Cartan string.*

Proof. Induction on $n \in \mathbb{N}$. Note first that, if A and B are compact sets in X with bases of Stein neighborhoods, then $A \cap B$ also has a basis of Stein neighborhoods (since intersections of Stein domains are again Stein). Consider first the case $n = 1$, i.e., we have a Cartan pair (A_0, A_1) , and we wish to prove that $(A_0 \cap B, A_1 \cap B)$ is also a Cartan pair. Clearly it satisfies (i) and (ii) in def. 4.1. Property (iii) follows from

4.4 Lemma. *If $D_0 \subset D_1$ and Ω are open Stein domains in a complex manifold X , and if D_0 is Runge in D_1 , then $D_0 \cap \Omega$ is Runge in $D_1 \cap \Omega$.*

Proof of Lemma 4.4. Choose any compact set $K \subset\subset D_0 \cap \Omega$. We denote by \widehat{K}_D the holomorphically convex hull of K with respect to a domain $D \subset X$

containing K . Clearly $\widehat{K}_{D_1 \cap \Omega} \subset \widehat{K}_{D_1} \cap \widehat{K}_\Omega$. Since D_0 is Runge in D_1 and both domains are Stein, we have $\widehat{K}_{D_0} = \widehat{K}_{D_1}$ ([Hör], Theorem 2.7.3), and therefore

$$\widehat{K}_{D_1 \cap \Omega} \subset \widehat{K}_{D_1} \cap \widehat{K}_\Omega = \widehat{K}_{D_0} \cap \widehat{K}_\Omega \subset\subset D_0 \cap \Omega.$$

It follows (Theorem 2.7.3 in [Hör]) that $D_0 \cap \Omega$ is Runge in $D_1 \cap \Omega$. □

This completes the Proof of Proposition 4.3 when $n = 1$. Suppose now that the result holds for some n . Let $(A_0, A_1, \dots, A_{n+1})$ be a Cartan string of length $n + 2$. To see that $(A_0 \cap B, \dots, A_{n+1} \cap B)$ is a Cartan string, we must verify that:

- (i) the pair of sets $(A_0 \cap B) \cup \dots \cup (A_n \cap B) = (A_0 \cup \dots \cup A_n) \cap B$ and $A_{n+1} \cap B$ is a Cartan pair. Since $(A_0 \cup \dots \cup A_n, A_{n+1})$ is a Cartan pair, this follows from the case $n = 1$ proved above;
- (ii) the strings $(A_0 \cap B, \dots, A_n \cap B)$ and $(A_0 \cap A_{n+1} \cap B, \dots, A_n \cap A_{n+1} \cap B)$ are Cartan strings of length $n + 1$. This follows immediately from Definition 4.3 and from the inductive hypothesis.

□

4.5 Corollary. *If $\mathcal{A} = \{A_0, A_1, A_2, \dots\}$ is a sequence of compact sets in a complex manifold X such that for each $n \in \mathbb{N}$ the pair $(A_0 \cup \dots \cup A_{n-1}, A_n)$ is a Cartan pair, then for each $n \in \mathbb{N}$ the string (A_0, A_1, \dots, A_n) is a Cartan string.*

Proof. This follows from Proposition 4.3 by induction on n . □

4.6 Theorem. *For each open covering $\mathcal{U} = \{U_j\}$ of a Stein manifold X there exists a Cartan covering $\mathcal{A} = \{A_i : i = 0, 1, \dots\}$ of X which is subordinate to \mathcal{U} , i.e., such that each set A_i is contained in U_j for some $j = j(i)$. Moreover, if $K \subset X$ is a compact $\mathcal{H}(X)$ -convex subset and $U_0 \subset X$ is an open set containing K , we can choose \mathcal{A} such that $K \subset A_0 \subset U_0$ and $A_i \cap K = \emptyset$ for $i \geq 1$.*

Proof. This was proved by Henkin and Leiterer in Sect. 2 of [HL]. We recall briefly the main idea. The conditions imply that there exists a smooth strongly plurisubharmonic exhaustion function $\rho : X \rightarrow \mathbb{R}$ with nondegenerate critical points (a Morse function) such that $K \subset \{\rho < 0\} \subset\subset U_0$ and 0 is a regular value of ρ . Set $A_0 = \{\rho \leq 0\}$. One can reach any higher sublevel set $\{\rho \leq c\}$, for $c > 0$ being a regular value of ρ , by successively adding (finitely many times) small strongly pseudoconvex domains A_k to the union of the previous sets $A^{k-1} = \cup_{i=0}^{k-1} A_i$ in such a way that for each k the pair (A^{k-1}, A_k) is a *special pseudoconvex bump* in the terminology of [HL]. It is clear from their definition that such a pair is also a Cartan pair as defined here. Moreover, we can insure that each set A_k for $k \geq 1$ is contained in a set U_j for some $j \geq 1$, and the resulting family is locally finite. Corollary 4.5 above implies that the covering $\mathcal{A} = \{A_0, A_1, A_2, \dots\}$ of X that one builds in this way is a Cartan covering.

If ρ has no critical values on an interval $[c_0, c_1] \subset \mathbb{R}$ then we can actually reach $\{\rho \leq c_1\}$ from $\{\rho \leq c_0\}$ by adding *convex bumps* (this is called a noncritical pseudoconvex extension). To cross the critical points of ρ one must attach more general pseudoconvex bumps. □

The following proposition provides a homotopy of complexes (a 1-prism) from the initial continuous section to a holomorphic complex over a Cartan covering of X .

4.7 Proposition. (Construction of the initial holomorphic complex) *Let X be a Stein manifold, $K \subset X$ a compact $\mathcal{H}(X)$ -convex subset and $h: Z \rightarrow X$ a holomorphic submersion onto X with the property that each point $x \in X \setminus K$ has an open neighborhood $U_x \subset X$ such that $h: Z|_{U_x} \rightarrow U_x$ admits a spray. Let $a: X \rightarrow Z$ be a continuous section which is holomorphic in an open set $U_0 \supset K$. Then there exists a Cartan covering $\mathcal{A} = \{A_i : i = 0, 1, \dots\}$ of X and a continuous $(\mathcal{K}(\mathcal{A}), 1)$ -prism $a_* = \{a_{*,s} : s \in [0, 1]\}$ with values in Z such that*

- (i) $K \subset A_0 \subset U_0, K \cap A_i = \emptyset$ for $i \geq 1$, and $a_{(0),s} = a|_{U_0}$ for each $s \in [0, 1]$,
- (ii) $a_{*,0}$ is the constant $\mathcal{K}(\mathcal{A})$ -complex given by the section $a: X \rightarrow Z$,
- (iii) $a_{*,1}$ is a holomorphic $\mathcal{K}(\mathcal{A})$ -complex,
- (iv) for each $j \geq 1$ the submersion $Z \rightarrow X$ admits a spray over an open set $U_j \supset A_j$.

Proof. Let $N = \dim Z = n + m$, where $n = \dim X$ and m is the dimension of the fibers $h^{-1}(x)$ ($x \in X$). Denote by P^N the unit open polydisc in \mathbb{C}^N with complex coordinates $\zeta = (\zeta', \zeta'')$, where $\zeta' \in \mathbb{C}^n$ and $\zeta'' \in \mathbb{C}^m$. Let $\pi: P^N \rightarrow P^n$ be the projection $\pi(\zeta', \zeta'') = \zeta'$. Since $h: Z \rightarrow X$ is a submersion, there exist for each point $z_0 \in Z$ open neighborhoods $V \subset Z$ of $z_0, U \subset X$ of $x_0 = h(z_0)$, and biholomorphic maps $\Phi: V \rightarrow P^N, \phi: U \rightarrow P^n$, such that $\pi \circ \Phi = \phi \circ h$ on V and $\Phi(z_0) = 0$. Such map Φ induces a linear structure on the fibers of $h|_V$ which lets us add sections of $h: V \rightarrow U$ and take their convex linear combinations.

If z_0 belongs to the graph of a , we can choose these neighborhoods and maps such that $a(U) \subset V$. In this case Φ maps $a(U)$ onto the graph of a section $\tilde{a}(\zeta') = (\zeta', a''(\zeta'))$ ($\zeta' \in P^n$) of the projection $\pi: P^N \rightarrow P^n$. The family $a_s: U \rightarrow V$, given by

$$a_s(x) = \Phi^{-1}(\phi(x), (1 - s)a''(\phi(x))) \quad (x \in U, 0 \leq s \leq 1),$$

is a homotopy of continuous sections of h over U such that $a_s(U) \subset V$ for each $s \in [0, 1]$, $a_0 = a|_U$, and the section a_1 is holomorphic. By shrinking U around x_0 and replacing V by $V \cap h^{-1}(U)$ we may insure in addition that the graph of the entire homotopy a_s stays in a prescribed open neighborhood of $\bar{a}(U)$.

Let $U_0 \subset X$ be an open set containing K such that a is holomorphic in U_0 . Set $V_0 = h^{-1}(U_0) \subset Z$ and $a_{(0),s} = a|_{U_0}$ for $s \in [0, 1]$. Using the above argument we can cover the graph of a outside of V_0 by open neighborhoods $V_j \subset Z$

($j = 1, 2, 3, \dots$) biholomorphic to P^N , with $U_j = h(V_j) \subset X$ biholomorphic to P^n , such that for each $j \in \mathbb{N}$ we have a homotopy of continuous sections $a_{(j),s}: U_j \rightarrow V_j$ ($0 \leq s \leq 1$) of h satisfying

- (a) $a_{(j),0} = a|U_j$,
- (b) the section $a_{(j),1}$ is holomorphic in U_j ,
- (c) $U_j \cap K = \emptyset$ and $Z|U_j$ admits a spray for each $j \geq 1$,
- (d) if $U_{(i,j)} = U_i \cap U_j \neq \emptyset$ for some $i, j \geq 0$ then $a_{(i),s}(U_{(i,j)}) \subset V_j$ for each $s \in [0, 1]$.

