

Noncritical holomorphic functions on Stein manifolds

by

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In memory of my father, Franc, and sister, Helena

1. Introduction

In 1967 Gunning and Narasimhan proved that every open Riemann surface admits a holomorphic function without critical points [GN], thus giving an affirmative answer to a long-standing question. Their proof was an ingenious application of the approximation methods of Behnke and Stein.

A complex manifold is called *Stein* (after Karl Stein [Ste], 1951) if it is biholomorphic to a closed complex submanifold of a complex Euclidean space \mathbf{C}^N . Open Riemann surfaces are precisely Stein manifolds of complex dimension one. In this paper we prove the following result.

THEOREM I. *Every Stein manifold admits a holomorphic function without critical points. More precisely, an n -dimensional Stein manifold admits $\lfloor \frac{1}{2}(n+1) \rfloor$ holomorphic functions with pointwise independent differentials, and this number is maximal for every n .*

For a more precise statement see Theorems 2.1 and 2.6. An example of Forster [Fo1] provides for each $n \in \mathbf{N}$ an n -dimensional Stein manifold which does not admit more than $\lfloor \frac{1}{2}(n+1) \rfloor$ holomorphic functions with independent differentials (Proposition 2.12 below).

The question on the existence of noncritical holomorphic functions on a Stein manifold has been open since the 1967 work of Gunning and Narasimhan [GN]; it was mentioned in Gromov's monograph [Gro3, p. 70]. Our proof, which also applies to Riemann surfaces, is conceptually different from the one in [GN]. It is much easier to construct noncritical smooth real functions on smooth open manifolds; see e.g. Lemma 1.15 in [God, p. 9].

The critical locus of a generically chosen holomorphic function on a Stein manifold is discrete. Conversely, we prove that for any discrete subset P in a Stein manifold X there exists a holomorphic function $f \in \mathcal{O}(X)$ whose critical locus equals P (Corollary 2.2).

Recall that a holomorphic map $f = (f_1, \dots, f_q): X \rightarrow \mathbf{C}^q$ is a *submersion* if its differential $df_x: T_x X \rightarrow T_{f(x)} \mathbf{C}^q \simeq \mathbf{C}^q$ is surjective for every $x \in X$. Equivalently, the differentials of its component functions must be linearly independent, i.e., $df_1 \wedge df_2 \wedge \dots \wedge df_q \neq 0$. Thus the differential of a holomorphic submersion $X \rightarrow \mathbf{C}^q$ induces a surjective complex vector bundle map $TX \rightarrow X \times \mathbf{C}^q$ of the tangent bundle of X onto the trivial bundle of rank q over X . Our main result is that, for $q < \dim X$, this necessary condition for the existence of a submersion $X \rightarrow \mathbf{C}^q$ is also sufficient.

THEOREM II. (The homotopy principle for holomorphic submersions.) *If X is a Stein manifold and $1 \leq q < \dim X$ then every surjective complex vector bundle map $TX \rightarrow X \times \mathbf{C}^q$ is homotopic to the differential of a holomorphic submersion $X \rightarrow \mathbf{C}^q$.*

The homotopy referred to above belongs to the space of surjective complex vector bundle maps $TX \rightarrow X \times \mathbf{C}^q$. Theorem II is the holomorphic analogue of the basic homotopy principle for submersions of smooth open manifolds to real Euclidean spaces, due to A. Phillips [Ph1] and M. Gromov [Gro1]. A more precise statement is given by Theorems 2.5 and 2.6 in §2. We don't know whether the same conclusion holds for $q = \dim X > 1$ (for open Riemann surfaces see [GN]).

By using the tools developed in this paper one can also prove the following. If $f_0, f_1: X \rightarrow \mathbf{C}^q$ are holomorphic submersions ($q < \dim X$) whose differentials df_0, df_1 are homotopic through a family of surjective complex vector bundle maps of TX onto the trivial bundle $X \times \mathbf{C}^q$ then there exists a homotopy of holomorphic submersions $f_\tau: X \rightarrow \mathbf{C}^q$ ($\tau \in [0, 1]$) connecting f_0 to f_1 . The details are included in the sequel to this paper, entitled 'Holomorphic submersions from Stein manifolds' (to appear in *Ann. Inst. Fourier*), in which we investigate the same problem for more general target manifolds.

Theorem I is a corollary of Theorem II and a result of Ramspott [Ra] to the effect that the cotangent bundle of an n -dimensional Stein manifold admits $\lfloor \frac{1}{2}(n+1) \rfloor$ independent sections, and these define a surjective complex vector bundle map $TX \rightarrow X \times \mathbf{C}^{\lfloor (n+1)/2 \rfloor}$. Ramspott's theorem combines the Lefschetz theorem [AF] with the standard method of constructing sections of fiber bundles over CW-complexes by stepwise extension over the skeleta. Our proof gives both results simultaneously and does not use Ramspott's theorem.

We give numerous applications to the existence of nonsingular holomorphic foliations on Stein manifolds. We prove that every complex vector subbundle $E \subset TX$ with trivial quotient TX/E is homotopic to the tangent bundle of a holomorphic foliation

(Corollary 2.9); the same is true if TX/E admits locally constant transition functions (Theorem 7.1). Analogous results for smooth foliations on open manifolds were proved by Gromov [Gro1] and Phillips [Ph2], [Ph3], [Ph4], and on closed manifolds by Thurston [Th1], [Th2]. Every n -dimensional Stein manifold admits nonsingular holomorphic submersion foliations of any dimension $\geq [\frac{1}{2}n]$, and if X has geometric dimension $k \leq n$ then it admits submersion foliations of any dimension $\geq [\frac{1}{2}k]$ (Corollary 2.7). We construct submersion foliations transverse to certain complex submanifolds of X (Corollaries 2.3 and 2.11) or containing it as a leaf (Corollaries 2.10 and 7.2).

Our construction depends on three main ingredients developed in this paper. We postpone the general discussion to §2 and mention at this point only the following *splitting lemma for biholomorphic maps* (Theorem 4.1): If $A, B \subset X$ is a *Cartan pair* in a complex manifold X then every biholomorphic map γ sufficiently uniformly close to the identity in a neighborhood of $A \cap B$ admits a decomposition $\gamma = \beta \circ \alpha^{-1}$, where α (resp. β) is a biholomorphic map close to the identity in a neighborhood of A (resp. of B).

This lemma is used to patch a pair of holomorphic submersions f, g to \mathbf{C}^q , defined in a neighborhood of A and B respectively, which are sufficiently uniformly close in a neighborhood of $A \cap B$, into a submersion \tilde{f} in a neighborhood of $A \cup B$. The map γ arises as a transition map satisfying $f = g \circ \gamma$ near $A \cap B$. From $\gamma = \beta \circ \alpha^{-1}$ we obtain $f \circ \alpha = g \circ \beta$, which gives \tilde{f} .

Our splitting lemma plays the analogous role in our construction of submersions as *Cartan's lemma* (on product splitting of holomorphic maps with values in a complex Lie group) does in Cartan's theory or in the Oka–Grauert theory. A key difference is that our lemma gives a compositional splitting of biholomorphic maps and is closer in spirit to Kolmogorov's work on compositions of functions [Ko]. We prove it by a rapidly convergent Kolmogorov–Nash–Moser-type iteration (§4).

Our proof of Theorem II breaks down for $q = \dim X > 1$ due to a possible Picard-type obstruction in the approximation problem (Lemma 3.4). Hence the following problem remains open.

Problem 1. Does a parallelizable Stein manifold of dimension $n > 1$ holomorphically immerse in \mathbf{C}^n (i.e., is it a Riemann domain over \mathbf{C}^n)?

This well-known problem (see [BN, p. 18] or [Gro3, p. 70]) was our main motivation for the present work. To find such an immersion it would suffice to obtain an affirmative answer to any of the following two problems.

Problem 2. Let B be an open convex set in \mathbf{C}^n for $n > 1$. Is every holomorphic immersion (=submersion) $B \rightarrow \mathbf{C}^n$ a uniform limit on compacts of entire immersions $\mathbf{C}^n \rightarrow \mathbf{C}^n$?

The analogous problem for mappings with constant Jacobian may be related to the *Jacobian problem* for holomorphic polynomial maps [BN, p. 21]. The situation is much better understood for biholomorphic maps: *If f is an injective holomorphic map from a convex open set $B \subset \mathbf{C}^n$ onto a Runge set $f(B) \subset \mathbf{C}^n$ then f can be approximated uniformly on compacts in B by holomorphic automorphisms of \mathbf{C}^n* [AL]. No comparable result seems to be known for noninjective immersions.

Problem 3. Let $f=(f_1, \dots, f_q): X \rightarrow \mathbf{C}^q$ be a holomorphic submersion for some $q < \dim X$. Given a $(1,0)$ -form θ_0 such that $df_1 \wedge \dots \wedge df_q \wedge \theta_0 \neq 0$ on X , find a homotopy of $(1,0)$ -form θ_t ($t \in [0, 1]$) such that $df_1 \wedge \dots \wedge df_q \wedge \theta_t \neq 0$ for all $t \in [0, 1]$ and $\theta_1 = dg$ for some $g \in \mathcal{O}(X)$. (The map $(f, g): X \rightarrow \mathbf{C}^{q+1}$ is then a submersion.)

Problem 1 has an affirmative answer if one can solve Problem 3 with $q=n-1$. Explicitly, given a holomorphic submersion $f: X^n \rightarrow \mathbf{C}^{n-1}$ such that $\ker df$ is a trivial line subbundle of TX , find a $g \in \mathcal{O}(X)$ whose restriction to every level set $\{f=c\}$ is noncritical.

2. The main results

Let X be a Stein manifold (for their general theory see [GR] and [Hö2]). Denote by $\mathcal{O}(X)$ the algebra of all holomorphic functions on X . A compact set $K \subset X$ is said to be $\mathcal{O}(X)$ -convex if for any point $x \in X \setminus K$ there exists $f \in \mathcal{O}(X)$ satisfying $|f(x)| > \max_K |f|$. An $\mathcal{O}(\mathbf{C}^n)$ -convex set is called *polynomially convex*. A function is holomorphic on a closed subset $K \subset X$ if it is holomorphic in some unspecified open neighborhood of K ; the set of all such functions (with the usual identification of functions which agree near K) is denoted $\mathcal{O}(K)$. We denote by $j_x^r(f)$ the r -jet of a function f at $x \in X$. The *critical set* of $f \in \mathcal{O}(X)$ is $\text{Crit}(f; X) = \{x \in X: df_x = 0\}$; a function without critical points will be called *noncritical*. We denote by $|z|$ the Euclidean norm of $z \in \mathbf{C}^n$.

(1) *Functions with prescribed critical locus.* Our first main result is

THEOREM 2.1. *Let X be a Stein manifold, $X_0 \subset X$ a closed complex subvariety of X and $K \subset X$ a compact $\mathcal{O}(X)$ -convex subset. Let $U \subset X$ be an open set containing $X_0 \cup K$ and $f \in \mathcal{O}(U)$ a holomorphic function with discrete critical set $P = \{p_1, p_2, \dots\} \subset X_0 \cup K$. For any $\varepsilon > 0$ and $r, n_1, n_2, \dots \in \mathbf{N}$ there exists an $\tilde{f} \in \mathcal{O}(X)$ satisfying $\text{Crit}(\tilde{f}; X) = P$, $|f(x) - \tilde{f}(x)| < \varepsilon$ for all $x \in K$, $j_x^r(f - \tilde{f}) = 0$ for all $x \in X_0$, and $j_{p_k}^{n_k}(f - \tilde{f}) = 0$ ($k=1, 2, \dots$). In particular, if f is noncritical on U then \tilde{f} is noncritical on X .*

Theorem 2.1 implies that any noncritical holomorphic function on a closed complex submanifold X_0 of a Stein manifold X extends to a noncritical holomorphic function

on X . Furthermore, there exist noncritical functions satisfying the axioms of a Stein manifold ([Hö2, p. 116, Definition 5.1.3]). Theorem 2.1 is proved in §5.

The critical locus of a generically chosen holomorphic function on a Stein manifold is discrete. Theorem 2.1 implies the following converse.

COROLLARY 2.2. *Let $P = \{p_1, p_2, p_3, \dots\}$ be a discrete set in a Stein manifold X and let f_k be a holomorphic function in a neighborhood of p_k with an isolated critical point at p_k for $k=1, 2, \dots$. For any choice of integers $n_k \in \mathbf{N}$ there exists an $f \in \mathcal{O}(X)$ with $\text{Crit}(f) = P$ such that $f - f_k$ vanishes at least to order n_k at p_k for every $k=1, 2, \dots$.*

(2) *Foliations by complex hypersurfaces.* We denote by TX the holomorphic tangent bundle of X and by T^*X its holomorphic cotangent bundle. For the general theory of foliations we refer to [God].

COROLLARY 2.3. *Every Stein manifold admits a nonsingular holomorphic foliation by closed complex hypersurfaces; in addition such a foliation may be chosen to be transverse to a given closed complex submanifold.*

Proof. A closed complex submanifold V of a Stein manifold X is itself Stein and hence admits a noncritical function $f \in \mathcal{O}(V)$ by Theorem 2.1. By Cartan's theorem f extends to a holomorphic function on X . Since the extension remains noncritical on X_0 , Theorem 2.1 gives a noncritical function $\tilde{f} \in \mathcal{O}(X)$ such that $\tilde{f}|_V = f$. The family of connected components of the levels sets $\{\tilde{f} = c\}$ ($c \in \mathbf{C}$) is a foliation of X by closed complex hypersurfaces transverse to V .

COROLLARY 2.4. *If V is a smooth closed complex hypersurface with trivial normal bundle in a Stein manifold X then V is a union of leaves in a nonsingular holomorphic foliation of X by closed complex hypersurfaces. This holds in particular if $H^2(V; \mathbf{Z}) = 0$, or if $X = \mathbf{C}^n$. Any smooth connected complex curve in a Stein surface is a leaf in a nonsingular holomorphic foliation.*

Proof. Choose a holomorphic trivialization of the normal bundle $N = TX|_V / TV \simeq V \times \mathbf{C}$. The projection $h: N \rightarrow \mathbf{C}$ on the second factor is a noncritical holomorphic function on N , and $N_0 = \{h=0\}$ is the zero-section of N . The Docquier–Grauert theorem [DG] (see also Theorem 8 in [GR, p. 257]) gives an open neighborhood $\Omega \subset X$ of V and an injective holomorphic map $\phi: \Omega \rightarrow N$ with $\phi(V) = N_0$. Then $f = h \circ \phi$ is a noncritical function on Ω with $\{f=0\} = V$. Applying Theorem 2.1 (with $X_0 = V$) we obtain a noncritical function $\tilde{f} \in \mathcal{O}(X)$ which vanishes on V , and the foliation $\{\tilde{f} = c\}$ clearly satisfies Corollary 2.4. The second statement follows from the isomorphism $\text{Pic}(V) = H^1(V; \mathcal{O}^*) \simeq H^2(V; \mathbf{Z})$; the latter group vanishes if V is an open Riemann surface. Since every divisor on \mathbf{C}^n is a principal divisor, the normal bundle of any complex hypersurface $V \subset \mathbf{C}^n$ is trivial.

(3) *Holomorphic submersions and foliations.* We now consider the existence of holomorphic submersions $f=(f_1, \dots, f_q): X \rightarrow \mathbf{C}^q$ for $q \leq n = \dim X$. The components of f are noncritical functions with pointwise independent differentials, i.e., $df_1 \wedge \dots \wedge df_q \neq 0$ on X . Hence an obvious necessary condition is that there exists a q -tuple $\theta=(\theta_1, \dots, \theta_q)$ of continuous differential $(1, 0)$ -forms on X satisfying $\theta_1 \wedge \dots \wedge \theta_q|_x \neq 0$ for all $x \in X$. Any such ordered q -tuple will be called a q -coframe on X . We may view θ as a *complex vector bundle epimorphism* $\theta: TX \rightarrow X \times \mathbf{C}^q$ of the tangent bundle TX onto the trivial bundle of rank q over X . Clearly we may speak of *holomorphic q -coframes*, *homotopies of q -coframes*, etc. If $\theta_j = df_j$ for some $f_j \in \mathcal{O}(X)$ ($j=1, \dots, q$) we shall write $\theta = df$ and call θ *exact holomorphic*. The following two theorems are our main results; they are proved in §6.

THEOREM 2.5. *Let X be a Stein manifold and $1 \leq q < \dim X$. For every q -coframe θ^0 on X there exists a homotopy of q -coframes θ^t ($t \in [0, 1]$) such that $\theta^1 = df$ where $f: X \rightarrow \mathbf{C}^q$ is a holomorphic submersion. Furthermore, if $X_0, K \subset X$ are as in Theorem 2.1, $r \in \mathbf{N}$, $\varepsilon > 0$, and if we assume that $\theta^0 = df^0$ is exact holomorphic in an open set $U \supset X_0 \cup K$, the homotopy may be chosen such that $\theta^t = df^t$ is exact holomorphic in a neighborhood of $X_0 \cup K$ for every $t \in [0, 1]$, $f^t - f^0$ vanishes to order r on X_0 , and $|f^t - f^0| < \varepsilon$ on K .*

Theorem 2.5 also holds for $q = \dim X = 1$ and is due in this case to Gunning and Narasimhan who proved that for every nonvanishing holomorphic one-form θ on an open Riemann surface there exists a holomorphic function w such that $e^w \theta = df$ is exact holomorphic [GN, p. 107]. The homotopy $\theta^t = e^{tw} \theta$ consisting of nonvanishing one-forms connects $\theta^0 = \theta$ to $\theta^1 = df$. We do not know whether Theorem 2.5 holds for $q = \dim X > 1$.

We state separately the case when the necessary condition on the existence of a q -coframe is automatically fulfilled due to topological reasons. Recall that any Morse critical point of a strongly plurisubharmonic function on an n -dimensional complex manifold has Morse index at most n [AF]. If X admits a strongly plurisubharmonic Morse exhaustion function $\varrho: X \rightarrow \mathbf{R}$ all of whose critical points have index $\leq k$ (and k is a minimal such), we say that X has *geometric dimension k* ; such an X is homotopically equivalent to a k -dimensional CW-complex [AF].

THEOREM 2.6. *Let $\varrho: X \rightarrow \mathbf{R}$ be a strongly plurisubharmonic Morse exhaustion function on an n -dimensional Stein manifold X . Assume that c is a regular value of ϱ and every critical point of ϱ in $\{x \in X: \varrho(x) > c\}$ has Morse index $\leq k$. If $q \leq q(k, n) := \min\{n - \lfloor \frac{1}{2}k \rfloor, n - 1\}$ then every holomorphic submersion $f: \{x \in X: \varrho(x) < c\} \rightarrow \mathbf{C}^q$ can be approximated uniformly on compacts by holomorphic submersions $\tilde{f}: X \rightarrow \mathbf{C}^q$. Every n -dimensional Stein manifold X admits a holomorphic submersion to $\mathbf{C}^{\lfloor (n+1)/2 \rfloor}$; if X has geometric dimension k then it admits a holomorphic submersion to $\mathbf{C}^{q(k, n)}$.*

Proposition 2.12 below shows that the submersion dimension in Theorem 2.6 is optimal for every n . Theorem 2.6 immediately gives

COROLLARY 2.7. *Every Stein manifold X of geometric dimension k admits nonsingular holomorphic foliations of any dimension $\geq [\frac{1}{2}k]$. If X is parallelizable, it admits a holomorphic submersion $X \rightarrow \mathbf{C}^{n-1}$ ($n = \dim X$) and nonsingular holomorphic foliations of any dimension ≥ 1 .*

The foliations in Corollary 2.7 are given by submersions to Euclidean spaces; hence all leaves are topologically closed and the normal bundle is trivial.

