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THE HOMOTOPY PRINCIPLE IN COMPLEX ANALYSIS: A SURVEY

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Introduction

We say that the *homotopy principle* holds for a certain analytic or geometric problem if a solution exists provided there are no topological (or homotopical, cohomological,...) obstructions. One of the principal examples is the theory of smooth immersions developed during 1958–61 by S. Smale ([Sm1], [Sm2]) and M. Hirsch ([Hi1], [Hi2]): *Immersions of a smooth manifold X to affine spaces $\mathbb{R}^N$ of dimension $N > \dim X$ are classified up to regular homotopy by their tangent maps*, and hence by vector bundle injections from the tangent bundle TX to the trivial bundle $X \times \mathbb{R}^N$. In particular, an immersion $X \to \mathbb{R}^N$ exists if and only if TX embeds in $X \times \mathbb{R}^N$. If X is an open manifold then the same holds also for $N = \dim X$.

Slightly earlier J. Nash ([N1], [N2]) proved that *every Riemannian manifold admits an isometric immersion into a Euclidean spaces with the flat metric.* In the process of doing this he discovered an important method for inverting certain classes of non-linear partial differential operators by using a suitably modified Newton's iteration to pass from solutions of the linearized problem to a solution of the non-linear problem. The *Nash-Moser-Kolmogorov implicit function theorem* became one of the key methods for proving the homotopy principle in problems involving underdetermined systems of partial differential equations.

The homotopy principle was investigated even earlier in complex analysis where the customary notion for this phenomenon is *the Oka principle.* In 1939 Kiyoshi Oka [Oka] studied the *second Cousin problem*: Given an open covering $\mathcal{U} = \{U_j\}$ of a complex manifold X and a collection of nowhere vanishing holomorphic functions $f_{ij} \in \mathcal{O}^*(U_{ij})$ satisfying the 1-cocycle condition ($f_{ii} = 1$, $f_{ij}f_{ji} = 1$, $f_{ij}f_{jk}f_{ki} = 1$), the problem is to find a collection of nonvanishing holomorphic functions $f_i \in \mathcal{O}^*(U_i)$ such that $f_i = f_{ij}f_j$ on $U_{ij} = U_i \cap U_j$. ($\mathcal{O}^*$ denotes the multiplicative sheaf of nonvanishing holomorphic functions on X.) Oka proved that, if X is a *domain of holomorphy* in $\mathbb{C}^n$, *a second Cousin problem can be solved by holomorphic functions f_i provided that it can be solved by continuous functions.* More precisely, the inclusion $\mathcal{O}^* \hookrightarrow \mathcal{C}^*$ of $\mathcal{O}^*$ into the sheaf $\mathcal{C}^*$ of nonvanishing continuous functions induces an isomorphism $H^1(X;\mathcal{O}^*) \simeq H^1(X;\mathcal{C}^*)$; the latter group is always isomorphic to $H^2(X;Z)$.

In 1951 K. Stein [Stn] introduced an important class of complex manifolds, now called *Stein manifolds*, on which the algebra of global holomorphic functions has similar properties as on domains of holomorphy. It soon became clear throught the work of Remmert [Rem] that Stein manifolds can be characterized as being biholomorphic to closed complex submanifolds of the affine complex spaces. (For more precise results see [Na1], [Na2] and [Bis]. For the general theory of Stein manifolds and Stein spaces we refer to the monographs [GR], [GRe] and [Hör].) H. Cartan proved that every coherent analytic sheaf on a Stein manifold is generated by global sections and has vanishing cohomology groups in all dimensions $q \geq 1$ (Theorems A and B); hence every analytic problem on a Stein manifold whose obstruction lies in such a group is solvable.

An equivalent formulation of Oka's result on the second Cousin problem is that two holomorphic line bundles on a Stein manifold are holomorphically isomorphic provided they are isomorphic as topological complex vector bundles. This problem has an immediate extension to vector bundles of rank $q > 1$. The holomorphic equivalence classes of such bundles are represented by the cohomology group $H^1(X; \mathcal{G}_q)$ with coefficients in the non-abelian sheaf of (germs of) holomorphic maps $X \to GL_q(\mathbb{C})$. Cartan's theory does not apply directly to such sheaves and one must in some sense linearize the problem. This was done by H. Grauert in seminal papers [Gra1], [Gra2] (1957-58) in which he proved that, *on a Stein space, the holomorphic and topological classification of principal G-bundles coincide for an arbitrary complex Lie group G*; furthermore, on such bundles, *the inclusion of the space of holomorphic sections into the space of continuous sections is a weak homotopy equivalence.* Expositions of Grauert's theory can be found in [Ca], [Ram], [Lei]. An equivariant version of Grauert's theorem was proved in [HK]. A different proof and extension to 1-convex manifolds was given in [HL1] and [HL2], and the result has recently been extended to 1-convex complex spaces [LV]. A converse to Grauert's theorem for domains in Stein spaces was proved by M. Putinar [Pu].

Progress during the 1960's brought improvements and extensions of the Hirsch–Smale theory in real geometry and of Grauert's theory in complex geometry. Phillips showed that the homotopy principle, analogous to the Hirsch-Smale theory of immersions, holds for smooth submersions and foliations of open manifolds ([Ph1], [Ph2], [Ph3]). Forster applied the Oka-Grauert principle to study holomorphic embeddings of Stein manifolds in low dimensional affine spaces ([Fs1], [Fs2]). Forster and Ramspott [FRa] proved the Oka principle in the problem of holomorphic complete intersections. In another direction, Gunning and Narasimhan [GN] constructed noncritical holomorphic functions on any open Riemann surface.

The homotopy principle in real differential topology in geometry was revolutionized by Mikhael Gromov in the period 1967–73. In his seminal paper [Gr1] Gromov presented the method of *convex integration of differential relations* which unified many seemingly unrelated geometric results (in particular the Smale-Hirsch theory of immersions and Phillips's result on submersions). Gromov's methods initiated rapid progress and new examples which fit into his framework are being found even today. For a comprehensive survey of this and other methods to prove the homotopy principle we refer to Gromov's 1986 monograph [Gro3] and to the more recent monographs of Springer [Sp] and Eliashberg and Mishachev [EM]. The convex integration method, together with other methods for solving global problems such as the *removal of singularities* (Gromov and Eliashberg [GE], [Gro3]), *continuous sheaves* [Gro3], *inversions of differential operators* (Nash [N1], [N2], Hamilton [Ham], Gromov [Gro3]), provides a cornestone of differential topology and geometry.

In this paper we survey the homotopy principle in complex analysis and geometry, drawing parallels with the real geometry where appropriate. Results of this type are commonly referred to (as instances of) the *Oka principle* when the underlying manifold is Stein. To our knowledge this notion has never been precisely defined, or at least there is no universal agreement on what the definition should be. In the monograph [GRe] of Grauert and Remmert one finds on p. 145 the following formulation: *Analytic problems (on Stein manifolds) which can be cohomologically formulated have only topological obstructions.* If 'cohomologically' is interpreted in the sense that the obstruction lies in a cohomology group with coefficients in a coherent analytic sheaf then this is just Cartan's Theorem B. The Oka-Grauert theory goes a step further by reducing holomorphic problems to problems in homotopy theory; hence one is tempted to include in the Oka principle all those analytic problems on Stein manifolds which can be homotopically formulated. There is a serious limitation to such attempts since certain analytic problems have no solution due to *hyperbolicity* (Picard's theorems, Kobayashi hyperbolicity, etc.).

What is then a sensible notion of the Oka principle which would adequately cover the known results? It's probably impossible to find one. For the purposes of this paper we adopt the convention that

The Oka principle = the homotopy principle in complex analysis.

We give precise definitions, conforming to Gromov's [Gro3], in Section 1.

It is not surprising that some of the most powerful methods to prove the homotopy principle in the smooth category do not extend to the holomorphic category. The absence of partitions of unity can be substituted to a large extent by Cartan's theory and the $\overline{\partial}$-methods. A more serious problem is that boundary values completely determine holomorphic objects; this disqualifies the convex integration method which is based on extending a solution by induction over the skeleta of a CW-complex. Fortunately some of the other methods mentioned above, such as the *elimination of singularities* method, remain applicable in Stein geometry.

In 1970's it became clear that progress on many questions in Stein geometry depended on extending the Oka-Grauert principle to sections of more general types of holomorphic fiber bundles, and even of non-locally trivial submersions. A crucial contribution was made by Henkin and Leiterer ([HL1], [HL2]) who reproved Grauert's theorem using the 'bumping method', thereby localizing the approximation problems which arise in the construction of global sections. This turned out to be a key point which opened the way to extensions. The potential was realized by Gromov in 1989 [Gro4] who introduced the concept of a *dominating spray* as a replacement of the exponential map in the linearization and patching problems which appear in the Oka-Grauert theory. The presence of a dominating spray on the fiber of a holomorphic bundle over a Stein manifold implies the Oka principle for its sections. Results on this topic are given in Section 3.

In Section 4 we look at the question of *removing intersections* of holomorphic maps from Stein source manifolds with closed complex subvarieties of the (not necessarily Stein) target manifolds. Progress in this direction was made possible by the techniques developed to prove Gromov's Oka principle mentioned above. In the classical case of *complete intersections* the Oka principle was proved in 1967 by Forster and Ramspott [FRa].

In Section 5 we survey the results on *embeddings and immersions of Stein manifolds into affine spaces.* By the results of Remmert [Rem], Bishop [Bis], and Narasimhan ([Na1], [Na2]), every Stein manifold X^n admits a proper holomorphic embedding in $\mathbb{C}^{2n+1}$ and immersion in $\mathbb{C}^{2n}$. For embeddings of smooth manifolds $X^n \to \mathbb{R}^N$ the general minimal dimension is $N = 2n$ (Whitney). However, an n-dimensional Stein manifold X^n is homotopic to an (at most) n-dimensional CW-complex which can be used to obtain holomorphic embeddings $X^n \hookrightarrow \mathbb{C}^N$ for smaller values of N. After the initial work of Forster ([Fs1], [Fs2]) the optimal embedding dimension $N = [\frac{3n}{2}] + 1$ was conjectured by Gromov and Eliashberg in 1971 [GE] and proved in 1992 [EG] (for odd n the proof was completed in [Sch]). The problem of embedding open Riemann surfaces into $\mathbb{C}^2$ is still open and we survey it in Section 6.

In Section 7 we survey the results from the recent work [F8] on the existence of holomorphic submersions of Stein manifolds to complex Euclidean spaces.

This survey is not meant to be comprehensive in any way. Among many topics which are not discussed, even though they would naturally belong here, is the existence and homotopy classification of Stein structures on smooth manifolds; see Eliashberg's paper [El] and the monographs by Gromov [Gro3] and Gompf and Stipsicz [GSt]. I apologize to all authors whose contributions may have been unjustly left out, and I hope to compensate in a more comprehensive future work on this subject.

&1. The homotopy principle and the Oka principle.

We begin by recalling from [Gro3] the notion of a *differential relation.* Consider a smooth submersion $h\colon Z \to X$ between smooth real manifolds. Let $Z^{(r)}$ denote the space of r-jets of (germs of) smooth sections $f\colon X \to Z$ for $r = 0, 1, 2, \ldots$. The 0-jet of f at $x \in X$ is its value $f(x) \in Z_x = h^{-1}(x)$. The r-jet $j^r_x(f) \in Z^{(r)}$ is determined in local coordinates near $x \in X$ resp. $f(x) \in Z$ by the partial derivatives of f of order $\le r$ at x.

