The Oka principle for multivalued sections of ramified mappings

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Abstract. We prove a basic version of the Oka principle for multivalued sections of ramified holomorphic mappings $h : Z \to X$ of a complex space Z onto an irreducible Stein space X. Assume that $X_0 \subset X$ is a closed complex subvariety of X such that h is an elliptic submersion over $X \backslash X_0$, in the sense that it admits a fiber-dominating spray over a small neighborhood of any point in $X \backslash X_0$. If F_0 is a d-valued continuous section of h which is holomorphic in a neighborhood of X_0, unramified over $X \backslash X_0$ and whose discriminant locus has Hausdorff $(2n - 1)$-dimensional measure zero, where $n = \dim X$, then F_0 can be homotopically deformed to a holomorphic d-valued section F of h. Since the graph $V(F)$ of such F is an effective analytic chain in Z such that $h : V(F) \to X$ is an analytic cover, we obtain an existence result for such chains. Our results apply to maps h whose generic fibers are parabolic or elliptic curves.

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0 Introduction

A central problem in the analysis of holomorphic mappings $h : Z \to X$ between complex manifolds (or complex spaces) is to find holomorphic sections, i.e., maps $f : X \to Z$ satisfying $h(f(x)) = x$ for all $x \in X$. If h is unramified (a holomorphic submersion), there exist plenty of local holomorphic sections passing through any point of Z, and the main question is the existence of global sections. The most one may hope for is that every continuous section can be homotopically deformed to a holomorphic section. This holds only for special classes of submersions and is commonly referred to as the *Oka principle* (or the Oka-Grauert principle) for sections of h. The strongest result in this direction [F3] extends the works of Oka [O], Grauert [Gr1, Gr2], Forster and Ramspott [FR] and Gromov [Gro]. (For proofs of Gromov's results see [FP1, FP2, FP3] and [L].)

In this paper we consider the analogous problem for *ramified mappings*. A point $z \in Z$ is a *ramification point* of $h : Z \to X$ if $dh_z : T_z Z \to T_{h(z)} X$ is not surjective, and the set of all such points is the *ramification locus* br_h. (For maps of complex spaces we include $Z_{\mathrm{sing}} \cup h^{-1}(X_{\mathrm{sing}})$ into br_h.) There need not exist any local sections,

not even continuous ones, passing through a ramification point (consider $h(z) = z^2$ at $z = 0$). Furthermore, the existence of a continuous section need not imply the existence of a local holomorphic section at such points. For instance, the function $h(z, w) = z^p w^q$ on $\mathbb{C}^2$ for coprime $p, q \in \mathbb{N}$ admits the Hölder-continuous section $f(re^{i\theta}) = (r^{1/2p}e^{ip'\theta}, r^{1/2q}e^{iq'\theta})$, where p', q' are integers with $pp' + qq' = 1$, but it admits no local single-valued holomorphic section with $f(0) = (0, 0)$.

The natural objects associated to ramified maps $h : Z \to X$ are complex subvarieties $V \subset Z$ such that $h|_V : V \to X$ is a proper finite projection onto X, i.e., an *analytic cover*. (For properties of analytic covers we refer to Whitney [W].) If the base X is a connected complex manifold or an irreducible complex space, there are an integer $d \in \mathbb{N}$ and a closed complex subvariety $\delta \subset X$ such that for all $x \in X \backslash \delta$ the fiber $V_x = V \cap h^{-1}(x)$ consists of precisely d distinct points while the exceptional fibers V_x over points $x \in \delta$ contain less than d points. The number d is the *degree* and δ is the *discriminant locus* of the analytic cover. If we consider the points in V_x with algebraic multiplicities then every fiber consists of precisely d points and we may think of V as the *graph of a holomorphic d-valued section F of h*. This notion can be made precise by considering F as a fiber preserving holomorphic map from X to the d-fold symmetric power Z^d_{sym} of Z. The latter point of view is especially convenient since it allows us to consider the topological objects in Z, with combinatorial properties analogous to analytic covers, as graphs of continuous (or smooth) multivalued sections of h. The graph of a holomorphic multivalued section of h is an *effective analytic chain* in Z, i.e., a formal combination $V = \sum_j m_j V_j$ of finitely many closed irreducible complex subvarieties $V_j \subset Z$ with coefficients $m_j \in \mathbb{N}$ such that $h|_{V_j} : V_j \to X$ is an analytic cover for each j.