Remark. The Oka–Grauert principle applies to q -coframes on a Stein manifold and shows that any q -coframe is homotopic to a holomorphic q -coframe, and any homotopy between a pair of holomorphic q -coframes can be deformed to a homotopy consisting of holomorphic q -coframes. This is seen by viewing q -coframes as sections of the holomorphic fiber bundle $V^q(T^*X) \rightarrow X$ whose fiber V_x^q is the Stiefel variety of all ordered q -tuples of \mathbf{C} -independent elements in T_x^*X . Since the Lie group $GL_n(\mathbf{C})$ ($n = \dim X$) acts transitively on V_x^q , the Oka–Grauert principle [Gra] applies to sections $X \rightarrow V^q(T^*X)$.

(4) *Existence of homotopies to integrable subbundles.* The components of a q -coframe on X are linearly independent sections of T^*X , which therefore span a trivial complex subbundle of rank q in T^*X . Conversely, every trivial rank- q subbundle $\Theta \subset T^*X$ is spanned by (the components of) a q -coframe. Different q -coframes θ, θ' spanning the same subbundle of T^*X are related by $\theta' = \theta \cdot A$ for some $A: X \rightarrow GL_q(\mathbf{C})$. A homotopy of q -coframes induces a homotopy of the associated subbundles of T^*X . Hence Theorem 2.5 implies

COROLLARY 2.8. *Let X be a Stein manifold. Every trivial complex subbundle $\Theta \subset T^*X$ of rank $q < \dim X$ is homotopic to a subbundle generated by independent holomorphic differentials df_1, \dots, df_q . If Θ is holomorphic then the homotopy can be chosen through holomorphic subbundles of T^*X .*

The last statement follows from the Oka–Grauert principle [Gra]. Corollary 2.8 admits the following dual formulation in terms of subbundles of TX (for a generalization see Theorem 7.1).

COROLLARY 2.9. *Let X be a Stein manifold of dimension n . Every complex subbundle $E \subset TX$ of rank $k \geq 1$ with trivial quotient bundle TX/E is homotopic to an integrable holomorphic subbundle $\ker df \subset TX$, where $f: X \rightarrow \mathbf{C}^{n-k}$ is a holomorphic submersion. If E is holomorphic then the homotopy may be chosen through holomorphic subbundles.*

Proof. The complex subbundle $\Theta = E^\perp \subset T^*X$ with fibers $\Theta_x = \{\lambda \in T_x^*X : \lambda(v) = 0 \text{ for all } v \in E_x\}$ (the conormal bundle of E) satisfies $\Theta \simeq (TX/E)^*$ and hence is trivial. Corollary 2.8 gives a homotopy $\Theta^t \subset T^*X$ ($t \in [0, 1]$) from $\Theta^0 = \Theta$ to a subbundle $\Theta^1 \subset T^*X$ spanned by $n-k$ independent holomorphic differentials df_1, \dots, df_{n-k} . The homotopy $E^t = (\Theta^t)^\perp \subset TX$ satisfies Corollary 2.9. The last statement follows from the Oka–Grauert principle.

We conclude with a couple of results on the existence of submersion foliations which either contain a given submanifold as a leaf, or else are transverse to it. Both depend on Theorem 2.5 and are proved in §6.

COROLLARY 2.10. *Let X be an n -dimensional Stein manifold and $V \subset X$ a closed complex submanifold. If TX admits a trivial complex subbundle N satisfying $TX|_V = TV \oplus N|_V$ then there is a holomorphic submersion $f: X \rightarrow \mathbf{C}^q$ ($q = n - \dim V$) such that V is a union of connected components of the fiber $f^{-1}(0)$. If $\dim V \geq [\frac{1}{2}n]$ then the above conclusion holds provided that V has a trivial normal bundle in X .*

COROLLARY 2.11. *Let X be a Stein manifold, $\iota: V \hookrightarrow X$ a closed complex submanifold, and $f = (f_1, \dots, f_q): V \rightarrow \mathbf{C}^q$ a holomorphic submersion. If there is a q -coframe $\theta = (\theta_1, \dots, \theta_q)$ on X satisfying $\iota^*\theta_j = df_j$ ($j = 1, \dots, q$) then there exists a holomorphic submersion $F: X \rightarrow \mathbf{C}^q$ with $F|_V = f$. Such an F always exists if $q \leq [\frac{1}{2}(n+1)]$, where $n = \dim X$.*

(5) *An example.* The following example shows that the submersion dimension in Theorem 2.6 is maximal for every n .

PROPOSITION 2.12. *Set $Y = \{[x:y:z] \in \mathbf{CP}^2 : x^2 + y^2 + z^2 \neq 0\}$ and*

$$X = \begin{cases} Y^m, & \text{if } n = 2m, \\ Y^m \times \mathbf{C}, & \text{if } n = 2m + 1. \end{cases}$$

Then X is an n -dimensional Stein manifold which does not admit a holomorphic submersion to $\mathbf{C}^{[(n+1)/2]+1}$.

Proof. These manifolds were considered by Forster [Fo1, p. 714], [Fo2, Proposition 3]. He showed that Y is a Stein surface which admits a strong deformation retraction onto the real projective plane $M = \{[x:y:z] : x, y, z \in \mathbf{R}\} \simeq \mathbf{RP}^2$ contained in Y as a totally real submanifold. Thus Y is a complexified \mathbf{RP}^2 , and X is homotopic to $(\mathbf{RP}^2)^m$. Using the fact that $TY|_M \simeq TM \oplus TM$ (as real bundles) Forster proved that the Stiefel–Whitney class $w_{2m}(TX)$ is the nonzero element of the group $H^{2m}(X; \mathbf{Z}_2) = H^{2m}((\mathbf{RP}^2)^m; \mathbf{Z}_2) = \mathbf{Z}_2$, and consequently the Chern class $c_m(TX)$ is the nonzero element of $H^{2m}(X; \mathbf{Z}) = \mathbf{Z}_2$. Hence $c_m(T^*X) = (-1)^m c_m(TX) \neq 0$ [MS, p. 168], which implies that T^*X does not contain a trivial complex subbundle of rank $n - m + 1 = [\frac{1}{2}(n+1)] + 1$. (Proof: if $T^*X = E \oplus E'$ where

E' is trivial then $0 \neq c_m(T^*X) = c_m(E)$ [MS, Lemma 14.3], which means that $\text{rank } E \geq m$ and consequently $\text{rank } E' \leq n - m$.) Hence there exists no submersion $X \rightarrow \mathbf{C}^{[(n+1)/2]+1}$. (The only essential property of X is that the Chern class of TX of order $[\frac{1}{2}n]$ does not vanish.)

Recall that holomorphic immersions of a Stein manifold X into Euclidean spaces of dimension $N > \dim X$ satisfy the following homotopy principle (Eliashberg and Gromov [Gro3, pp. 65–75]): *Every injective complex vector bundle map $TX \rightarrow X \times \mathbf{C}^N$ is homotopic to the differential of a holomorphic immersion $X \rightarrow \mathbf{C}^N$.* In particular, every n -dimensional Stein manifold admits a holomorphic immersion in $\mathbf{C}^{[3n/2]}$, and the manifold X in Proposition 2.12 does not immerse in $\mathbf{C}^{[3n/2]-1}$ [Fo2, p. 183]. A comparison with Theorem 2.6 shows that the *submersion dimension* $q(n)$ and the *immersion dimension* $N(n)$, respectively, are symmetric with respect to $n = \dim X$:

$$q(n) = [\frac{1}{2}(n+1)] = n - [\frac{1}{2}n], \quad N(n) = n + [\frac{1}{2}n].$$

If X has geometric dimension at most k and $k \geq 2$ then X admits a submersion to $\mathbf{C}^{n-[k/2]}$ and immersion in $\mathbf{C}^{n+[k/2]}$, and both bounds are sharp (an example is the manifold $Y^{[k/2]} \times \mathbf{C}^{n-2[k/2]}$ where Y is as in Proposition 2.12).

(6) *Remarks on parallelizable Stein manifolds.* By Grauert [Gra] the tangent bundle of a Stein manifold X is holomorphically trivial if and only if it is topologically trivial (as a complex vector bundle). The question whether every such manifold immerses in \mathbf{C}^n with $n = \dim X$ remains open for $n > 1$. Every closed complex submanifold $X \subset \mathbf{C}^N$ with trivial normal bundle is parallelizable [Fo1, p. 712]. (Triviality of the normal bundle is equivalent to X being a holomorphic complete intersection in some open neighborhood.) In particular, every closed complex hypersurface in \mathbf{C}^{n+1} is parallelizable [Fo1, Corollary 2], but it is unknown whether these immerse into \mathbf{C}^n . J. J. Loeb [BN, p. 19] found explicit holomorphic immersions $X \rightarrow \mathbf{C}^n$ of algebraic hypersurfaces $X \subset \mathbf{C}^{n+1}$ of the following type:

$$X = \{(z_0, z_1, \dots, z_k) : z_0^d + P_1(z_1) + \dots + P_k(z_k) = 1\} \subset \mathbf{C}^{n+1},$$

where $z_0 \in \mathbf{C}$, $z_j \in \mathbf{C}^{n_j}$, P_j is a homogeneous polynomial of some degree d_j on \mathbf{C}^{n_j} for every $j = 1, \dots, k$, and $n_1 + \dots + n_k = n$. These manifolds are even algebraically parallelizable but do not admit algebraic immersions to \mathbf{C}^n . An example of this type is the complex n -sphere $\Sigma^n = \{z \in \mathbf{C}^{n+1} : \sum z_j^2 = 1\}$.

In another direction, Y. Nishimura [N] found explicit holomorphic immersions $\mathbf{CP}^2 \setminus C \rightarrow \mathbf{C}^2$ where C is an irreducible cuspidal cubic in \mathbf{CP}^2 . Further examples and remarks on parallelizable Stein manifolds can be found in [Fo1].

Unlike Σ^n , the real n -sphere $S^n = \Sigma^n \cap \mathbf{R}^{n+1}$ (which is a maximal totally real submanifold of Σ^n) is parallelizable only for $n=1, 3, 7$. By Thurston ([Th1], [Th2]) S^3 and S^7 admit C^∞ -foliations of all dimensions. However, a simply-connected closed real-analytic manifold (such as S^n for $n>1$) does not admit any real-analytic foliations of codimension one (Haefliger [Ha1]).

(7) *Comparison with smooth immersions and submersions.* The homotopy classification of smooth immersions $X \rightarrow \mathbf{R}^q$ was discovered by Smale [Sm] and Hirsch [Hi1], [Hi2]. Subsequently analogous results were proved for submersions (Phillips [Ph1] and Gromov [Gro1]), k -mersions (Feit [Fe]), and maps of constant rank [Ph5]. Gromov's monograph [Gro3] offers a comprehensive survey; see also the more recent monographs [Sp] and [EM]. Our Theorem 2.5 is a holomorphic analogue of the basic homotopy principle for smooth submersions $X \rightarrow \mathbf{R}^q$ which holds for all $1 \leq q \leq \dim_{\mathbf{R}} X$ provided that X is a smooth open manifold (see [Hi2], [Ph1], [Gro1] and [Ha2]).

The differential relation controlling smooth immersions of positive codimension is *ample in the coordinate directions*, and the corresponding homotopy principle follows from the *convex integration lemma* of M. Gromov (see the discussion and references in Subsection (2) of §6 below). The smooth submersion relation is not ample in the coordinate directions (Example 2 in [EM, p.168]), and the homotopy principle for smooth submersions is obtained by exploiting the invariance of the submersion condition under local diffeomorphisms and reducing the problem to a subpolyhedron in the given manifold. On the other hand, we shall see that the complex (holomorphic) submersion relation is ample in the coordinate directions on any totally real submanifold, and this is exploited to obtain a maximal rank extension of the map across a totally real handle (Lemma 6.5). The invariance under local biholomorphisms is also strongly exploited in the approximation and patching of submersions.

The results of homotopy principle type on Stein manifolds are traditionally referred to as (instances of) the *Oka principle*; see the recent survey [F3].

(8) *Outline of proof of the main theorems.* Our construction relies on three main ingredients developed in this paper.

The first one is a new technique for approximating a noncritical holomorphic function f on a compact polynomially convex subset $K \subset \mathbf{C}^n$ by entire noncritical functions (§3). We exploit the invariance of the maximal rank condition under biholomorphisms. Choose a preliminary approximation of f on K by a holomorphic polynomial h with finite critical set $\Sigma = \text{Crit}(h)$ disjoint from K . When $n>1$, the main step is to find an injective holomorphic map $\phi: \mathbf{C}^n \rightarrow \mathbf{C}^n \setminus \Sigma$ (a Fatou–Bieberbach map) which is close to the identity map on K and whose range avoids Σ . Such a ϕ can be obtained as a

limit of holomorphic automorphisms of \mathbf{C}^n using methods developed by Andersén and Lempert [A], [AL], and Rosay and the author [FR], [F1], [F2]. Then $\tilde{f}=h\circ\phi$ is noncritical on \mathbf{C}^n and approximates f uniformly on K . For $n=1$ we give a different proof using Mergelyan's theorem. Similar methods are developed for submersions $\mathbf{C}^n\rightarrow\mathbf{C}^q$ for $q<n$.

The second ingredient concerns patching of holomorphic submersions. Let $A, B\subset X$ be compact sets in a complex manifold X such that $A\cup B$ has a basis of Stein neighborhoods and $\overline{A\setminus B}\cap\overline{B\setminus A}=\emptyset$. For any biholomorphic (=injective holomorphic) map $\gamma:V\rightarrow X$, sufficiently close to the identity map in a neighborhood $V\subset X$ of $C=A\cap B$, we obtain a compositional splitting $\gamma=\beta\circ\alpha^{-1}$, where α (resp. β) is a biholomorphic map close to the identity in a neighborhood of A (resp. B) (Theorem 4.1). If f (resp. g) is a submersion to \mathbf{C}^q in a neighborhood of A (resp. B) and g is sufficiently uniformly close to f in a neighborhood of C then $f=g\circ\gamma$ for a biholomorphic map γ close to the identity; splitting $\gamma=\beta\circ\alpha^{-1}$ as above we obtain $f\circ\alpha=g\circ\beta$ near C ; this gives a submersion \tilde{f} in a neighborhood of $A\cup B$ which approximates f on A .

Remark. The standard $\bar{\partial}$ -method for patching f and g would be to take $h=f+\chi(g-f)$ and $\tilde{f}=h-T(\bar{\partial}h)$, where T is a bounded solution operator to the $\bar{\partial}$ -equation in a neighborhood of $A\cup B$ and χ is a smooth function which equals zero in a neighborhood of $\overline{A\setminus B}$ and one in a neighborhood of $\overline{B\setminus A}$. Since $\bar{\partial}h=(g-f)\bar{\partial}\chi$, the correction term $T(\bar{\partial}h)$ is controlled by $|f-g|$, and hence \tilde{f} is noncritical in a neighborhood of A provided that $|g-f|$ is sufficiently small in a neighborhood of $A\cap B$. However, to insure that \tilde{f} is also noncritical in a neighborhood of B we would need the pointwise estimate $|d(T(\bar{\partial}h))|<|dg|$. Since we obtain g by a Runge approximation of f on $A\cap B$ (and we have no control on its differential on $B\setminus A$), such an estimate is impossible.

In the construction of submersions $X\rightarrow\mathbf{C}^q$ for $q>1$ another nontrivial problem is the crossing of the critical levels of a strongly plurisubharmonic Morse exhaustion function ρ on X . We combine three ingredients (§6):

- a *convex integration lemma* of Gromov, or Thom's *jet transversality theorem* when $q\leq[\frac{1}{2}(n+1)]$, to obtain a smooth extension across a handle;
- holomorphic approximation on certain handlebodies;
- the construction of an increasing family of smooth strongly pseudoconvex neighborhoods of a handlebody, passing over the critical level of ρ .

We globalize the construction using the 'bumping method' similar to the one in [Gro4], [HL3], [FP1], [FP2], [FP3]. We exhaust X by an increasing sequence $A_0\subset A_1\subset A_2\subset\dots\subset\bigcup_{k=1}^{\infty}A_k=X$ of compact $\mathcal{O}(X)$ -convex sets such that the initial function (or submersion) $f=f_0$ is defined on A_0 , and for each $k\geq 0$ we have $A_{k+1}=A_k\cup B_k$ where (A_k, B_k) is a *special Cartan pair*. This enables us to approximate a noncritical function

f_k on A_k by a noncritical function f_{k+1} on A_{k+1} . The limit $\tilde{f} = \lim_{k \rightarrow \infty} f_k$ is a noncritical function on X . The details are given in §5 for functions and in §6 for submersions.

In §7 we construct holomorphic sections transverse to certain holomorphic foliations, thus generalizing Corollaries 2.9 and 2.10.

3. Approximation of noncritical functions and submersions

This section uses the Andersén–Lempert theory of holomorphic automorphisms of \mathbf{C}^n [A], [AL] as developed further in [FR], [F1], [F2]. The following is one of the main steps in our construction of noncritical holomorphic functions.

THEOREM 3.1. *Let K be a compact polynomially convex subset of \mathbf{C}^n . Let f be a holomorphic function in an open set $U \supset K$ satisfying $df \neq 0$. Given $\varepsilon > 0$ there exists a $g \in \mathcal{O}(\mathbf{C}^n)$ satisfying $dg \neq 0$ on \mathbf{C}^n and $\sup_K |f - g| < \varepsilon$.*

Proof. Choose a compact polynomially convex set $L \subset U$ with smooth boundary and containing K in the interior. Such an L may be obtained as a regular sublevel set of a strongly plurisubharmonic exhaustion function on \mathbf{C}^n which is negative on K and positive on $\mathbf{C}^n \setminus U$ [Hö2, Theorem 2.6.11].

Consider first the case $n=1$. Since $L \subset \mathbf{C}$ is smoothly bounded and polynomially convex, it is a union $L = \bigcup_{j=1}^m L_j$ of finitely many compact, connected and simply-connected sets L_j . Since $f'(z) \neq 0$ for $z \in U$, there is a holomorphic function h in a neighborhood of L such that $f'(z) = e^{h(z)}$ for each z .

For every $j=2, \dots, m$ we connect L_1 to L_j by a simple smooth arc C_j contained in $\mathbf{C} \setminus L$ except for its endpoints $a_j \in L_1$, $b_j \in L_j$. Furthermore we choose the arcs C_j to be pairwise disjoint. The sets $S := L \cup C_2 \cup \dots \cup C_m$ and $\mathbf{C} \setminus S$ are connected, and h can be extended to a smooth function on C_j satisfying $\int_{C_j} e^{h(\zeta)} d\zeta = f(b_j) - f(a_j)$ for $j=2, \dots, m$ (where C_j is oriented from a_j to b_j). By Mergelyan's theorem we can approximate h uniformly on S as close as desired by a holomorphic polynomial \tilde{h} . Choose a point $a \in L_1$ and define $g(z) = f(a) + \int_a^z e^{\tilde{h}(\zeta)} d\zeta$. The integral does not depend on the choice of the path, and hence g is an entire function on \mathbf{C} satisfying $g'(z) = e^{\tilde{h}(z)} \neq 0$ for each $z \in \mathbf{C}$. If $z \in L$, we can choose the path of integration from a to z entirely contained in S and with length bounded from above independently of z . (If $z \in L_j$ for $j > 1$, we include the arc C_j in the path of integration.) It follows that g approximates f uniformly on L . This completes the proof for $n=1$.