We have natural projections $p^r\colon Z^{(r)} \to Z$ and $p^s_r\colon Z^{(s)} \to Z^{(r)}$ for $s > r \ge 0$, where $Z^{(0)} = Z$. The jet bundles $Z^{(r)}$ carry natural smooth structures, as well as affine structures in fibers (see [Gro3] for more details). When X and Z are complex manifolds and $h\colon Z \to X$ is a holomorphic submersion, we shall denote by $Z^{(r)}$ the space of r-jets of holomorphic sections $f\colon X \to Z$.

Note that for every section $g\colon X \to Z^{(r)}$ we get a corresponding 'base point' section $f = p^r(g)\colon X \to Z$. In general g need not equal $j^r(f)$; when $g = j^r(f)$ we say that the section g is *holonomic.*

1.1 Definition. [Gro3, p. 2]) *A* **differential relation** *of order r is a subset $\mathcal{R} \subset Z^{(r)}$ of the r-jet bundle $Z^{(r)}$. A $\mathcal{C}^r$ section $f\colon X \to Z$ is said to satisfy (or to be a solution of) $\mathcal{R}$ if $j^r(f)\colon X \to Z^{(r)}$ has values in $\mathcal{R}$ (i.e., $j^r_x(f)$ belongs to the fiber $\mathcal{R}_{f(x)} = (p^r)^{-1}(f(x))$ of $\mathcal{R}$ over the point $f(x) \in Z$).*

The relation $\mathcal{R} \subset Z^{(r)}$ is said to be *open* (resp. *closed*) when $\mathcal{R}$ is an open (resp. closed) subset of the jet bundle $Z^{(r)}$. Natural examples of closed relations which arise in geometric problems are unions of submanifolds (or subvarieties) of the jet bundle $Z^{(r)}$, and open relations as complements of submanifolds (or subvarieties). Differential equations are examples of closed differential relations.

1.2 Definition. *(a)* ([Gro3, p. 3]) *Let r be a nonnegative integer and let $s \in \{r, r+1, \ldots, \infty\}$. We say that solutions of class $\mathcal{C}^s$ of a differential relation $\mathcal{R} \subset Z^{(r)}$ satisfy the* **basic h-principle** *if every continuous section $\phi_0 \colon X \to \mathcal{R}$ is homotopic through sections $\phi_t \colon X \to \mathcal{R}$ ($t \in [0,1]$) to a holonomic section $\phi_1 = j^r(f)$ for some $\mathcal{C}^s$ section $f \colon X \to Z$.*
(b) Assume that $h \colon Z \to X$ is a holomorphic submersion. We say that sections $X \to Z$ of h satisfy the **basic Oka principle** *if every continuous section is homotopic to a holomorphic section. (For the parametric Oka principle see Definition 2.1.)*
(c) ([Gro3, p. 66]; assumptions as in (b).) *For $r \geq 1$ we say that a differential relation $\mathcal{R} \subset Z^{(r)}$ satisfies the* **holomorphic h-principle** *if every* holomorphic *section $\phi_0 \colon X \to \mathcal{R}$ is homotopic through holomorphic sections $\phi_t \colon X \to \mathcal{R}$ to a holonomic holomorphic section $\phi_1 = j^r(f) \colon X \to \mathcal{R}$ (where $f \colon X \to Z$ is a holomorphic section of $h \colon Z \to X$). We say that $\mathcal{R}$ satisfies the* **basic Oka principle** *if every* continuous *section $\phi_0 \colon X \to \mathcal{R}$ is homotopic through a family of continuous sections of $\mathcal{R}$ to a holomorphic holonomic section of $\mathcal{R}$.*

1.3 Remarks. (a) In the $\mathcal{C}^s$-smooth case one usually takes the *fine* $\mathcal{C}^s$ topology on the space of $\mathcal{C}^s$ sections $X \to Z$. In the holomorphic case one must use the weaker compact-open topology to obtain meaningful results.

(b) One can introduce more refined notions such as the *parametric h-principle*, the *h-principle with approximation* (or *interpolation*), the *relative h-principle*, etc. We refer the reader to [Gro3]. In section two we introduce some of these notions in the holomorphic case (for relations of order zero).

(c) The problem of deforming a continuous section $\phi_0 \colon X \to \mathcal{R}$ to a holomorphic holonomic section can be treated in two steps:

- first deform ϕ_0 through continuous sections of $\mathcal{R}$ to a holomorphic section $\phi_1 \colon X \to \mathcal{R}$ (the ordinary Oka principle for sections $X \to \mathcal{R}$);
- deform a (non-holonomic) holomorphic section $\phi_1 \colon X \to \mathcal{R}$ through a homotopy of holomorphic sections $\phi_t \colon X \to \mathcal{R}$ ($t \in [1,2]$) to a holomorphic holonomic section $\phi_2 = j^r(f)$ (the holomorphic h-principle).

1.4 Examples. (a) Mappings $X \to Y$. An open differential relation of order zero is specified by an open subset $\Omega \subset Y$, and the h-principle requires that every continuous map $X \to \Omega$ is homotopic to a smooth (real-analytic, holomorphic) map through a homotopy with range in Ω. For smooth maps this follows from Whitney's approximation theorem. The problem is highly nontrivial in the holomorphic case (Sections 2 and 3).

(b) Smooth immersions. Let X be a smooth manifold. A map $f = (f_1, \ldots, f_q) \colon X \to \mathbb{R}^q$ is an immersion if its differential $df_x \colon T_x X \to T_{f(x)}\mathbb{R}^q \simeq \mathbb{R}^q$ is a *injective* for every $x \in X$. The pertinent differential relation (of order one) consists of all points (x, y, λ) where $x \in X$, $y \in \mathbb{R}^q$ and $\lambda \in \mathrm{H}om(T_x X, \mathbb{R}^q)$ with λ *injective*. Clearly the value $f(x)$ is unimportant due to translation invariance, and we can reduce the problem to the relation $\mathcal{R}$ whose sections are *injective vector bundle maps* $TX \to X \times \mathbb{R}^q$ from the tangent bundle of X into the trivial bundle $X \times \mathbb{R}^q$. (Alternatively, we can consider the relation whose sections are q-tuples of differential 1-forms $\theta = (\theta_1, \ldots, \theta_q)$ on X which together span the cotangent space $T_x^* X$ at each point $x \in X$.) The h-principle of Smale ([Sm1], [Sm2]) and Hirsch ([Hi1], [Hi2]) asserts that if either $q > \dim X$, or if $q = \dim X$ and X is open, then *the regular homotopy classes of smooth immersions $X \to \mathbb{R}^q$ are in one-to-one correspondence with the homotopy classes of vector bundle injections $TX \to X \times \mathbb{R}^q$*. In particular, an immersion $X \to \mathbb{R}^q$ exists if and only if the cotangent bundle T^*X is generated by q sections.

(c) Holomorphic immersions. The Oka principle for holomorphic immersions $X \to \mathbb{C}^q$ of Stein manifolds to affine spaces of dimension $q > \dim X$ was proved by Eliashberg and Gromov [Gro3] (Section 5 below). The problem is open in the critical dimension $q = \dim X$ except for a positive result in dimension $n = 1$ due to Gunning and Narasimhan [GN]. The Oka principle also holds for *relative immersions* (maps $g\colon X \to \mathbb{C}^n$ such that $f = b \oplus g\colon X \to \mathbb{C}^{m+n}$ is a holomorphic immersion, where $b\colon X \to \mathbb{C}^m$ is a fixed holomorphic map).

(d) Smooth submersions. These are smooth maps $X \to Y$ of rank equal to $\dim Y$ at each point of X (hence $\dim Y \leq \dim X$). The tangent map of a submersion $X^n \to \mathbb{R}^q$ induces a surjective vector bundle map $TX \to X \times \mathbb{R}^q$. The homotopy principle due to Phillips ([Ph1], [Ph3]) asserts that *for any open manifold X, the regular homotopy classes of submersions $X \to \mathbb{R}^q$ are in one-to-one correspondence with surjective vector bundle maps $TX \to X \times \mathbb{R}^q$*. In particular, a submersion $X \to \mathbb{R}^q$ exists if the tangent bundle TX admits a trivial subbundle of rank q. (See also Gromov [Gro3], p. 26 and p. 53.) In fact the h-principle holds for smooth submersions of open manifolds to arbitrary manifolds. It also holds for smooth maps $X \to Y$ of constant rank k [Ph2] or rank $\geq k$ ([Fe], [Gro3, p. 27]).

(e) Holomorphic submersions. In 1967 Gunning and Narasimhan proved that *every open Riemann surface admits a holomorphic function without critical points* [GN]. Very recently it was proved in [F8] that the same holds on every Stein manifold X. Moreover, *holomorphic submersions $X \to \mathbb{C}^q$ satisfy the Oka principle when $q < \dim X$* (Section 7 below). In the maximal dimension $q = \dim X$ the problem is still open: *Does every Stein manifold X^n with trivial tangent bundle admit a locally biholomorphic map $f\colon X \to \mathbb{C}^n$?*

(f) Foliations. A foliation $\mathcal{F}$ of rank q of an n-dimensional manifold X is determined by an integrable rank q subbundle $E \subset TX$ whose fiber E_x is the tangent space to the leaf of $\mathcal{F}$ through $x \in X$. The corresponding homotopy principle was proved for smooth open manifolds by Phillips [Ph2] and Gromov [Gro1], and for closed manifolds by Thurston ([Th1], [Th2]) (see also [Gro3], p. 102 and p. 106): *If a smooth subbundle $E \subset TX$ has a trivial normal bundle TX/E then E is homotopic to an integrable smooth subbundle in TX* (which therefore determines a smooth foliation on X). The same holds if $N = TX/E$ admits locally constant transition functions. In [F8] the analogous results are proved for holomorphic foliations on Stein manifolds. The h-principle fails for real-analytic foliations on *closed* manifolds since *no closed simply connected real-analytic manifold admits a real-analytic foliation of codimension one* (Haefliger [Hae]). For example, the seven-sphere S^7 admits a smooth codimension one foliation but no real-analytic one.

(g) Totally real immersions and embeddings. Let S be a smooth manifold and X a complex (or almost complex) manifold. An immersion $f\colon S \to X$ is *totally real* if for each $p \in S$ the image $\Lambda_p = df_p(T_pS) \subset T_{f(p)}X$ is a totally real linear subspaces of $T_{f(p)}X$, i.e, $\Lambda_p \cap J(\Lambda_p) = \{0\}$ where $J \in \mathrm{End}(TX)$ is the almost complex structure on X. The pertinent differential relation is the set of tripples

(p, x, λ) where $p \in S$, $x \in X$ and $\lambda \in Hom(T_pS, T_xX)$ is an injective $\mathbb{R}$-linear map whose image is a totally real subspace of T_xX. When $X = \mathbb{C}^n$ we can reduce the problem to a relation whose sections are injective $\mathbb{R}$-linear vector bundle maps $\iota\colon TS \to S \times \mathbb{C}^n$ with $\iota(T_pS) \cap i\iota(T_pS) = \{0\}$ for all $p \in S$. The homotopy principle holds for totally real immersions and embeddings of any smooth manifold into any complex manifold. An important point is that the *Whitney trick can be performed through totally real submanifolds.* See [Gro3], [EH], [F1], [Au], and for dimension two also [F2], [F7] and [Ne].

(h) Isometric immersions and embeddings. The fundamental work of Nash ([Na1], [Na2]) initiated a rich and complex theory. An interesting feature of Nash's discovery is that every smooth Riemannian manifold (X^n, g) admits local isometric immersions of class $\mathcal{C}^1$ in $\mathbb{R}^{n+2}$ (codimension two!), but $\mathcal{C}^\infty$ isometric immersions require larger codimension. For $q \geq \frac{1}{2}(n+2)(n+3)$ *free isometric immersions* $X^n \to \mathbb{R}^q$ *satisfy the homotopy principle* [Gro3, p. 12]. Important results in this field were obtained by R. Greene and collaborators [Gn], [GW], [GSh]. We refer to [Gro3] for further results and references.