The property (d) insures that for each pair of indices $i, j \in \mathbf{Z}_+$ such that $U_{(i,j)} \neq \emptyset$ there is a homotopy of sections $a_{(i,j),s}(t): U_{(i,j)} \rightarrow Z$, depending continuously on $t, s \in [0, 1]$, such that $a_{(i,j),s}(0) = a_{(i),s}|U_{(i,j)}$, $a_{(i,j),s}(1) = a_{(j),s}|U_{(i,j)}$, $a_{(i,j),0}(t) = a|U_{(i,j)}$, and the section $a_{(i,j),1}(t)$ is holomorphic on $U_{(i,j)}$ for each $t \in [0, 1]$. We get $a_{(i,j),s}(t)$ by taking the convex linear combinations in $t \in [0, 1]$ of the sections $a_{(i),s}$ and $a_{(j),s}$, restricted to $U_{(i,j)}$, where the combinations are taken with respect to the linear structure on the fibers of $h|V_i$ induced by Φ_i (or with respect to the linear structure on the fibers of $h|V_j$ induced by Φ_j). If one of the indices is zero, say $i = 0$ and $j > 0$, we must use the linear structure on V_j induced by Φ_j since there is no such structure on V_0 .

Likewise, if $U_{(i,j,k)} \neq \emptyset$ for some multiindex $J = (i, j, k)$, we can use the linear structure on the fibers in any one of the sets V_i, V_j, V_k to get a homotopy of sections $a_{J,s}(t): U_J \rightarrow Z$, with t belonging to the standard 2-simplex $\Delta^2 \subset \mathbb{R}^2$, whose restriction to the sides of the simplex equals the respective homotopy obtained in the previous step. Continuing this way we build a 1-prism a_* on the covering $\mathcal{U} = \{U_0, U_1, U_2, \dots\}$ of X .

By Theorem 4.6 there is a Cartan covering $\mathcal{A} = \{A_0, A_1, A_2, \dots\}$ of X subordinate to \mathcal{U} , with $K \subset A_0 \subset U_0$. Then a_* induces in a natural way a $(\mathcal{K}(\mathcal{A}), 1)$ -prism with the required properties. This proves Proposition 4.7. \square

5 Modifying holomorphic prisms over Cartan strings

This section contains the heart of the Proof of Theorems 1.2 and 1.5. In Proposition 5.1 we show by induction on n how to modify a holomorphic $\mathcal{K}(\mathcal{A})$ -complex f_* over any finite subcomplex $\mathcal{K}^n = \mathcal{K}(A_0, \dots, A_n)$ into a holomorphic section, defined over a neighborhood of $A^n = A_0 \cup A_1 \cup \dots \cup A_n$, provided that (A_0, \dots, A_n) is a Cartan string. Since the inductive procedure requires us to solve the problem for parametrized families of complexes, we consider holomorphic prisms from the outset. The initial case is $n = 1$ when the string (A_0, A_1) is a *Cartan pair*; here we need the analytic tools from [FP1].

5.1 Proposition. *Let $h: Z \rightarrow X$ be a holomorphic submersion. Let (A_0, \dots, A_n) be a Cartan string in X and $\mathcal{K}^n = \mathcal{K}(A_0, \dots, A_n)$ its nerve. Assume that for each $i = 1, \dots, n$ there is an open set $U_i \supset A_i$ in X such that $h: Z|U_i \rightarrow U_i$*

admits a spray. If f_* is a holomorphic (\mathcal{K}^n, k) -prism with values in Z which is sectionally constant on a nice compact set $Y \subset [0, 1]^k$ (def. 1.4), there is a homotopy f_*^u ($0 \leq u \leq 1$) of holomorphic (\mathcal{K}^n, k) -prisms such that

- (i) $f_*^0 = f_*$ is the given prism,
- (ii) the prism f_*^1 is sectionally constant,
- (iii) for each $y \in [0, 1]^k$ and $u \in [0, 1]$, the section $f_{(0),y}^u$ approximates $f_{(0),y}$ on A_0 as well as desired,
- (iv) $f_{*,y}^u = f_{*,y}$ for all $y \in Y$ and $u \in [0, 1]$ (i.e., the homotopy is fixed on Y).

Moreover, if the restriction $f_*|_{\mathcal{K}^{n-1}}$ to the subcomplex $\mathcal{K}^{n-1} = \mathcal{K}(A_0, \dots, A_{n-1})$ is sectionally constant, the homotopy f_*^u can be chosen such that, in addition to the above, the prism $f_*^u|_{\mathcal{K}^{n-1}}$ is sectionally constant for each $u \in [0, 1]$ and the corresponding sections $f_{*,y}^u|_{\mathcal{K}^{n-1}}$ for $y \in [0, 1]^k$, which are holomorphic in a neighborhood of A^{n-1} , approximate $f_{*,y}|_{\mathcal{K}^{n-1}}$ uniformly on A^{n-1} .

Proof. Replacing X by a suitable Stein neighborhood of A^n we may assume it is Stein. The proof is by induction on $n \geq 0$, and for $n = 0$ there is nothing to prove.

The case $n = 1$. Our data consists of a Cartan pair (A_0, A_1) in X and a holomorphic $(\mathcal{K}(A_0, A_1), k)$ -prism f_* which is sectionally constant on a nice compact set $Y \subset [0, 1]^k$. Such an object f_* is determined by the following data:

- (a) a pair of open sets $U_0 \supset A_0$ and $U_1 \supset A_1$,
- (b) families of holomorphic sections $a_y = f_{(0),y}: U_0 \rightarrow Z$, $b_y = f_{(1),y}: U_1 \rightarrow Z$, depending continuously on $y \in [0, 1]^k$,
- (c) a family of holomorphic sections

$$c_{y,t} = f_{(0,1),y}(t): U_{(0,1)} = U_0 \cap U_1 \rightarrow Z$$

depending continuously on $t \in [0, 1]$ and $y \in [0, 1]^k$,

such that $a_y|_{U_{(0,1)}} = c_{y,0}$, $b_y|_{U_{(0,1)}} = c_{y,1}$, and for each $y \in Y$ the section $c_{y,t}$ is independent of $t \in [0, 1]$. Hence for $y \in Y$ the family $\{c_{y,t}: t \in [0, 1]\}$ determines a holomorphic section $c_y: U_0 \cup U_1 \rightarrow Z$ such that $c_y|_{U_0} = a_y$ and $c_y|_{U_1} = b_y$. We shall write $f_* = (a_*, b_*, c_*)$, where $*$ indicates the missing parameters.

Our goal is to construct a homotopy $f_*^u = (a_*^u, b_*^u, c_*^u)$ ($0 \leq u \leq 1$) of holomorphic $(\mathcal{K}(A_0, A_1), k)$ -prisms over smaller sets $U_0 \supset A_0$ and $U_1 \supset A_1$ such that $f_*^0 = f_*$ and f_*^1 is a constant prism, which really means that f_*^1 is a collection of holomorphic sections $f_y^1: U_0 \cup U_1 \rightarrow Z$ (we have eliminated the t parameter!). Moreover, the homotopy must be fixed for $y \in Y$ and it must approximate the sections a_y over A_0 for each $y \in [0, 1]^k$. We shall denote the data in the homotopy f_*^u by the same letters as above, adding the upper index u .

The homotopy f_*^u will be constructed in two steps. For convenience we use the parameter interval $u \in [0, 2]$ and later rescale it to $[0, 1]$. In the first step

we apply the h-Runge theorem (Theorem 4.2 in [FP1]) to get a homotopy f_*^u , $0 \leq u \leq 1$, from $f_*^0 = f_*$ to another prism f_*^1 such that we do not move the section a_y (i.e., $a_y^u = a_y : U_0 \rightarrow Z$ for all u and y), and such that the section $b_y^1 : U_1 \rightarrow Z$ approximates a_y in a neighborhood of $A_0 \cap A_1$ for each $y \in [0, 1]^k$. In the second step we apply the gluing lemma, Theorem 5.5 in [FP1], to obtain homotopies of sections

$$a_y^u : U_0 \rightarrow Z, \quad b_y^u : U_1 \rightarrow Z, \quad c_{y,t}^u : U_0 \cap U_1 \rightarrow Z \quad (1 \leq u \leq 2)$$

such that at $u = 2$, $a_y^2 = b_y^2$ over $U_0 \cap U_1$ for each $y \in [0, 1]^k$; hence these two sections define a single holomorphic section $f_y^2 : U_0 \cup U_1 \rightarrow Z$. Of course we shrink the sets U_0 and U_1 again when gluing. Moreover, both homotopies will be fixed on Y .

Consider the first step. Since $A_0 \cap A_1$ is Runge in A_1 and the submersion $h : Z \rightarrow X$ admits a spray over a neighborhood of A_1 , the h-Runge theorem with parameters (Theorem 4.2 in [FP1] and the remark following it) shows that, after shrinking the sets U_0 and U_1 , there exists a homotopy of holomorphic sections $g_{y,t}^s : U_0 \cap U_1 \rightarrow Z$ ($0 \leq s \leq 1$), depending continuously on t, s, y , satisfying

- (i) $g_{y,t}^0 = c_{y,t}$ for each y and t ,
- (ii) $g_{y,1}^s = c_{y,1} = b_y | U_0 \cap U_1$ for each s and y ,
- (iii) $g_{y,t}^1$ extends to a holomorphic section over U_1 for each y and t ,
- (iv) the homotopy is fixed on Y , i.e., for $y \in Y$ we have $g_{y,t}^s = c_y | U_0 \cap U_1$ for each s and t , and
- (v) $g_{y,t}^s$ approximates $c_{y,t}$ in a neighborhood of $A_0 \cap A_1$ as close as desired, uniformly with respect to all parameters.

We define $f_*^u = (a_*^u, b_*^u, c_*^u)$ for $0 \leq u \leq 1$ by

$$a_y^u = a_y, \quad b_y^u = g_{y,1-u}^1,$$

$$c_{y,t}^u = \begin{cases} c_{y,t} & \text{if } 0 \leq t \leq 1 - u; \\ g_{y,1-u}^{(t+u-1)/u} & \text{if } 1 - u < t \leq 1. \end{cases}$$

The reader will easily verify that this satisfies all required properties. In particular, at $u = 1$, $b_y^1 = g_{y,0}^1$ approximates $c_{y,0} = a_y$ in a neighborhood of $A_0 \cap A_1$.