Assume now $n \geq 2$. Since L is polynomially convex, there exists for any $\varepsilon > 0$ a holomorphic polynomial h on \mathbf{C}^n satisfying $\sup_L |f - h| < \frac{1}{2}\varepsilon$. If ε is chosen sufficiently small then $dh \neq 0$ on K . For a generic choice of h its critical set $\Sigma = \{z \in \mathbf{C}^n : dh_z = 0\} \subset$

$\mathbf{C}^n \setminus K$ is finite (since it is given by n polynomial equations $\partial h / \partial z_j = 0$, $j=1, \dots, n$). To complete the proof we need

PROPOSITION 3.2. *Let K be a compact polynomially convex subset of \mathbf{C}^n ($n \geq 2$). Given a finite set $\Sigma \subset \mathbf{C}^n \setminus K$ and $\delta > 0$ there is a biholomorphic map ϕ of \mathbf{C}^n onto a subset $\Omega \subset \mathbf{C}^n \setminus \Sigma$ such that $|\phi(z) - z| < \delta$ for all $z \in K$.*

Recall that a biholomorphic map of \mathbf{C}^n onto a proper subset of \mathbf{C}^n is called a *Fatou–Bieberbach map*. Thus ϕ is a Fatou–Bieberbach map whose restriction to K is close to the identity map and whose range avoids Σ .

Assume for a moment that Proposition 3.2 holds. Let $c = \sup_{z \in L} |dh_z|$. Choose $\delta < \min\{\text{dist}(K, \mathbf{C}^n \setminus L), \varepsilon/2c\}$. Let $g = h \circ \phi \in \mathcal{O}(\mathbf{C}^n)$ where ϕ is furnished by Proposition 3.2. Then $dg_z = dh_{\phi(z)} \cdot d\phi_z \neq 0$ for every $z \in \mathbf{C}^n$ (since $\phi(z) \in \Omega \subset \mathbf{C}^n \setminus \Sigma$ and $dh \neq 0$ on $\mathbf{C}^n \setminus \Sigma$). For every $z \in K$ we have

$$|g(z) - h(z)| = |h(\phi(z)) - h(z)| \leq c|\phi(z) - z| < c\delta < \frac{1}{2}\varepsilon$$

and hence $|g(z) - f(z)| < \varepsilon$. This proves Theorem 3.1.

Proof of Proposition 3.2. Choose $\varepsilon \in (0, 1)$. Let B denote the closed unit ball centered at the origin in \mathbf{C}^n and rB its dilation by $r > 0$. Choose a compact set $L \subset \mathbf{C}^n \setminus \Sigma$ containing K in its interior. Let $r_1 > 1$ be chosen such that $L \subset (r_1 - 1)B$. Set $r_k = r_1 + k - 1$ and $\varepsilon_k = 2^{-k-1}\varepsilon$ for $k=1, 2, 3, \dots$.

Consider the holomorphic flow on a neighborhood of $L \cup \Sigma$ in \mathbf{C}^n which rests near L and moves the finite set Σ out of the ball r_1B . Since the trace of this flow is polynomially convex, the time-one map can be approximated uniformly on L by holomorphic automorphisms of \mathbf{C}^n according to Theorem 1.1 in [FR]. This gives a holomorphic automorphism ψ_1 of \mathbf{C}^n satisfying $|\psi_1(z) - z| < \varepsilon_1$ for $z \in L$ and $\psi_1(\Sigma) \cap r_1B = \emptyset$. (That is, we pushed Σ out of the ball r_1B by a holomorphic automorphism of \mathbf{C}^n which is ε_1 -close to the identity map on L .)

Set $\Sigma_1 = \psi_1(\Sigma)$. By the same argument there is an automorphism ψ_2 of \mathbf{C}^n satisfying $|\psi_2(z) - z| < \varepsilon_2$ for $z \in r_1B$ and $\psi_2(\Sigma_1) \cap r_2B = \emptyset$.

Continuing inductively we obtain a sequence of automorphisms ψ_k of \mathbf{C}^n such that $|\psi_k(z) - z| < \varepsilon_k$ on $r_{k-1}B$ and $\psi_k(\Sigma_{k-1}) \cap r_kB = \emptyset$ for each $k=1, 2, \dots$. By Proposition 5.1 in [F2] (which is entirely elementary) the sequence of compositions $\psi_k \circ \psi_{k-1} \circ \dots \circ \psi_1$ converges as $k \rightarrow \infty$ to a biholomorphic map $\psi: \Omega \rightarrow \mathbf{C}^n$ from an open set $\Omega \subset \mathbf{C}^n$ onto \mathbf{C}^n . By construction we have $L \subset \Omega \subset \mathbf{C}^n \setminus \Sigma$ and $|\psi(z) - z| < \varepsilon$ for $z \in L$. The inverse map $\phi = \psi^{-1}: \mathbf{C}^n \rightarrow \Omega$ is biholomorphic onto $\Omega \subset \mathbf{C}^n \setminus \Sigma$ and is uniformly close to the identity on K . Choosing ε sufficiently small we can insure that $|\phi(z) - z| < \delta$ for $z \in K$. This proves Proposition 3.2.

To construct holomorphic submersions $X \rightarrow \mathbf{C}^q$ for $1 < q < n = \dim X$ we need a suitable analogue of Theorem 3.1 for submersions $f: U \rightarrow \mathbf{C}^q$, where U is an open set in \mathbf{C}^n containing a given compact polynomially convex set $K \subset \mathbf{C}^n$. The initial approximation of f gives a polynomial map $h: \mathbf{C}^n \rightarrow \mathbf{C}^q$ for which $\Sigma := \{z \in \mathbf{C}^n : \text{rank } dh_z < q\}$ is an algebraic subvariety of \mathbf{C}^n of complex dimension $q-1$ (which is at most $n-2$). To conclude the proof as above we would need a Fatou–Bieberbach map whose range contains K but omits Σ . Unfortunately we have been unable to construct such a map, and we even have some doubts about its existence due to the possible linking of K and Σ . Instead we prove a result of this kind only for very special pairs (K, Σ) which suffices for the application at hand.

PROPOSITION 3.3. *Let $x=(z, w)$ be complex coordinates on $\mathbf{C}^n = \mathbf{C}^r \times \mathbf{C}^s$. Let $D \subset \mathbf{C}^r$ and $K \subset \mathbf{C}^n$ be compact polynomially convex sets such that $D \times \{0\}^s \subset K \subset D \times \mathbf{C}^s$ and each fiber $K_z = \{w \in \mathbf{C}^s : (z, w) \in K\}$ ($z \in D$) is convex. Assume that $s \geq 2$ and $q \leq r+1$. Let $f: U \rightarrow \mathbf{C}^q$ be a holomorphic submersion in an open set $U \subset \mathbf{C}^n$ containing K . Given $\varepsilon > 0$ and a compact set $L \subset D \times \mathbf{C}^s$ containing K , there exists a holomorphic submersion $g: V \rightarrow \mathbf{C}^q$ in an open set $V \supset L$ satisfying $\sup_K |f-g| < \varepsilon$.*

Remark. Proposition 3.3 is only used in the proof of Proposition 6.1 (§6) with $r = n-2$, $s=2$; hence the only condition on q is $q \leq n-1$.

Proof. Denote by $\pi: \mathbf{C}^n \rightarrow \mathbf{C}^r$ the projection $\pi(z, w) = z$. We can approximate f uniformly on a neighborhood of K by a polynomial map $h = (h_1, \dots, h_q): \mathbf{C}^n \rightarrow \mathbf{C}^q$. A generic choice of h insures that the set $\Sigma := \{x \in \mathbf{C}^n : \text{rank } dh_x < q\}$ is an algebraic subvariety of dimension $q-1 \leq r$ which does not intersect K and the projection $\pi|_\Sigma: \Sigma \rightarrow \mathbf{C}^r$ is proper. (This follows from the jet transversality theorem: Σ is the common zero-set of all maximal minors of the complex $(q \times n)$ -matrix $(\partial h_j / \partial x_i)$; at each point at least $n-(q-1)$ of these equations are independent. Hence for a generic choice of h the set Σ has dimension $q-1$. For a complete proof see Proposition 2 in [Fo2]. The properness of $\pi|_\Sigma$ is easily satisfied by a small rotation of coordinates.) We may assume that $L = D \times B$ for some closed ball $B \subset \mathbf{C}^s$. To complete the proof we take $g = h \circ \psi$ where ψ is furnished by the following lemma.

LEMMA 3.4. (Hypotheses as above.) *For every $\delta > 0$ there exists a holomorphic automorphism ψ of \mathbf{C}^n of the form $\psi(z, w) = (z, \beta(z, w))$ such that $\psi(L) \cap \Sigma = \emptyset$ and $\sup_{x \in K} |\psi(x) - x| < \delta$.*

Remark. If $q = n$ then Σ is a hypersurface in \mathbf{C}^n . In this case Lemma 3.4 fails since the complement $\mathbf{C}^n \setminus \Sigma$ may be Kobayashi hyperbolic, which would imply that any entire map $\psi: \mathbf{C}^n \rightarrow \mathbf{C}^n \setminus \Sigma$ is constant.

To prove Lemma 3.4 we shall need a version of Theorem 1.1 (or Theorem 2.1) from [FR] with a holomorphic dependence on parameters. Recall that a vector field is *complete* if its flow exists for all times and all initial conditions. We shall consider holomorphic vector fields on \mathbf{C}^n of the form

$$V(z, w) = \sum_{j=1}^s a_j(z, w) \frac{\partial}{\partial w_j}, \quad z \in \mathbf{C}^r, w \in \mathbf{C}^s, \quad (3.1)$$

where the a_j 's are entire (or polynomial) functions on $\mathbf{C}^n = \mathbf{C}^r \times \mathbf{C}^s$. Its flow remains in the level sets $\{z = \text{const}\}$, and V is complete on \mathbf{C}^n if and only if $V(z, \cdot)$ is complete on \mathbf{C}^s for each $z \in \mathbf{C}^r$.

LEMMA 3.5. *If $s \geq 2$ then every polynomial vector field of type (3.1) on $\mathbf{C}^r \times \mathbf{C}^s$ is a finite sum of complete polynomial fields of the same type.*

Proof. We can write a polynomial field (3.1) as a finite sum $V(z, w) = \sum_{\alpha} z^{\alpha} V_{\alpha}(w)$ where $V_{\alpha}(w) = \sum_{j=1}^s a_{\alpha, j}(w) \partial / \partial w_j$ and $z^{\alpha} = z_1^{\alpha_1} \dots z_r^{\alpha_r}$. By [AL] every polynomial holomorphic vector field on \mathbf{C}^s for $s \geq 2$ is a finite sum of complete polynomial fields (see the Appendix in [F1] for a short proof). Hence each $V_{\alpha}(w)$ is a finite sum of complete polynomial fields. The products of such fields with z^{α} are complete on \mathbf{C}^n , which proves the result.

Remark. For a more general result in this direction see Lemma 2.5 in [V] and the recent preprint [Ku].

Lemma 3.5 implies that the time- t map of any entire holomorphic vector field (3.1) can be approximated, uniformly on any compact set on which it exists, by holomorphic automorphisms of \mathbf{C}^n of the form $(z, w) \rightarrow (z, \varphi(z, w))$ (Lemma 1.4 in [FR]). The same holds for time-dependent entire holomorphic vector fields of the form (3.1). From this one obtains the following parametric version of Theorem 2.1 from [FR].

COROLLARY 3.6. *Assume that $\phi_t: \Omega_0 \rightarrow \Omega_t$ ($t \in [0, T]$) is a smooth isotopy of biholomorphic maps between domains in \mathbf{C}^n , with ϕ_0 the identity map on Ω_0 , where $n = r + s$, $s \geq 2$, and each ϕ_t is of the form $\phi_t(z, w) = (z, \varphi_t(z, w))$ ($z \in \mathbf{C}^r$, $w \in \mathbf{C}^s$). If $M \subset \Omega_0$ is a compact polynomially convex set such that $\phi_t(M)$ is polynomially convex in \mathbf{C}^n for every $t \in [0, T]$ then ϕ_T can be approximated, uniformly on M , by automorphisms of \mathbf{C}^n of the form $(z, w) \rightarrow (z, \varphi(z, w))$.*

Proof of Lemma 3.4. Let Σ , K and L be as in the lemma. The set $\Sigma' := \Sigma \cap (D \times \mathbf{C}^s)$ is polynomially convex, $\Sigma' \cap K = \emptyset$ and $M := K \cup \Sigma'$ is also polynomially convex. Let $\theta_t(z, w) = (z, e^t w)$. Since the fibers K_z ($z \in D$) are convex and contain the origin, the

subvariety $\theta_t(\Sigma) \subset \mathbf{C}^n$ is disjoint from K and $M_t = K \cup \theta_t(\Sigma')$ is polynomially convex for every $t \geq 0$. Clearly $\theta_T(\Sigma) \cap L = \emptyset$ for a sufficiently large $T > 0$.

Consider the flow ϕ_t which rests on a neighborhood of K and equals θ_t on a neighborhood of Σ' . Since $\phi_t(M) = M_t$ is polynomially convex for every $t \geq 0$, Corollary 3.6 gives an automorphism $\phi(z, w) = (z, \varphi(z, w))$ approximating ϕ_T uniformly on a neighborhood of M . Thus ϕ is close to the identity on a neighborhood of K and $\phi(\Sigma') \cap L = \emptyset$. Since ϕ maps each of the affine planes $\{z\} \times \mathbf{C}^s$ to itself, it follows that $\phi(\Sigma) \cap L = \emptyset$. The inverse $\psi = \phi^{-1}$ clearly satisfies Lemma 3.4.

4. Compositional splitting of biholomorphic mappings

Let X be a complex manifold of dimension n . An injective holomorphic map $\gamma: V \rightarrow X$ in an open set $V \subset X$ will be called *biholomorphic*. Set $\Delta = \{\zeta \in \mathbf{C} : |\zeta| < 1\}$. Suppose that \mathcal{F} is a nonsingular holomorphic foliation of X of dimension p and codimension $q = n - p$. Every $x \in X$ is contained in a *distinguished chart* (U, ϕ) , where $U \subset X$ is an open subset containing x and $\phi: U \rightarrow \Delta^n \subset \mathbf{C}^n$ is a biholomorphic map onto the open unit polydisc in \mathbf{C}^n such that, in the coordinates (z, w) on $\Delta^n = \Delta^p \times \Delta^q$ ($z \in \Delta^p, w \in \Delta^q$), $\phi(\mathcal{F}|_U)$ is given by $\{w = c\}$, $c \in \Delta^q$. Fix a number $0 < r < 1$. For any distinguished chart (U, ϕ) on X let $U' \subset U$ be defined by $\phi(U') = (r\Delta^p) \times \Delta^q$. Given any relatively compact set $V \subset X$, there exists a finite collection of distinguished charts $\mathcal{U} = \{(U_j, \phi_j) : 1 \leq j \leq N\}$ such that $\bar{V} \subset \bigcup_{j=1}^N U'_j$ and \mathcal{U} is \mathcal{F} -regular in the sense of Definition 1.5 in [God, p. 72] (this means that for every $U_i, U_j \in \mathcal{U}$ the set $\bar{U}_i \cap \bar{U}_j$ is contained in a distinguished chart).

Definition. A biholomorphic map $\gamma: V \rightarrow X$ is said to be an \mathcal{F} -map if there exists \mathcal{U} as above such that for every $(U_j, \phi_j) \in \mathcal{U}$ the restriction of γ to $V \cap U'_j$ has range in U_j and is of the form $(z, w) \rightarrow (c_j(z, w), w)$ in the distinguished holomorphic coordinates on U_j .

Thus an \mathcal{F} -map preserves the leaves of \mathcal{F} and does not permute the connected components of a global leaf intersected with any of the distinguished sets U_j . The definition is good since the transition map between a pair of distinguished charts preserves this form of the map. Any γ preserving the leaves of \mathcal{F} (in the sense that x and $\gamma(x)$ belong to the same leaf) which is close to the identity map in the *fine topology* on X , defined by \mathcal{F} , is of this form. (The restriction of the fine topology to any distinguished local chart $U \simeq \Delta^p \times \Delta^q$ is the product of the usual topology on Δ^p and the discrete topology on Δ^q . For further details see [God, pp. 2–3 and pp. 71–75].)

THEOREM 4.1. *Let A and B be compact sets in a complex manifold X such that $D = A \cup B$ has a basis of Stein neighborhoods in X and $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$. Given an open set $\tilde{C} \subset X$ containing $C := A \cap B$ there exist open sets $A' \supset A, B' \supset B, C' \supset C$, with $C' \subset$*

$A' \cap B' \subset \tilde{C}$, satisfying the following. For every biholomorphic map $\gamma: \tilde{C} \rightarrow X$ which is sufficiently uniformly close to the identity on \tilde{C} there exist biholomorphic maps $\alpha: A' \rightarrow X$, $\beta: B' \rightarrow X$, uniformly close to the identity on their respective domains and satisfying

$$\gamma = \beta \circ \alpha^{-1} \quad \text{on } C'.$$

If \mathcal{F} is a holomorphic foliation of X and γ is an \mathcal{F} -map on \tilde{C} then we can choose α and β to be \mathcal{F} -maps on A' and B' , respectively. If X_0 is a closed complex subvariety of X such that $X_0 \cap C = \emptyset$ then we can choose α and β as above such that they are tangent to the identity map to any given finite order along X_0 .

Theorem 4.1, which is a key ingredient in our construction of noncritical holomorphic functions and submersions, is proved in this section by a Kolmogorov–Nash–Moser-type rapidly convergent iteration. (We shall only need it for the trivial foliation with X as the only leaf, but we prove the extended version for possible future applications.) It will be used in Propositions 5.2 and 6.1 below to patch pairs of noncritical functions or submersions (we have already explained in the introduction why the standard $\bar{\partial}$ -theory does not suffice). Similar decompositions have been used in the theory of quasiconformal mappings and in complex dynamics. For example, a theorem of Pfluger [Pf] from 1961 asserts that every orientation-preserving quasiconformal homeomorphism $\gamma: \mathbf{R} \rightarrow \mathbf{R}$ is the restriction to \mathbf{R} of the composition $\beta \circ \alpha^{-1}$, where α and β are conformal maps of the upper and lower half-plane, respectively, to itself which map \mathbf{R} to \mathbf{R} . (See also [LV, p. 92].)

We begin with preparatory results. We fix once and for all a complete distance function $d: X \times X \rightarrow \mathbf{R}_+$ induced by a smooth Riemannian metric on TX . Given a subset $A \subset X$ and an $r > 0$ we set

$$A(r) = \{x \in X: d(x, y) < r \text{ for some } y \in A\}.$$

If A is a (relatively) compact, smoothly bounded domain in X then for all sufficiently small $r > 0$ the set $A(r)$ is a smoothly bounded open domain.

We say that the subsets $A, B \subset X$ are *separated* if $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$.

LEMMA 4.2. *Given $A, B \subset X$ and $r > 0$ then we have $(A \cup B)(r) = A(r) \cup B(r)$ and $(A \cap B)(r) \subset A(r) \cap B(r)$. If A and B are (relatively) compact and separated in X then for all sufficiently small $r > 0$ we also have $(A \cap B)(r) = A(r) \cap B(r)$, and the sets $\overline{A(r)}$ and $\overline{B(r)}$ are separated.*

Proof. The first two properties are immediate. Now write $A = (A \setminus B) \cup (A \cap B)$, $B = (B \setminus A) \cup (A \cap B)$ and apply the first property to get

$$A(r) = (A \setminus B)(r) \cup (A \cap B)(r), \quad B(r) = (B \setminus A)(r) \cup (A \cap B)(r).$$

If $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$ then for all sufficiently small $r > 0$ we have $(A \setminus B)(r) \cap (B \setminus A)(r) = \emptyset$ (in fact, even the closures of $(A \setminus B)(r)$ and $(B \setminus A)(r)$ are disjoint). Hence the previous display gives $A(r) \cap B(r) = (A \cap B)(r)$ as well as the separation property for the pair $\overline{A(r)}, \overline{B(r)}$.