&2. The Oka principle for mappings: first examples.

Let X and Y be complex manifolds. We say that mappings $X \to Y$ satisfy the **basic Oka principle** if every continuous map is homotopic to a holomorphic map. For mappings between Riemann surfaces a complete answer on the validity of the Oka principle was given by Winkelmann [Wi]. For manifolds of higher dimension only partial results are known.

A compact set K in a Stein manifold X is *holomorphically convex* if for every point $p \in X \backslash K$ there is an $f \in \mathcal{O}(X)$ such that $f(p_0) > \sup_{x \in K} |f(x)|$. Let P be a compact Hausdorff space (the parameter space) and $P_0 \subset P$ be a compact subset which is a strong deformation retraction of some neighborhood of P_0 in P. We shall consider the following stronger versions of the Oka principle (see the notion of Ell_∞ fibrations in [Gro4], as well as Theorem 1.5 in [FP2].)

2.1 Definition. *(a) Maps $X \to Y$ satisy the* **parametric Oka principle** *if for any continuous map $f\colon X \times P \to Y$ such that $f_p = f(\cdot\,, p)\colon X \to Y$ is holomorphic for each $p \in P_0$ there is a homotopy of continuous maps $f^t\colon X \times P \to Y$ $(t \in [0,1])$ such that (i) $f^0 = f$, (ii) the map $f^1_p := f^1(\cdot\,, p)\colon X \to Y$ is holomorphic for each $p \in P$, and (iii) $f^t_p = f_p$ for each $p \in P_0$ and $t \in [0,1]$.*
(b) Let X be a Stein manifold and let ρ be a metric on the complex manifold Y inducing the manifold topology. Maps $X \to Y$ satisfy the **parametric Oka principle with approximation** *if for every compact holomorphically convex subset $K \subset X$ and for every $f\colon X \times P \to Y$ as in (a) such that all maps f_p $(p \in P)$ are holomorphic in an open neighborhood of K in X, there is for every $\epsilon > 0$ a homotopy $f^t\colon X \times P \to Y$ satisfying (a) and also $\rho(f^t(x,p), f(x,p)) < \epsilon$ $(x \in K,\ p \in P)$.*
(c) Maps $X \to Y$ satisfy the **parametric Oka principle with interpolation** *if for every closed complex subvariety $X_0 \subset X$ and for every $f\colon X \times P \to Y$ as in (a) such that all maps f_p $(p \in P)$ are holomorphic in an open neighborhood of X_0 in X, there is for every $r \in \mathbb{N}$ a homotopy $f^t\colon X \times P \to Y$ as in (a) such that f^t_p agrees with f_p to order r on X_0 for every $t \in [0,1]$ and $p \in P$.*

The above notions clearly extend to sections $f\colon X \to Z$ of a holomorphic submersion $h\colon Z \to X$ (see [Gro4] or [FP2]).

2.2 Remark. The parametric Oka principle for maps $X \to Y$ implies that the inclusion

$$\iota\colon \mathrm{Holo}(X;Y) \hookrightarrow \mathrm{Cont}(X;Y) \tag{2.1}$$

of the space of holomorphic maps into the space of continuous maps is a *weak homotopy equivalence*, i.e., it induces isomorphisms of the corresponding homotopy groups of the two spaces which are equipped with the compact-open topology [FP1]. (In some papers this is the definition of the parametric Oka principle.) In particular, each connected component of $\mathrm{Cont}(X;Y)$ contains precisely one connected component of $\mathrm{Holo}(X;Y)$ which means that (a) every continuous map is homotopic to a holomorphic map, and (b) every homotopy between a pair of holomorphic maps can be continuously deformed to a homotopy consisting of holomorphic maps.

The basic Oka principle can hold for a trivial reason that either of the manifolds X or Y is contractible (hence every map is homotopic to constant). However, topological contractibility does not necessarily imply the parametric (or any other) Oka principle. In all nontrivial situations where the Oka principle has been established for all Stein source manifolds X, it was actually proved in the strongest form (parametric, with interpolation and approximation). This is no coincidence since at least the approximation is built into all the known proofs.

We collect positive results on the Oka principle for maps (and sections) in Section 3 below. In the remainder of this section we give examples illustrating the failure of the Oka principle for maps from Stein source manifolds. In most examples the reason for the failure is *hyperbolicity* of the target manifold.

2.3 Example. If either X or Y is contractible then the Oka principle for maps $X \to Y$ trivially holds. However, the Oka principle with approximation fails already for self-maps of the unit disc $U = \{z \in \mathbb{C}\colon |z| < 1\}$: If $a \in U\backslash\{0\}$ and $0 < r < 1-|a|$, the translation $f(z) = z + a$ maps $\{|z| < r\}$ holomorphically into U and it extends to a smooth a map $U \to U$, but it cannot be approximated uniformly on any neighborhood of the origin $0 \in U$ by a holomorphic map $g\colon U \to U$ since this would give $g'(0) \approx f'(0) = 1$ in contradiction to the Schwarz lemma.

2.4 Example. This and the next example can be found in [Gro4] (see also [Wi]). Let $X = \{1 < |z| < r\}$ and $Y = \{1 < |z| < R\}$ be annuli in $\mathbb{C}$. The space of homotopy classes of maps $X \to Y$ equals $\pi_1(Y) = Z$. However, if $1 < R < r$ then every holomorphic map $X \to Y$ is homotopic to constant and hence the Oka principle fails. Furthermore, for any choice of values of $r, R > 1$ only finitely many homotopy classes of maps $X \to Y$ are represented by holomorphic maps. To see this, observe that the infimum of the Kobayashi length of closed curves in X or Y which generate the respective fundamental group is positive, and holomorphic maps do not increase the length. On the other hand, the Oka principle holds for maps $X \to \mathbb{C}\backslash\{0\}$ from any Stein manifold X (Section 3).

2.5 Example. The argument in Example 2.4 extends to any Kobayashi hyperbolic target manifold. Recall that a complex manifold Y is *Kobayashi hyperbolic* if for any point $y \in Y$ and tangent vector $v \in T_yY \in \{0\}$ the set of all numbers $\lambda \in \mathbb{C}$ of the form $f'(0) = \lambda v$ for some holomorphic map $f\colon U \to Y$, $f(0) = y$, is bounded: $|\lambda| \le M$ for some $M = M(y,v) < +\infty$. For instance, the twice punctured plane $\mathbb{C}\backslash\{0,1\}$ is hyperbolic by Picard's theorem. If we take as before X to be an annulus then even the basic Oka principle fails for maps $X \to Y = \mathbb{C}\backslash\{0,1\}$ which is seen by the following argument from [Gro4, p. 853]. Take a circle $S = \{|z| = \rho\} \subset X$

and wrap is sufficiently many times around each of the points $0, 1$ by a smooth map $f: S \to Y$. Since the minimal Kobayashi length of closed curves representing a given homotopy class in $\pi_1(Y)$ increases to $+\infty$ when we increase the number of rotations around the two punctures, the length of $f(S)$ in Y will exceed the length of S in X for any f representing a suitably chosen class in $\pi_1(Y)$. Since holomorphic maps do not increase the Kobayashi length, it follows that such f is not homotopic to any holomorphic map $X \to Y$. In fact only finitely many classes in $\pi_1(Y)$ can be represented by holomorphic maps $X \to Y$.

2.6 Example. The following example, due to J.-P. Rosay (private communication), is an improvement of Proposition 2.2 in [ČF]. It shows that holomorphic graphs over the unit disc cannot avoid even fairly simple complex curves in $\mathbb{C}^2$. Let (z, w) be complex coordinates on $\mathbb{C}^2$. For $k \in \mathbb{C}$ let

$$\Sigma_k = \{w = 0\} \cup \{w = 1\} \cup \{w = kz\} \cup \{zw = 1\} \subset \mathbb{C}^2.$$

Proposition. (J.-P. Rosay) *There is a $k > 0$ such that the graph of any holomorphic function $f: U = \{|z| < 1\} \to \mathbb{C}$ intersects Σ_k. (Indeed this is true for every sufficiently large $|k|$.) On the other hand, for any k there exists a smooth function $U \to \mathbb{C}$ whose graph avoids Σ_k.*

This should be compared with Examples 3.4 and 3.5 below on avoiding subvarieties of codimension at least two. This is also in strong contrast to the situation for *holomorphic motions*, i.e., *disjoint unions of holomorphic graphs* over the disc, which can always be extended to maximal motions according to Slodkowski ([Sl1], [Sl2]).

Proof. The last statement is a simple topological exercise. Suppose now that $f: U \to \mathbb{C}\backslash\{0, 1\}$ is a holomorphic function omitting 0 and 1. Denote by l the length of the circle $C = \{|z| = 1/2\}$ with respect to the Kobayashi (=Poincaré) metric on U. Denote by d the Kobayashi distance function on $\mathbb{C}\backslash\{0, 1\}$. Let

$$k_0 = \sup\{|\zeta| \in \mathbb{C}\backslash\{0,1\}: \inf_\theta d(\zeta, 2e^{i\theta}) \le l\}.$$

We have $k_0 < +\infty$ since $\mathbb{C}\backslash\{0, 1\}$ is complete hyperbolic. Observe that the Kobayashi length of $f(C) \subset \mathbb{C}\backslash\{0, 1\}$ is at most l. We consider two cases.
Case 1. There exists a $\theta \in \mathbb{R}$ such that $|f(e^{i\theta}/2)| \le 2$. Then for all $\gamma \in \mathbb{R}$ we have $|f(e^{i\gamma}/2)| \le k_0$ by the choice of k_0. Rouché's theorem shows that for every $k > 2k_0$ the equation $kz - f(z) = 0$ has a solution with $|z| < 1/2$, and at this point the graph of f intersects Σ_k.
Case 2. For every $\theta \in \mathbb{R}$ we have $|f(e^{i\theta}/2)| > 2$. Since f has values in $\mathbb{C}\backslash\{0, 1\}$, so does $g = 1/f$, and the above gives $|g(e^{i\theta}/2)| < 1/2$ for every θ. Rouché's theorem implies that $z - g(z) = z - \frac{1}{f(z)}$ has one zero with $|z| < 1/2$, which means that $zf(z) = 1$ has a solution with $|z| < 1/2$. At this point the graph of f intersects Σ_k. This completes the proof.

2.7 Example. This example is taken from [FP3]. *For every $n \in \mathbb{N}$ there exists a discrete subset $P \subset \mathbb{C}^n$ such that the basic Oka principle fails for maps $X \to \mathbb{C}^n\backslash P$ from some Stein manifold X.* In fact this holds for any discrete set P which is *unavoidable* in the sense of Rosay and Rudin [RR], i.e., such that any entire map $\mathbb{C}^n \to \mathbb{C}^n$ of generically maximal rank intersects P infinitely often. Alternatively, any entire map $\mathbb{C}^k \to \mathbb{C}^n\backslash P$ has rank $< n$ at each point (here k may be different from n). The same holds for maps $X \to \mathbb{C}^n\backslash P$ for any X covered by an affine space.