Next we apply Theorem 5.5 in [FP1] (on gluing parametrized families of holomorphic sections over Cartan pairs) to get homotopies of sections $a_y^u : U_0 \rightarrow Z$ and $b_y^u : U_1 \rightarrow Z$ for $1 \leq u \leq 2$ such that a_y^u approximates $a_y^1 = a_y$ on A_0 for each $u \in [1, 2]$ and $a_y^2 = b_y^2$ in $U_0 \cap U_1$. Moreover, over a neighborhood of the set A_0 , the graphs of all sections a_y^u, b_y^u ($1 \leq u \leq 2$) and $c_{y,t}^1$ ($0 \leq t \leq 1$) lie in small neighborhood of the (graph of) the section a_y in Z , and such a neighborhood is biholomorphic to a neighborhood of the zero section in a holomorphic vector bundle over U_0 (Lemma 5.3 in [FP1]). Using the resulting vector bundle structure

on this neighborhood, we see that the triangle of homotopies formed by these families is contractible, meaning that it can be filled by a 2-parameter homotopy $c_{y,t}^u$ ($0 \leq t \leq 1, 1 \leq u \leq 2$) over a neighborhood of $A_0 \cap A_1$. This completes the Proof of Proposition 5.1 for $n = 1$.

The induction step $n \Rightarrow n + 1$. Suppose that Proposition 5.1 holds for all Cartan strings of length $n + 1$ for some $n \geq 1$ and for all $k \geq 0$. Let $\mathcal{A} = (A_0, \dots, A_{n+1})$ be a Cartan string of length $n + 2$ with the nerve $\mathcal{K}^{n+1} = \mathcal{K}(\mathcal{A})$, and let f_* be a holomorphic (\mathcal{K}^{n+1}, k) -prism with values in Z which is sectionally constant on a nice compact subset $Y \subset [0, 1]^k$. Let $\mathcal{K}^n = \mathcal{K}(A_0, \dots, A_n) \subset \mathcal{K}^{n+1}$. The proof consists of the following three steps, each of which is accomplished by constructing a suitable homotopy of prisms.

- Step 1:* Reduction to the case when $f_*|_{\mathcal{K}^n}$ is a sectionally constant prism.
- Step 2:* Reduction to the case when f_* represents a $(k + 1)$ -prism over the Cartan pair (A^n, A_{n+1}) , where $A^n = A_0 \cup A_1 \cup \dots \cup A_n$.
- Step 3:* Applying the case $n = 1$ to the prism in Step 2 to get a sectionally constant (\mathcal{K}^{n+1}, k) -prism.

We begin by some general considerations. We denote the coordinates on \mathbb{R}^{n+1} by $t = (t', t_{n+1})$, where $t' = (t_1, \dots, t_n) \in \mathbb{R}^n$, and identify \mathbb{R}^n with the coordinate hyperplane $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. The body K^{n+1} of the nerve \mathcal{K}^{n+1} can be represented as the union of certain faces of the standard simplex $\Delta^{n+1} \subset \mathbb{R}^{n+1}$. (In fact $K^{n+1} = \Delta^{n+1}$ if and only if $A_0 \cap A_1 \cap \dots \cap A_{n+1} \neq \emptyset$.) The body $K^n \subset \mathbb{R}^n$ of the subcomplex $\mathcal{K}^n = \mathcal{K}(A_0, \dots, A_n) \subset \mathcal{K}^{n+1}$ is precisely the base $K^{n+1} \cap \{t_{n+1} = 0\}$ of K^{n+1} . We shall also need the complex

$$K_1^n = \mathcal{K}(A_0 \cap A_{n+1}, \dots, A_n \cap A_{n+1}) \subset K^n. \tag{5.1}$$

Note that $K_1^n = \{J \in \mathcal{K}^n : (J, n + 1) \in \mathcal{K}^{n+1}\}$. Its body K_1^n is a subset of K^n which equals $(K^{n+1} \setminus K^n) \cap K^n$. Moreover, for each $0 < s < 1$, the section $K^{n+1} \cap \{t_{n+1} = s\}$ is homeomorphic to K_1^n . The map

$$r : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}, \quad r(t, s) = (t(1 - s), s) \quad (t \in \mathbb{R}^n, s \in \mathbb{R})$$

maps the prism $\Delta^n \times [0, 1] \subset \mathbb{R}^{n+1}$ onto the standard simplex Δ^{n+1} (it is homeomorphic outside the level $s = 1$), and it maps $K_1^n \times \{s\}$ homeomorphically onto the section $K^{n+1} \cap \{t_{n+1} = s\}$ for each $s \in (0, 1)$.

Proof of Step 1. Since $\tilde{f}_*^0 = f_*|_{\mathcal{K}^n} = \{f_J : J \in \mathcal{K}^n\}$ is a k -prism over a Cartan string of length $n + 1$, the induction hypothesis provides a holomorphic homotopy $\tilde{f}_*^u = \{\tilde{f}_*^u : u \in [-1, 0]\}$ such that each \tilde{f}_*^u is a (\mathcal{K}^n, k) -prism, the homotopy is fixed for all $y \in Y$, and the prism \tilde{f}_*^{-1} is sectionally constant. The parameter space of the prism f_* is $K^{n+1} \times [0, 1]^k$ while the parameter space of \tilde{f}_* is $K' \times [0, 1]^k$, where

$$K' = \{(t', u) \in \mathbb{R}^n \times \mathbb{R} : t' \in K^n, -1 \leq u \leq 0\}.$$

Note that f_* and \tilde{f}_* agree on the intersections of their domains $K^n \times [0, 1]^k$ and hence define a collection of sections, parametrized by the set $L \times [0, 1]^k$, where $L = K^{n+1} \cup K' \subset \mathbb{R}^{n+1}$. We denote this collection by $\{g_y(t) : t \in L, y \in [0, 1]^k\}$.

For each $s \in [0, 1]$ we denote by $L_s \subset \mathbb{R}^{n+1}$ the set

$$L_s = (K^{n+1} \setminus K^n) \cup \{(t', t_{n+1}) : t' \in K_1^n, -s \leq t_{n+1} \leq 0\} \cup \{(t', -s) : t' \in K^n\}.$$

Intuitively speaking, L_s is obtained by pushing the base K^n of K^{n+1} for s units in the negative t_{n+1} direction and then adding to this the vertical sides $K_1^n \times [-s, 0]$. Clearly $L_0 = K^{n+1}$, and L_s is homeomorphic to K^{n+1} for each $s \in [0, 1]$. In fact, there is a continuous family of homeomorphisms $\Theta_s : K^{n+1} \rightarrow L_s$ ($0 \leq s \leq 1$) such that Θ_0 is the identity, each Θ_s preserves the top vertex $(0, \dots, 0, 1) \in K^{n+1}$ and the cellular structure of the two sets, and Θ_s maps K^n (the base of K^{n+1}) onto $K^n \times \{-s\}$ (the base of L_s) by a downward shift for s units. By ‘respecting the cellular structure’ we mean the following. Each face $J \in \mathcal{K}_1^n$ determines a face $\tilde{J} = (J, n + 1) \in \mathcal{K}^{n+1}$, and Θ_s maps its body $|\tilde{J}| \subset K^{n+1}$ onto the set $|\tilde{J}| \cup \{(t', t_{n+1}) : t' \in |J|, -s \leq t_{n+1} \leq 0\} \subset L_s$. Using this family we define a homotopy H_*^u ($0 \leq u \leq 1$) of (\mathcal{K}^{n+1}, k) -prisms by

$$H_J^u(t) = g_y(\Theta_u(t)) \quad (J \in \mathcal{K}^{n+1}, t \in |J| \subset K^{n+1}).$$

Clearly $H_*^0 = f_*$ and $H_*^1|_{\mathcal{K}^n} = \tilde{f}_*^{-1}|_{\mathcal{K}^n}$ is sectionally constant. This proves Step 1.

Proof of Step 2: By Step 1 we may assume that the prism f_* is such that $f_*|_{\mathcal{K}^n}$ is sectionally constant. The next step is to modify f_* by a homotopy of holomorphic prisms into another prism which is sectionally constant also in the direction of the last variable t_{n+1} . Let \mathcal{K}_1^n be the complex (5.1). We associate to f_* a holomorphic $(\mathcal{K}_1^n, k + 1)$ -prism

$$F_* = \{F_{J,(y,s)} : |J| \rightarrow \mathcal{O}_h(U_{(J,n+1)}, Z), \quad J \in \mathcal{K}_1^n, y \in [0, 1]^k, s \in [0, 1]\}$$

where

$$F_{J,(y,s)}(t) = f_{(J,n+1),y}(r(t, s)), \quad (t \in |J|, y \in [0, 1]^k, s \in [0, 1]).$$

The set

$$Y_1 = (Y \times [0, 1]) \cup ([0, 1]^k \times \{0, 1\}) \subset [0, 1]^{k+1}$$

is a nice compact subset of $[0, 1]^{k+1}$ (def. 1.4). Since $f_*|_{\mathcal{K}^n}$ is sectionally constant, the prism F_* (which is associated to the complex \mathcal{K}_1^n of length $n + 1$) satisfies the induction hypothesis with respect to the set Y_1 . Hence there is a homotopy F_*^u ($u \in [0, 1]$) of holomorphic $(\mathcal{K}_1^n, k + 1)$ -prisms, beginning with $F_*^0 = F_*$, such that the homotopy is fixed for $(y, s) \in Y_1$ and such that F_*^1 is sectionally constant. This means that for each fixed $(y, s) \in [0, 1]^{k+1}$ the \mathcal{K}_1^n -complex $F_{*,(y,s)}^1$

is constant, i.e., it represents a holomorphic section $F_y^1(s): V \rightarrow Z$ of $Z \rightarrow X$ over an open set $V \supset A^n \cap A_{n+1}$. In particular, since the homotopy is fixed for $s = 0$ and $s = 1$, the section $F_y^1(0)$ coincides with the section represented by the constant complex $f_{*,y}|K^n$, and the section $F_y^1(1)$ coincides with the section $f_{(0,\dots,0,1),y}$ associated to the set A_{n+1} .