Our main result (Theorem 1.1) is a basic version of the Oka principle for multivalued sections of certain ramified maps over an irreducible Stein base. It applies in particular to maps h whose regular fibers are compact complex curves of genus zero or one (Corollary 1.2). We also prove the Oka principle for liftings of holomorphic maps (Theorem 1.3 and Corollary 1.4). We hope that these results will be useful in the study of ramified mappings which arise naturally in analytic and algebraic geometry.

1 The results

All complex spaces are assumed to be reduced and finite dimensional. If X is a complex space then X_{sing} denotes its singular locus and $X_{\mathrm{reg}} = X \backslash X_{\mathrm{sing}}$ its regular locus. Given a holomorphic map $h : Z \to X$, we denote by $\mathrm{br}_h \subset Z$ its *ramification locus*, consisting of all points $z \in Z$ such that Z is singular at z or X is singular at $x = h(z)$ or the derivative $dh_z : T_z Z \to T_x X$ is not surjective.

For $d \in \mathbb{N}$ we denote by Z^d_{sym} the d-fold *symmetric power* of Z, the quotient of the Cartesian power Z^d by the action of the permutation group on d elements permuting the entries of $(z_1, \ldots, z_d) \in Z^d$. Thus a point in Z^d_{sym} is an unordered d-tuple of points in Z. Z^d_{sym} inherits from Z^d the structure of a complex space and hence we may speak of continuous resp. holomorphic maps $X \to Z^d_{\mathrm{sym}}$. For details we refer to section 4 below and to [W].

Definition 1. A holomorphic (resp. continuous) *d-valued section* of h is a holomorphic (resp. continuous) map $F : X \to Z_{\text{sym}}^d$ such that $F(x) \subset h^{-1}(x)$ for each $x \in X$. The number d is called the *degree* of F. F is *unramified* at $x \in X$ if its restriction to some small open neighborhood $U \subset X$ of x is a union of d continuous (resp. holomorphic) single-valued sections of h. We denote by br_F the *ramification locus* of F consisting of all points $x \in X$ such that F is not a union of single-valued sections in any neighborhood of x.

Let $\#F(x)$ denote the number of distinct points in $F(x)$. Set

$$\mu_F = \max\{\#F(x) : x \in X\} \le d, \quad \delta_F = \{x \in X : \#F(x) < \mu_F\}.$$

The set δ_F is called the *discriminant locus* of F. Clearly $\text{br}_F \subset \delta_F$ and both sets are closed when F is continuous. If F is holomorphic then br_F and δ_F are closed complex subvarieties of X.

Definition 2. A holomorphic map $h : Z \to X$ between complex spaces is an *elliptic submersion* over an open subset $\Omega \subset X_{\text{reg}}$ if the restriction $h : h^{-1}(\Omega) \to \Omega$ is a submersion of complex manifolds and each point $x \in \Omega$ has an open neighborhood U such that there exist a holomorphic vector bundle $p : E \to Z|_U := h^{-1}(U)$ and a holomorphic map $s : E \to Z|_U$ satisfying the following conditions for each $z \in Z|_U$:

(i) $s(E_z) \subset Z_{h(z)}$ (equivalently, $hs = hp$),

(ii) $s(0_z) = z$, and

(iii) the derivative $ds : T_{0_z}E \to T_z Z$ maps the subspace $E_z \subset T_{0_z}E$ surjectively onto the vertical tangent space $VT_z(Z) := \ker dh_z$.