A simple argument shows that any holomorphic map $X \to \mathbb{C}^n\backslash P$ of rank $< n$ is homotopic to the constant map in $\mathbb{C}^n\backslash P$. However, for certain X covered by an affine space there exist homotopically nontrivial smooth maps $X \to \mathbb{C}^n\backslash P$ and hence the Oka principle fails. To obtain such an example let $n = 2$ and $X = (\mathbb{C}\backslash\{0\})^3$ (which is universally covered by $\mathbb{C}^3$). There is a smooth contraction of X onto the standard torus $T^3 \subset \mathbb{C}^3$. Let $f\colon T^3 \to \mathbb{C}^2\backslash P$ be an embedding of T^3 onto a small hypersurface torus surrounding a point $p_0 \in P$. Composing f with the contraction $X \to T^3$ we get a nontrivial smooth map $X \to \mathbb{C}^2\backslash P$ which is not homotopic to any holomorphic map.

Note that the infinitesimal Kobayashi pseudometric on $Y = \mathbb{C}^n\backslash P$ is totally degenerate, but the Kobayashi-Eisenmann volume form on Y is nontrivial (when P is unavoidable).

&3. Mappings of Stein manifolds into subelliptic manifolds.

In this section we present results on the Oka principle for maps $X \to Y$ from Stein source manifolds, as well as for section of submersions onto a Stein base. Our main references are the papers by Grauert ([Gra1], [Gra2]), Cartan [Ca], Gromov [Gro4], and [FP1], [FP2], [FP3], [F5].

By definition every Stein manifold admits plenty of holomorphic maps to complex affine spaces $\mathbb{C}^q$. The basic idea introduced by Gromov [Gro4] is the following. Suppose that a complex manifold Y admits sufficiently many *dominating holomorphic maps* $s\colon \mathbb{C}^q \to Y$, where the domination property means s is a submersion outside a subvariety of $\mathbb{C}^q$. Then there also exist plenty of holomorphic maps $X \to Y$ from any Stein X. (In some sense the idea is to factor maps $X \to Y$ as $X \to \mathbb{C}^q \to Y$.) What is needed in the proofs is a family of dominating maps $s_y\colon \mathbb{C}^q \to Y$, depending holomorphically on the point $y = s_y(0) \in Y$. This leads to the following concept of a *dominating spray* introduced by Gromov [Gro4]. The notion of a *dominating family of sprays* and of *subelliptic manifolds* was introduced in [F5].

3.1 Definition. *A* **spray** *on a complex manifold Y is a holomorphic map $s\colon E \to Y$, defined on the total space of a holomorphic vector bundle $p\colon E \to Y$, such that $s(0_y) = y$ for every $y \in Y$. The spray is* **dominating at** *y if its differential $ds_{0_y}\colon T_{0_y}E \to T_yY$ maps E_y (which is a linear subspace of $T_{0_y}(E)$) onto T_yY; it is* **dominating** *if this holds at every point $y \in Y$. A* **dominating family of sprays** *is a collection of sprays $s_j\colon E_j \to Y$ $(j = 1, 2, \ldots, k)$ such that for every $y \in Y$ we have*

$$(ds_1)_{0_y}(E_{1,y}) + (ds_2)_{0_y}(E_{2,y}) \cdots + (ds_k)_{0_y}(E_{k,y}) = T_yY. \tag{3.1}$$

A manifold Y is called **elliptic** *if it admits a dominating spray, and* **subelliptic** *if it admits a finite dominating family of sprays.*

3.2 Theorem. (The Oka principle for maps to subelliptic manifolds.) *If X is a Stein manifold and Y is a subelliptic manifold then mappings $X \to Y$ satisfy the parametric Oka principle with interpolation and approximation. Furthermore, the Oka principle holds (in all forms) for sections $X \to Z$ of any holomorphic fiber bundle $h\colon Z \to X$ with subelliptic fiber $Z_x = h^{-1}(x)$.*

We emphasize that there is no restriction on the structure group of $Z \to X$ (we may use the entire group of holomorphic automorphisms of the fiber). Theorem 3.2 includes the results of Grauert ([Gra1], [Gra2]) and Gromov [Gro4, Sec. 2]. Observe that (sub)ellipticity of a complex manifold eliminates Kobayashi or Eisenman hyperbolicity.

Theorem 3.2 is proved constructively by approximation and gluing of holomorphic maps to Y (resp. of sections $X \to Z$) defined on holomorphically convex subsets of X. The main steps have been developed in the papers cited above and in [FP1], [FP2], [FP3]. In [F5] the result was proved in the final form as stated here. For an extension to sections of subelliptic submersions see Theorem 3.8 below.

We now give examples of sprays and (sub)elliptic manifolds; thus the Oka principle holds for maps from any Stein manifold to any manifold on this list. Most of them can be found in [Gro4].

3.3 Example: Complex homogeneous manifolds. Let G be a complex Lie group which acts holomorphically and transitively on a complex manifold Y by holomorphic automorphisms. Let $\mathbf{g} = T_eG$ denote its Lie algebra and $\exp\colon \mathbf{g} \to G$ the associated exponential map. The map $Y \times \mathbf{g} \to Y$, $(y,t) \to e^t y$, is a dominating spray on Y and hence Y is elliptic. The Oka principle for maps $X \to Y$ is due to Grauert [Gra1], [Gra2].

3.4 Example: Sprays induced by complete vector fields. (See [Gro4] and [FP1].) Let $V_1, \ldots, V_k$ be holomorphic vector fields on Y which are complete in complex time. Denote by $\theta_j \colon Y \times \mathbb{C} \to Y$ the flow of V_j. The superposition of these flows (in any order) gives a spray $s\colon Y \times \mathbb{C}^k \to Y$ which is dominating at a point $y \in Y$ if the vectors $V_1(y), \ldots, V_k(y)$ span the tangent space T_yY. For example, if $A \subset \mathbb{C}^n$ is an algebraic subvariety of complex codimension at least two then the complement $Y = \mathbb{C}^n \backslash A$ admits a dominating polynomial spray of this kind. (This fails for most complex hypersurfaces A, see Example 2.6.) A complex Lie group admits sprays of this kind induced by left (or right) invariant vector fields spanning the Lie algebra.

3.5 Example: Complements of projective subvarieties. *If A is a closed complex (=algebraic) subvariety of complex codimension at least two in the complex projective space $\mathbb{CP}^n$ then the manifold $Y = \mathbb{CP}^n \backslash A$ is subelliptic. The same holds if we replace $\mathbb{CP}^n$ by a complex Grassmanian.* The proof proceeds as follows (see Proposition 1.2 in [F5]; the idea can be found in [Gro4]). Removing a complex hyperplane L from $\mathbb{CP}^n$ we are left with $\mathbb{C}^n \backslash A$ which admits a dominating algebraic (polynomial) spray defined on a trivial bundle $E = (\mathbb{C}^n \backslash A) \times \mathbb{C}^k \to \mathbb{C}^n \backslash A$ (Example 3.4). Let $[L] = \mathcal{O}_{\mathbb{CP}^n}(1)$ denote the line bundle on $\mathbb{CP}^n$ determined by the divisor of L (the 'hyperplane section bundle'). For sufficiently large $m > 0$ the spray s extends to an algebraic spray $\widetilde{s}\colon E \otimes [L]^{-m} \to \mathbb{CP}^n$ which is dominating over $\mathbb{CP}^n \backslash (L \cup A)$. Repeating this with $n+1$ hyperplanes in general position we obtain a finite dominating family of sprays on Y. The bundles of these sprays are nontrivial (in fact negative) and hence we cannot combine them into a single dominating spray as we did with the flows θ_j in Example 3.4. It is not known whether $\mathbb{CP}^n \backslash A$ is elliptic for every such A.

3.6 Example: Matrix-valued maps with nonzero determinant. Let X be a Stein manifold and $g_1, \ldots, g_k \colon X \to \mathbb{C}^n$ holomorphic maps (with $1 \le k < n$) such that the vectors $g_1(x), \ldots, g_k(x) \in \mathbb{C}^n$ are $\mathbb{C}$-linearly independent for every $x \in X$. The problem is to find holomorphic maps $g_{k+1}, \ldots, g_n \colon X \to \mathbb{C}^n$ such that the matrix $g(x) = (g_1(x), \dots, g_n(x))$ with columns $g_j(x)$ satisfies $\det g(x) \neq 0$ (or even $\det g(x) = 1$) for every $x \in X$. The Oka principle holds in this problem (in all forms); in particular, *a holomorphic solution exists provided there exists a continuous solution.* To see this we consider the manifold

$$Z = \{(x, v_{k+1}, \dots, v_n) \colon x \in X,\ v_j \in \mathbb{C}^n \text{ for } j = k+1, \dots, n, \\ \det(g_1(x), \dots, g_k(x), v_{k+1}, \dots, v_n) \neq 0\}$$

with the projection $h\colon Z \to X$ onto the first factor. A solution to the problem is a section $X \to Z$ of this fibration. The Oka principle follows from the observation that $Z \to X$ is a holomorphic fiber bundle whose fiber $GL_{n-k}(\mathbb{C}) \times \mathbb{C}^{k(n-k)}$ is a complex Lie group. For maps into $SL_n(\mathbb{C})$ is suffices to divide one of the columns by the (nonvanishing) determinant function.

Theorem 3.2 extends to sections of *subelliptic submersions* which we now introduce. Let $h\colon Z \to X$ be a holomorphic submersion onto X. For $U \subset X$ we write $Z|_U = h^{-1}(U)$. For $z \in Z$ we denote by VT_zZ the kernel of dh_z (which equals the tangent space to the fiber of Z at z) and call it the *vertical tangent space* of Z at z. The space $VT(Z) \to Z$ with fibers VT_zZ is a holomorphic vector subbundle of the tangent bundle TZ.

If $p\colon E \to Z$ is a holomorphic vector bundle we denote by $0_z \in E$ the base point in the fiber $E_z = p^{-1}(z)$. At each point $z \in Z$ (=the zero section of E) we have a natural splitting $T_{0_z}E = T_zZ \oplus E_z$.

3.7 Definition. [Gro4, sec. 1.1.B] *A* **spray** *associated to a holomorphic submersion $h\colon Z \to X$ (an h-spray) is a triple (E, p, s), where $p\colon E \to Z$ is a holomorphic vector bundle and $s\colon E \to Z$ is a holomorphic map such that for each $z \in Z$ we have $s(0_z) = z$ and $s(E_z) \subset Z_{h(z)}$. The spray s is dominating at the point $z \in Z$ if the derivative $ds\colon T_{0_z}E \to T_zZ$ maps E_z (which is a linear subspace of $T_{0_z}E$) surjectively onto $VT_zZ = \ker dh_z$. The submersion $h\colon Z \to X$ is called* **subelliptic** *if each point in X has an open neighborhood $U \subset X$ such that $h\colon Z|_U \to U$ admits finitely many h-sprays (E_j, p_j, s_j) for $j = 1, \dots, k$ satisfying*

$$(ds_1)_{0_z}(E_{1,z}) + (ds_2)_{0_z}(E_{2,z}) \cdots + (ds_k)_{0_z}(E_{k,z}) = VT_zZ \qquad (3.2)$$

for each $z \in Z|_U$. A collection of sprays satisfying (3.1) is said to be dominating at z. A submersion h is **elliptic** *if the above holds with $k = 1$.*

Comparing with Definition 3.1 we see that a spray on a manifold Y is the same thing as a spray associated to the trivial submersion $Y \to$ *point*. By definition every elliptic submersion is also subelliptic, but the converse is not known. A holomorphic fiber bundle $Z \to X$ is (sub)elliptic if and only if the fiber has this property (since a spray on E induces an h-spray on the product bundle $h\colon U \times E \to U$).