Proof of Step 3: We may consider the family of sections

$$F_*^1 = \{F_y^1(s): V \rightarrow Z \mid s \in [0, 1], y \in [0, 1]^k\},$$

obtained in Step 2, as a holomorphic k -prism over the complex $K' = K(A^n, A_{n+1})$ determined by the Cartan pair (A^n, A_{n+1}) . Here the parameter $s \in [0, 1]$ represents the variable in the body $|K'| = [0, 1]$. For each $y \in [0, 1]^k$ the section $F_y^1(0)$ extends holomorphically to a neighborhood of A^n and $F_y^1(1)$ extends to a neighborhood of A_{n+1} .

The case $n = 1$ of Proposition 5.1 gives us a homotopy G_*^u ($0 \leq u \leq 1$) of holomorphic (K', k) -prisms such that $G_*^0 = F_*^1$, G_y^1 is a constant K' -complex for each $y \in [0, 1]^k$ (i.e., a holomorphic section over an open neighborhood of $A^{n+1} = A_0 \cup \dots \cup A_{n+1}$), and the homotopy is fixed for $y \in Y$ (where $G_y^0 = F_y^1$ is already a section over A^{n+1}). Moreover, for each $u \in [0, 1]$ the section $G_y^u(0)$ (which is holomorphic over a neighborhood of A^n) approximates the section $F_y^1(0) = f_{*,y}|K^n$ on the set A^n .

We can interpret the families $\{F_*^u: 0 \leq u \leq 1\}$ and $\{G_*^u: 0 \leq u \leq 1\}$ as homotopies of holomorphic (K^{n+1}, k) -prisms. By connecting these homotopies F_*^u and G_*^u (in this order) we obtain a homotopy f_*^u ($0 \leq u \leq 1$) of holomorphic (K^{n+1}, k) -prisms, beginning at $u = 0$ with f_* and ending at $u = 1$ with the sectionally constant prism G_*^1 .

If we assume in addition that the restriction $f_*|K^n$ is sectionally constant on $[0, 1]^k$ (so $f_{*,y}|K^n$ is a holomorphic section in a neighborhood of A^n for each $y \in [0, 1]^k$), we can skip the initial step in the proof of the inductive step. Note that, by construction, the restriction $F_*^u|K^n$ is independent of $u \in [0, 1]$ (since the homotopy F_*^u is fixed on the set Y_1), and the homotopy G_*^u is such that the complex $G_{*,y}^u|K^n$ is represented by a holomorphic section in a neighborhood of A^n which approximates the section $F_{*,y}^1|K^n = f_{*,y}|K^n$ on A^n , uniformly with respect to $u \in [0, 1]$ and $y \in [0, 1]^k$. Hence the section $f_{*,y}^u|K^n$ approximates $f_{*,y}|K^n$ on A^n , the approximation being uniform with respect to $u \in [0, 1]$ and $y \in [0, 1]^k$. This concludes the proof of the induction step. □

The next proposition shows that a holomorphic 1-prism can be extended from a finite subcomplex to the entire complex such that the 0-level of the prism matches a given complex. This does not require any analytic tools and hence the result applies to any locally finite family \mathcal{A} .

5.2 Proposition. *Let $\mathcal{A} = \{A_0, A_1, A_2, \dots\}$ be a locally finite family of compact sets in a complex manifold X . Denote its nerve by $K(\mathcal{A})$ and let $K^n =$*

$\mathcal{K}(A_0, \dots, A_n) \subset \mathcal{K}(\mathcal{A})$ for each $n \in \mathbb{N}$. Assume that $h: Z \rightarrow X$ is a holomorphic submersion onto X , f_* is a holomorphic $\mathcal{K}(\mathcal{A})$ -complex with values in Z , and g_* is a holomorphic $(\mathcal{K}^n, 1)$ -prism for some $n \in \mathbb{N}$ such that $g_{*,0} = f_*$. Then there exists a holomorphic $(\mathcal{K}(\mathcal{A}), 1)$ -prism G_* such that $G_{*,0} = f_*$ and $G_*|_{\mathcal{K}^n} = g_*$.

Remark. A similar result holds if f_* is a holomorphic $(\mathcal{K}(\mathcal{A}), k)$ -prism and g_* is a holomorphic $(\mathcal{K}^n, k + 1)$ -prism with base $f_*|_{\mathcal{K}^n}$: such g_* extends to a holomorphic $(\mathcal{K}(\mathcal{A}), k + 1)$ -prism G_* .

Proof. We choose representatives of f_* and g_* defined on a faithful open neighborhood \mathcal{U} of \mathcal{A} (see def. 3.2). Write $A^n = A_0 \cup \dots \cup A_n$ as before. Let $m \geq n$ be the smallest integer such that $A_k \cap A^n = \emptyset$ for each $k \geq m$. We represent the set $K^m = K(A_0, \dots, A_m)$ (the body of the subcomplex $\mathcal{K}^m \subset \mathcal{K}(\mathcal{A})$) as a subset of \mathbb{R}^m . We denote the coordinates on \mathbb{R}^{m+1} by (t, s) , with $t \in \mathbb{R}^m$ and $s \in \mathbb{R}$, and identify \mathbb{R}^m with $\mathbb{R}^m \times \{0\} = \{s = 0\} \subset \mathbb{R}^{m+1}$. Similarly we identify a set $K \subset \mathbb{R}^m$ with $K \times \{0\} \subset \mathbb{R}^{m+1}$ and write $K \times [0, 1] = \{(t, s) : t \in K, s \in [0, 1]\}$. For each face $J \in \mathcal{K}^m$ we denote by $b|J| \subset K^m$ the boundary of its body $|J|$.

5.3 Lemma. *There exists a retraction*

$$r: K^m \times [0, 1] \rightarrow K^m \cup (K^n \times [0, 1]) \subset \mathbb{R}^{m+1}$$

such that for each face $J \in \mathcal{K}^m \setminus \mathcal{K}^n$ we have

- (i) $r(|J| \times [0, 1]) \subset |J| \cup (b|J| \times [0, 1])$,
- (ii) if $|J| \cap K^n = \emptyset$ then $r(t, s) = t$ for each $t \in |J|$ and $s \in [0, 1]$.

Proof. We first define r over those faces $J \in \mathcal{K}^m$ for which either $|J| \subset K^n$ (we let r be the identity on $|J| \times [0, 1]$) or $|J| \cap K^n = \emptyset$ (we let $r(t, s) = t$ for $t \in |J|$). We also define r to be the identity map on the bottom side $K^m = K^m \times \{0\}$. On the remaining faces $J \in \mathcal{K}^m$ we define r inductively with respect to the dimension of J . Suppose that r has already been defined on all faces of dimension $< k$ and let $J = (j_0, \dots, j_k) \in \mathcal{K}^m$. Then r is defined on $|J| \cup (b|J| \times [0, 1])$ and satisfies (i), and it satisfies (ii) on those sides of $b|J|$ which are disjoint from K^n . Moreover, r is the identity on $|J| = |J| \times \{0\}$. It is now clear that r extends from $|J| \cup (b|J| \times [0, 1])$ to $|J| \times [0, 1]$ so that (i) holds. □

Let r be as in Lemma 5.3. Write $r(t, s) = (r_0(t, s), u(t, s))$, where $r_0(t, s) \in K^m$ and $u(t, s) \in [0, 1]$. We define a holomorphic $(\mathcal{K}^m, 1)$ -prism G_* by setting for each $J \in \mathcal{K}^m$, $t \in |J|$ and $s \in [0, 1]$

$$G_{J,s}(t) = \begin{cases} f_J(r_0(t, s)) & \text{if } u(t, s) = 0; \\ g_{J,u(t,s)}(r_0(t, s)) & \text{if } u(t, s) > 0. \end{cases}$$

Property (i) in Lemma 5.3 implies that the section $G_{J,s}(t)$ for $t \in |J|$ is defined (and holomorphic) in the set U_J (it may be holomorphic in a larger set if $r_0(t, s) \in$

$b|J|$, but in such case we restrict it to U_J). The collection $G_* = \{G_{J,s} : J \in \mathcal{K}^m, s \in [0, 1]\}$ is then a holomorphic $(\mathcal{K}^m, 1)$ -prism which extends g_* and satisfies $G_{*,0} = f_*$.

The property (ii) of the retraction r lets us extend G_* to a prism over the entire complex $\mathcal{K}(\mathcal{A})$ by observing that for those faces $J \in \mathcal{K}(\mathcal{A})$ which do not belong to \mathcal{K}^m we have $|J| \cap K^n = \emptyset$ (by definition of m) and therefore $r(t, s) = t$ for $t \in |J| \cap K^m$. Thus we can simply take $G_{J,s}(t) = f_J(t)$ for $t \in |J|$ and $s \in [0, 1]$. This completes the Proof of Proposition 5.2. □

6 Proof of Theorem 1.5

In this section we prove Theorem 1.5 using the tools developed in Sect. 5. We shall concentrate on the case of a single section; the proof in the general parametric case is essentially the same. Thus, we are given a continuous section $a : X \rightarrow Z$ which is holomorphic in an open set $U_0 \supset K$, where $K \subset X$ is $\mathcal{H}(X)$ -convex; our goal is to construct a homotopy $H_s : X \rightarrow Z$ ($0 \leq s \leq 1$) of continuous sections such that $H_0 = a$, the section H_1 is holomorphic on X , and for each $s \in [0, 1]$ the section H_s is holomorphic near K and it approximates a on K .