A triple (E, p, s) as above is called a (fiber-) *dominating spray* associated to the submersion $h : Z|_U \to U$, (iii) being the domination property of s. To our knowledge sprays were first introduced by M. Gromov in [Gro] as a weaker replacement of the exponential map on a Lie group which was used in the proof of the classical Oka-Grauert principle.

The notation $\mathscr{H}^k(A) = 0$ means that A has the k-dimensional Hausdorff measure zero; this makes sense for subsets of analytic spaces.

1.1 Theorem (Oka's principle for multivalued sections). *Let $h : Z \to X$ be a holomorphic map of a complex space Z onto an irreducible n-dimensional Stein space X. Assume that X_0 is a closed complex subvariety of X containing $h(\text{br}_h)$ such that h is an elliptic submersion over $X \backslash X_0$. Let F be a continuous d-valued section of h which is holomorphic in a neighborhood of X_0, unramified over $X \backslash X_0$ and satisfies $\mathscr{H}^{2n-1}(\delta_F) = 0$. Then there is a homotopy $F_t : X \to Z_{\text{sym}}^d$ $(t \in [0, 1])$ of continuous d-valued sections of h such that $F_0 = F$, each F_t is holomorphic in a neighborhood of X_0, unramified over $X \backslash X_0$ and satisfies $F_t(x) = F(x)$ for $x \in X_0$, and F_1 is holomorphic on X.*

We have already mentioned that the graph of a holomorphic multivalued section of h is an effective n-dimensional analytic chain in Z with proper finite projection onto X (Proposition 4.2), and hence Theorem 1.1 may be viewed as an existence result for such chains. The problem of extending a holomorphic d-valued section of h defined locally near X_0 to a continuous d-valued section over X can be treated by methods of *obstruction theory* (see for instance [Wd]).

Remarks concerning Theorem 1.1. 1. The space Z need not be Stein (the fibers of h may even be compact). Theorem 1.1 is new even for $d = 1$ since the known results only apply to single-valued sections of unramified maps. Under suitable hypotheses the result also holds over a reducible Stein space X (for single-valued sections no extra hypotheses are needed).

2. Theorem 1.1 holds with the usual additions described in [Gro, FP2, FP3]. For instance, if F is holomorphic in a neighborhood of $K \cup X_0$ for some compact, holomorphically convex subset $K \subset X$ and if h is elliptic over $X \backslash (K \cup X_0)$ then the homotopy F_t in Theorem 1.1 may be chosen such that each F_t is holomorphic in a neighborhood of $K \cup X_0$, it approximates F uniformly on K, and it agrees with F to a given finite order along X_0. (See Theorem 1.4 in [FP3] for a precise statement of such a result for single-valued sections.) In addition, Theorem 1.1 holds under the weaker condition that h is a *subelliptic submersion* over $\tilde{X} = X \backslash X_0$ (this is explained in the subsequent paper [F3]). The same applies to Theorems 1.3 and 2.1 below.

3. The condition $\mathscr{H}^{2n-1}(\delta_F) = 0$ is satisfied for any holomorphic multivalued section since δ_F is a proper complex subvariety of X. It implies that δ_F is nowhere dense in X and $X \backslash \delta_F$ is path connected and locally path connected (provided that X is irreducible) which guarantees a unique decomposition of F into irreducible components by Proposition 4.1. This is no longer the case if $\mathscr{H}^{2n-1}(\delta_F) > 0$. For instance, the 2-valued map $F : \mathbb{C} \to \mathbb{C}^2_{\text{sym}}$ given by $F(x + iy) = [|x|, -|x|]$ (with $\delta_F = \{x = 0\}$ of Hausdorff dimension one) has two splittings into single-valued continuous maps: (a) $F_1(z) = |x|$, $F_2(z) = -|x|$; (b) $F_1(z) = x$, $F_2(z) = -x$.