3.8 Theorem. *If $h\colon Z \to X$ is a subelliptic submersion onto a Stein manifold X then sections $f\colon X \to Z$ satisfy the parametric Oka principle with interpolation and approximation.*

For elliptic submersions Theorem 3.8 coincides with Gromov's Main Theorem in [Gro4, Sec. 4.5]. The result is proved in [FP1] for fiber bundles with elliptic fibers, in [FP2] for elliptic submersions but without interpolation, in [FP3] for elliptic submersions with interpolation, and the extension to subelliptic submersions is obtained in [F5]. A version of the Oka principle for multi-valued sections of ramified holomorphic maps can be found in [F6]. F. Lárusson [Lár] explained this result from the homotopy theory point of view.

Historical comments. The so-called *classical case* of Theorem 3.8 (for sections of principal fiber bundles with complex homogeneous fibers) is due to Grauert theorem [Gra1, Gra2, Ca, Ram]. A very important develoment which opened the way to generalizations was the work of Henkin and Leiterer in 1984 [HL1] (published in 1998 [HL2]) where they introduced the *bumping method* to this problem. Its main advantage over the original method of Grauert is that one only needs the Oka-Weil approximation theorem for sections of $Z \to X$ over small subsets of the base manifold X. The second main contribution was made by Gromov [Gro4] who replaced the exponential map on the fiber (which was used in Grauert's proof to linearize the problems and to patch local sections) by the more flexible notion of a dominating spray. The idea of using several sprays instead of one is implicitly present in [Gro4], but the condition which we call subellipticity was not formulated there explicitly.

3.9 Example: Avoiding subvarieties with algebraic fibers. Let $h\colon Z \to X$ be a fiber bundle with fiber $Z_x = h^{-1}(x) \simeq \mathbb{CP}^n$ or a complex Grassmanian. Assume that A is a closed complex subvariety of Z whose fiber $A_x = A \cap Z_x$ has complex codimension at least two in Z_x for every $x \in X$. (A_x is algebraic by Chow's theorem.) Then the restricted submersion $h\colon Z\backslash A \to X$ is subelliptic and hence Theorem 3.8 applies (Proposition 1.2 (b) in [F5]). If Z is obtained from a holomorphic vector bundle $E \to X$ by taking projective closure of each fiber (i.e., $Z_x \simeq \mathbb{CP}^n$ is obtained by adding to $E_x \simeq \mathbb{C}^n$ the hyperplane at infinity $\Lambda_x \simeq \mathbb{CP}^{n-1}$ for every $x \in X$), the restricted submersion $h\colon E\backslash A \to X$ is even elliptic (Corollary 1.8 in [FP2]). The Oka principle for such submersions is used in the constructions of proper holomorphic immersion and embeddings of Stein manifolds in affine spaces of minimal dimension (Section 5).

3.10 Ellipticity versus subellipticity. By definition every elliptic complex manifold is also subelliptic. It is not known whether there exist subelliptic manifolds which are not elliptic. Natural candidates for a possible counterexample are the complements $Y = \mathbb{CP}^n\backslash A$ of generic algebraic subvarieties $A \subset \mathbb{CP}^n$ of codimension at least two (and of sufficiently large degree); see Example 3.5. Such Y admits a finite dominating family of algebraic sprays defined on *negative* line bundles over $\mathbb{CP}^n$, but we don't see how to obtain a dominating spray. (See [F5] for more.)

3.11 Problem. It is not known whether the Oka principle holds for maps from Stein manifolds into the complement $Y = \mathbb{C}^n\backslash K$ of any infinite compact set $K \subset \mathbb{C}^n$ (this is unknown even if K is a closed ball). Such complements have no Kobayashi-Eisenman hyperbolicity. If K is convex then every point $y \in Y = \mathbb{C}^n\backslash K$ is contained in a Fatou-Bieberbach domain $\Omega_y \subset Y$. However, it is not known whether $Y = \mathbb{C}^n\backslash K$ admits any nontrivial sprays. (It is easily seen that there are no sprays $s\colon Y \times \mathbb{C}^N \to Y$ from a trivial bundle.) A good test case for the validity of the Oka principle might be the holomorphic map

$$\mathrm{SL}(2,\mathbb{C}) \to \mathbb{C}^2\backslash\{0\}, \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to (\alpha,\beta) \qquad (\alpha\delta - \beta\gamma = 1).$$

This map is clearly homotopic to a smooth map into $Y = \{z \in \mathbb{C}^2 \colon |z| > 1\}$ but it is unknown whether it is homotopic to a holomorphic map to Y. (This problem has been mentioned in [FP3].)

The main problem in this connection is the following. Suppose that $C \subset B \subset \mathbb{C}^m$ is a pair of compact convex sets. Let K be a closed ball in $\mathbb{C}^n$ and let $f\colon \widetilde{C} \to \mathbb{C}^n \backslash K$ be a holomorphic map on a neighborhood of C whose range avoids K. Is it possible to approximate f uniformly on C by a holomorphic map $\widetilde{f}\colon \widetilde{B} \to \mathbb{C}^n \backslash K$ defined on a neighborhood of B ?

3.12 Problem. For any Stein manifold Y Theorem 3.2 has the following converse: *If the Oka principle holds for maps $X \to Y$ from any Stein manifold X, with second order interpolation on any closed complex submanifold $X_0 \subset X$, then Y admits a dominating spray* (see [Gro4] and [FP3]). In [Gro4] the reader can find some further examples of target manifolds Y for which this holds, but it is not known whether it holds for all manifolds.

&4. Removing intersections with complex subvarieties.

Let X and Y be complex manifolds and $A \subset Y$ a closed complex subvariety of Y. Given a holomorphic map $f\colon X \to Y$ we write $f^{-1}(A) = \{x \in X\colon f(x) \in A\}$ and call it the *intersection set of f with A*. The question is to what extent is it possible to prescribe the intersection set if f is allowed to vary within a homotopy class of maps $X \to Y$. More precisely, we consider the following

4.1 Problem. Suppose that $f^{-1}(A) = X_0 \cup X_1$, where $X_0, X_1 \subset X$ are disjoint complex subvarieties of X. When is it possible to remove X_1 from $f^{-1}(A)$ by homotopy of holomorphic maps $f_t\colon X \to Y$ $(t \in [0,1])$ which is fixed on X_0 and satisfies $f_0 = f$ and $f_1^{-1}(A) = X_0$?

In the simplest case when $X = \mathbb{C}$ and A consists of d points in $Y = \mathbb{CP}^1$ the answer changes when passing from $d = 2$ to $d = 3$: One can prescribe the pull-back of any two points in $\mathbb{CP}^1$ by a holomorphic map $f\colon \mathbb{C} \to \mathbb{CP}^1$ (and there are infinitely many such maps), but when $d \geq 3$ the pull-back divisor f^*A completely determines the map f. Similar situation occurs when A consists of d hyperplanes in general position in $Y = \mathbb{CP}^n$: we have flexibility (=infinitely many maps) for $d \leq n+1$ and rigidity (few maps) for $d \geq n+2$.

We say that the Oka principle holds if the existence of a homotopy of continuous maps $X \to Y$ (which remain holomorphic near X_0 and remove X_1 from the preimage) implies the existence of a holomorphic homotopy with the required properties. We break down the problem as follows.

Step 1: Find a homotopy $f_t\colon X \to Y$ $(0 \leq t \leq 1/2)$ with $f_0 = f$ such that each f_t equals f in an open neighborhood of X_0 in X and $f_{1/2}^{-1}(A) = X_0$. This is a homotopy theoretical problem.

Step 2: With $f_{1/2}$ as in Step 1, find a homotopy $f_t\colon X \to Y$ $(1/2 \leq t \leq 1)$ such that each f_t is holomorphic near X_0 and matches f on X_0, $f_t^{-1}(A) = X_0$ for each t, and f_1 is holomorphic on X. A solution is given by Theorem 4.2 below when X is Stein and $Y \backslash A$ is subelliptic.

Step 3: Deform the combined homotopy f_t $(0 \leq t \leq 1)$ from Steps 1 and 2, with fixed f_0 and f_1, to a holomorphic homotopy $\widetilde{f}_t\colon X \to Y$ $(t \in [0,1])$ such that the resulting two-parameter homotopy is fixed along X_0. This is possible if X is Stein and Y is subelliptic (Theorem 4.4 below).

The conclusion is that *the Oka principle holds in Problem 4.1 provided that X is Stein and the manifolds Y and $Y \backslash A$ are subelliptic.* Examples of such pairs $A \subset Y$

are algebraic subvarieties of codimension at least two (or those with homogeneous complement) in $\mathbb{C}^n$, $\mathbb{CP}^n$, or in a complex Grassmanian manifold.

We now present the results mentioned above (see [F4] and [F5] for more). Thus Theorem 4.2 provides a solution to Step 2 and Theorem 4.4 provides a solution to Step 3.

4.2 Theorem. *Let A be a closed complex subvariety of a complex manifold Y. Let X be a Stein manifold, $K \subset X$ a compact $\mathcal{O}(X)$-convex subset, and $f\colon X \to Y$ a continuous map which is holomorphic in an open set $U_0 \subset X$ containing $f^{-1}(A) \cup K$. If $Y \backslash A$ is subelliptic then for any $r \in \mathbb{N}$ there are an open set $U \supset f^{-1}(A) \cup K$ and a homotopy $f_t\colon X \to Y$ $(t \in [0,1])$ of continuous maps such that $f_0 = f$, f_t is holomorphic in U and tangent to f to order r along $f_t^{-1}(A) = f^{-1}(A)$ for each $t \in [0,1]$, and f_1 is holomorphic on X.*

4.3 Corollary. *The conclusion of Theorem 4.2 holds in the following cases:*

(a) *Y is an affine space $\mathbb{C}^n$, a projective space $\mathbb{CP}^n$ or a complex Grassmanian and $A \subset Y$ is an algebraic subvariety of codimension at least two.*

(b) *$Y = \mathbb{CP}^n$ and A consists of at most $n+1$ hyperplanes in general position.*

(c) *A complex Lie group acts transitively on $Y \backslash A$.*

In any of these cases $Y \backslash A$ is subelliptic by the results stated in section three. Note that (b) is a special case of (c).

4.4 Theorem. (The Oka principle for removing intersections.) *Assume that $f\colon X \to Y$ is a holomorphic map of a Stein manifold X to a subelliptic manifold Y, $A \subset Y$ is a complex subvariety such that $Y \backslash A$ is subelliptic, and $f^{-1}(A) = X_0 \cup X_1$ where X_0 and X_1 are unions of connected components of $f^{-1}(A)$ with $X_0 \cap X_1 = \emptyset$. If there exists a homotopy $\widetilde{f}_t\colon X \to Y$ $(t \in [0,1])$ of continous maps satisfying $\widetilde{f}_0 = f$, $\widetilde{f}_1^{-1}(A) = X_0$, and $\widetilde{f}_t|_U = f|_U$ for some open set $U \supset X_0$ and for all $t \in [0,1]$ then for each $r \in \mathbb{N}$ there exists a homotopy of holomorphic maps $f_t\colon X \to Y$ such that $f = f_0$, $f_1^{-1}(A) = X_0$, and for each $t \in [0,1]$ the map f_t agrees to order r with f along X_0.*

Note that X_0 is a union of connected components of $f_t^{-1}(A)$ for every $t \in [0,1]$. In plain language Theorem 4.4 says the following. Suppose that we can remove X_1 from $f^{-1}(A) = X_0 \cup X_1$ by a homotopy of continuous maps $X \to Y$ which agree with f in a neighborhood of X_0. If Y and $Y \backslash A$ are both subelliptic then X_1 can also be removed from the preimage of A by a homotopy of holomorphic maps $X \to Y$ which agree with f to any given order on X_0.

Theorem 4.4 applies if Y is any of the manifolds $\mathbb{C}^n$, $\mathbb{CP}^n$ or a complex Grassmanian (these are complex homogeneous and therefore elliptic) and $A \subset Y$ is as in Corollary 4.3. When $Y = \mathbb{C}^d$ and $Y \backslash A$ is elliptic, Theorem 4.4 coincides with Theorem 1.3 in [F5].