Let $\mathcal{A} = \{A_0, A_1, \dots\}$ be a Cartan covering of X given by Theorem 4.6 such that $K \subset A_0 \subset U_0$ and $K \cap A_i = \emptyset$ for $i \geq 1$. Also let $a_* = \{a_{*,s} : 0 \leq s \leq 1\}$ be a continuous $(\mathcal{K}(\mathcal{A}), 1)$ -prism provided by Proposition 4.7. Thus the complex $a_{*,0}$ is constant and represents the section a , $a_{*,1}$ is a holomorphic $\mathcal{K}(\mathcal{A})$ -complex, and $a_{(0),s} = a|_{U_0}$ for each $s \in [0, 1]$.

Let d be a complete metric on Z compatible with the manifold topology. Fix an $\epsilon > 0$. We inductively construct a sequence of holomorphic $\mathcal{K}(\mathcal{A})$ -complexes f_*^n and holomorphic $(\mathcal{K}(\mathcal{A}), 1)$ -prisms $G_*^n = \{G_{*,s}^n : 0 \leq s \leq 1\}$ for $n = 0, 1, 2, \dots$, satisfying the following properties:

- (a) $f_*^0 = a_{*,1}$,
- (b) $G_{*,0}^n = f_*^n$ and $G_{*,1}^n = f_*^{n+1}$ for each $n \in \mathbf{Z}_+$,
- (c) for each $k \in \mathbf{Z}_+, n \geq k$ and $s \in [0, 1]$ the complexes $f_*^n|_{\mathcal{K}^k}$ and $G_{*,s}^n|_{\mathcal{K}^k}$ are constant, i.e., they are given by holomorphic sections denoted by f^n resp. G_s^n in an open neighborhood of $A^k = A_0 \cup \dots \cup A_k$,
- (d) (approximation) for each $n \in \mathbf{Z}_+$ and $s \in [0, 1]$ we have

$$d(G_s^n(x), f^n(x)) < \epsilon/2^{n+1} \quad (x \in A^n).$$

In particular we have $d(f^{n+1}(x), f^n(x)) < \epsilon/2^{n+1}$ for $x \in A^n$.

In (d) we are using the notation for sections established in (c). The property (d) implies that the sequence of sections $f^n : A^n \rightarrow Z$ ($n = 0, 1, 2, \dots$) converges uniformly on compacts in X to a holomorphic section $f^\infty = \lim_{n \rightarrow \infty} f^n$:

$X \rightarrow Z$ which satisfies

$$d(f^\infty(x), a(x)) = d(f^\infty(x), f^0(x)) < \epsilon \quad (x \in A_0).$$

To construct a homotopy $H_s: X \rightarrow Z$ ($0 \leq s \leq 1$) between $H_0 = a$ and $H_1 = f^\infty$ we first construct a continuous $(\mathcal{K}(\mathcal{A}), 1)$ -prism h_* such that $h_{*,0} = a$ and $h_{*,1} = f^\infty$. To do this we simply collect all individual 1-prisms a_* and G_*^n ($n \in \mathbf{Z}_+$) into a single 1-prism as follows. For each $n \in \mathbf{Z}_+$ set $I_n = [1 - 2^{-n}, 1 - 2^{-n-1}]$ and let $\lambda_n: I_n \rightarrow [0, 1]$ be the linear bijection $\lambda_n(s) = 2^{n+1}(s - 1 + 2^{-n})$. Then $\cup_{n=0}^\infty I_n = [0, 1)$. For $s \in [0, 1)$ we define

$$h_{*,s} = \begin{cases} a_{*,2s} & \text{if } s \in I_0 = [0, 1/2]; \\ G_{*,\lambda_n(s)}^{n-1} & \text{if } s \in I_n, n \geq 1. \end{cases}$$

The two definitions of $h_{*,s}$ at the values $s = 1 - 2^{-n}$ ($n \in \mathbf{Z}_+$) are compatible by (b). Properties (c) and (d) imply that $\lim_{s \rightarrow 1} h_{*,s} = f^\infty$ uniformly on compacts in X . Indeed, each compact set $L \subset\subset X$ is contained in some A^m , and for $n \geq m$ the complex $G_{*,s}^n$ is constant on A^n , i.e., represented by a holomorphic section. Hence for $1 - 2^{-n-1} \leq s < 1$ the complex $h_{*,s}|_{\mathcal{K}^n}$ is a holomorphic section in a neighborhood of A^n . As $s \rightarrow 1$, these sections converge uniformly on A^n (and hence on L) to f^∞ . This proves that, if we set $h_{*,1} = f^\infty$, the collection $h_* = \{h_{*,s}: 0 \leq s \leq 1\}$ is indeed a continuous $(\mathcal{K}(\mathcal{A}), 1)$ -prism. Notice also that the restriction of h_* to A_0 (more precisely, to the complex $\mathcal{K}(A_0)$ represented by the first set A_0) is in fact a homotopy of holomorphic sections h_s ($0 \leq s \leq 1$) in a neighborhood of A_0 , connecting $h_0 = a$ to $h_1 = f^\infty$ and such that all sections in the family satisfy $d(h_s(x), a(x)) < \epsilon$ for $x \in A_0$.

To complete the Proof of Theorem 1.5 we apply the version of Proposition 5.1 for continuous prisms to modify the 1-prism h_* by a homotopy of 1-prisms (keeping the ends $s = 0$ and $s = 1$ fixed) into a 1-prism H_* which is *sectionally constant*, i.e., such that H_* represents a homotopy of continuous sections $\{H_s: X \rightarrow Z: 0 \leq s \leq 1\}$. Moreover we can achieve that in a neighborhood of A_0 the two sections H_s and h_s agree.

This concludes the Proof of Theorem 1.5 in the case without parameters. The parametric case is proved by the same tools by introducing the parameter space P into the definition of holomorphic (and continuous) complexes and prisms and repeating the same arguments in this setting. The analytic tools used in the proof, namely the h-Runge theorem and the gluing theorem, have been established in this generality in [FP1].

A final remark. We have described a procedure in which the final holomorphic section of $Z \rightarrow X$ is obtained as a locally uniform limit of holomorphic sections, defined over increasingly larger compacts in X . Often one need less: To modify a given continuous section, which is holomorphic over a Stein compactum $K \subset X$, into another section that is holomorphic over a larger Stein compactum $L \supset K$

in X . Our proof accomplishes this in a *finite number of steps*, provided that L is a strongly pseudoconvex extension of K . This means that K and L are two regular sublevel sets of a smooth strongly plurisubharmonic function defined in a neighborhood of $\bar{L} \setminus \bar{K}$. In such case the construction in [HL] gives a finite Cartan string (A_0, A_1, \dots, A_n) in X with $A_0 = K$ and $A^n = \bigcup_{0 \leq j \leq n} A_j = L$. It suffices to apply our proof on this finite string.

7 Proof of Theorem 1.6

Proof of Theorem 1.6 (a). For each nonzero vector $v \in \mathbb{C}^q$ we denote by $[v] \in \mathbb{C}\mathbb{P}^{q-1}$ the complex line in \mathbb{C}^q spanned by v , and we denote by $\pi_v: \mathbb{C}^q \rightarrow v^\perp = \mathbb{C}^{q-1}$ the orthogonal projection onto v^\perp with $\pi(v) = 0$.

7.1 Lemma. (Existence of sprays.) *Let $U \subset \mathbb{C}^n$ be an open set ($n \geq 1$) and let $\Sigma \subset U \times \mathbb{C}^q$ for $q \geq 2$ be a closed analytic subset such that each fiber $\Sigma_x = \{w \in \mathbb{C}^q : (x, w) \in \Sigma\}$ has complex codimension at least two in \mathbb{C}^q (it may be empty). Assume that there exists a nonempty open set $\Omega \subset \mathbb{C}\mathbb{P}^{q-1}$ such that for each $[v] \in \Omega$ the linear projection $\tilde{\pi}_v: U \times \mathbb{C}^q \rightarrow U \times \mathbb{C}^{q-1}$,*

$$\tilde{\pi}_v(x, w) = (x, \pi_v(w)) \quad (x \in U, w \in \mathbb{C}^q),$$

is proper when restricted to Σ . Then the submersion $h: (U \times \mathbb{C}^q) \setminus \Sigma \rightarrow U$, $h(x, w) = x$, admits a spray.

Observe that condition (1.1) in part (a) of Theorem 1.6 implies that for each vector $v \in \mathbb{C}^q$ sufficiently close to $e_q = (0, \dots, 0, 1)$, the projection π_v is proper on Σ . Granted Lemma 7.1 we get a spray on $\mathbb{C}^q \setminus \Sigma$, and hence part (a) of Theorem 1.6 follows from Theorem 1.2.

Proof of Lemma 7.1. We denote the coordinates on $U \times \mathbb{C}^q$ by $z = (x, w)$. Fix a point $z_0 = (x_0, w_0) \in (U \times \mathbb{C}^q) \setminus \Sigma$. For each line $[v] \in \Omega$ such that the affine complex line $\{w_0 + tv : t \in \mathbb{C}\} \subset \mathbb{C}^q$ does not intersect Σ_{x_0} , the projection $\tilde{\pi}_v$ satisfies $\tilde{\pi}_v(z_0) \notin \tilde{\pi}_v(\Sigma)$. By the codimension condition on Σ this holds for all directions $[v]$ outside a proper subvariety of Ω . For each such $[v]$ there exists a holomorphic function $g: U \times \mathbb{C}^{q-1} \rightarrow \mathbb{C}$ which vanishes on the subvariety $\tilde{\pi}_v(\Sigma)$ and equals one at the point $\tilde{\pi}_v(z_0)$. The holomorphic vector field $V(z) = g(\tilde{\pi}_v(z))v$ (a shear field) is \mathbb{C} -complete and vertical on $U \times \mathbb{C}^q$, with the flow $\theta_t(z) = z + tg(\tilde{\pi}_v(z))v$ ($t \in \mathbb{C}$). By the choice of g the field V vanishes on Σ , and hence its restriction to $(U \times \mathbb{C}^q) \setminus \Sigma$ is also a complete field. We have $V(z_0) = v$.