LEMMA 4.3. *Let $A, B \subset X$ be compact sets in a complex manifold X satisfying $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$. Assume that $A \cup B$ has a basis of Stein neighborhoods. Given open sets $\tilde{A} \supset A, \tilde{B} \supset B, \tilde{C} \supset C = A \cap B, \tilde{D} \supset A \cup B$, there exist compact sets $A', B' \subset X$ satisfying*

$$(a) \quad A \subset A' \subset \tilde{A}, \quad B \subset B' \subset \tilde{B}, \quad A' \cap B' \subset \tilde{C};$$

$$(b) \quad \overline{A' \setminus B'} \cap \overline{B' \setminus A'} = \emptyset;$$

(c) *the set $D' = A' \cup B' \subset \tilde{D}$ is the closure of a smoothly bounded strongly pseudoconvex Stein domain in X .*

Proof. If $r > 0$ is chosen sufficiently small then by Lemma 4.2 we have $A(r) \subset \subset \tilde{A}$, $B(r) \subset \subset \tilde{B}$, $A(r) \cap B(r) = C(r) \subset \subset \tilde{C}$, and the sets $\overline{A(r)}, \overline{B(r)}$ are separated. By assumption there is a closed strongly pseudoconvex Stein domain $D' \subset X$ with $A \cup B \subset D' \subset A(r) \cup B(r)$. It is easily verified that the sets $A' = \overline{A(r)} \cap D'$, $B' = \overline{B(r)} \cap D'$ satisfy the stated properties.

Due to Lemma 4.3 it suffices to prove Theorem 4.1 under the assumption that X is a Stein manifold, $A, B \subset X$ is pair of separated compact subsets and $D = A \cup B$ is the closure of a smoothly bounded strongly pseudoconvex domain. We assume this to be the case for the rest of this section.

Let \mathcal{F} be a holomorphic foliation of X with leaves \mathcal{F}_x ($x \in X$). By Cartan's Theorem A the tangent bundle $T\mathcal{F} \subset TX$ of \mathcal{F} is spanned by finitely many holomorphic vector fields L_1, L_2, \dots, L_m on X . (We may have to shrink X a bit.) Denote by $\theta_t^j(x)$ the flow of L_j for time $t \in \mathbf{C}$, solving $(\partial/\partial t)\theta_t^j(x) = L_j(\theta_t^j(x))$ and $\theta_0^j(x) = x$. The map θ^j is defined and holomorphic for (x, t) in an open neighborhood of $X \times \{0\}$ in $X \times \mathbf{C}$. Their composition

$$\theta(x, t) = \theta(x, t_1, \dots, t_m) := \theta_{t_m}^m \circ \dots \circ \theta_{t_2}^2 \circ \theta_{t_1}^1(x) \in X$$

is a holomorphic map on an open neighborhood $U \subset X \times \mathbf{C}^m$ of the zero-section $X \times \{0\}^m$, satisfying $\theta(x, t) \in \mathcal{F}_x$ for all $(x, t) \in U$ and

$$\theta(x, 0) = x, \quad \frac{\partial}{\partial t_j} \theta(x, t)|_{t=0} = L_j(x), \quad x \in X, \quad 1 \leq j \leq m.$$

Hence $\Theta := \partial_t \theta|_{t=0}$ maps the trivial bundle $X \times \mathbf{C}^m$ surjectively onto the tangent bundle $T\mathcal{F}$ of \mathcal{F} . Splitting $X \times \mathbf{C}^m = E \oplus \ker \Theta$ we see that $\Theta: E \rightarrow T\mathcal{F}$ is an isomorphism of holomorphic vector bundles. In any holomorphic vector bundle chart on E we have a

Taylor expansion

$$\theta(x, t_1, \dots, t_m) = x + \sum_{j=1}^m t_j L_j(x) + O(|t|^2) \quad (4.1)$$

where the remainder $O(|t|^2)$ is uniform on any compact subset of the base set.

Choose a Hermitian metric $|\cdot|_E$ on E . Given an open set $V \subset X$ and a section $c: V \rightarrow E|_V$ we shall write $\|c\|_V = \sup_{x \in V} |c(x)|_E$. By the construction of θ and E , $x \rightarrow \theta(x, c(x))$ is an \mathcal{F} -map provided that $\|c\|_V$ is sufficiently small.

Given a map $\gamma: V \rightarrow X$ we define $\|\gamma - \text{id}\|_V = \sup_{x \in V} d(\gamma(x), x)$, and we say that γ is ε -close to the identity on V if $\|\gamma - \text{id}\|_V < \varepsilon$. The following lemma follows from the implicit function theorem.

LEMMA 4.4. *For every open relatively compact set $V \subset X$ there exist constants $M_1 \geq 1$ and $\varepsilon_0 > 0$ satisfying the following property. For every \mathcal{F} -map $\gamma: V \rightarrow X$ with $\|\gamma - \text{id}\|_V < \varepsilon_0$ there is a unique holomorphic section $c: V \rightarrow E$ of $E|_V \rightarrow V$ such that for every $x \in V$ we have $\theta(x, c(x)) = \gamma(x)$ and*

$$M_1^{-1}|c(x)| \leq d(\gamma(x), x) \leq M_1|c(x)|.$$

If \mathcal{F} is the trivial foliation with X as the only leaf, Lemma 4.4 asserts that every biholomorphic map $\gamma: V \rightarrow X$ sufficiently close to the identity map has the form $\gamma(x) = \theta(x, c(x))$ for some holomorphic section $c: V \rightarrow TV$.

We shall write the composition $\gamma \circ \alpha$ simply as $\gamma\alpha$. From now on all our sets in X will be assumed contained in a fixed relatively compact set for which Lemma 4.4 holds with a constant M_1 . Recall that $V(\delta)$ denotes the open δ -neighborhood of $V \subset X$ with respect to the distance function d .

LEMMA 4.5. *Let $V \subset X$. There are constants $\delta_0 > 0$ (small) and $M_2 > 0$ (large) with the following property. Let $0 < \delta < \delta_0$ and $0 < 4\varepsilon < \delta$. Assume that $\alpha, \beta, \gamma: V(\delta) \rightarrow X$ are \mathcal{F} -maps which are ε -close to the identity on $V(\delta)$. Then $\tilde{\gamma} := \beta^{-1}\gamma\alpha: V \rightarrow X$ is a well-defined \mathcal{F} -map on V . Write*

$$\begin{aligned} \alpha(x) &= \theta(x, a(x)), & \beta(x) &= \theta(x, b(x)), \\ \gamma(x) &= \theta(x, c(x)), & \tilde{\gamma}(x) &= \theta(x, \tilde{c}(x)), \end{aligned}$$

where a, b, c are sections of $E|_{V(\delta)} \rightarrow V(\delta)$ and \tilde{c} is a section of $E|_V \rightarrow V$ given by Lemma 4.4. Then

$$\|\tilde{c} - (c + a - b)\|_V \leq M_2 \delta^{-1} \varepsilon^2. \quad (4.2)$$

If $c = b - a$ on $V(\delta)$ then $\|\tilde{c}\|_V \leq M_2 \delta^{-1} \varepsilon^2$ and $\|\tilde{\gamma} - \text{id}\|_V \leq M_1 M_2 \delta^{-1} \varepsilon^2$.

Proof. The conditions imply that $\gamma\alpha$ maps V biholomorphically onto a subset of $V(2\varepsilon)$. Since β is ε -close to the identity map on $V(\delta)$, the degree theory shows that

its range contains $V(\delta-\varepsilon)$. Hence β^{-1} is defined on $V(\delta-\varepsilon)$ and is ε -close to the identity on this set. Since $4\varepsilon < \delta$, it follows that $\tilde{\gamma} = \beta^{-1}\gamma\alpha$ is defined on V and maps V biholomorphically onto a subset of $V(3\varepsilon) \subset \subset V(\delta)$.

To prove the estimate (4.2) we choose a holomorphic vector bundle chart on $\pi: E \rightarrow X$ over an open set $\tilde{U} \subset X$ and let $U \subset \subset \tilde{U}$. We shall use the expansion (4.1) for θ on $\pi^{-1}(U) \subset E$; this suffices since $\overline{V(\delta)}$ can be covered by finitely many such sets U . We replace the fiber variable t in (4.1) by one of the functions $a(x)$, $b(x)$ or $c(x)$. These are bounded on $V(\delta)$ by $M_1\varepsilon$ where M_1 is the constant from Lemma 4.4. This gives for $x \in U \cap V(\delta)$,

$$\begin{aligned}\alpha(x) &= x + \sum_{j=1}^m a_j(x)L_j(x) + O(\varepsilon^2), \\ \beta(x) &= x + \sum_{j=1}^m b_j(x)L_j(x) + O(\varepsilon^2), \\ \gamma(x) &= x + \sum_{j=1}^m c_j(x)L_j(x) + O(\varepsilon^2),\end{aligned}$$

where the remainder term $O(\varepsilon^2)$ is uniform with respect to $x \in U \cap V(\delta)$. For $x \in U \cap V$ this gives

$$\begin{aligned}\gamma(\alpha(x)) &= \alpha(x) + \sum_{j=1}^m c_j(\alpha(x))L_j(\alpha(x)) + O(\varepsilon^2) \\ &= x + \sum_{j=1}^m (a_j(x) + c_j(x))L_j(x) + \sum_{j=1}^m (c_j(\alpha(x))L_j(\alpha(x)) - c_j(x)L_j(x)) + O(\varepsilon^2).\end{aligned}$$

To estimate terms in the last sum we fix j and write $g(x) = c_j(x)L_j(x)$ for $x \in U \cap V(\delta)$. Since $\|c_j\|_{V(\delta)} < M_1\varepsilon$ and $4\varepsilon < \delta$, the Cauchy estimates imply $\|dc_j\|_{U \cap V(\varepsilon)} = O(\varepsilon/\delta)$ (here dc_j denotes the differential of c_j). Since L_j is holomorphic in a neighborhood of $\overline{V(\delta)}$, we may assume that its expression in the local coordinates on U is uniformly bounded and has uniformly bounded differential. This gives $\|dg\|_{U \cap V(\varepsilon)} = O(\varepsilon/\delta)$. Since $d(x, \alpha(x)) < \varepsilon$, there is a smooth arc $\lambda: [0, 1] \rightarrow U$, of length comparable to ε , such that $\lambda(0) = x$ and $\lambda(1) = \alpha(x)$. Then

$$|g(\alpha(x)) - g(x)| \leq \int_0^1 |dg(\lambda(\tau)) \cdot |\lambda'(\tau)| d\tau \leq O(\delta^{-1}\varepsilon^2)$$

(the extra ε is contributed by the length of λ). This gives for $x \in U \cap V$,

$$\gamma(\alpha(x)) = x + \sum_{j=1}^m (a_j(x) + c_j(x))L_j(x) + O(\delta^{-1}\varepsilon^2).$$

The same argument holds for the composition of several maps provided that ε is sufficiently small in comparison to δ ; the error term remains of order $O(\delta^{-1}\varepsilon^2)$.

It remains to find the Taylor expansion of β^{-1} on the set $U \cap V(2\varepsilon)$ where U is a local chart as above. Set $\tilde{\beta}(x) = x - \sum_{j=1}^m b_j(x)L_j(x)$ for $x \in U \cap V(\delta)$. Assuming that $\beta(x) \in U \cap V(2\varepsilon)$ we obtain

$$\begin{aligned} \tilde{\beta}(\beta(x)) &= \beta(x) - \sum_{j=1}^m b_j(\beta(x))L_j(\beta(x)) \\ &= x + \sum_{j=1}^m (b_j(x)L_j(x) - b_j(\beta(x))L_j(\beta(x))) + O(\varepsilon^2) \\ &= x + O(\delta^{-1}\varepsilon^2). \end{aligned}$$

We have estimated the terms in the parentheses on the middle line by $O(\delta^{-1}\varepsilon^2)$ in exactly the same way as above, using the Cauchy estimates and integrating over an arc of length comparable to ε . Writing $\beta(x) = y \in U \cap V(2\varepsilon)$, $x = \beta^{-1}(y)$, the above gives $\tilde{\beta}(y) = \beta^{-1}(y) + O(\delta^{-1}\varepsilon^2)$ and therefore

$$\beta^{-1}(y) = y - \sum_{j=1}^m b_j(y)L_j(y) + O(\delta^{-1}\varepsilon^2).$$

The same argument as before gives

$$\tilde{\gamma}(x) = (\beta^{-1}\gamma\alpha)(x) = x + \sum_{j=1}^m (c_j(x) + a_j(x) - b_j(x))L_j(x) + O(\delta^{-1}\varepsilon^2)$$

for $x \in U \cap V$. This proves the estimate (4.2).

Remark. The proof of Lemmas 4.4 and 4.5 shows that for each fixed open set $V_0 \subset\subset X$ the constants M_1, M_2, δ_0 may be chosen independent of V for any open set $V \subset V_0$. In this case, $O(\delta^{-1}\varepsilon^2)$ means $\leq C\delta^{-1}\varepsilon^2$, with C independent of ε, δ and V .

LEMMA 4.6. *Let $E \rightarrow X$ be a holomorphic vector bundle over a Stein manifold X . Let $U, V \subset X$ be open sets such that $\overline{U} \setminus \overline{V} \cap \overline{V} \setminus \overline{U} = \emptyset$ and $D = U \cup V$ is a relatively compact, smoothly bounded, strongly pseudoconvex domain in X . Set $W = U \cap V$. There is a constant $M_3 \geq 1$ such that for every bounded holomorphic section $c: W \rightarrow E|_W$ there exist bounded holomorphic sections $a: U \rightarrow E|_U, b: V \rightarrow E|_V$ satisfying*

$$c = b|_W - a|_W, \quad \|a\|_U < M_3\|c\|_W, \quad \|b\|_V < M_3\|c\|_W.$$

Such a and b are given by bounded linear operators between the spaces of bounded holomorphic sections of E on the respective sets. The constant M_3 can be chosen uniform

for all such pairs (U, V) in X close to an initial pair (U_0, V_0) provided that $D=U\cup V$ is sufficiently \mathcal{C}^2 -close to $D_0=U_0\cup V_0$. If X_0 is a closed complex subvariety of X and $X_0\cap\overline{W}=\emptyset$ then for every $s\in\mathbf{N}$ we can insure in addition that a and b vanish to order s on X_0 .

Proof. This is a standard application of the solvability of the $\bar{\partial}$ -equation. We give a brief sketch for the sake of completeness.

Condition (b) insures that there is a smooth function $\chi: X\rightarrow[0, 1]$ which equals zero in a neighborhood of $\overline{U}\setminus\overline{V}$ and equals one in a neighborhood of $\overline{V}\setminus\overline{U}$. Since $D=U\cup V$ is a relatively compact strongly pseudoconvex domain in X , there exists a bounded linear solution operator T for the $\bar{\partial}$ -equation associated to sections of $E\rightarrow X$ over D . Precisely, for any bounded $\bar{\partial}$ -closed E -valued $(0, 1)$ -form g on D we have $\bar{\partial}_E(T(g))=g$ and $\|T(g)\|_D\leq\text{const}\|g\|_D$, and the constant can be chosen uniform for all domains in X which are sufficiently \mathcal{C}^2 -close to an initial strongly pseudoconvex domain. (For functions this can be found in [HL1, p. 82]; the problem for sections of a vector bundle E can be reduced to that for functions by embedding E as a subbundle of a trivial bundle over X .)

Observe that χc extends to a bounded smooth section of E over U , and $(\chi-1)c$ extends to a bounded section over V . Since $\text{supp}(\bar{\partial}\chi)\cap D\subset W=U\cap V$, the bounded $(0, 1)$ -form $g=\bar{\partial}(\chi c)=\bar{\partial}((\chi-1)c)=c\bar{\partial}\chi$ on W extends to a bounded $(0, 1)$ -form on D which is zero outside of W . It is immediate that the pair of sections

$$a=-\chi c+T(g)|_U, \quad b=(1-\chi)c+T(g)|_V$$

satisfies Lemma 4.6. The last statement (regarding the interpolation on X_0) follows in the case of functions from [FP2, Lemma 3.2]; the same proof applies to sections of $E\rightarrow X$ by embedding E into a trivial bundle over X .

LEMMA 4.7. *Let $A, B\subset X$ be compact sets such that $\overline{A}\setminus\overline{B}\cap\overline{B}\setminus\overline{A}=\emptyset$ and $D=A\cup B$ is a closed, smoothly bounded, strongly pseudoconvex domain in X . Let \mathcal{F} be a holomorphic foliation of X and let X_0 be a closed complex subvariety of X with $X_0\cap C=\emptyset$, where $C=A\cap B$. Then there are constants $r_0>0$, $\delta_0>0$ (small) and $M_4, M_5>1$ (large) satisfying the following. Let $0<r\leq r_0$, $0<\delta\leq\delta_0$ and $s\in\mathbf{N}$. For every \mathcal{F} -map $\gamma:C(r+\delta)\rightarrow X$ satisfying $4M_4\|\gamma-\text{id}\|_{C(r+\delta)}<\delta$ there exist \mathcal{F} -maps $\alpha:A(r+\delta)\rightarrow X$ and $\beta:B(r+\delta)\rightarrow X$, tangent to the identity map to order s along X_0 , such that $\tilde{\gamma}:=\beta^{-1}\gamma\alpha$ is an \mathcal{F} -map on $C(r)$ satisfying*

$$\|\tilde{\gamma}-\text{id}\|_{C(r)}<M_5\delta^{-1}\|\gamma-\text{id}\|_{C(r+\delta)}^2. \quad (4.3)$$

Proof. If r_0 and δ_0 are chosen sufficiently small, the set $D(t)$ is a small \mathcal{C}^2 -perturbation of the strongly pseudoconvex domain $D=A\cup B$ for every $t\in[0, r_0+\delta_0]$, and hence

we can use the same constant as a bound on the sup-norm of an operator solving the $\bar{\partial}$ -problem on $D(t)$.

Let $\varepsilon = \|\gamma - \text{id}\|_{C(r+\delta)}$. By Lemma 4.4 there is a holomorphic section $c: C(r+\delta) \rightarrow E$, with $\|c\|_{C(r+\delta)} \leq M_1 \varepsilon$, such that $\gamma(x) = \theta(x, c(x))$. (Here we can use the constant M_1 for the set $D(r_0 + \delta_0)$.) Write $c = b - a$ where a is a section of E over $A(r+\delta)$ and b is a section of E over $B(r+\delta)$ furnished by Lemma 4.6. The sup-norms of a and b on their respective domains are bounded by $M_1 M_3 \varepsilon$, where the constant M_3 from Lemma 4.6 can be chosen independent of r and δ . Set

$$\begin{aligned}\alpha(x) &= \theta(x, a(x)), & x \in A(r+\delta), \\ \beta(x) &= \theta(x, b(x)), & x \in B(r+\delta).\end{aligned}$$

By Lemma 4.4 we have $\|\alpha - \text{id}\|_{A(r+\delta)} < M_1^2 M_3 \varepsilon$ and $\|\beta - \text{id}\|_{B(r+\delta)} < M_1^2 M_3 \varepsilon$. Set $M_4 = M_1^2 M_3$. If $0 < 4M_4 \varepsilon < \delta$ then by Lemma 4.5 the composition $\tilde{\gamma} = \beta^{-1} \gamma \alpha$ is an \mathcal{F} -map on $C(r)$ satisfying the estimate (4.3) with $M_5 = M_2 M_4^2 = M_1^4 M_2 M_3^2$. This completes the proof.

Proof of Theorem 4.1. By Lemma 4.3 we may assume that $D = A \cup B$ is the closure of a smooth strongly pseudoconvex domain in X and $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$. Choose a sufficiently small number $0 < r_0 < 1$ such that the initial \mathcal{F} -map γ is defined on the set $C_0 := C(r_0)$ and Lemma 4.7 holds for all $\delta, r > 0$ with $\delta + r \leq r_0$. For each $k = 0, 1, 2, \dots$ we set

$$r_k = r_0 \prod_{j=1}^k (1 - 2^{-j}), \quad \delta_k = r_k - r_{k+1} = r_k 2^{-k-1}.$$

The sequence $r_k > 0$ is decreasing, $r^* = \lim_{k \rightarrow \infty} r_k > 0$, $\delta_k > r^* 2^{-k-1}$ for all k , and $\sum_{k=0}^{\infty} \delta_k = r_0 - r^*$. Set $A_k = A(r_k)$, $B_k = B(r_k)$ and $C_k = C(r_k)$. We choose $r_0 > 0$ sufficiently small such that $C_k = A_k \cap B_k$ for all k (Lemma 4.2).