4.5 Example: The Oka principle for complete intersections. When A is the origin in $Y = \mathbb{C}^d$, Theorem 4.4 implies the following result of Forster and Ramspott [FRa] (1967). Let X be a Stein manifold. Assume that $X_0 \subset X$ is a closed complex subvariety of pure codimension d which is a *complete intersection* in an open set $U \subset X$ containing X_0, i.e., there exist functions $f_1, \ldots, f_d \in \mathcal{O}(U)$ which together generate the ideal of X_0 at each point of U. If these functions admit continuous extensions to X with no additional common zeros then X_0 is a complete intersection in X. The analogous result holds for set-theoretic complete intersections.

4.6 Example: Smooth versus holomorphic complete intersections. In [F4] it was proved that *there exists a three dimensional closed complex submanifold X in $\mathbb{C}^5$ which is a smooth (even real-analytic) complete intersection but which is not a holomorphic complete intersection.* More precisely, given any compact orientable two dimensional surface M of genus $g \geq 2$, there is a three dimensional Stein manifold X which is homotopy equivalent to M and whose tangent bundle TX is trivial as a real vector bundle but is nontrivial as a complex vector bundle over X. The image of any proper holomorphic embedding of X in $\mathbb{C}^5$ (or in $\mathbb{C}^7$) is a smooth complete intersection in $\mathbb{C}^5$ (resp. $\mathbb{C}^7$) but is not a holomorphic complete intersection in any open neighborhood of X (since its normal bundle is nontrivial as a complex vector bundle on X). The following problem remains open.

Problem: *Let $X \subset \mathbb{C}^n$ be a closed complex submanifold such that (i) X is a smooth complete intersection in $\mathbb{C}^n$, and (ii) its normal bundle $T\mathbb{C}^n|_X/TX$ is trivial as a complex vector bundle (hence X is a holomorphic complete intersection in an open neighborhood $U \subset \mathbb{C}^n$). Is X a holomorphic complete intersection in $\mathbb{C}^n$?*

4.7 Example: Unavoidable discrete sets. Theorem 4.4 fails if $Y = \mathbb{C}^n$ and A is any unavoidable discrete subset of $\mathbb{C}^n$ (see Example 2.7 above). To see this, write $A = \{p\} \cup A_1$ for some $p \in A$. Then A_1 is still unavoidable and consequently every entire map $F\colon \mathbb{C}^n \to \mathbb{C}^n \backslash A_1$ has rank $< n$ at each point. Take $X = \mathbb{C}^n$, $f = Id\colon \mathbb{C}^n \to \mathbb{C}^n$, $X_0 = \{p\}$ and $X_1 = A_1$. The conditions of Theorem 4.4 are clearly satisfied but its conclusion fails since the rank condition for holomorphic maps $F\colon \mathbb{C}^n \to \mathbb{C}^n \backslash A_1$ implies that $F^{-1}(p)$ contains no isolated points, and hence $X_0 = \{p\}$ cannot be a connected component of $F^{-1}(p)$.

&5. Embeddings and immersions of Stein manifolds.

In this section we collect results on holomorphic immersions and embeddings of Stein manifolds into affine complex spaces. There has been considerable progress on this subject since the 1990 survey of Bell an Narasimhan [BN].

In 1956 Remmert announced that every Stein manifold of dimension $n \geq 1$ admits a proper holomorphic embedding in $\mathbb{C}^{2n+1}$ and a proper holomorphic immersion in $\mathbb{C}^{2n}$ [Rem]. Further more precise results were obtained by Narasimhan ([Na1], [Na2]), Bishop [Bis], Ramspott [Ram], Forster ([Fs1], [Fs2]), Schaft [Sht], and others. The following optimal result, due to Eliashberg and Gromov, was announced in 1971 [GE] and proved in 1992 [EG]. An improvement of the embedding dimension for odd n is due to Schürmann [Sch] (1997).

5.1 Theorem. *Every Stein manifold X of dimension $n > 1$ admits a proper holomorphic embedding in $\mathbb{C}^{[3n/2]+1}$ and a proper holomorphic immersion in $\mathbb{C}^{[(3n+1)/2]}$.*

Also there exists a (not necessarily proper) holomorphic immersion $X \to \mathbb{C}^{[3n/2]}$ (see Theorem 5.9 below). Schürmann [Sch] also proved an optimal embedding theorem for Stein spaces with singularities which have uniformly bounded local embedding dimension. Recently J. Prezelj [Pr2] constructed *proper weakly holomorphic embeddings of Stein spaces with isolated singularities in Euclidean spaces of minimal dimension.* These results strongly depend on Lefschetz's theorem to the effect that a Stein manifold is homotopically equivalent to a CW-complex of real dimension at most $\dim_{\mathbb{C}} X$ [AF]. The following example of Forster [Fs1] shows that the dimensions in Theorem 5.1 cannot be lowered.

5.2 Example. Let $Y = \{[x\colon y\colon z] \in \mathbb{CP}^2 \colon x^2 + y^2 + z^2 \neq 0\}$ and

$$X = \begin{cases} Y^m, & \text{if } n = 2m; \\ Y^m \times \mathbb{C}, & \text{if } n = 2m+1. \end{cases}$$

Clearly X is a Stein manifold of dimension n. Forster proved ([Fo1], Proposition 3) that the Stiefel-Whitney class $w_{2m}(TX)$ is the nonzero element of the group $H^{2m}(X; Z_2) = H^{2m}((\mathbb{RP}^2)^m; Z_2) = Z_2$, and consequently the Chern class $c_m(TX)$ is the nonzero element of $H^{2m}(X; Z) = Z_2$. It follows that X does not embed in $\mathbb{C}^{[3n/2]}$ and does not immerse in $\mathbb{C}^{[3n/2]-1}$ (see [Hu, p. 263]).

The proof of Theorem 5.1 in [EG] and [Sch] relies on the elimination of singularities method and on the Oka principle for sections of certain submersions onto Stein manifolds (Example 3.9 above). We describe the main idea. One begins by choosing a generic almost proper holomorphic map $b\colon X^n \to \mathbb{C}^n$ constructed by Bishop [Bis]. This means that the b-preimage of any compact set in $\mathbb{C}^n$ has (at most countably many) compact connected components. We then try to find a map $g\colon X \to \mathbb{C}^q$ which 'desingularizes b' in the sense that $f = (b,g)\colon X \to \mathbb{C}^{n+q}$ is a proper holomorphic embedding (resp. immersion). Properness is easily achieved by choosing g sufficiently large on a certain sequence of compact sets in X (here we need that b is almost proper).

To insure that $f = (b,g)$ is an immersion we must choose g such that its differential dg_x is nondegenerate on the kernel of db_x at each $x \in X$. To obtain injectivity we must choose g to separate points on the fibers of b. Both requirements can be satisfied if $q \geq [n/2] + 1$ and this number is determined by topological restrictions. (The immersion condition requires $q \geq [n/2]$.) One proves this by a finite induction. We stratify X by a descending finite chain of closed complex subvarieties $X = X_0 \supset X_1 \supset X_2 \ldots \supset X_m = \emptyset$ such that the kernel of db_x has constant dimension on each stratum $S_k = X_k \backslash X_{k+1}$ (which is chosen to be nonsingular), and the number of distinct points in $b^{-1}(x)$ is constant for $x \in S_k$. (In the actual proof we must replace X by a suitable subset $B \subset X$ which is mapped by b properly onto a bounded domain in $\mathbb{C}^n$; in the end we perform an induction by increasing B to X.) Furthermore, once we have a map $g_k\colon X \to \mathbb{C}^q$ satisfying these conditions along X_k, we choose $g_{k-1}\colon X \to \mathbb{C}^q$ such that it satisfies both condition on the next stratum S_{k-1} and agrees with g_k to second order along X_k (so that g_{k-1} does not destroy what g_k has achieved). A suitable g_{k-1} is obtained by the Oka principle (Section 3) provided there are no topological obstructions, and this is so when $q \geq [n/2] + 1$.

Although Theorem 5.1 gives the optimal result for the entire collection of n-dimensional Stein manifolds, the method does not give a better result for 'simple' Stein manifolds which are expected to embed in lower dimensional space. For instance, it is not known what is the minimal proper embedding dimension of the polydisc or the ball in $\mathbb{C}^n$. Globevnik proved by a different method (using shear automorphisms of $\mathbb{C}^n$) that there are arbitrarily small perturbations of the polydisc in $\mathbb{C}^n$ which embed in $\mathbb{C}^{n+1}$ [Gl2].

5.3 Question. *What is the proper holomorhic embedding dimension of the ball? The polydisc? A general convex domains in $\mathbb{C}^n$? How does it depend on the topology and geometry of the domain?*

We now consider the existence of *relative embeddings*. The following result was proved in [ABT] following the method of Narasimhan [Na2].

5.4 Theorem. *Suppose that X is a Stein manifold of dimension n, $Y \subset X$ is a closed complex submanifold in X and $f\colon Y \to \mathbb{C}^N$ is a proper holomorphic embedding. If $N \geq 2n+1$ then there exists a proper holomorphic embedding $\widetilde{f}\colon X \to \mathbb{C}^N$ such that $\widetilde{f}|_Y = f$.*

It is not known whether Theorem 5.4 is valid for $N = 2n$, but it is false for $N \leq 2n-1$ by Corollary 5.7 below which follows from the following interpolation results for holomorphic embeddings from [BFn] and [F3].

5.5 Theorem. *Let Σ be a discrete subset of $\mathbb{C}^N$ for some $N > 1$. If a Stein manifold X admits a proper holomorphic embedding $f_0\colon X \to \mathbb{C}^N$ then X also admits an embedding $f\colon X \to \mathbb{C}^N$ whose image $f(X)$ contains Σ. In addition we may choose f such that for every entire map $\psi\colon \mathbb{C}^d \to \mathbb{C}^N$ whose rank equals $d = N - \dim X$ at most points of $\mathbb{C}^d$ the set $\psi(\mathbb{C}^d) \cap f(X)$ is infinite. If $d = 1$, we may insure that $\mathbb{C}^N \backslash f(X)$ is Kobayashi hyperbolic.*

5.6 Corollary. *Let $n, d \geq 1$, $N = n + d$. There exists a proper holomorphic embedding $f\colon \mathbb{C}^n \to \mathbb{C}^N$ such that every entire map $\psi\colon \mathbb{C}^d \to \mathbb{C}^N$ of rank d intersects $f(\mathbb{C}^n)$ at infinitely many points. For $d = 1$ we may choose f such that $\mathbb{C}^{n+1} \backslash f(\mathbb{C}^n)$ is Kobayashi hyperbolic.*

The proofs in [BFn] and [F3] use results on holomorphic automorphisms of $\mathbb{C}^N$ obtained in [And], [AL], [FRo]. The first result in this direction [FGR] was that *there exist holomorphically embedded complex lines in $\mathbb{C}^2$ which are not equivalent to the standard embedding $\mathbb{C} \to \mathbb{C} \times \{0\} \subset \mathbb{C}^2$ by automorphisms of $\mathbb{C}^2$*. This is in strong contrast to the situation for algebraic (polynomial) embeddings $\mathbb{C} \to \mathbb{C}^2$ which are all equivalent to the standard embedding by polynomial automorphisms of $\mathbb{C}^2$ according to Abhyankar and Moh [AM]. Using such 'twisted' holomorphic embeddings of $\mathbb{C}$ in $\mathbb{C}^2$ Derksen and Kutzschebauch constructed nonlinearizable periodic holomorphic automorphisms of $\mathbb{C}^4$ [DK].