This shows that the complete vertical fields on $(U \times \mathbb{C}^q) \setminus \Sigma$ generate the vertical tangent space at each point. It remains to see that we can do the same with finitely many fields. We may assume that U is connected. We begin by choosing complete vertical fields V_1, \dots, V_q as above which generate $VT_z(U \times \mathbb{C}^q)$ at

one point of $(U \times \mathbb{C}^q) \setminus \Sigma$. Hence the same fields generate the tangent space at each point outside a proper analytic subset $A \subset (U \times \mathbb{C}^q) \setminus \Sigma$. Let $A = \cup_j A_j$ be the (finite or countable) decomposition of A into irreducible components. Choose a point $z_j \in A_j$ in each component and consider the set $\Omega_j \subset \Omega$ of all complex directions $[v] \in \Omega$ for which v belongs to the linear span of the vectors $V_k(z_j)$ ($1 \leq k \leq q$) or $\tilde{\pi}_v(z_j) \in \tilde{\pi}_v(\Sigma)$. Clearly Ω_j is a proper analytic subset of Ω and hence $\cup_j \Omega_j$ is a set of the first category in Ω . Choose any direction $[v] \in \Omega \setminus \cup_j \Omega_j$. By construction we then have $\tilde{\pi}_v(z_j) \notin \tilde{\pi}_v(\Sigma)$ for each j . Let g_1, \dots, g_k be holomorphic functions in $U \times \mathbb{C}^{q-1}$ whose common zero set is precisely $\tilde{\pi}_v(\Sigma)$. If we add the corresponding complete vertical vector fields $W_i(z) = g_i(\tilde{\pi}_v(z))v$ for $i = 1, \dots, k$ to the previous collection V_1, \dots, V_q , we increase the dimension of the linear span by at least one at each point z_j , and hence at each point outside a proper subvariety of each irreducible component A_j of A . An induction on dimensions of the span and of the exceptional set completes the Proof of Lemma 7.1.

Proof of Theorem 1.6 (b). According to Rosay and Rudin [RRu] there is for each $q \in \mathbb{N}$ a discrete set $\Sigma \subset \mathbb{C}^q$ such that any entire holomorphic map $f: \mathbb{C}^n \rightarrow \mathbb{C}^q \setminus \Sigma$ has complex rank at most $q - 1$ at each point. The same is then true for holomorphic maps $f: X \rightarrow \mathbb{C}^q \setminus \Sigma$ from any Stein manifold X whose universal cover is a Euclidean space, such as $X = (\mathbb{C}^*)^n$. We claim that any such map is holomorphically homotopic to a constant map of X into $\mathbb{C}^q \setminus \Sigma$. To see this, note that the rank condition on f implies that $f(X)$ is contained in a countable union $A = \cup_{j=1}^\infty A_j \subset \mathbb{C}^q \setminus \Sigma$, where each A_j is a compact subset contained in a local analytic set of complex dimension $\leq q - 1$ in \mathbb{C}^q [Chi]. We denote by $C_z(A_j)$ the real cone on A_j with vertex at $z \in \mathbb{C}^q$. Since Σ is discrete and $C_z(A_j)$ has dimension at most $2q - 1$, we have $C_z(A_j) \cap \Sigma = \emptyset$ for an open and dense set of points $z \in \mathbb{C}^q$. By Baire’s theorem we can choose $z \in \mathbb{C}^q \setminus \Sigma$ such that the above holds for all j and hence $C_z(A) \cap \Sigma = \emptyset$. The homotopy $f_t(x) = tz + (1 - t)f(x)$ ($0 \leq t \leq 1$) contracts the initial map $f = f_0$ to the constant map $f_1(x) = z$ in $\mathbb{C}^q \setminus \Sigma$, establishing our claim.

On the other hand, when $n = 2q - 1$, there exist continuous (even real-analytic) maps $f: X = (\mathbb{C}^*)^n \rightarrow \mathbb{C}^q \setminus \Sigma$ which are not homotopic to constant: we contract $(\mathbb{C}^*)^n$ onto the torus T^n and embed T^n as a real hypersurface in $\mathbb{C}^q \setminus \Sigma$ so that at least one point of Σ is contained in the bounded component of $\mathbb{C}^q \setminus T^n$. Thus the Oka principle fails for maps $(\mathbb{C}^*)^{2q-1} \rightarrow \mathbb{C}^q \setminus \Sigma$.

Proof of Theorem 1.6 (c). In [BFo] and [Fo1] it was shown that for any pair of integers $1 \leq k < q$ there exist proper holomorphic embeddings $g: \mathbb{C}^k \rightarrow \mathbb{C}^q$ such that every entire map $f: \mathbb{C}^n \rightarrow D = \mathbb{C}^q \setminus g(\mathbb{C}^k)$ has complex rank $< q - k$ at each point. The same proof as in case (b) shows that, if X is any Stein manifold which is universally covered by a Euclidean space, any holomorphic map $X \rightarrow D = \mathbb{C}^q \setminus g(\mathbb{C}^k)$ is contractible to a point in D , while for some such X there exist

nontrivial real-analytic maps into D . For instance, choose a point $z \in g(\mathbb{C}^k)$ and let $\Lambda \subset \mathbb{C}^q$ be the normal plane to $g(\mathbb{C}^k)$ at z (of complex dimension $q - k$). We can embed the torus T^n , with $n = 2(q - k) - 1$, into Λ so that z is contained in the bounded component of $\Lambda \setminus T^n$ in Λ . Since T^n is a retraction of $(\mathbb{C}^*)^n$, we get a map $(\mathbb{C}^*)^{2(q-k)-1} \rightarrow D$ which is not zero homotopic in D . \square

8 Sections of vector bundles avoiding analytic subsets

Proof of Theorem 1.7. If $X_0 = \emptyset$, Theorem 1.7 follows from Theorem 1.5 and Lemma 7.1 which may be seen as follows. To apply Theorem 1.5 we need to show that for each $x \in X \setminus K$ there is a neighborhood $U \subset X$ of x such that the submersion $\pi^{-1}(U) \setminus \Sigma \rightarrow U$ admits a spray. Let U and Φ be chosen as in Theorem 1.5. Set $U' = \pi^{-1}(U)$ and $\Sigma' = \Phi(\Sigma \cap U') \subset U \times \mathbb{C}^q$. Condition (1.2) in Theorem 1.7 implies that for each $v \in \mathbb{C}^q$ sufficiently close to $e_q = (0, \dots, 0, 1)$, the projection $\tilde{\pi}_v: U \times \mathbb{C}^q \rightarrow U \times \mathbb{C}^{q-1}$, given by $\tilde{\pi}_v(x, w) = (x, \pi_v(w))$, is proper on Σ' . By Lemma 7.1 the submersion $(U \times \mathbb{C}^q) \setminus \Sigma' \rightarrow U$ admits a spray s' defined on a trivial bundle over the given set. Then $s = \Phi^{-1} \circ s'$ is a spray associated to $U' \setminus \Sigma \rightarrow U$ and hence Theorem 1.5 applies.

Interpolation along X_0 needs additional work. A direct approach would require modifications in the approximation theorems and in the gluing lemma in [FP1]. While it would be possible and not overly difficult to carry out these modifications, we present here an alternative approach which requires patching only on subsets that do not intersect $X_0 \cup K$.

The section $f_0: X \rightarrow V$ is assumed to be holomorphic in a neighborhood of $X_0 \cup K$. This set has a basis of Stein neighborhoods.

8.1 Lemma. (Notation as above.) *We can write f_0 in the form*

$$f_0 = \phi + \sum_{j=1}^m h_j g_j^0, \tag{8.1}$$

where $\phi: X \rightarrow V$ is global holomorphic section of V , $g_j^0: X \rightarrow V$ are continuous sections which are holomorphic in an open set $U_0 \supset X_0 \cup K$, and the functions $h_1, \dots, h_m \in \mathcal{H}(X)$ vanish to order k on X_0 and satisfy $X_0 = \{x \in X: h_j(x) = 0, 1 \leq j \leq m\}$. (The graph of ϕ may intersect Σ outside a neighborhood of X_0 .)

Proof. This is a straightforward application of the Oka-Cartan theory, but we include the proof for completeness. We begin by choosing finitely many functions $h_1, \dots, h_m \in \mathcal{H}(X)$ with the stated properties. We denote by $\mathcal{O} = \mathcal{O}_X$ the sheaf of germs of holomorphic functions on X and by \mathcal{V} the sheaf of germs of holomorphic sections of $V \rightarrow X$. Furthermore, let $\mathcal{J} \subset \mathcal{O}$ denote the sheaf of ideals in \mathcal{O} generated by the functions h_1, \dots, h_m , and let $\mathcal{V}_0 = \mathcal{J}\mathcal{V} \subset \mathcal{V}$ be the

sheaf of germs of sections of V which can be locally written as $\sum_{j=1}^m h_j g_j$ for some $g_j \in \mathcal{V}$, $1 \leq j \leq m$. We have a short exact sequence of coherent analytic sheaves on X :

$$0 \rightarrow \mathcal{V}_0 \hookrightarrow \mathcal{V} \rightarrow \mathcal{K} \rightarrow 0.$$

The quotient sheaf $\mathcal{K} = \mathcal{V}/\mathcal{V}_0$ is trivial on $X \setminus X_0$ by the choice of the h_j ’s. Hence the section $f_0: X \rightarrow V$, which is holomorphic in an open set containing X_0 , determines a global holomorphic section $\tilde{f}_0: X \rightarrow \mathcal{K}$. By Cartan’s Theorem B we have $H^1(X; \mathcal{V}_0) = 0$, and hence there is a holomorphic section $\phi: X \rightarrow \mathcal{V}$ whose image in \mathcal{K} equals \tilde{f}_0 . We identify ϕ with the corresponding section of $V \rightarrow X$; then $f_0 - \phi$ is a holomorphic section of $V \rightarrow X$ over an open set $U_1 \supset X_0 \cup K$ (which we may take to be Stein), and by construction $f_0 - \phi$ vanishes to order k on X_0 .