Let $\varepsilon_0 := \|\gamma - \text{id}\|_{C_0}$. Assuming that $4M_4 \varepsilon_0 < \delta_0 = \frac{1}{2} r_0$, Lemma 4.7 gives \mathcal{F} -maps $\alpha_0: A_0 \rightarrow X$ and $\beta_0: B_0 \rightarrow X$ such that $\gamma_1 = \beta_0^{-1} \gamma \alpha_0: C_1 \rightarrow X$ is an \mathcal{F} -map defined on C_1 , satisfying

$$\|\gamma_1 - \text{id}\|_{C_1} < M_5 \delta_0^{-1} \varepsilon_0^2 < 2M \varepsilon_0^2,$$

where we have set $M = M_5 / r^*$. Define $\varepsilon_1 = \|\gamma_1 - \text{id}\|_{C_1}$, so $\varepsilon_1 < 2M \varepsilon_0^2$. Assuming for a moment that $4M_4 \varepsilon_1 < \delta_1$, we can apply Lemma 4.7 to obtain a pair of \mathcal{F} -maps $\alpha_1: A_1 \rightarrow X$, $\beta_1: B_1 \rightarrow X$ such that $\gamma_2 = \beta_1^{-1} \gamma_1 \alpha_1: C_2 \rightarrow X$ is an \mathcal{F} -map satisfying

$$\varepsilon_2 := \|\gamma_2 - \text{id}\|_{C_2} < M_5 \delta_1^{-1} \varepsilon_1^2 < 2^2 M \varepsilon_1^2.$$

Continuing inductively we obtain sequences of \mathcal{F} -maps

$$\alpha_k: A_k \rightarrow X, \quad \beta_k: B_k \rightarrow X, \quad \gamma_k: C_k \rightarrow X$$

such that $\gamma_{k+1} = \beta_k^{-1} \gamma_k \alpha_k: C_{k+1} \rightarrow X$ is an \mathcal{F} -map satisfying

$$\varepsilon_{k+1} := \|\gamma_{k+1} - \text{id}\|_{C_{k+1}} < M_5 \delta_k^{-1} \varepsilon_k^2 < 2^{k+1} M \varepsilon_k^2. \tag{4.4}$$

The necessary condition for the induction step is that $4M_4 \varepsilon_k < \delta_k$ holds for each k . Since $\delta_k > r^* 2^{-k-1}$, it suffices to have

$$4M_4 \varepsilon_k < r^* 2^{-k-1}, \quad k = 0, 1, 2, \dots \tag{4.5}$$

In order to obtain convergence of this process we need

LEMMA 4.8. *Let $M, M_4 \geq 1$. Let the sequence $\varrho_k > 0$ be defined recursively by $\varrho_0 = \varepsilon_0 > 0$ and $\varrho_{k+1} = 2^{k+1} M \varrho_k^2$ for $k = 0, 1, \dots$. If $\varepsilon_0 < r^*/32MM_4$ then $\varrho_k < (4M\varepsilon_0)^{2^k} < (\frac{1}{8})^{2^k}$ and $4M_4 \varrho_k < r^* 2^{-k-1}$ for all $k = 0, 1, 2, \dots$.*

Assuming Lemma 4.8 we complete the proof of Theorem 4.1 as follows. From (4.4) we see that $\varepsilon_k \leq \varrho_k$ where ϱ_k is the sequence from Lemma 4.8. From the assumption $\varepsilon_0 < r^*/32MM_4$ we obtain $q := 4M\varepsilon_0 < r^*/8M_4 < \frac{1}{8}$ (since $0 < r^* < 1$ and $M_4 \geq 1$). Hence the sequence $\varepsilon_k = \|\gamma_k - \text{id}\|_{C_k} < q^{2^k} < (\frac{1}{8})^{2^k}$ converges to zero very rapidly as $k \rightarrow \infty$. The second estimate on ϱ_k in Lemma 4.8 insures that (4.5) holds, and hence the induction described above works.

Setting $\tilde{\alpha}_k = \alpha_0 \alpha_1 \dots \alpha_k: A_k \rightarrow X$, $\tilde{\beta}_k = \beta_0 \beta_1 \dots \beta_k: B_k \rightarrow X$, we have $\gamma_{k+1} = \tilde{\beta}_k^{-1} \gamma \tilde{\alpha}_k$ on C_{k+1} for $k = 0, 1, 2, \dots$. Our construction insures that, as $k \rightarrow \infty$, the sequences $\tilde{\alpha}_k$ and $\tilde{\beta}_k$ converge, uniformly on $A(r^*)$ and $B(r^*)$, respectively, to \mathcal{F} -maps $\alpha: A(r^*) \rightarrow X$ and $\beta: B(r^*) \rightarrow X$, respectively. Furthermore, the sequence γ_k converges uniformly on $C(r^*)$ to the identity map according to (4.4) and Lemma 4.8. In the limit we obtain $\beta^{-1} \gamma \alpha = \text{id}$ on $C(r^*)$, and hence $\gamma = \beta \alpha^{-1}$ on $\alpha(C(r^*))$. If $\varepsilon_0 > 0$ is chosen sufficiently small (for a fixed r_0) then the latter set contains a neighborhood C' of C . This completes the proof of Theorem 4.1, provided that Lemma 4.8 holds.

Proof of Lemma 4.8. The sequence is of the form $\varrho_k = 2^{a_k} M^{b_k} \varepsilon_0^{c_k}$ where the exponents satisfy the recursive relations

$$\begin{aligned} a_{k+1} &= 2a_k + k + 1, & a_0 &= 0; \\ b_{k+1} &= 2b_k + 1, & b_0 &= 0; \\ c_{k+1} &= 2c_k, & c_0 &= 1. \end{aligned}$$

The solutions are $a_k = 2^k \sum_{j=1}^k j 2^{-j} < 2^{k+1}$, $b_k = 2^k - 1$, $c_k = 2^k$. Thus

$$\varrho_k < 2^{2^{k+1}} M^{2^k} \varepsilon_0^{2^k} = (4M\varepsilon_0)^{2^k},$$

which proves the first required estimate. From the assumption $\varepsilon_0 < r^*/32MM_4$ we get $q := 4M\varepsilon_0 < r^*/8M_4 < \frac{1}{8}$. Hence $\varrho_k < q^{2^k} < (\frac{1}{8})^{2^k}$ and

$$4M_4\varrho_k < (4M_4q)q^{2^k-1} = (4M_4 \cdot 4M\varepsilon_0)(\frac{1}{8})^{2^k-1} < (\frac{1}{2}r^*)2^{-k} = r^*2^{-k-1}$$

for all $k \geq 0$, which proves the second estimate. Lemma 4.8 is proved.

Remark. The above construction actually gives nonlinear operators \mathcal{A}, \mathcal{B} on the set of \mathcal{F} -maps γ which are sufficiently uniformly close to the identity on a fixed neighborhood of C such that the pair of \mathcal{F} -maps $\alpha = \mathcal{A}(\gamma), \beta = \mathcal{B}(\gamma)$ satisfies $\gamma = \beta\alpha^{-1}$. This yields the analogous result for families $\{\gamma_p : p \in P\}$ of \mathcal{F} -maps which depend continuously on a parameter p in a compact Hausdorff space P and which are sufficiently close to the identity map on a neighborhood of C .

5. Construction of noncritical holomorphic functions

A compact set K in a complex manifold X is said to be a *Stein compactum* if it has a basis of open Stein neighborhoods. Let d be a distance function on X induced by a smooth Riemannian metric on TX . We shall use the terminology introduced in §4. Recall that $\|\gamma - \text{id}\|_V = \sup_{x \in V} d(\gamma(x), x)$.

LEMMA 5.1. *Let K be a Stein compactum in a complex manifold X . Let $U \subset X$ be an open set containing K , and $f : U \rightarrow \mathbf{C}^q$ a holomorphic submersion for some $q \leq \dim X$. Then there exist constants $\varepsilon_0 > 0, M > 0$ and an open set $V \subset X$, with $K \subset V \subset U$, satisfying the following property. Given $\varepsilon \in (0, \varepsilon_0)$ and a holomorphic submersion $g : U \rightarrow \mathbf{C}^q$ with $\sup_{x \in U} |f(x) - g(x)| < \varepsilon$ there is a biholomorphic map $\gamma : V \rightarrow X$ satisfying $f = g \circ \gamma$ on V and $\|\gamma - \text{id}\|_V < M\varepsilon$.*

Proof. We may assume that U is Stein. Hence $TX|_U = \ker df \oplus E$ for some trivial rank- q holomorphic subbundle $E \subset TX|_U$. Thus E is spanned by q independent holomorphic vector fields on U . Denote by $\theta(x, t_1, \dots, t_q)$ the composition of their local flows (see the construction of θ in (4.1)). The map θ is defined in an open set $\Omega \subset U \times \mathbf{C}^q$ containing $U \times \{0\}^q$. For $x \in U$ write $\Omega_x = \{t \in \mathbf{C}^q : (x, t) \in \Omega\}$. After shrinking Ω we may assume that for each $x \in U$ the fiber Ω_x is connected and $F_x := \{\theta(x, t) : t \in \Omega_x\} \subset X$ is a local complex submanifold of X which intersects the level set $\{f = f(x)\}$ transversely at x (since $T_x F_x = E_x$ is complementary to the kernel of df_x). By the implicit function theorem we may assume that (after shrinking Ω) the map $t \in \Omega_x \rightarrow f(\theta(x, t)) \in \mathbf{C}^q$ maps Ω_x biholomorphically onto a neighborhood of the point $f(x)$ in \mathbf{C}^q . The same holds for the map $t \in \Omega_x \rightarrow g(\theta(x, t))$ provided that $g : U \rightarrow \mathbf{C}^q$ is sufficiently uniformly close to f

and we restrict x to a compact subset of U . It follows that, if $V \subset \subset U$ and g is sufficiently close to f on U , there is for every $x \in V$ a unique point $c(x) \in \Omega_x$ such that $g(\theta(x, c(x))) = f(x)$. Clearly $c: V \rightarrow \mathbf{C}^q$ is holomorphic and the map $\gamma(x) = \theta(x, c(x)) \in X$ ($x \in V$) satisfies Lemma 5.1.

Definition. An ordered pair of compact sets (A, B) in an n -dimensional complex manifold X is said to be a *special Cartan pair* if

- (i) the sets $A, B, C := A \cap B, A \cup B$ are Stein compacta (see above);
- (ii) $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$;
- (iii) there is an open set $U \supset B$ and an injective holomorphic map $\psi: U \rightarrow \mathbf{C}^n$ such that $\psi(C) \subset \mathbf{C}^n$ is polynomially convex.

The following is the main step in the proof of Theorem 2.1.

PROPOSITION 5.2. *Let (A, B) be a special Cartan pair in a complex manifold X and let $f \in \mathcal{O}(A)$ be a function whose critical set P is finite and does not meet $C := A \cap B$. Given $\varepsilon > 0$ there exists $\tilde{f} \in \mathcal{O}(A \cup B)$ with the same critical set P such that $\sup_A |\tilde{f} - f| < \varepsilon$. If X_0 is a closed complex subvariety of X with $X_0 \cap C = \emptyset$ then for any $r \in \mathbf{N}$ we can choose \tilde{f} as above such that $\tilde{f} - f$ vanishes to order r on $X_0 \cap A$. In particular, if f is noncritical on A then \tilde{f} is noncritical on $A \cup B$.*

Proof. We use the notation from (iii) in the definition of a special Cartan pair. The function $f' = f \circ \psi^{-1}$ is defined and noncritical in an open set $\tilde{C} \subset \mathbf{C}^n$ containing $\psi(C)$. Choose a compact polynomially convex set K with $\psi(C) \subset \text{int } K \subset K \subset \tilde{C}$. By Theorem 3.1 we can approximate f' uniformly on K by a noncritical holomorphic function $g' \in \mathcal{O}(\mathbf{C}^n)$. Thus $g = g' \circ \psi$ is noncritical in a neighborhood of B , and it approximates f uniformly in a neighborhood of C . If the approximation is sufficiently close then by Lemma 5.1 there is a biholomorphic map γ , uniformly close to the identity map in a neighborhood of C , satisfying $f = g \circ \gamma$. By Theorem 4.1 we have $\gamma = \beta \circ \alpha^{-1}$, where α is a biholomorphic map close to the identity in a neighborhood of A and β is a map with the analogous properties in a neighborhood of B . Furthermore we insure that α agrees with the identity map to a sufficiently high order at each point of $(X_0 \cup P) \cap A$. From $f = g \circ \gamma = g \circ \beta \circ \alpha^{-1}$ (which holds in a neighborhood of C) we obtain $f \circ \alpha = g \circ \beta$. The two sides define a holomorphic function $\tilde{f} \in \mathcal{O}(A \cup B)$ with the stated properties.

Proof of Theorem 2.1. We first consider the simplest case when f is noncritical on U and $X_0 = \emptyset$. By Corollary 2.8 in [HL3] there is a sequence of compact $\mathcal{O}(X)$ -convex subsets $A_0 \subset A_1 \subset \dots \subset \bigcup_{k=0}^{\infty} A_k = X$ such that

- (i) $K \subset \text{int } A_0 \subset A_0 \subset U$;
- (ii) for every $k = 0, 1, 2, \dots$ we have $A_{k+1} = A_k \cup B_k$ where (A_k, B_k) is a special Cartan pair in X .

Fix $\varepsilon > 0$. Write $f_0 = f$, $A_{-1} = K$. Choose a sufficiently small number $\varepsilon_0 \in (0, \frac{1}{2}\varepsilon)$ such that every $g \in \mathcal{O}(A_0)$ with $\sup_{A_0} |g - f_0| < 2\varepsilon_0$ is noncritical on K . Proposition 5.2 gives a noncritical function $f_1 \in \mathcal{O}(A_1)$ satisfying $\sup_{A_0} |f_1 - f_0| < \varepsilon_0 < \frac{1}{2}\varepsilon$. Now choose $\varepsilon_1 \in (0, \frac{1}{2}\varepsilon_0)$ such that every function $g \in \mathcal{O}(A_1)$ with $\sup_{A_1} |g - f_1| < 2\varepsilon_1$ is noncritical on A_0 . Proposition 5.2 gives a noncritical function $f_2 \in \mathcal{O}(A_2)$ such that $\sup_{A_1} |f_2 - f_1| < \varepsilon_1 < \frac{1}{4}\varepsilon$. Continuing inductively we obtain a sequence of noncritical functions $f_k \in \mathcal{O}(A_k)$ and a decreasing sequence $\varepsilon_k > 0$ with $\sum_{k=0}^{\infty} \varepsilon_k < \varepsilon$ such that $\sup_{A_k} |f_{k+1} - f_k| < \varepsilon_k < \varepsilon 2^{-k-1}$ for every $k = 0, 1, 2, \dots$. The sequence f_k converges uniformly on compacts in X to $\tilde{f} \in \mathcal{O}(X)$ satisfying $\sup_K |\tilde{f} - f| < \varepsilon$ and $\sup_{A_k} |\tilde{f} - f_k| < 2\varepsilon_k$ for every $k = 0, 1, 2, \dots$. By the choice of ε_k this insures that \tilde{f} is noncritical on A_{k-1} . Since this holds for every k , \tilde{f} is noncritical on X .

Consider now the general case. Let $P = \{p_1, p_2, \dots\}$ denote the (discrete) critical set of $f \in \mathcal{O}(U)$. We replace X_0 by $X_0 \cup P$. For each $j \in \mathbf{N}$ we choose a sufficiently large integer $n_j \in \mathbf{N}$ such that for every germ of a holomorphic function g which vanishes to order n_j at p_j , the germ of $f + g$ still has an isolated critical point at p_j . In the sequel we shall often use the following elementary fact. Given a pair of compact sets $K \subset L$ in the domain of f , with $K \subset \text{int } L$, we can choose $\eta > 0$ such that for every $g \in \mathcal{O}(L)$ which vanishes to order n_j at every point $p_j \in P \cap K$ and satisfies $\sup_L |g| < \eta$, the critical set of $f + g$ in K equals $P \cap K$.

Denote by $\mathcal{J} \subset \mathcal{O}_X$ the coherent analytic sheaf of ideals consisting of all germs of holomorphic functions on X which vanish to order r on X_0 and to order n_j at $p_j \in P$ for every $j \in \mathbf{N}$. We can replace f by a function holomorphic on X such that the difference of the two functions is a section of \mathcal{J} near X_0 and is uniformly small on K (see Lemma 8.1 in [FP1]). The new function (which we still denote f) may have additional critical points, but there is a neighborhood $U \supset X_0 \cup K$ such that $\text{Crit}(f; U) = P$. Choose a compact $\mathcal{O}(X)$ -convex set $L \subset X$ containing K in its interior. Fix an $\eta > 0$. We claim that there exists an $f' \in \mathcal{O}(L)$ satisfying

- (i) $\text{Crit}(f'; L) = P \cap L$;
- (ii) $f' - f$ is a section of \mathcal{J} over L ;
- (iii) $|f' - f| < \eta$ on K .

Proof. By Lemma 8.4 in [FP3] there is a finite sequence $A_0 \subset A_1 \subset \dots \subset A_{k_0} = L$ of compact $\mathcal{O}(X)$ -convex subsets such that for each $k = 0, 1, \dots, k_0 - 1$ we have $A_{k+1} = A_k \cup B_k$, where (A_k, B_k) is a special Cartan pair in X and

- (a) $K \cup (X_0 \cap L) \subset A_0 \subset \subset U$;
- (b) $B_k \cap X_0 = \emptyset$ for $k = 0, 1, \dots, k_0 - 1$.

(Our notation differs from [FP3]: the set A_k in [FP3] is denoted B_{k-1} in this paper, while the set A_k in this paper is the same as $\bigcup_{l=0}^k A_l$ in [FP3].) Assume inductively that for

some $k < k_0$ we already have a function $f_k \in \mathcal{O}(A_k)$ satisfying the above properties (i)–(iii) (with f' replaced by f_k). Since $B_k \cap X_0 = \emptyset$, f_k is noncritical in a neighborhood of $A_k \cap B_k$, and hence Proposition 5.2 furnishes a function $f_{k+1} \in \mathcal{O}(A_{k+1})$ satisfying (i)–(iii) on its domain. After k_0 steps we obtain the desired function $f' \in \mathcal{O}(L)$, thus proving the claim.

In order to complete the induction step we show that there exists $h \in \mathcal{O}(X)$ such that $h - f$ is a section of \mathcal{J} , h approximates f' uniformly on L , and there is a neighborhood $\tilde{U} \supset X_0 \cup L$ such that $\text{Crit}(h; \tilde{U}) = P$. By Cartan's Theorem A the sheaf \mathcal{J} is finitely generated on the compact set L , say by functions $\xi_l \in \mathcal{O}(X)$ ($l=1, 2, \dots, m$). Since $f' - f$ is a section of \mathcal{J} over a neighborhood of L , we have $f' = f + \sum_{j=1}^m \xi_j g_j$ for some $g_j \in \mathcal{O}(L)$. Since L is $\mathcal{O}(X)$ -convex, we can approximate g_j uniformly on a neighborhood of L by $\tilde{g}_j \in \mathcal{O}(X)$. The function $h = f + \sum_{j=1}^m \xi_j \tilde{g}_j \in \mathcal{O}(X)$ satisfies the stated properties provided that the approximation of g_j by \tilde{g}_j was sufficiently close for every j .