5.7 Corollary. *For every $n \geq 2$ there exists a proper holomorphic embedding $f\colon \mathbb{C}^{n-1} \to \mathbb{C}^{2n-1}$ which does not admit an injective holomorphic extension $\widetilde{f}\colon \mathbb{C}^n \to \mathbb{C}^{2n-1}$.*

Proof. Choose $f\colon \mathbb{C}^{n-1} \to \mathbb{C}^{2n-1}$ as in Corollary 5.6 such that the range of any entire map $\mathbb{C}^n \to \mathbb{C}^{2n-1}$ of generic rank n intersects $f(\mathbb{C}^{n-1})$. If $\widetilde{f}\colon \mathbb{C}^n \to \mathbb{C}^{2n-1}$ is an injective holomorphic extension of $f\colon \mathbb{C}^{n-1} \times \{0\} \to \mathbb{C}^{2n-1}$ then

$$\psi(z) = \psi(z_1, \ldots, z_n) = \widetilde{f}(z_1, \ldots, z_{n-1}, e^{z_n}) \qquad (z \in \mathbb{C}^n)$$

is an entire map which has rank n at a generic point of $\mathbb{C}^n$ and whose image misses $f(\mathbb{C}^{n-1}) \subset \mathbb{C}^{2n-1}$ in contradiction to the assumption on f.

In Theorem 5.5 the image $f(X) \subset \mathbb{C}^N$ contains a given discrete set $\{p_j\} \subset \mathbb{C}^N$, but we don't specify the points in X which correspond to the points p_j under the embedding f. The following more precise interpolation theorem was proved recently by J. Prezelj [Pr1].

5.8 Theorem. *Let X be a Stein manifold of dimension $n \geq 1$. Define $q(n) = \min\{[\frac{n+1}{2}]+1, 3\}$. Then for any $N \geq n+q(n)$ and any pair of discrete sets $\{a_k\} \subset X$, $\{b_k\} \subset \mathbb{C}^N$ there exists a proper holomorphic embedding $f\colon X \to \mathbb{C}^N$ satisfying*

$f(a_k) = b_k$ for every $k = 1, 2, 3, \ldots$. The analogous conclusion holds for proper holomorphic immersions $X \to \mathbb{C}^N$ when $N \geq [\frac{3n+1}{2}]$.

Comparing with Theorem 5.1 we see that the embedding dimension is minimal for even n and is off by at most one for odd n. Prezelj's proof in [Pr1] uses an improved version of the scheme from [EG] and [Sch]. By entirely different methods (using holomorphic automorphisms) J. Globevnik proved that the conclusion of Theorem 5.8 also holds for proper holomorphic embeddings of the unit disc in $\mathbb{C}^2$ [Gl3]. A Carleman type embedding theorem (approximating a given smooth proper embedding $\mathbb{R} \to \mathbb{C}^n$ in the fine $\mathcal{C}^\infty$ topology on $\mathbb{R}$ by proper holomorphic embeddings $\mathbb{C} \to \mathbb{C}^n$) was proved in [BF].

It is not known whether *proper* holomorphic immersions or embeddings of Stein manifolds satisfy the Oka principle. However, non-proper holomorphic immersions of Stein manifolds do satisfy the following Oka principle (Gromov and Eliashberg [GE]; see also section 2.1.5. in [Gro3]).

5.9 Theorem. *If the cotangent bundle T^*X of a Stein manifold is generated by q differential $(1,0)$-forms $\theta_1, \ldots, \theta_q$ for some $q > \dim X$ then there exists a holomorphic immersion $X \to \mathbb{C}^q$. More precisely, every such q-tuple $(\theta_1, \ldots, \theta_q)$ can be changed by a homotopy (through q-tuples generating T^*X) to a q-tuple $(df_1, \ldots, df_q)$ where $f = (f_1, \ldots, f_q)\colon X \to \mathbb{C}^q$ is a holomorphic immersion. Every n-dimensional Stein manifold admits a holomorphic immersion in $\mathbb{C}^{[3n/2]}$.*

Example 5.2 above shows that the immersion dimension $[\frac{3n}{2}]$ is the best possible for every n. The idea of the proof of Theorem 5.9 is the following. By the Oka-Grauert principle we may assume that θ_j are holomorphic 1-forms. In the first step one of the forms, say θ_q, is replaced by the differential df_q of a holomorphic function on X such that $\theta_1, \ldots, \theta_{q-1}, df_q$ still generate T^*X. Since $q > \dim X$, we may assume that the forms $\theta_1, \ldots, \theta_{q-1}$ already generate T^*X outside a proper complex subvariety $\Sigma \subset X$, and f_q must satisfy an essentially algebraic condition on its jet along Σ. Once f_q has been chosen one proceeds in the same way and replaces θ_{q-1} with an exact differential. In finitely many steps all forms are replaced with differentials. The technical details of the proof are considerable.

&6. Embeddings of open Riemann surfaces in the affine plane.

In this section we describe the state of knowledge on the following problem.

6.1 Problem. *Does every open Riemann surface admit a proper holomorphic embedding in $\mathbb{C}^2$? Is the algebra of global holomorphic functions on such a surface always doubly generated?*

Open Riemann surfaces are precisely Stein manifolds of dimension one, and in view of Theorem 5.1 (on embedding n-dimensional Stein manifolds in $\mathbb{C}^{[3n/2]+1}$ for $n > 1$) one might expect that they embed in $\mathbb{C}^2$. (For comparison we recall that every compact Riemann surface embeds in $\mathbb{CP}^3$ but most of them don't embed in $\mathbb{CP}^2$ [FK].) The proof of Theorem 5.1 only gives embeddings into $\mathbb{C}^3$, the reason being that for embeddings $X \hookrightarrow \mathbb{C}^2$ it runs into a hyperbolicity obstruction (Example 2.6 in Section 2). The main difficulty is to find injective holomorphic maps to $\mathbb{C}^2$; this is essentially equivalent to the algebra of holomorphic functions being doubly generated. Here are some Riemann surfaces which are known to embed in $\mathbb{C}^2$:

- the disc $U = \{z \in \mathbb{C} \colon |z| < 1\}$ (Kasahara and Nishino [Ste]);
- annuli $\{1 < |z| < r\}$ (Laufer [Lau]),
- punctured disc $U \backslash \{0\}$ (Alexander [Ale]),
- all finitely connected planar domains $\Omega \subset \mathbb{C}$ different from $\mathbb{C}$ whose boundary contains no isolated points (Globevnik and Stensønes [GS]).

We now consider the embedding problem for *bordered Riemann surfaces.* Let $\mathcal{R}$ be a compact, orientable, smooth real surfaces whose boundary $b\mathcal{R} = \cup_{j=1}^{m} C_j$ consists of finitely many curves and no isolated points. Such $\mathcal{R}$ is a sphere with g handles (g is the *geometric genus* of $\mathcal{R}$) and $m \geq 1$ holes (removed discs). A *complex structure* on $\mathcal{R}$ is determined by an endomorphism J of the tangent bundle $T\mathcal{R}$ satisying $J^2 = -Id$ (Gauss-Ahlfors-Bers). We may assume that J is Hölder continuous of class $\mathcal{C}^{\alpha}(\mathcal{R})$ for a fixed $\alpha \in (0,1)$. A differentiable function $f \colon \mathcal{R} \to \mathbb{C}$ is J-holomorphic if $df \circ J = \sqrt{-1}\, df$. Two complex structures J_0 and J_1 are *equivalent* if there exists a diffeomorphism $\phi \colon \mathcal{R} \to \mathcal{R}$ of class $\mathcal{C}^{1,\alpha}(\mathcal{R})$ satisfying $d\phi \circ J_0 = J_1 \circ d\phi$. The set of equivalence classes of complex structures on $\mathcal{R}$ is the *moduli space* $\mathcal{M}(\mathcal{R})$. The following result from [ČF] shows that *there are no topological obstructions for embedding finite bordered Riemann surfaces in* $\mathbb{C}^2$.

6.2 Theorem. *For every smooth bordered surface $\mathcal{R}$ there exists a nonempty open set $\Omega \subset \mathcal{M}(\mathcal{R})$ such that for every complex structure J on $\mathcal{R}$ with $[J] \in \Omega$ the open Riemann surface $\mathring{\mathcal{R}} = \mathcal{R} \backslash b\mathcal{R}$ admits a proper J-holomorphic embedding in $\mathbb{C}^2$.*

Theorem 6.2 follows from the following result in [ČF] which sems to contain all known results on embeddings in $\mathbb{C}^2$ except for the punctured disc. (For planar domains see [GS] and [ČG]). Let $U = \{z \in \mathbb{C} \colon |z| < 1\}$.

6.3 Theorem. *Let $(\mathcal{R}, J)$ be a finite bordered Riemann surface of genus g with m boundary components, where J is of class $\mathcal{C}^{\alpha}(\mathcal{R})$ for some $\alpha \in (0,1)$. Assume that there exists an injective immersion $f = (f_1, f_2) \colon \mathcal{R} \to \overline{U} \times \mathbb{C}$ of class $\mathcal{C}^2$ which is J-holomorphic in $\mathring{\mathcal{R}}$, $|f_1| = 1$ on $b\mathcal{R}$, and the generic fiber of f_1 contains at least $2g + m - 1$ points. Then $\mathring{\mathcal{R}}$ admits a proper J-holomorphic embedding in $\mathbb{C}^2$. Furthermore, for every complex structure $\widetilde{J}$ sufficiently $\mathcal{C}^{\alpha}$ close to J the surface $\mathring{\mathcal{R}}$ also admits a proper $\widetilde{J}$-holomorphic embedding in $\mathbb{C}^2$.*

6.4 Corollary. *The following Riemann surfaces admit a proper holomorphic embedding in $\mathbb{C}^2$:*

(i) *finitely connected domains in $\mathbb{C}$ without isolated boundary points,*
(ii) *every complex torus with one hole,*
(iii) *every bordered Riemann surface whose double is hyperelliptic.*

The proof of Theorems 6.2 and 6.3 in [ČF] is based partly on the method developed by Globevnik and Stenønes [GS] who proved the result for planar domains. In this case the conditions in Theorem 6.3 are satisfied if we take $g(z) = z$ and f an inner function of degree $\geq m - 1$, where m is the number of boundary components of the domain.

A hyperelliptic (compact) Riemann surface X is the normalization of a curve in $\mathbb{CP}^2$ given by

$$y^2 = \prod_{j=0}^{g} (x - \alpha_j)(1 - \overline{\alpha}_j x) \tag{6.1}$$

for distinct points $\alpha_j \in U$ $(0 \le j \le g)$, where $g = g(X)$ is the genus of X. If X is the double of a bordered Riemann surface $\mathcal{R}$ then $g(X) = 2g(\mathcal{R}) + m - 1$ where m is the number of boundary curves of $\mathcal{R}$ (which equals either 1 or 2 in this case), and the representation (6.1) can be chosen such that $\mathcal{R} = \{(x,y) \in X\colon |x| \le 1\}$. The pair of functions

$$f_1 = y/\prod_{j=0}^{g}(1 - \overline{\alpha}_j x), \quad f_2 = x$$

provides an embedding $f = (f_1, f_2)\colon \mathcal{R} \to \overline{U}^2$ which maps $b\mathcal{R}$ into the torus $(bU)^2$. Clearly g has multiplicity two. From $f_1^2 = \prod_{j=0}^{g}(f_2 - \alpha_j)/(1 - \overline{\alpha}_j f_2)$ which follows from (6.1) we see that f_1 has multiplicity $g + 1 = 2g(\mathcal{R}) + m$ and hence Theorem 6.3 applies. Sikorav gave a slightly different proof for embedding of complex tori with one hole (unpublished); these are all hyperelliptic.