Consider the sheaf epimorphism $\mathcal{V}^m \rightarrow \mathcal{V}_0, (g_1, \dots, g_m) \rightarrow \sum_{j=1}^m h_j g_j$. By Cartan’s Theorem B [GRo] we can lift each global holomorphic section of \mathcal{V}_0 over the Stein manifold U_1 to a holomorphic section of $\mathcal{V}^m \rightarrow U_1$. Applying this to $f_0 - \phi$ we obtain holomorphic sections $g_j^0: U_1 \rightarrow V$ ($1 \leq j \leq m$) satisfying (8.1) over U_1 .

We denote by $\widehat{V} = V^m \rightarrow X$ the direct (Whitney) sum of m copies of the bundle $V \rightarrow X$. Then $G_0 = (g_1^0, \dots, g_m^0): U_1 \rightarrow \widehat{V}$ is a holomorphic section of $\widehat{V} \rightarrow U_1$. Using a partition of unity we can modify G_0 outside a smaller neighborhood $U_0 \subset U_1$ of $X_0 \cup K$ (without changing it on U_0) and extend it continuously to X such that (8.1) holds everywhere. □

Denote by $\Theta: \widehat{V} \rightarrow V$ the map

$$\Theta(x; v_1, \dots, v_m) = (x; \phi(x) + \sum_{j=1}^m h_j(x)v_j), \tag{8.2}$$

where $x \in X$ and $v_1, \dots, v_m \in V_x$. Clearly Θ is a submersion over $X \setminus X_0$ and is degenerate over X_0 . Set $\widehat{\Sigma} = \Theta^{-1}(\overline{\Sigma}) \subset \widehat{V}$. Then the graph of a section $G = (g_1, \dots, g_m): X \rightarrow \widehat{V}$ avoids $\widehat{\Sigma}$ if and only if the graph of the associated section $f = \Theta \circ G = \phi + \sum_{j=1}^m h_j g_j: X \rightarrow V$ avoids $\overline{\Sigma}$. Notice that every section of this type agrees with f_0 to order k along X_0 . By construction we have $G_0(X) \cap \widehat{\Sigma} = \emptyset$, and G_0 is holomorphic in U_0 . To complete the Proof of Theorem 1.7 we need the following.

8.2 Proposition. (Notation as above.) *Let $G_0: X \rightarrow \widehat{V} \setminus \widehat{\Sigma}$ be a continuous section which is holomorphic in an open set $U_0 \supset X_0 \cup K$, where $K \subset X$ is $\mathcal{H}(X)$ -convex. For each compact $\mathcal{H}(X)$ -convex subset $L \subset X$ containing K there are an open set U'_0 , with $X_0 \cup K \subset U'_0 \subset U_0$, and a homotopy $G_t: X \rightarrow \widehat{V} \setminus \widehat{\Sigma}$ ($t \in [0, 1]$) of continuous sections which are holomorphic in U'_0 such that G_t approximates G_0 uniformly on K for each $t \in [0, 1]$ and the section G_1 is holomorphic in an open set $W_0 \supset X_0 \cup L$.*

Granted Proposition 8.2 we can complete the Proof of Theorem 1.7 as follows. We exhaust X by a sequence $\{L_j : j \in \mathbf{Z}_+\}$ of compact $\mathcal{H}(X)$ -convex sets such that $L_0 = K$ and $X = \cup_{j=0}^\infty L_j$. Applying Proposition 8.2 inductively on each pair (L_j, L_{j+1}) we obtain a sequence of sections $G_j : X \rightarrow \widehat{V} \setminus \widehat{\Sigma}$ ($j \in \mathbf{Z}_+$) such that for each j , G_{j+1} is holomorphic in a neighborhood of $X_0 \cup L_{j+1}$ and is homotopic to G_j by a homotopy $G_t : X \rightarrow \widehat{V} \setminus \widehat{\Sigma}$ ($t \in [j, j + 1]$) which is holomorphic in a neighborhood $X_0 \cup L_j$ and which approximates G_j uniformly on L_j . If these approximations are sufficiently close, the sequence G_j converges to a holomorphic section $G_\infty = \lim_{j \rightarrow \infty} G_j : X \rightarrow \widehat{V} \setminus \widehat{\Sigma}$. By reparametrizing the homotopy G_t ($t \in [0, +\infty)$) as in the Proof of Theorem 1.5 (Sect. 6) we get a homotopy G_t ($t \in [0, 1]$) from G_0 to $G_1 = G_\infty$. The associated homotopy $f_t = \Theta \circ G_t : X \rightarrow V$ ($t \in [0, 1]$) then satisfies Theorem 1.7. \square

To prove Proposition 8.2 we shall carry out the modification procedure, described in the Proof of Theorem 1.5, so that we only glue sections over Cartan pairs (A, B) in X for which $B \cap (X_0 \cup K) = \emptyset$. We need the following lemmas.

8.3 Lemma. *The submersion $\widehat{\pi} : \widehat{V} \setminus \widehat{\Sigma} \rightarrow X$ admits a spray over a neighborhood of any point $x \in X' = X \setminus (X_0 \cup K)$.*

Proof. Since the map Θ (8.2) is an affine linear submersion of holomorphic vector bundles over $X \setminus X_0$, it is locally (over small sets $U \subset X \setminus X_0$) equivalent to the projection $(x; v, w) \rightarrow (x; v)$ of trivial bundles. In such coordinates on $\widehat{V}|_U$ resp. $V|_U$ the set $\widehat{\Sigma} \cap \widehat{\pi}^{-1}(U)$ is defined by the same equations as $\Sigma \cap \pi^{-1}(U)$, and the additional $(m - 1)q$ fiber coordinates are not present in these equations. It is now immediate that the validity of (1.2) for $\Sigma \cap U'$ implies the analogous condition for $\widehat{\Sigma} \cap \widehat{U}$. Lemma 7.1 shows as before that the submersion $\widehat{\pi} : \widehat{V} \setminus \widehat{\Sigma} \rightarrow X$ admits a spray over a neighborhood of any point $x \in X'$. \square

8.4 Lemma. *Let X be a Stein manifold, X_0 a closed analytic subvariety of X and $K, L \subset X$ a pair of compact $\mathcal{H}(X)$ -convex subsets, with $K \subset \text{Int}L$. Let $\mathcal{U} = \{U_j\}_{j=0}^\infty$ be an open covering of X such that $X_0 \cup K \subset U_0$ and $(X_0 \cup K) \cap U_j = \emptyset$ for each $j \geq 1$. Then there is a Cartan string (A_0, A_1, \dots, A_n) in X such that A_0 is $\mathcal{H}(X)$ -convex and the following hold:*

- (i) $K \cup (X_0 \cap L) \subset A_0 \subset U_0$;
- (ii) for $j = 1, 2, \dots, n$ we have $A_j \cap (X_0 \cup K) = \emptyset$ and $A_j \subset U_k$ for some $k = k(j) \geq 1$;
- (iii) $L = \bigcup_{0 \leq j \leq n} A_j$.

Proof. Choose a compact $\mathcal{H}(X)$ -convex set $K' \subset X$ with $L \subset \text{Int}K'$. Then $S = (X_0 \cap K') \cup K$ is also $\mathcal{H}(X)$ -convex. By Theorem 4.6 there exists a Cartan string $(A'_0, A'_1, \dots, A'_r)$ in X satisfying the following:

- (a) $S \subset A'_0 \subset U_0$;

- (b) if $1 \leq j \leq r$ then $A'_j \cap S = \emptyset$;
- (c) if $A_j \cap X_0 = \emptyset$ then $A'_j \subset U_{k(j)}$ for some $k(j) \geq 1$;
- (d) if $A'_j \cap X_0 \neq \emptyset$ for some $1 \leq j \leq r$ then $A'_j \cap L = \emptyset$;
- (e) $L \subset \bigcup_{0 \leq j \leq r} A'_j$.

Let A_0, \dots, A_n denote the nonempty sets in the string $A'_j \cap L, 0 \leq j \leq r$ (in the given order). Since L is $\mathcal{H}(X)$ -convex, Proposition 4.3 implies that (A_0, A_1, \dots, A_n) is a Cartan string in X . It is clear that (i) and (iii) in Lemma 8.3 hold, and (ii) holds because no set A'_j for $j \geq 1$ intersects both X_0 and L at the same time according to (d). This proves Lemma 8.3. □

Proof of Proposition 8.2. By Lemma 8.3 there is an open covering $\mathcal{U} = \{U_j\}_{j=0}^\infty$ of X such that the submersion $\widehat{\pi} : \widehat{V} \setminus \widehat{\Sigma} \rightarrow X$ admits a spray over each set U_j for $j \geq 1$. Furthermore, we may choose the sets in the covering such that Proposition 4.7 applies to G_0 , i.e., we can modify G_0 into a holomorphic $\mathcal{K}(\mathcal{U})$ -complex.

Let $\mathcal{A} = (A_0, A_1, \dots, A_n)$ be a Cartan string provided by Lemma 8.4, subordinate to \mathcal{U} and satisfying $L = \bigcup_{j=0}^n A_j$. Applying Proposition 4.7 we deform G_0 over a neighborhood of L into a holomorphic $\mathcal{K}(\mathcal{A})$ -complex H_* . Since G_0 is holomorphic on $U_0 \supset A_0$, we may (and do) take the section $H_{(0)}$ in the complex H_* to be G_0 , restricted to a neighborhood of A_0 .

By the process described in section 5 (see especially the final remark at the end of Sect. 5) we can modify the complex H_* in a finite sequence of steps into a holomorphic section $H_1 : W \rightarrow \widehat{V} \setminus \widehat{\Sigma}$ over an open set $W \supset L$ such that H_1 approximates G_0 uniformly on a neighborhood $U_1 \supset A_0$. In addition, shrinking U_1 around A_0 if necessary, the construction in Sect. 5 gives a homotopy of sections $H_t : W \rightarrow \widehat{V} \setminus \widehat{\Sigma}$ ($t \in [0, 1]$) which are holomorphic in U_1 and approximate G_0 there, with $H_0 = G_0$.