Note that h satisfies the same properties on a neighborhood of $L \cup X_0$ as f did on a neighborhood of $K \cup X_0$. The proof of Theorem 2.1 is completed by an obvious induction over a sequence of compact $\mathcal{O}(X)$ -convex sets $L_1 \subset L_2 \subset \dots$ exhausting X (compare with the noncritical case given above).

6. Construction of holomorphic submersions

In this section we prove Theorems 2.5 and 2.6 and Corollaries 2.10 and 2.11. We begin with Theorems 2.5 and 2.6. Since the proof is fairly long, we first explain the outline and then treat each of the main ingredients in a separate subsection.

We are given a q -coframe $\theta = (\theta_1, \dots, \theta_q)$ on X such that $\theta|_U = df$ in an open set $U \supset K$ where $f: U \rightarrow \mathbf{C}^q$ is a holomorphic submersion. Our task is to deform θ to the differential $d\tilde{f}$ where $\tilde{f}: X \rightarrow \mathbf{C}^q$ is a holomorphic submersion which approximates f uniformly on K . (We shall deal with interpolation along a subvariety $X_0 \subset X$ in Subsection (5).)

Let $\varrho: X \rightarrow \mathbf{R}$ be a smooth strongly plurisubharmonic Morse exhaustion function such that $\varrho < 0$ on K and $\varrho > 0$ on $X \setminus U$ [Hö2, Theorem 5.1.6]. Each sublevel set $\{\varrho \leq c\}$ is compact and $\mathcal{O}(X)$ -convex; if $c \in \mathbf{R}$ is a regular value of ϱ then $\{\varrho < c\}$ is a smooth strongly pseudoconvex domain. The set of critical values of ϱ is discrete in \mathbf{R} and hence at most countable, and each critical level contains a unique critical point.

It suffices to explain how to approximate a submersion f defined in a neighborhood of $\{\varrho \leq c_0\}$ (and with df homotopic to θ through q -coframes) by a submersion \tilde{f} with similar properties defined in neighborhood of $\{\varrho \leq c_1\}$, where $c_0 < c_1$ is any pair of regular values of ϱ . The construction is then completed by an obvious induction as in Theorem 2.1. Using a smooth cut-off function in the parameter of the homotopy from df to θ we can deform the q -coframe θ at each step to insure that $\theta = df$ in a neighborhood of $\{\varrho \leq c_0\}$.

The construction of the extension breaks into two distinct arguments:

- (i) going through noncritical values of ϱ (mainly complex analysis);
- (ii) crossing a critical value (mainly topology and ‘convex analysis’).

If ϱ has no critical values in $[c_0, c_1]$ then $\{\varrho \leq c_1\}$ is obtained from $\{\varrho \leq c_0\}$ by finitely many attachings of small *convex bumps*. In each step we approximately extend f over the bump by Proposition 3.3, and we patch the two pieces using Theorem 4.1. In finitely many steps we obtain a submersion f in a neighborhood of $\{\varrho \leq c_1\}$ (Subsection (1)).

Crossing a critical value of ϱ relies on a combination of three techniques:

- smooth extension across a handle attached to $\{\varrho \leq c_0\}$ (Subsection (2));
- approximation by a holomorphic submersion defined in a neighborhood of a handlebody (Subsection (3));
- applying the noncritical case with a different strongly plurisubharmonic function to extend across the critical level of ϱ (Subsection (4)).

The proof of Theorem 2.5 is completed immediately after Lemma 6.7 in Subsection (4), with the exception of the interpolation along a subvariety $X_0 \subset X$ which is explained in Subsection (5). There we also prove Corollaries 2.10 and 2.11.

(1) *The noncritical case.* A compact set $\tilde{A} \subset X$ in a complex manifold X is a *noncritical strongly pseudoconvex extension* of a compact set $A \subset \tilde{A}$ if there is a smooth strongly plurisubharmonic function ϱ in an open set $\Omega \supset \tilde{A} \setminus A$ which has no critical points on Ω and satisfies

$$A \cap \Omega = \{x \in \Omega : \varrho(x) \leq 0\}, \quad \tilde{A} \cap \Omega = \{x \in \Omega : \varrho(x) \leq 1\}.$$

Note that for each $t \in [0, 1]$ the set $A_t = A \cup \{\varrho \leq t\} \subset X$ is a smooth (closed) strongly pseudoconvex domain in X , and the family smoothly increases from $A = A_0$ to $\tilde{A} = A_1$. We say that X is a *noncritical strongly pseudoconvex extension* of A if there exists a smooth exhaustion function $\varrho: X \rightarrow \mathbf{R}$ such that $A = \{\varrho \leq 0\}$ and ϱ is strongly plurisubharmonic and without critical points on $\{\varrho \geq 0\} = X \setminus \text{int } A$.

PROPOSITION 6.1. *Let X be a Stein manifold and $\tilde{A} \subset X$ a noncritical strongly pseudoconvex extension of $A \subset \tilde{A}$. If $f: A \rightarrow \mathbf{C}^q$ is a holomorphic submersion with $q < \dim X$ then for every $\varepsilon > 0$ there exists a holomorphic submersion $\tilde{f}: \tilde{A} \rightarrow \mathbf{C}^q$ satisfying $\sup_A |f - \tilde{f}| < \varepsilon$.*

COROLLARY 6.2. (a) *If X is a noncritical strongly pseudoconvex extension of $A \subset X$ then every holomorphic submersion $f: A \rightarrow \mathbf{C}^q$ ($q < \dim X$) can be approximated uniformly on A by holomorphic submersions $\tilde{f}: X \rightarrow \mathbf{C}^q$.*

(b) *Let $\Omega \subset \mathbf{C}^n$ be a convex open set. Any holomorphic submersion $f: \Omega \rightarrow \mathbf{C}^q$ ($q < n$) can be approximated uniformly on compacts by submersions $\mathbf{C}^n \rightarrow \mathbf{C}^q$.*

Proof of Proposition 6.1. Let $z=(z_1, \dots, z_n)=(x_1+iy_1, \dots, x_n+iy_n)$ denote the coordinates on \mathbf{C}^n . Let

$$P = \{z \in \mathbf{C}^n : |x_j| < 1, |y_j| < 1, j = 1, \dots, n\}$$

denote the open unit cube. Set $P' = \{z \in P : y_n = 0\}$.

Let $A, B \subset X$ be compact sets in X . We say that B is a *convex bump* on A if there exist an open set $U \subset X$ containing B , a biholomorphic map $\phi: U \rightarrow P$ onto $P \subset \mathbf{C}^n$, and smooth strongly concave functions $h, \tilde{h}: P' \rightarrow [-a, a]$ for some $a < 1$ such that $h \leq \tilde{h}$, $h = \tilde{h}$ near the boundary of P' , and

$$\begin{aligned} \phi(A \cap U) &= \{z \in P : y_n \leq h(z_1, \dots, z_{n-1}, x_n)\}, \\ \phi((A \cup B) \cap U) &= \{z \in P : y_n \leq \tilde{h}(z_1, \dots, z_{n-1}, x_n)\}. \end{aligned}$$

Suppose now that $A \subset \tilde{A}$ is a noncritical strongly pseudoconvex extension in X . By an elementary geometric argument, using Narasimhan's lemma on local convexification of strongly pseudoconvex domains, there is a finite sequence $A = A_0 \subset A_1 \subset \dots \subset A_{k_0} = \tilde{A}$ of compact strongly pseudoconvex domains in X such that for every $k = 0, 1, \dots, k_0 - 1$ we have $A_{k+1} = A_k \cup B_k$, where B_k is a convex bump on A_k as defined above. (For details see Lemma 12.3 in [HL2]. Similar 'bumping constructions' had been introduced by Grauert and were used in the Oka–Grauert theory; see [Gro4], [HL3], [FP1], [FP2], [FP3].) Hence Proposition 6.1 follows immediately from

LEMMA 6.3. *Assume that X is a Stein manifold, $A \subset X$ is a smooth compact strongly pseudoconvex domain, and $B \subset X$ is a convex bump on A . Given a holomorphic submersion $f: A \rightarrow \mathbf{C}^q$ ($q < \dim X$), there exists for every $\varepsilon > 0$ a holomorphic submersion $\tilde{f}: A \cup B \rightarrow \mathbf{C}^q$ satisfying $\sup_{A \cap B} |f - \tilde{f}| < \varepsilon$. If $X_0 \subset X$ is a closed complex subvariety such that $X_0 \cap B = \emptyset$, we can choose \tilde{f} such that it agrees with f to a given finite order along $X_0 \cap A$.*

Proof. We use the notation introduced above. Recall that h and \tilde{h} have range in $[-a, a]$ for some $a < 1$. Choose $c \in (a, 1)$ sufficiently close to 1 such that the (compact) support of $\tilde{h} - h$ is contained in cP' . Let $L := c\bar{P} \subset \mathbf{C}^n$ and $\tilde{L} := \phi^{-1}(L) \subset U$. Increasing $c < 1$ towards 1 we may assume that $B \subset \tilde{L}$. Set $\tilde{K} = A \cap \tilde{L}$ and $K = \phi(\tilde{K}) \subset P$. The pair of compact sets $K, L \subset \mathbf{C}^n$ satisfies the hypothesis of Proposition 3.3 with respect to the splitting $z = (z', z'') \in \mathbf{C}^n$, with $z' = (z_1, \dots, z_{n-2}) \in \mathbf{C}^{n-2}$ and $z'' = (z_{n-1}, z_n) \in \mathbf{C}^2$. Applying Proposition 3.3 (with $r = n - 2$, $s = 2$) we obtain a holomorphic submersion g from a neighborhood of \tilde{L} to \mathbf{C}^q which approximates f uniformly in a neighborhood of \tilde{K} . Since $B \subset \tilde{L}$ and $A \cap B \subset A \cap \tilde{L} = \tilde{K}$, g is defined in a neighborhood of B and it approximates f uniformly in a neighborhood of $A \cap B$. By Lemma 5.1 we have $f = g \circ \gamma$ for a

biholomorphic map γ close to the identity in a neighborhood of $A \cap B$ in X . Splitting $\gamma = \beta \circ \alpha^{-1}$ by Theorem 4.1 we obtain $f \circ \alpha = g \circ \beta$ in a neighborhood of $A \cap B$, and hence the two sides define a holomorphic submersion $\tilde{f}: A \cup B \rightarrow \mathbf{C}^q$. The same proof applies with interpolation on X_0 .

In the remainder of this section we treat the critical case. Let p be a critical point of ϱ , with Morse index k . If $k=0$ then ϱ has a local minimum at p , and a new connected component appears in $\{\varrho < c\}$ as c passes $\varrho(p)$. We can trivially extend f to this new component by taking any local submersions to \mathbf{C}^q near p . In the sequel we only treat the case $k \geq 1$. It is no loss of generality to assume $\varrho(p) = 0$. Choose $c_0 > 0$ such that p is the only critical point of ϱ in $[-c_0, 3c_0]$. In the following three subsections we explain how to approximately extend a submersion f from $\{\varrho \leq -c_0\}$ to $\{\varrho \leq +c_0\}$.

(2) *Smooth extension across a handle.* Recall that $k \in \{1, \dots, n\}$ is the index of p . Write $z = (z', z'') = (x' + iy', x'' + iy'')$, where $z' \in \mathbf{C}^k$ and $z'' \in \mathbf{C}^{n-k}$. Denote by $P \subset \mathbf{C}^n$ the open unit polydisc. By Lemma 3 in [HW2, p. 166] (see also Lemma 2.5 in [HL]) there is a neighborhood $U \subset X$ of p and a biholomorphic coordinate map $\phi: U \rightarrow P$, with $\phi(p) = 0$, such that the function $\tilde{\varrho}(z) := \varrho(\phi^{-1}(z))$ is given by

$$\tilde{\varrho}(z) = Q(y', z'') - |x'|^2, \quad Q(y', z'') = \langle Ay', y' \rangle + \langle By'', y'' \rangle + |x''|^2. \quad (6.1)$$

Here $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and A, B are positive definite symmetric matrices such that all eigenvalues of A are larger than 1 (thus $A > I$ and $B > 0$). Furthermore one may diagonalize A and B .

We may assume that $c_0 < 1$. Choose $c \in (0, c_0)$. By the noncritical case we may assume that f has already been extended to $\{\varrho < -\frac{1}{2}c\}$. The set $E \subset U$ defined by

$$\phi(E) = \{(x' + iy', z'') : y' = 0, z'' = 0, |x'|^2 \leq c\} \quad (6.2)$$

is a k -dimensional handle attached from the outside to $\{\varrho \leq -c\}$ along the $(k-1)$ -sphere $bE \subset \{\varrho = -c\}$.

In a neighborhood of E we may consider f as a function of z . We identify $x \in \mathbf{R}^n$ with $x + i0 \in \mathbf{C}^n$. The components θ_j of the q -coframe θ are expressed in the z -coordinates by $\theta_j(z) = \sum_{l=1}^n \theta_{j,l}(z) dz_l$, where $\theta_{j,l}$ are continuous functions and the $(q \times n)$ -matrix $\tilde{J} = (\theta_{j,l})$ has maximal complex rank q at each point. For $x \in E$ near bE we have $\theta_{j,l}(x) = \partial f_j / \partial z_l(x) = \partial f_j / \partial x_l(x)$.

Denote by $M_{q,n} \simeq \mathbf{C}^{q \times n}$ the set of all complex $(q \times n)$ -matrices, and let $M_{q,n}^*$ consist of all matrices of rank q in $M_{q,n}$.

LEMMA 6.4. *There is a $c' \in (\frac{1}{2}c, c)$ such that f and all its partial derivatives $\partial f / \partial z_i$ extend smoothly to $\{\varrho \leq -c'\} \cup E$ (without changing their values on $\{\varrho \leq -c'\}$) such that the Jacobian matrix $J(f) = (\partial f_j / \partial z_i)$ of the extension has complex rank q at each point of E , and $J(f)$ can be connected to $\tilde{J} = (\theta_{j,i})$ by a homotopy of maps from E to $M_{q,n}^*$ which is fixed on $\{\varrho \leq -c'\} \cap E$.*

Lemma 6.4 is obtained from a *convex integration lemma* due to Gromov [Gro2, Lemma 3.1.3]. We state the special case which is needed. Fix numbers $0 < r < R$, $\delta > 0$, and let

$$D = \{x \in \mathbf{R}^n : |x'| \leq R, |x''| \leq \delta\}, \quad A = \{x \in \mathbf{R}^n : r \leq |x'| \leq R, |x''| \leq \delta\}.$$

LEMMA 6.5. *Assume that $f = (f_1, \dots, f_q) : A \rightarrow \mathbf{C}^q$ ($q \leq n$) is a smooth map whose Jacobian $J(f) = (\partial f_j / \partial x_i)$ has complex rank q at each point. If there exists a continuous map $\tilde{J} : D \rightarrow M_{q,n}^*$ with $\tilde{J}|_A = J(f)$ then there is a smooth map $\tilde{f} : D \rightarrow \mathbf{C}^q$ such that*

- (i) $\tilde{f}|_A = f$;
- (ii) the Jacobian $J(\tilde{f})$ has range in $M_{q,n}^*$;
- (iii) $J(\tilde{f})$ is homotopic to \tilde{J} through maps $D \rightarrow M_{q,n}^*$ which are fixed on A .

If $q \leq n - [\frac{1}{2}k]$ then such \tilde{J} and \tilde{f} always exist.

Proof. We have $M_{q,n}^* = M_{q,n} \setminus \Sigma$ where Σ consists of all matrices of rank less than q . We claim that Σ is an algebraic subvariety of complex codimension $n - q + 1$ in $M_{q,n} \simeq \mathbf{C}^{q \times n}$. Assume that $B \in \Sigma$ has rank $q - 1$. Choose $1 \leq j_1 < j_2 < \dots < j_{q-1} \leq n$ such that the corresponding columns of B are linearly independent. Locally near B the set Σ is defined by vanishing of the determinants obtained by adding to the columns j_1, \dots, j_{q-1} any of the remaining $n - q + 1$ columns of B . Locally this gives $n - q + 1$ independent polynomial equations for Σ . A similar argument holds when B has rank less than $q - 1$. (See also Proposition 2 in [Fo2].)

We are looking for an extension $\tilde{f} : D \rightarrow \mathbf{C}^q$ of f whose Jacobian $J(\tilde{f})$ misses Σ . If $k < 2(n - q + 1)$ (which is equivalent to $q \leq n - [\frac{1}{2}k]$) Thom's jet transversality theorem ([Tho] or [GG, p. 54]) gives a maximal rank extension of f and its full one-jet from A to the k -dimensional disc $D_k = \{(x', 0) : |x'| \leq R\}$, and hence to an open neighborhood $V \subset \mathbf{R}^n$ of $A \cup D_k$. Clearly there exists a diffeomorphism $\psi : D \rightarrow \psi(D) \subset V$ which equals the identity on A . Then $\tilde{f} = f \circ \psi$ has the desired properties.

The general case of Lemma 6.5 follows from Gromov's *convex integration lemma* [Gro2, Lemma 3.1.3]. (This can also be found in §2.4 of [Gro3]; see especially (D) and (E) in [Gro3, 2.4.1]. Another source is §18.2 of [EM]; see especially Corollary 18.2.2.) To apply Gromov's lemma we consider $M_{q,n}$ as the space of all one-jets of smooth maps $D \rightarrow \mathbf{C}^q$ at any point $x \in D$ (that is, the space of all first-order partial derivatives at x ,

ignoring the image point). The open set $\Omega = M_{q,n}^* \subset M_{q,n}$ defines a differential relation of order one which is *ample in the coordinate directions* (see [Gro2] or §18.1 in [EM] for a definition of this notion), and the stated results follow from the convex integration lemma.

Ampleness of Ω in the coordinate directions means the following. Choose $l \in \{1, \dots, n\}$ and fix in an arbitrary way the entries of a $(q \times n)$ -matrix which do not belong to the column l (these represent the partial derivatives $\partial f_j / \partial x_k$ for $k \neq l$ at some point x). Let $\Omega' \subset \mathbf{C}^q$ consist of all vectors whose insertion in the l th column gives a matrix of maximal rank q (thus belonging to Ω). Ω is ample in the coordinate directions if every such set Ω' is either empty or else the convex hull of each of its connected components equals \mathbf{C}^q . In our case, Ω' is either empty, the complement of a complex hyperplane in \mathbf{C}^q , or all of \mathbf{C}^q , depending on the rank of the initial $(q \times (n-1))$ -matrix. This completes the proof of Lemma 6.5.

Proof of Lemma 6.4. Let $A \subset D$ be subsets of $U \subset X$ defined by

$$\begin{aligned} \phi(D) &= \{(x' + i0', x'' + i0'') : |x'|^2 \leq c, |x''| \leq \delta\}, \\ \phi(A) &= \{z \in \phi(D) : r \leq |x'|^2 \leq c\}. \end{aligned}$$

Choosing $\delta > 0$ sufficiently small and $r < c$ sufficiently close to c we insure that $A \subset \{\varrho < -\frac{1}{2}c\}$, and hence $f|_A : A \rightarrow \mathbf{C}^q$ is a well-defined smooth map with differential of maximal complex rank q . Lemma 6.5 gives the desired smooth extension to D as well as a homotopy of q -coframes which is fixed on A . If $c' < c$ is chosen sufficiently close to c then $D \cap \{\varrho \leq -c'\} \subset A$, and hence Lemma 6.4 holds for such a c' .