The proof of Theorem 6.3 goes as follows. Let $P = (2U) \times (RU) \subset \mathbb{C}^2$ for some $R > \sup|f_2|$. Globevnik and Stensønes ([GS], [Gl1]) found arbitrarily small smooth perturbations $S \subset \mathbb{C}^2$ of the cylinder $S_0 = bU \times \mathbb{C}$ such that the connected component Ω_S of $P\backslash S$ containing the origin is of the form $\Omega_S = \widetilde{\Omega}_S \cap P$ for some Fatou-Bieberbach domain $\widetilde{\Omega}_S \subset \mathbb{C}^2$. Let $\phi_S\colon \widetilde{\Omega}_S \to \mathbb{C}^2$ be a biholomorphic (Fatou-Bieberbach) map. If $f^S = (f_1^S, f_2^S)\colon \mathcal{R} \to \overline{\Omega}_S$ is a continuous map which maps $\mathring{\mathcal{R}}$ holomorphically into Ω_S and maps $b\mathcal{R}$ into $S \cap P$ then clearly $\phi_S \circ f^S\colon \mathring{\mathcal{R}} \to \mathbb{C}^2$ is a proper holomorphic embedding. A map f^S with these properties is obtained from the initial map f satisfying the hypothesis of Theorem 6.3 by solving a Riemann-Hilbert boundary value problem on $\mathcal{R}$ (in fact we only need to perturb the first component of f).

6.5 Open problems.

(a) *Does every finite Riemann surface with boundary consisting of finitely many closed curves and isolated points embed in $\mathbb{C}^2$?*

(b) *Does every planar domain with finitely many punctures embed in $\mathbb{C}^2$?*

&7. Noncritical holomorphic functions and submersions.

In 1967 Gunning and Narasimhan [GN] proved that every open Riemann surface admits a holomorphic function without critical points, thus giving an affirmative answer to a long standing question. Open Riemann surfaces are precisely Stein manifolds of complex dimension one. In the recent paper [F8] the result of [GN] was extended to Stein manifolds of any dimension.

7.1 Theorem. (Theorem I in [F8].) *Every Stein manifold admits a holomorphic function without critical points. More precisely, an n-dimensional Stein manifold admits $[\frac{n+1}{2}]$ holomorphic functions with pointwise independent differentials and this number is maximal for every n.*

In fact the manifold X in Example 5.2 above does not admit $[\frac{n+1}{2}]+1$ functions with independent differentials, the reason being that $c_m(X) \neq 0$ for $m = [\frac{n}{2}]$ (Proposition 2.12 in [F8]).

It is furthermore proved in [F8] (Theorem 2.1 and Corollary 2.2) that *for any discrete subset P in a Stein manifold X there exists a holomorphic function $f \in \mathcal{O}(X)$ whose critical set equals P, and one can prescribe the finite order jet of f at each point of P.* Any noncritical holomorphic function on a closed complex submanifold X_0 in a Stein manifold X extends to a noncritical holomorphic function on X, and hence X admits a nonsingular hypersurface foliation transverse to the submanifold X_0.

The main result of [F8] is a *homotopy principle for holomorphic submersions of Stein manifolds to Euclidean spaces of lower dimension.* A holomorphic map $f\colon X \to \mathbb{C}^q$ is a *submersion* if $df_x\colon T_xX \to T_{f(x)}\mathbb{C}^q \simeq \mathbb{C}^q$ is surjective for every $x \in X$, and hence its differential induces a surjective complex vector bundle map of the tangent bundle TX onto the trivial bundle $X \times \mathbb{C}^q$. For $q < \dim X$ this necessary condition for the existence of a submersion $X \to \mathbb{C}^q$ is also sufficient. The following is Theorem II from [F8].

7.2 Theorem. *If X is a Stein manifold and $1 \le q < \dim X$ then every surjective complex vector bundle map $TX \to X \times \mathbb{C}^q$ is homotopic (in the space of surjective complex vector bundle maps) to the differential of a holomorphic submersion $X \to \mathbb{C}^q$. If $f_0, f_1\colon X \to \mathbb{C}^q$ are holomorphic submersions whose differentials are homotopic through a family of surjective complex vector bundle maps $\theta_t\colon TX \to X \times \mathbb{C}^q$ ($t \in [0,1]$) then there exists a regular homotopy of holomorphic submersions $f_t\colon X \to \mathbb{C}^q$ ($t \in [0,1]$) connecting f_0 to f_1.*

For a more precise result see Theorem 2.5 in [F8]. It is not known whether the same conclusion holds for $q = \dim X$, except on open Riemann surfaces where it was proved in [GN]. The corresponding homotopy principle for submersions of smooth open manifolds to $\mathbb{R}^q$ is due to Phillips ([Ph1], [Ph3]) and Gromov [Gr1]. For an extension to smooth (or real-analytic) foliations see [Ph2], [Th1], [Th2].

In [F8] the reader can find numerous applications to the existence of nonsingular holomorphic foliations on Stein manifolds. For instance we have the following (Corollary 2.9 and Theorem 7.1 in [F8]).

7.3 Corollary. *Let X be a Stein manifold of dimension n and $E \subset TX$ a complex subbundle of rank $k \ge 1$. If the quotient bundle TX/E is trivial or admits locally constant transition functions then E is homotopic (through complex rank k subbundles of TX) to the tangent bundle of a nonsingular holomorphic foliation of X. If E is holomorphic then the homotopy may be chosen through holomorphic subbundles.*

When TX/E is trivial (of rank $q = n-k$) the foliation in Corollary 7.3 is given by a holomorphic submersion $X \to \mathbb{C}^q$.

To any complex subbundle $E \subset TX$ we associate its conormal bundle $\Theta = E^\perp \subset T^*X$ with fibers $\Theta_x = \{\omega \in T^*_xX\colon \omega(v) = 0 \text{ for all } v \in E_x\}$. Then $\Theta = (TX/E)^*$, and the first part of Corollary 7.3 can be equivalently expressed as follows: *Any trivial complex subbundle $\Theta \subset T^*X$ of rank $q < \dim X$ is homotopic to a subbundle generated by holomorphic differentials $df_1, \ldots, df_q$.*

By a theorem of Lefschetz [AF] an n-dimensional Stein manifold X is homotopic to a CW-complex of real dimension at most n. Elementary homotopy theory implies that every complex vector bundle of rank $m \ge [\frac{n}{2}]+1$ on X admits a nonvanishing section. Applying this inductively we see that T^*X admits a trivial complex subbundle of rank $[\frac{n+1}{2}]$. Hence Corollary 7.3 implies the following (compare with Theorem 7.1):

7.4 Corollary. *Every n-dimensional Stein manifold X admits nonsingular holomorphic foliations of any dimension $k \ge [\frac{n}{2}]$. If X is holomorphically parallelizable, it admits a holomorphic submersion $X \to \mathbb{C}^{n-1}$ and holomorphic foliations of any dimension $1, 2, \ldots, n-1$.*

It is conjectured that every parallelizable Stein manifold of dimension n admits a holomorphic submersion (=immersion) in $\mathbb{C}^n$ (see 7.5 below). The construction of submersions $X \to \mathbb{C}^q$ in [F8] breaks down for $q = \dim X$ due to a *Picard type obstruction* in the approximation problem for submersions on Euclidean spaces. For example, the complex n-sphere $\Sigma^n = \{z \in \mathbb{C}^{n+1} \colon \sum_{j=0}^{n} z_j^2 = 1\}$ is (algebraically) parallelizable for any $n \in \mathbb{N}$, and an explicit holomorphic immersion $\Sigma^n \to \mathbb{C}^n$ was found by J. J. Loeb (see [BN, p.18]). The complement $\mathbb{CP}^2 \backslash C$ of any smooth cubic curve $C \subset \mathbb{CP}^2$ is also algebraically parallelizable, but is is not known whether they all immerse in $\mathbb{C}^2$.

The construction of noncritical holomorphic functions and submersions in [F8] depends on three main ingredients. The first is a method for approximating noncritical holomorphic functions on polynomially convex subsets of $\mathbb{C}^n$ by entire noncritical functions. For $n > 1$ this uses the theory of holomorphic automorphisms of affine complex spaces from [And], [AL], [FRo]. A similar method is used for submersions $\mathbb{C}^n \to \mathbb{C}^q$ with $1 < q < n$, but with a weaker conclusion.

The second crucial new ingredient is a tool for patching holomorphic maps preserving the maximal rank condition. If $A, B \subset X$ is a *Cartan pair* in a complex manifold X then every biholomorphic map γ sufficiently uniformly close to the identity in a neighborhood of $A \cap B$ admits a decomposition $\gamma = \beta \circ \alpha^{-1}$, where α (resp. β) is a biholomorphic map close to the identity in a neighborhood of A (resp. of B); see Theorem 4.1 in [F8]. The map γ arises as a 'transition map' between a pair of holomorphic submersions f resp. g defined in a neighborhood of A (resp. of B), satisfying $f = g \circ \gamma$ near $A \cap B$. From $\gamma = \beta \circ \alpha^{-1}$ we obtain $f \circ \alpha = g \circ \beta$ which gives a submersion in a neighborhood of $A \cup B$.

The construction of holomorphic submersions $X \to \mathbb{C}^q$ for $q > 1$ in [F8] also uses a new device for crossing the critical points of a strongly plurisubharmonic exhaustion function $\rho \colon X \to \mathbb{R}$. (This is not needed for $q = 1$.) Let $p \in X$ be a Morse critical point of ρ. Using a special case of Gromov's *convex integration lemma* one obtains a smooth maximal rank extension of the map f from a sublevel set $\{\rho \le c\} \subset X$ (where $c < \rho(p)$ is close to $\rho(p)$) to a totally real handle $E \subset X$ attached to $\{\rho \le c\}$ which describes the change of topology of the sublevel sets of ρ at p. This forms an intrinsic link between the smooth and holomorphic submersions, the main point being that the complex submersion differential relation is *ample in the coordinate directions* on any totally real submanifold. One then constructs a smooth family of increasing strongly pseudoconvex neigborhoods of $\{\rho \le c\} \cup E$ such that the largest one contains the sublevel set $\{\rho < c'\}$ for some $c' > \rho(p)$. This reduces the extension problem to the noncritical case for a different strongly plurisubharmonic function and lets us pass the critical level of ρ at p.

7.5 Open problems.

1. Let X be a parallelizable Stein manifold of dimension $n > 1$. Does there exist a holomorphic immersion (=submersion) $X^n \to \mathbb{C}^n$? For $n = 1$ the affirmative answer was given by Gunning and Narasimhan [GN] in 1967. Not much is known for $n > 1$ (see e.g. [Na3] and [BN]). A positive answer would follow from the positive answer to any of the following problems.
2. Let B be an open convex set in $\mathbb{C}^n$. Is every holomorphic immersion (=submersion) $B \to \mathbb{C}^n$ a limit of entire immersions $\mathbb{C}^n \to \mathbb{C}^n$, uniformly on compacts in B ? The same question may be asked for maps with

constant Jacobian; compare with the *Jacobian problem* for polynomial maps [BN, p. 21].

3. Let $f\colon X^n \to \mathbb{C}^{n-1}$ be a holomorphic submersion and let $\mathcal{F}$ denote the foliation $\{f = c\}$, $c \in \mathbb{C}^{n-1}$. Assuming that the tangent bundle $T\mathcal{F}$ is trivial, find a $g \in \mathcal{O}(X)$ which is noncritical on every leaf of $\mathcal{F}$. (The map $(f,g)\colon X \to \mathbb{C}^n$ is then locally biholomorphic.)

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