Since L is $\mathcal{H}(X)$ -convex, we may approximate H_1 (which is holomorphic in W) uniformly on L by a global holomorphic section $G : X \rightarrow V$. For $t \in [1, 2]$ set $H_t = (2-t)H_1 + (t-1)G : X \rightarrow \widehat{V}$. At $t = 1$ this coincides with the section H_1 defined earlier. We also have $H_2 = G$, and for each $t \in [1, 2]$ the section H_t is holomorphic in W and it approximates G_0 in $U_1 \supset A_0$. For convenience of notation we replace $t \in [0, 2]$ by $t/2 \in [0, 1]$, thus reparametrizing $\{H_t : t \in [0, 2]\}$ to the interval $t \in [0, 1]$.

By shrinking the set $U_0 \supset X_0 \cup A_0$ we may assume that U_0 is Stein. We can now approximate the homotopy $H_t : U_1 \rightarrow \widehat{V}$, uniformly on $A_0 \subset U_1$, by a holomorphic homotopy $\widetilde{H}_t : U_0 \rightarrow \widehat{V}$ ($t \in [0, 1]$) satisfying $\widetilde{H}_0 = G_0$ and $\widetilde{H}_1 = G$. This can be done for instance by applying the h-Runge approximation theorem to the family H_t twice. First we apply it with the pair $A_0 \subset U_0$ and the initial section G_0 to obtain a family $H_t^{(1)} : U_0 \rightarrow \widehat{V}$ ($t \in [0, 1]$) with $H_0^{(1)} = G_0$; the second time we apply it with the pair $A_0 \subset X$ and the ‘initial’ section H_1 to get a family $H_t^{(2)} : X \rightarrow \widehat{V}$ ($t \in [0, 1]$) with $H_1^{(2)} = G$. All sections $H_t^{(1)}$ and

$H_t^{(2)}$ approximate H_t , and hence G_0 , uniformly on A_0 . Finally we take

$$\tilde{H}_t = (1 - t)H_t^{(1)} + tH_t^{(2)} \quad (t \in [0, 1])$$

(restricted to U_0 when $t < 1$).

The homotopy \tilde{H}_t is holomorphic in U_0 . Recall that, over A_0 , both H_t and \tilde{H}_t approximate G_0 as close as desired, and $\widehat{\Sigma}$ has no points over X_0 . It follows that, if the approximations are sufficiently close, there is an open set $U'_1 \subset X$, with $X_0 \cup A_0 \subset U'_1 \subset U_0$, such that for any $x \in U'_1$ and $t, \tau \in [0, 1]$ we have

$$\tau \tilde{H}_t(x) + (1 - \tau)H_t(x) \notin \widehat{\Sigma}. \tag{8.3}$$

Furthermore the graph of $H_1 = G$ avoids $\widehat{\Sigma}$ over an open set $W'_0 \supset X_0 \cup L$. Choose a smaller open set $U'_0 \subset X$, with $X_0 \cup A_0 \subset U'_0$ and $\overline{U'_0} \subset U'_1$, and choose a smooth function $\tau : X \rightarrow [0, 1]$ satisfying $\tau = 1$ on $\overline{U'_0}$ and $\text{supp } \tau \subset U'_1$. Define a new homotopy by

$$G'_t(x) = \tau(x)\tilde{H}_t(x) + (1 - \tau(x))H_t(x) \quad (x \in U'_0 \cup W, t \in [0, 1]).$$

The first term \tilde{H}_t is only defined on U'_1 , but since $\tau = 0$ on $X \setminus U'_1$, we can extend this terms to X . Likewise H_t is only defined on W , but $1 - \tau(x) = 0$ on $U'_0 \supset X_0$ and hence the second term extends to $U'_0 \cup W$. Thus G'_t is defined on $U'_0 \cup W$ and it satisfies the following:

- (i) $G'_0 = G_0$ (since $\tilde{H}_0 = H_0 = G_0$), and likewise $G'_1 = G$ (since $\tilde{H}_1 = H_1 = G$);
- (ii) for each $t \in [0, 1]$ the section G'_t is holomorphic in U'_0 ;
- (iii) shrinking $W \supset L$ if necessary we have $G'_t(x) \notin \widehat{\Sigma}$ for any $x \in U'_0 \cup W$ and $t \in [0, 1]$. For $x \in U'_0 \cup (W \cap U'_1)$ this holds by (8.3), while for $x \in W \setminus U'_1$ we have $\tau(x) = 0$ and hence $G'_t(x) = H_t(x) \notin \widehat{\Sigma}$.

It remains to extend the homotopy G'_t to X . Choose an open set $W_0 \subset X$, with $X_0 \cup L \subset W_0$ and $\overline{W_0} \subset U'_0 \cup W$, and choose a smooth function $\chi : X \rightarrow [0, 1]$ which equals one on W_0 and zero on $X \setminus (U'_0 \cup W)$. The homotopy

$$G_t(x) = G'_{\chi(x)t}(x) \quad (x \in X, t \in [0, 1])$$

then satisfies Proposition 8.2. □

Proof of Theorem 1.9. The scheme is the same as in Theorem 1.7. Using the notation established above, we construct a section G_0 of the submersion $\widehat{\pi} : \widehat{V} \setminus \widehat{\Sigma} \rightarrow X$ such that $\Theta \circ G_0 = f_0$ is the given section of $V \setminus \Sigma \rightarrow X$, where Θ is the map (8.2). Our goal is to establish Proposition 8.2 in this situation, and for this we must show that Lemma 8.3 holds in the current setting, i.e., our submersion admits a spray over a small neighborhood U of any point $x \in X \setminus X_0$.

Since $\widehat{\pi}(\widehat{\Sigma}) = X_1 \setminus X_0$, we only need to consider points $x \in X_1 \setminus X_0$. For $U \subset X$ we set $U' = \pi^{-1}(U) \subset V$ and $\widehat{U} = \widehat{\pi}^{-1}(U) \subset \widehat{V}$. In the Proof of

Lemma 8.2 we saw that, over a small neighborhood U of x , there is a fiber preserving biholomorphic map $\Phi : \widehat{U} \rightarrow U' \times \mathbb{C}^{(m-1)q}$ which maps $\widehat{\Sigma} \cap \widehat{U}$ onto $(\Sigma \cap U') \times \mathbb{C}^{(m-1)q}$. By the assumption, if U is chosen sufficiently small, there is a holomorphic action of a complex Lie group G on U' which preserves the fibers of π and acts transitively on $V_y \setminus \Sigma_y$ for any $y \in U \cap X_1$. The Lie group $G_1 = G \times \mathbb{C}^{(m-1)q}$ then acts on $U' \times \mathbb{C}^{(m-1)q}$ by

$$(g, w) \cdot (z, w') = (g \cdot z, w + w') \quad (g \in G, z \in U', w, w' \in \mathbb{C}^{(m-1)q}).$$

Conjugating by Φ we get an associated action of G_1 on \widehat{U} which preserves the fibers of $\widehat{\pi}$ and acts transitively on $\widehat{V}_y \setminus \widehat{\Sigma}_y$ for $y \in U \cap X_1$.

Let \mathfrak{g}_1 denote the Lie algebra of G_1 . The holomorphic map $s_1 : E_1 = \widehat{U} \times \mathfrak{g}_1 \rightarrow \widehat{U}$, defined by $s_1(z; g) = \exp(g) \cdot z$ for $z \in \widehat{U}$ and $g \in \mathfrak{g}_1$, satisfies all conditions for a spray over $\widehat{U} \setminus \widehat{\Sigma}$ (def. 1.1), except that its vertical derivative at the zero section need not be surjective at points $z \in \widehat{U}$ for which $\widehat{\pi}(z) \in U \setminus X_1$. This is easily corrected as follows. Choose holomorphic functions b_1, \dots, b_k in U such that $X_1 \cap U = \{x \in U : b_j(x) = 0, 1 \leq j \leq k\}$. Also choose \mathbb{C} -complete vertical holomorphic vector fields V_1, \dots, V_{mq} on \widehat{U} which generate the vertical tangent space at each point. (Since $\widehat{U} \cong U \times \mathbb{C}^{mq}$, we may take the constant fields in the mq coordinate directions on \mathbb{C}^{mq} .) Let $\{\phi_l^t : 1 \leq l \leq N\}$ be the flows of the $N = kmq$ complete fields $(b_j \circ \widehat{\pi})V_i$ for $1 \leq i \leq mq$ and $1 \leq j \leq k$. Set $E = \widehat{U} \times (\mathfrak{g}_1 \oplus \mathbb{C}^N) \rightarrow \widehat{U}$ and let $s : E \rightarrow \widehat{U}$ be the map

$$s(z; g, t_1, \dots, t_N) = \phi_1^{t_1} \circ \dots \circ \phi_N^{t_N}(\exp(g) \cdot z), \tag{8.2}$$

where $z \in \widehat{U}$, $g \in \mathfrak{g}_1$ and $t_j \in \mathbb{C}$ for $1 \leq j \leq N$. It is immediate that s , restricted to the part of the bundle E over $\widehat{U} \setminus \widehat{\Sigma}$, is a spray on $\widehat{\pi} : \widehat{U} \setminus \widehat{\Sigma} \rightarrow U$. □

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Addendum. This paper was completed in February 1999. In the mean time its sequel [FP2], which depends on the methods developed here, has been published. [FP2] contains the Oka principle with interpolation of sections on closed complex subvarieties of the base manifold. Recently F. Lárusson [Lar] gave a homotopy theoretic proof of the Oka principle, granted the analytic results on approximation and patching of holomorphic sections proved in [FP1]. The papers [Pre1, Pre2] contain new applications of the Oka principle to the embedding problem. In [Fo2] and [Fo3] these methods were used to study removability of intersections of holomorphic maps from Stein manifolds with complex subvarieties of the target manifold. The paper [Fo4] contains a version of the Oka principle for multi-valued sections of certain ramified holomorphic maps over a Stein base. In [Fo5] the Oka principle is extended to a wider class of target manifolds and submersions.

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