(3) *Holomorphic approximation.* Let f be given by Lemma 6.4. In this subsection we prove

LEMMA 6.6. *For every $\eta > 0$ there exist an open neighborhood $\Omega \subset X$ of the set $K = \{\varrho \leq -c\} \cup E$ and a holomorphic submersion $\tilde{f} : \Omega \rightarrow \mathbf{C}^q$ such that $\|f - \tilde{f}\|_K < \eta$, $\|df - d\tilde{f}\|_E < \eta$, and $d\tilde{f}$ is q -coframe homotopic to θ .*

Here $\|f\|_K$ is the uniform norm of f on K , and $\|df\|_E$ is the norm of its differential on E , measured in a fixed Hermitian metric on TX .

Proof. We need an improved version of Theorem 4.1 from [HöW]. We first show that K is $\mathcal{O}(X)$ -convex and hence admits a basis of Stein neighborhoods. We use the notation from Subsection (2). Choose $L \subset U$ such that $\phi(L) = r\bar{P}$ for some $r < 1$ very close to 1. Then $\phi(K \cap L) = \{z \in r\bar{P} : \tilde{\varrho}(z) \leq -c\} \cup \phi(E)$. Clearly each of the sets $\{\tilde{\varrho} \leq -c\} \cap r\bar{P}$ and $\phi(E)$ is polynomially convex in \mathbf{C}^n . The holomorphic polynomial $h(z) = z_1^2 + \dots + z_k^2$ maps $\phi(E)$ to the interval $[0, c]$, $\phi(bE)$ to the point c , and from (6.1) we easily see

that $\operatorname{Re} h > c$ on $\{\bar{\varrho} \leq -c\} \setminus \phi(E)$. Thus h separates the two sets, and hence their union is polynomially convex (Lemma 29.21 in [Sto]). $\mathcal{O}(X)$ -convexity of $\{\varrho \leq -c\} \cup E$ follows by a usual patching argument, using strong plurisubharmonicity of ϱ (see Lemma 1 in [Ro]).

Choose a constant $\tilde{c} \in (c', c)$. By Lemma 4.3 in [HöW] there is a smooth map $g: X \rightarrow \mathbf{C}^q$ satisfying

- (i) $g = f$ on $\{\varrho \leq -\tilde{c}\} \cup E$;
- (ii) $dg_x = df_x$ for each $x \in E$;
- (iii) g is $\bar{\partial}$ -flat on E , i.e., $D^r(\bar{\partial}g)|_E = 0$ for all $r \in \mathbf{N}$.

Here D^r denotes the total derivative of order r . The cited lemma is proved in [HöW] for $X = \mathbf{C}^n$, but the result is local and holds for any smooth totally real submanifold E in a complex manifold. (One may use partitions of unity along E which are $\bar{\partial}$ -flat on E ; see Lemma 2.3 in [FLØ].) If E is of class \mathcal{C}^m then (iii) holds for $r \leq m-1$.

Fix an integer $m \geq n+1$. Let $\Omega_\varepsilon = \{x \in X: d(x, K) < \varepsilon\}$. In the proof of Theorem 4.1 in [HöW] on pp. 15–16 the authors obtained for each sufficiently small $\varepsilon > 0$ a map $w_\varepsilon: \Omega_\varepsilon \rightarrow \mathbf{C}^q$ satisfying $\bar{\partial}w_\varepsilon = \bar{\partial}g$ in Ω_ε and $\|w_\varepsilon\|_{L^2(\Omega_\varepsilon)} = o(\varepsilon^m)$ as $\varepsilon \rightarrow 0$. (The proof in [HöW] remains valid in any Stein manifold by applying the appropriate $\bar{\partial}$ -results from [Hö1].) On $\Omega_{\varepsilon/2}$ this gives a uniform estimate $|w_\varepsilon| = o(\varepsilon^{m-n})$ [HöW, p. 16] as well as $|D^r w_\varepsilon| = o(\varepsilon^{m-n-r})$ (Lemma 3.2 in [FLØ]). By construction the map $f_\varepsilon = g - w_\varepsilon: \Omega_\varepsilon \rightarrow \mathbf{C}^q$ is holomorphic and satisfies $|f_\varepsilon - f| = o(\varepsilon)$, $|df_\varepsilon - df| = o(1)$ on $\Omega_{\varepsilon/2}$ as $\varepsilon \rightarrow 0$. Hence for sufficiently small $\varepsilon > 0$ the map f_ε is a holomorphic submersion in an open neighborhood Ω of K , with df_ε close to df and hence homotopic to θ . This proves Lemma 6.6.

Remark. More precise approximation results on totally real submanifolds have been obtained by integral kernels; see [HW1], [RS] and [FLØ]. The paper [FLØ] contains optimal results on approximation of $\bar{\partial}$ -flat functions in tubes around totally real submanifolds.

(4) *Extension across the critical level.* The purpose of this subsection is to approximately extend a submersion, furnished by Lemma 6.6, across the critical level $\{\varrho = 0\}$ by applying the noncritical case (Proposition 6.1) with a different strongly plurisubharmonic function τ given by Lemma 6.7 below. Once this is done, we switch back to ϱ (perhaps sacrificing some of the gained territory) and continue (by the noncritical case) to its next critical level.

We shall use the notation established in Subsection (1). Let $\phi: U \rightarrow P \subset \mathbf{C}^n$ be a coordinate map as in the proof of Proposition 6.1 such that $\bar{\varrho} = \varrho \circ \phi^{-1}$ is given by (6.1). Let $c_0 > 0$ be the constant chosen in the paragraph preceding Subsection (2). By the noncritical case we can decrease $c_0 > 0$ to insure that

$$\{(x' + iy', z'') \in \mathbf{C}^n: |x'|^2 \leq c_0, Q(y', z'') \leq 4c_0\} \subset P.$$

Denote by E the handle (6.2) but with c replaced by c_0 (thus $\varrho = -c_0$ on bE). Let $\lambda_1 > 1$ denote the smallest eigenvalue of the matrix A . Choose a number $1 < \mu < \lambda_1$ and set $t_0 = (1 - 1/\mu)^2 c_0$.

LEMMA 6.7. *There exists a smooth strongly plurisubharmonic function τ on $\{\varrho < 3c_0\} \subset X$ which has no critical values in $(0, 3c_0) \subset \mathbf{R}$ and satisfies*

- (i) $\{\varrho \leq -c_0\} \cup E \subset \{\tau \leq 0\} \subset \{\varrho \leq -t_0\} \cup E$;
- (ii) $\{\varrho \leq c_0\} \subset \{\tau \leq 2c_0\} \subset \{\varrho < 3c_0\}$.

Using Lemma 6.7 we complete the crossing of the critical level $\{\varrho = 0\}$ as follows. By Lemma 6.6 (applied with $c = t_0$) there are an open set $\Omega \subset X$ containing the handlebody $\{\varrho \leq -t_0\} \cup E$ and a holomorphic submersion $f: \Omega \rightarrow \mathbf{C}^q$. Consider the family of sublevel sets $\{\tau \leq c\}$ as c increases from 0 to $2c_0$. Property (i) in Lemma 6.7 implies that for sufficiently small $c > 0$ we have $\{\tau \leq c\} \subset \Omega$. By Proposition 6.1 (the noncritical case) we can approximate f uniformly on $\{\tau \leq c\}$ by a submersion \tilde{f} defined in a neighborhood of $\{\tau \leq 2c_0\}$. By Lemma 6.7 (ii), \tilde{f} is defined on $\{\varrho \leq c_0\}$ and $d\tilde{f}$ is q -coframe homotopic to θ . Since $c_0 > 0$, this completes the extension across the critical level $\{\varrho = 0\}$. Hence Theorems 2.5 and 2.6 are proved except for the interpolation on a subvariety (see Subsection (5)).

In the proof of Lemma 6.7 we shall need a criterion for strong plurisubharmonicity of certain functions modeled on (6.1).

LEMMA 6.8. *Let $A > 0$ be a symmetric real $(n \times n)$ -matrix with the smallest eigenvalue $\lambda_1 > 0$. If a \mathcal{C}^2 -function $h: I \subset \mathbf{R}_+ \rightarrow \mathbf{R}$ satisfies*

$$\dot{h} < \lambda_1 \quad \text{and} \quad 2t\ddot{h} + \dot{h} < \lambda_1, \quad t \in I, \quad (6.3)$$

then the function $\tau(z) = \langle Ay, y \rangle - h(|x|^2)$ is strongly plurisubharmonic on $\{z = x + iy \in \mathbf{C}^n : |x|^2 \in I\}$.

Proof. Let $A = (a_{jl})$. A calculation gives

$$\begin{aligned} -\tau_{z_j} &= x_j \dot{h} + i \sum_{s=1}^n a_{js} y_s, \\ -2\tau_{z_j \bar{z}_l} &= \begin{cases} 2x_j^2 \ddot{h} + \dot{h} - a_{jj}, & \text{if } j = l, \\ 2x_j x_l \ddot{h} - a_{jl}, & \text{if } j \neq l. \end{cases} \end{aligned}$$

Thus the complex Hessian

$$H_\tau = \left(\frac{\partial^2 \tau}{\partial z_j \partial \bar{z}_l} \right)$$

of τ satisfies

$$-2H_\tau = 2\ddot{h} \cdot xx^t + \dot{h}I - A,$$

where xx^t is the matrix product of the column $x \in \mathbf{R}^n$ with the row x^t , and I denotes the identity matrix. For any $v \in \mathbf{R}^n$ we have $\langle (xx^t)v, v \rangle = v^t xx^t v = |\langle x, v \rangle|^2$, which lies between 0 and $|x|^2|v|^2$. Hence $0 \leqq xx^t \leqq |x|^2 I$. (Here we write $A \leqq B$ if $B - A$ is nonnegative definite.) At points $|x|^2 = t$ where $\ddot{h}(t) \geqq 0$ we thus get $-2H_\tau \leqq (2t\ddot{h} + \dot{h})I - A < \lambda_1 I - A \leqq 0$ and hence $H_\tau > 0$ (we used the second inequality in (6.3)). At points where $\ddot{h} < 0$ we can omit $2\ddot{h}xx^t \leqq 0$ to get $-2H_\tau \leqq \dot{h}I - A \leqq (\dot{h} - \lambda_1)I < 0$, so $H_\tau > 0$. Thus H_τ is positive definite, which proves Lemma 6.8.

Proof of Lemma 6.7. Recall that $1 < \mu < \lambda_1$ and $t_0 = (1 - 1/\mu)^2 c_0$. We shall find a smooth convex increasing function $h: \mathbf{R} \rightarrow [0, +\infty)$ satisfying

- (i) $h(t) = 0$ for $t \leqq t_0$;
- (ii) $h(t) = t - t_1$ for $t \geqq c_0$, where $t_1 = c_0 - h(c_0) \in (t_0, c_0)$;
- (iii) for all $t \geqq t_0$ we have $0 \leqq \dot{h} \leqq 1$, $2t\ddot{h} + \dot{h} < \lambda_1$ and $t - t_1 \leqq h(t) \leqq t - t_0$.

We first consider the function

$$\xi(t) = \begin{cases} 0, & \text{if } t \leqq t_0, \\ \mu(\sqrt{t} - \sqrt{t_0})^2, & \text{if } t_0 \leqq t \leqq c_0, \\ t - c_0(1 - 1/\mu), & \text{if } c_0 \leqq t. \end{cases}$$

On $[t_0, c_0]$, ξ solves the initial-value problem $2t\ddot{\xi} + \dot{\xi} = \mu$, $\xi(t_0) = \dot{\xi}(t_0) = 0$. It is \mathcal{C}^1 and piecewise \mathcal{C}^2 , with discontinuities of $\ddot{\xi}$ at t_0 and c_0 . The value of t_0 is chosen such that $\dot{\xi}(c_0) = 1$. We have $\ddot{\xi}(t) = \mu\sqrt{t_0}/2\sqrt{t}^3 > 0$ for $t \in [t_0, c_0]$, $\ddot{\xi}(t) = 0$ for t outside this interval, and $\int_{t_0}^{c_0} \ddot{\xi}(t) dt = 1$.

Choose a smooth function $\chi \geqq 0$ which vanishes outside $[t_0, c_0]$, equals $\ddot{\xi} + \varepsilon$ on $[t_0 + \delta, c_0 - \delta]$ for small $\varepsilon, \delta > 0$, and interpolates between 0 and $\ddot{\xi}$ on the intervals $[t_0, t_0 + \delta]$ and $[c_0 - \delta, c_0]$. We can find $\delta, \varepsilon > 0$ arbitrarily small such that $\int_{t_0}^{c_0} \chi(t) dt = \int_{t_0}^{c_0} \ddot{\xi}(t) dt = 1$. The function $h: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ obtained by integrating χ twice with the initial conditions $h(t_0) = \dot{h}(t_0) = 0$ will satisfy the properties (i)–(iii) provided that ε and δ were chosen sufficiently small (since h is then \mathcal{C}^1 -close to ξ and $\ddot{h} \leqq \ddot{\xi} + \varepsilon$). In particular, $t_1 = c_0 - h(c_0) \approx c_0 - \xi(c_0) = (1 - 1/\mu)c_0$ and hence $t_0 < t_1 < c_0$.

By Lemma 6.8 the function

$$\tilde{\tau}(z) = \langle Ay', y' \rangle - h(|x'|^2) + \langle By'', y'' \rangle + |x''|^2 = Q(y', z'') - h(|x'|^2)$$

is strongly plurisubharmonic on \mathbf{C}^n . Recall that $\tilde{\varrho}(z) = Q(y', z'') - |x'|^2$. The properties of h imply

- (a) $\tilde{\varrho} \leqq \tilde{\tau} \leqq \tilde{\varrho} + t_1$;
- (b) $\tilde{\varrho} + t_0 \leqq \tilde{\tau}$ on the set $\{|x'|^2 \geqq t_0\}$;
- (c) $\tilde{\tau} = \tilde{\varrho} + t_1$ on $\{|x'|^2 \geqq c_0\}$.

Let $V = \{\varrho < 3c_0\} \subset X$. We define $\tau: V \rightarrow \mathbf{R}$ by $\tau = \tilde{\tau} \circ \phi$ on $U \cap V$ and $\tau = \varrho + t_1$ on $V \setminus U$. Property (c) implies that both definitions agree on $U \cap V \cap \{|x'|^2 \geq c_0\}$, and hence τ is strongly plurisubharmonic. The stated properties of τ follow immediately. This completes the proof of Lemma 6.7.

(5) *Interpolation along a complex subvariety.* In this subsection we prove the interpolation statement in Theorem 2.5. Recall the situation:

- X_0 is a closed complex subvariety of a Stein manifold X ;
- K is a compact $\mathcal{O}(X)$ -convex subset of X ;
- $U \subset X$ is an open set containing $K \cup X_0$;
- $f: U \rightarrow \mathbf{C}^q$ is a holomorphic submersion such that $df = \theta|_U$ for some q -coframe θ defined on X .

Let c be a regular value of ϱ such that $L = \{\varrho \leq c\}$ contains K in its interior. Our task is to find a holomorphic submersion \tilde{f} from an open neighborhood of $L \cup X_0$ to \mathbf{C}^q which approximates f uniformly on K , interpolates f along X_0 to order $r \in \mathbf{N}$, and such that $d\tilde{f}$ is q -coframe homotopic to θ . The desired submersion $X \rightarrow \mathbf{C}^q$ is then obtained by the usual limiting process. For convenience of notation we take $c = 0$ and $L = \{\varrho \leq 0\}$.

The set $K' := (K \cup X_0) \cap \{\varrho \leq 1\}$ is $\mathcal{O}(X)$ -convex, and hence there exists a smooth strongly plurisubharmonic exhaustion function $\tau: X \rightarrow \mathbf{R}$ such that $\tau < 0$ on K' and $\tau > 0$ on $X \setminus U$. We may assume that 0 is a regular value of τ and the hypersurfaces $\{\varrho = 0\}$ and $\{\tau = 0\}$ intersect transversely. The set $D_0 = \{\tau \leq 0\}$ is a smooth strongly pseudoconvex domain contained in the domain U of f . The following lemma provides the main step.

LEMMA 6.9. *For each $\varepsilon > 0$ there exists a holomorphic submersion $g: \tilde{L} \rightarrow \mathbf{C}^q$ in an open set $\tilde{L} \supset L$ such that $|g - f| < \varepsilon$ on $D_0 \cap L$ and $g - f$ vanishes to order r on $X_0 \cap \tilde{L}$.*

Assuming Lemma 6.9 we complete the proof of Theorem 2.5 as follows. Cartan's theory gives $f' \in \mathcal{O}(X)^q$ such that $f' - f$ vanishes to order r on X_0 , and finitely many functions $\xi_j \in \mathcal{O}(X)$ ($j = 1, 2, \dots, m$) which vanish to order r on X_0 and generate the corresponding sheaf of ideals $\mathcal{J}_{X_0}^r$ on L (but not necessarily on X). Since $g - f' \in \mathcal{O}(L)^q$ vanishes to order r on $X_0 \cap L$, we have $g = f' + \sum_{j=1}^m \xi_j g_j$ for some $g_j \in \mathcal{O}(L)^q$. Since L is $\mathcal{O}(X)$ -convex, we can approximate each g_j uniformly on a neighborhood of L by $\tilde{g}_j \in \mathcal{O}(X)^q$. The map $\tilde{f} = f' + \sum_{j=1}^m \xi_j \tilde{g}_j: X \rightarrow \mathbf{C}^q$ is holomorphic, $|\tilde{f} - g|$ is small on a neighborhood of L (hence $|\tilde{f} - f|$ is small on $D_0 \cap L$), and $\tilde{f} - f$ vanishes to order r along X_0 . If the approximations are sufficiently close then \tilde{f} is a submersion in a neighborhood of $L \cup X_0$. This completes the induction step.

Proof of Lemma 6.9. Set $\varrho_t = \tau + t(\varrho - \tau) = (1 - t)\tau + t\varrho$ and let

$$D_t = \{\varrho_t \leq 0\} = \{\tau \leq t(\tau - \varrho)\}, \quad t \in [0, 1].$$

We have $D_0 = \{\tau \leq 0\}$, $D_1 = \{\varrho \leq 0\} = L$ and $D_0 \cap D_1 \subset D_t$ for all $t \in [0, 1]$. Let

$$\Omega = \{\varrho < 0, \tau > 0\} \subset D_1 \setminus D_0 \quad \text{and} \quad \Omega' = \{\varrho > 0, \tau < 0\} \subset D_0 \setminus D_1.$$

Since $\tau - \varrho > 0$ in Ω and $\tau - \varrho < 0$ in Ω' it follows that, as t increases from 0 to 1, the sets $D_t \cap L$ monotonically increase to $D_1 = L$ while $D_t \setminus L \subset D_0$ decrease to \emptyset . All hypersurfaces $\{\varrho_t = 0\} = bD_t$ intersect along the real codimension-two submanifold $S = \{\varrho = 0\} \cap \{\tau = 0\}$. Since $d\varrho_t = (1-t)d\tau + t d\varrho$ and the differentials $d\tau, d\varrho$ are linearly independent along S , each hypersurface bD_t is smooth near S . Since ϱ_t is a convex linear combination of strongly plurisubharmonic functions, it is itself strongly plurisubharmonic, and hence D_t is strongly pseudoconvex at every smooth point of bD_t .

We investigate more closely the nonsmooth points of $bD_t = \{\varrho_t = 0\}$ inside Ω . The defining equation of $D_t \cap \Omega$ can be written as $\tau \leq t(\tau - \varrho)$ and, after dividing by $\tau - \varrho > 0$, as

$$D_t \cap \Omega = \left\{ x \in \Omega : h(x) = \frac{\tau(x)}{\tau(x) - \varrho(x)} \leq t \right\}.$$

The equation $dh = 0$ for critical points is equivalent to

$$(\tau - \varrho)d\tau - \tau(d\tau - d\varrho) = \tau d\varrho - \varrho d\tau = 0.$$

A generic choice of ϱ and τ insures that there are at most finitely many solutions $p_1, \dots, p_m \in \Omega$ and no solution on $b\Omega$. A calculation shows that at each critical point the complex Hessians satisfy $(\tau - \varrho)^2 H_h = \tau H_\varrho - \varrho H_\tau$. Since $\tau > 0$ and $-\varrho > 0$ on Ω , we conclude that $H_h > 0$ at such points. By a small modification of h near each p_j we can therefore assume that it is of the form (6.1) in some local holomorphic coordinates.

If $c \in [0, 1)$ is a regular value of $h|_\Omega$ then for $c' > c$ sufficiently close to c (depending only on h) the domain $D_{c'}$ can be obtained from $D_c \cap D_{c'}$ by finitely many attachings of *convex bumps* (Subsection (1)). Indeed, the boundaries bD_c and $bD_{c'}$ intersect transversely at very small angles along S and are locally convexifiable. We begin by attaching small convex bumps to $D_c \cap D_{c'}$ along S in order to enlarge $D_c \cap L$ to $D_{c'} \cap L$ locally near S while keeping unchanged the part of the set outside of L (which equals $D_{c'} \setminus L$). Each of the bumps may be chosen disjoint from X_0 , and with finitely many bumps we can reach $D_{c'}$. By Lemma 6.3 every submersion defined in a neighborhood of D_c can be approximated uniformly on $D_c \cap D_{c'}$ by a submersion defined in a neighborhood of $D_{c'}$ such that the two maps agree to order r on X_0 . (We use the interpolation version of Theorem 4.1.) If $0 \leq c_0 < c_1 \leq 1$ are such that $h|_\Omega$ has no critical values in $[c_0, c_1]$, we can subdivide $[c_0, c_1]$ into finitely many subintervals on which the above procedure applies. This explains the noncritical case.

We have seen that the (finitely many) critical points of $h|_{\Omega}$ are of the form (6.1), and hence the method developed in Subsections (2)–(4) can be applied to cross every critical level of h . Lemma 6.6 with interpolation on X_0 (which does not intersect the handle E) is a trivial addition.

Together these two methods show that we can approximate a holomorphic submersion f , defined in a neighborhood of D_0 , uniformly on $L \cap D_0 \supset K$ by a submersion g defined in a neighborhood of L such that $g - f$ vanishes to order r on X_0 . This completes the proof of Lemma 6.9.

Proof of Corollary 2.10. The hypothesis implies that the normal bundle of V in X is isomorphic to $N|_V$ and hence is trivial. By the Docquier–Grauert theorem [DG] there exist functions $g_1, \dots, g_q \in \mathcal{O}(U)$ whose common zero-set equals V and whose differentials dg_1, \dots, dg_q are linearly independent along V . If dg extends to a q -coframe on X then Theorem 2.5 furnishes a submersion $f: X \rightarrow \mathbf{C}^q$ such that $f - g$ vanishes to second order along V . This implies that V is a union of connected components of $f^{-1}(0)$.

In general we must replace g_1, \dots, g_q by a different set of defining functions for V to insure the q -coframe extendability. Choose a complex subbundle $E \subset TX$ such that $TX = E \oplus N$ and $E = \ker dg$ in a neighborhood of V (in particular, $E|_V = TV$). Let $\Theta \subset T^*X$ be the conormal bundle with fibers $\Theta_x = \{\omega \in T_x^*X : \omega(v) = 0 \text{ for all } v \in E_x\}$. From $\Theta \simeq (TX/E)^* \simeq N^*$ we see that Θ is trivial. Hence there exists a q -coframe $\theta = (\theta_1, \dots, \theta_q)$ on X which spans Θ and is holomorphic near V . By construction the differentials dg_1, \dots, dg_q also span Θ near V , and hence $\theta_j = \sum_{k=1}^q a_{jk} dg_k$ for some holomorphic functions a_{jk} in a neighborhood of V . Set $h_j = \sum_{k=1}^q a_{jk} g_k$ for $j = 1, \dots, q$. Then $dh_j = \theta_j$ at points of V (since the term obtained by differentiating a_{jk} is multiplied by g_k which vanishes on V). Let χ be a smooth function on X which equals one in a small neighborhood of V and equals zero outside of a slightly larger neighborhood. If these neighborhoods are chosen sufficiently small then $\tilde{\theta} = \chi dh + (1 - \chi)\theta$ is a q -coframe on X which equals dh near V . Hence we can apply Theorem 2.5 to h as explained above.

Assume now $\dim V \leq [\frac{1}{2}n]$, so that the rank of its (trivial) normal bundle N_V is at most $[\frac{1}{2}(n+1)]$. It suffices to show that N_V extends to a trivial subbundle $N \subset TX$. To see this, recall that the pair (X, V) is homotopy equivalent to a relative CW-complex of dimension at most n [AF]. The standard topological method of extending sections over cells gives the following: *If $E \rightarrow X$ is a complex vector bundle of rank $k > \frac{1}{2}n$ then a nonvanishing section of E over V extends to a nonvanishing section of E over X .* Indeed the obstruction to extending a section from the boundary of an m -cell to its interior lies in the homotopy group $\pi_{m-1}(S^{2k-1})$, which vanishes if $m < 2k$. Our complex only contains cells of dimension $\leq n$, which gives the stated result. Using this inductively we see that the linearly independent sections generating $N_V \subset TX|_V$ extend to linearly

independent sections over X generating a trivial subbundle $N \subset TX$.

Proof of Corollary 2.11. By [DG] there exists an open set $U \subset X$ containing V and a holomorphic submersion $\pi: U \rightarrow V$ which retracts U onto V . Choose a holomorphic subbundle $H \subset TU$ such that $TU = H \oplus \ker d\pi$ and $H|_V = TV$. The map $f^0 := f \circ \pi: U \rightarrow \mathbf{C}^q$ is a holomorphic submersion with $f^0|_V = f$. By the assumption there is a q -coframe $\theta = (\theta_1, \dots, \theta_q)$ on X satisfying $\iota^* \theta_j = df_j$ for $j=1, \dots, q$. Choose a smooth cut-off function $\chi: X \rightarrow [0, 1]$ with support in U such that $\chi=1$ in a smaller open neighborhood $U_1 \subset U$ of V . The $(1, 0)$ -forms $\tilde{\theta}_j := \chi df_j^0 + (1-\chi)\theta_j$ ($j=1, \dots, q$) are well defined on X and are \mathbf{C} -linearly independent, except perhaps on the set where $0 < \chi < 1$. However, if we choose χ to be supported in a sufficiently thin neighborhood of V then these forms are also independent there since the H -components of the q -coframes θ and df^0 agree on $H|_V = TV$, and hence are close to each other over an open neighborhood of V . It remains to apply Theorem 2.5 to obtain a submersion $F: X \rightarrow \mathbf{C}^q$ extending f . If $q \leq [\frac{1}{2}(n+1)]$ then the q -coframe df^0 extends from a small neighborhood of V to all of X by the same argument as in the proof of Corollary 2.10, using the fact that the pair (X, V) is homotopic to a relative CW-complex of dimension $\leq \dim X$.

7. Holomorphic sections transverse to a foliation

A complex vector bundle $\pi: N \rightarrow X$ of rank q admits *locally constant transition functions* if there is an open covering $\{U_i\}_{i \in \mathbf{N}}$ of X and fiber-preserving homeomorphisms $\phi_i: N|_{U_i} = \pi^{-1}(U_i) \rightarrow U_i \times \mathbf{C}^q$ with transition maps

$$\phi_{ij}(x, z) = \phi_i \circ \phi_j^{-1}(x, z) = (x, h_{ij}(z)), \quad x \in U_i \cap U_j, z \in \mathbf{C}^q,$$

where $h_{ij} \in GL_q(\mathbf{C})$ is independent of the base point $x \in U_i \cap U_j$. The *structure group* $\Gamma \subset GL_q(\mathbf{C})$ of N , generated by all h_{ij} 's, is totally disconnected but not necessarily discrete. (Such an N , also called a *flat bundle*, is determined by a representation $\alpha: \pi_1(X) \rightarrow GL_q(\mathbf{C})$; its pull-back to the universal covering \tilde{X} of X is a trivial bundle over \tilde{X} . This will not be used in the sequel.)

THEOREM 7.1. *Let X be a Stein manifold. If E is a complex subbundle of the tangent bundle TX such that $N = TX/E$ admits locally constant transition functions then E is homotopic (through complex subbundles of TX) to the tangent bundle of a nonsingular holomorphic foliation of X .*

Theorem 7.1 extends Corollary 2.9 in which $N = TX/E$ was assumed to be trivial. The analogous result concerning smooth foliations on smooth open manifolds was proved

by Gromov [Gro1] and Phillips [Ph2], [Ph3], [Ph4], and on closed manifolds by Thurston [Th1], [Th2]. (See also [God, pp. 65–66] and [Gro3, p. 102].) The smooth analogue of Theorem 7.1 applies to any smooth codimension-one subbundle $E \subset TX$ (since any real line bundle admits a totally disconnected structure group). On the other hand, a complex line bundle $N \rightarrow X$ over a Stein manifold admits such a structure group only if its first Chern class $c_1(N) \in H^2(X; \mathbf{Z})$ is a torsion element of this group.

Proof of Theorem 7.1. Since the transition functions h_{ij} do not depend on the base point, the product foliations over the sets $U_j \in \mathcal{U}$ define a global holomorphic foliation \mathcal{H} of N such that the zero-section of N is a union of leaves (one for each connected component of X). More precisely, if $U_i \cap U_j \neq \emptyset$ and $z \in \mathbf{C}^q$ then $\phi_i^{-1}(U_i \times \{h_{ij}(z)\})$ and $\phi_j^{-1}(U_j \times \{z\})$ belong to the same leaf of \mathcal{H} . The tangent bundle of N decomposes as $TN = H \oplus V$ where the *horizontal component* $H := T\mathcal{H}$ is the tangent bundle of \mathcal{H} and the *vertical component* V is the tangent bundle of the foliation $N_x = \pi^{-1}(x)$ ($x \in X$). Denote by $\tau: TN \rightarrow V$ the projection onto V with kernel H . Observe that V is just the pull-back of the vector bundle $N \rightarrow X$ to the total space by the projection map π , and for every section $f: X \rightarrow N$ of π we have $f^*V = N$.

If $f: X \rightarrow N$ is a holomorphic section transverse to the foliation \mathcal{H} (this requires $q \leq n = \dim X$) then the intersection of $f(X) \subset N$ with \mathcal{H} defines a holomorphic foliation \mathcal{H}_f of X , of dimension $k = n - q$, whose tangent bundle $T\mathcal{H}_f \subset TX$ has fibers $(T\mathcal{H}_f)_x = \{\xi \in T_x X : \tau \circ df_x(\xi) = 0\}$. Transversality of f to \mathcal{H} means that the vector bundle map

$$f' := f^* \circ \tau \circ df: TX \rightarrow f^*V = N$$

is surjective and hence induces an isomorphism of $TX/T\mathcal{H}_f$ onto N . In particular, N is the normal bundle of any such foliation \mathcal{H}_f .

To prove Theorem 7.1 we construct a holomorphic section $f: X \rightarrow N$ transverse to \mathcal{H} and a complex vector bundle injection $\iota: N \rightarrow TX$ (not necessarily holomorphic) such that the subbundle $T\mathcal{H}_f \subset TX$ is homotopic to E and $f' \circ \iota: N \rightarrow N$ is a complex vector bundle automorphism homotopic to the identity through complex vector bundle automorphisms of N .

On every sufficiently small open set $U \subset X$ we have $N|_U \simeq U \times \mathbf{C}^q$, and the restriction of \mathcal{H} to N_U has leaves $U \times \{z\}$ ($z \in \mathbf{C}^q$). Any such U will be called *admissible*. A section of N over such a U is of the form $f(x) = (x, \tilde{f}(x))$ where $\tilde{f}: U \rightarrow \mathbf{C}^q$, and f is transverse to \mathcal{H} if and only if \tilde{f} is a submersion to the fiber \mathbf{C}^q . This reduces every local problem in the construction of a transverse section to the corresponding problem for submersions.

Choose a strongly plurisubharmonic Morse exhaustion function $\varrho: X \rightarrow \mathbf{R}$ and an initial embedding $\tau: N \rightarrow TX$ such that $TX = E \oplus \iota(N)$. Suppose that f is a transverse holomorphic section, defined on a sublevel set of ϱ , such that $\ker(\tau \circ df)$ is complementary

to $\iota(N)$ and $f' \circ \iota$ is homotopic to the identity over the domain of f . We inductively enlarge the domain of f as in the proof of Theorem 2.5. Whenever we change f the injection ι is changed accordingly (by a homotopy of injections $N \rightarrow TX$) such that $f' \circ \iota$ remains homotopic to the identity on N . We must explain the following two steps.

(a) Suppose that (A, B) is a special Cartan pair in X such that B is a convex bump on A contained in an admissible set $U \subset X$ (Subsection (1) of §6). Given a transverse section $f: \tilde{A} \rightarrow N$ in a neighborhood of A , find a transverse section F in a neighborhood of $A \cup B$ which approximates f uniformly on A . (The homotopy conditions trivially extend from A to $A \cup B$.) A solution to this problem will complete the proof in the noncritical case (compare with Proposition 6.1).

(b) Extend a transverse section across a critical level of ϱ . At this step we shall need the homotopy condition on $f' \circ \iota$.

Part (a) is proved as in Proposition 6.1 with one minor change. On $\tilde{A} \cap U$ we have $f(x) = (x, \tilde{f}(x))$ where \tilde{f} is a submersion to \mathbf{C}^q . We approximate \tilde{f} uniformly in a neighborhood of $A \cap B$ by a submersion $\tilde{g}: \tilde{B} \rightarrow \mathbf{C}^q$ defined in a neighborhood of B , find a transition map γ such that $\tilde{f} = \tilde{g} \circ \gamma$ in a neighborhood of $A \cap B$, and split $\gamma = \beta \circ \alpha^{-1}$ by Theorem 4.1. This gives $\tilde{f} \circ \alpha = \tilde{g} \circ \beta$ in a neighborhood of $A \cap B$ which defines a transverse holomorphic section F in a neighborhood of $(A \cup B) \cap U$ (actually we have to shrink the domain a bit so that the image of α remains in U). It remains to show that F extends holomorphically to a neighborhood of A . From

$$F(x) = (x, \tilde{f}(\alpha(x))), \quad f(\alpha(x)) = (\alpha(x), \tilde{f}(\alpha(x)))$$

we see that these two points belong to the same leaf of \mathcal{H} . Hence $F(x)$ is the unique point of N_x obtained from $f(\alpha(x)) \in N_{\alpha(x)}$ by a parallel transport along the leaf of \mathcal{H} through $f(\alpha(x))$. (More precisely, we take the nearest intersection point of the leaf with the fiber N_x .) Since α is a biholomorphism close to the identity in a neighborhood of A , this gives a well-defined holomorphic extension of F to a neighborhood of $A \cup B$ which is transverse to \mathcal{H} .

Consider now the problem (b). Let $p \in X$ be a critical point of ϱ and assume that f is already defined on $\{\varrho \leq c\}$ for some $c < \varrho(p)$ close to $\varrho(p)$. The crossing of the critical level is localized in an admissible coordinate neighborhood $U \subset X$ of p , except for the last step (Subsection (4) of §6) which uses the noncritical case (a). We must explain how to extend f smoothly across the handle $E \subset U$ attached to $\{\varrho \leq c\}$ (see Subsection (2) of §6 for the details). We identify U with an open subset of \mathbf{C}^n . Using a trivialization $N|_U \simeq U \times \mathbf{C}^q$ we have the following situation:

(i) $f(x) = (x, \tilde{f}(x))$ where \tilde{f} is a holomorphic submersion from a neighborhood of $U \cap \{\varrho \leq c\}$ to \mathbf{C}^q ;

(ii) $\iota: N|_U \rightarrow TX|_U = U \times \mathbf{C}^n$ equals $\iota(x, v) = (x, A_x v)$ ($v \in \mathbf{C}^q$) where A_x is a complex $(n \times q)$ -matrix of rank q depending continuously on $x \in U$;

(iii) $x \rightarrow J\tilde{f}(x) \cdot A_x \in GL_q(\mathbf{C})$ is homotopic to the constant map $x \rightarrow I_q$ in a neighborhood of $U \cap \{\varrho \leq c\}$. (Here I_q is the identity matrix.)

Note that (iii) is just the condition on $f' \circ \iota$ expressed in local coordinates. An elementary consequence of (iii) is that the Jacobian matrix $J\tilde{f}$ admits a smooth extension across the handle $E \subset \mathbf{R}^k$ to a map \tilde{J} into the space of complex $(n \times q)$ -matrices of rank q such that $x \rightarrow \tilde{J}(x) \cdot A_x \in GL_q(\mathbf{C})$ remains homotopic to the constant map $x \rightarrow I_q$ on the set $(\{\varrho \leq c\} \cup E) \cap U$.

Let D be a domain in $\mathbf{R}^n = \mathbf{R}^n + i0 \subset \mathbf{C}^n$ containing the handle E as in Lemma 6.4. Let Ω denote the differential relation of order one whose holonomic sections are smooth maps $h: D \rightarrow \mathbf{C}^q$ whose Jacobian satisfies the condition $Jh(x) \cdot A_x \in GL_q(\mathbf{C})$. We see as in Lemma 6.5 above that Ω is ample in the coordinate directions. Hence Gromov's convex integration lemma from [Gro3, 2.4.1] (or [EM, §18.2]) gives a smooth extension of \tilde{f} across the handle E such that $x \rightarrow J\tilde{f}(x) \cdot A_x \in GL_q(\mathbf{C})$ is homotopic to the constant map $x \rightarrow I_q$ on $(\{\varrho \leq c\} \cup E) \cap U$, thereby insuring that the extended section $f(x) = (x, \tilde{f}(x))$ is transverse to \mathcal{H} also over E and $f' \circ \iota$ remains homotopic to the identity on N . (See Lemma 6.5 for the details.) The remaining steps of the proof are the same as for submersions. Theorem 7.1 is proved.

COROLLARY 7.2. *Let V be a closed complex submanifold of a Stein manifold X . If the tangent bundle TX admits a complex vector subbundle N with locally constant transition functions such that $TX|_V = TV \oplus N|_V$ then V is a union of leaves in a nonsingular holomorphic foliation of X .*

Proof. Let \mathcal{H} be a foliation of N as in the proof of Theorem 7.1. By the Docquier–Grauert theorem [DG] there are an open neighborhood $U \subset X$ of V , a holomorphic retraction $\pi: U \rightarrow V$ and an injective holomorphic map $\phi: U \rightarrow N|_V$ such that $\phi(x) \in N_{\pi(x)}$ for each $x \in U$, and $\phi(x) = 0_x$ if and only if $x \in V$. The point $\phi(x)$ corresponds to a unique point $f(x) \in \pi^*(N|_V)_x$ via the pull-back map π^* . Shrinking U if necessary we have $N|_U \simeq \pi^*(N|_V)$. Using this identification we see that $f: U \rightarrow N|_U$ is a holomorphic section which intersects the zero-section of N transversely along V . Shrinking U again we conclude that f is transverse to \mathcal{H} and V is a leaf of the associated foliation \mathcal{F}_f of U . It remains to find a global transverse section $\tilde{f}: X \rightarrow N$ which agrees with f to second order along V . This is done as in the proof of Theorem 2.5 (Subsection (5) of §6), with the modifications explained above.

Remark. A closed connected complex submanifold V of a Stein manifold X is a leaf in a nonsingular holomorphic foliation defined in an open neighborhood of V if and only

if the normal bundle of V in X admits locally constant transition functions. The proof is essentially the same as for smooth foliations: the direct part is due to Ehresmann (see e.g. [God, p. 5]); for the converse part we transfer the above foliation \mathcal{H} of the normal bundle N to a neighborhood of V in X by the Docquier–Grauert theorem [DG]. Corollary 7.2 gives a sufficient condition for the existence of a *global foliation* of X with the same property.

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