4. A jet-transversality argument shows that for a generic smooth multivalued section the set $\delta \cap X_{\text{reg}}$ is a smooth real submanifold of real codimension at least two and hence $\mathscr{H}^{2n-1}(\delta) = 0$. However, a generic perturbation may introduce additional ramification points as the next example shows.

Example 1. Let $\chi : \mathbb{C} \to \mathbb{R}_+$ be a smooth function which vanishes precisely on $D = \{|z| \leq 1\}$. Consider the maps $F_\varepsilon : \mathbb{C} \to \mathbb{C}^2_{\text{sym}}$ defined by

$$F_\varepsilon(z) = [(\chi(z) + \varepsilon)\sqrt{z}, -(\chi(z) + \varepsilon)\sqrt{z}], \quad z \in \mathbb{C}.$$

Clearly F_0 is unramified and $\delta_{F_0} = D$, but for any $\varepsilon > 0$ the map F_ε is ramified at $z = 0$ and satisfies $\delta_{F_\varepsilon} = \{0\}$. There exists no unramified perturbation G of F_0 satisfying $\mathscr{H}^1(\delta_G) = 0$ which can be seen as follows. If such G exists, its normalized graph (obtained by separating the self-intersections as in the proof of Lemma 5.1) is a

covering space over $\mathbb{C}$ and hence trivial. This is a contradiction since the graph of F over any circle $\{|z| = r\}$ for $r > 1$ is a nontrivial covering space which remains non-trivial after a perturbation. ♠

1.2 Corollary. *Let $h : Z \to X$ be a holomorphic map of a complex space Z onto an irreducible n-dimensional Stein space X. Assume that X_0 is a closed complex subvariety of X such that $h : \tilde{Z} = Z \backslash h^{-1}(X_0) \to \tilde{X} = X \backslash X_0$ is a submersion of complex manifolds. Then Theorem 1.1 applies in each of the following cases:*

(a) *Each connected component of the fiber $Z_x = h^{-1}(x)$ for $x \in \tilde{X}$ is either a rational curve $(\mathbb{CP}^1)$ or an elliptic curve (a complex torus).*

(b) *The restriction $h : \tilde{Z} \to \tilde{X}$ is locally trivial (a holomorphic fiber bundle) and the fiber Z_x is a complex Lie group or a complex homogeneous space.*

(c) *$\tilde{Z} = V \backslash \Sigma$ where $h : V \to \tilde{X}$ is a holomorphic vector bundle over $\tilde{X}$ of rank $k \geq 2$ and Σ is complex subvariety of the associated bundle $\bar{V} \to \tilde{X}$ with fibers $\bar{V}_x \simeq \mathbb{CP}^k$ such that $\dim \Sigma_x \leq k - 2$ for all $x \in \tilde{X}$.*

Proof. In each case the restricted submersion $h : \tilde{Z} \to \tilde{X}$ is elliptic and hence Theorem 1.1 applies. Case (b) was considered in [Gr1, Gr2] and [FP2], and case (c) was considered in [Gro] and, more explicitly, in [FP2, Theorem 1.7]. In case (c) the fibers Σ_x are algebraic subvarieties of $\bar{V}_x \simeq \mathbb{CP}^k$ ($\bar{V}_x$ is obtained by compactifying $V_x \simeq \mathbb{C}^k$ by a hyperplane at infinity).

In case (a) the connected components of the fiber $Z_x = h^{-1}(x)$ for $x \in \tilde{X}$ are all of the same type (either $\mathbb{CP}^1$ or elliptic). In the first case the complex structure on Z_x is independent of x and hence $\tilde{Z} \to \tilde{X}$ is a fiber bundle with complex homogeneous fiber $\mathbb{CP}^1$, so the result is a special case of (b). If the components of Z_x are elliptic curves $C_{x,j}$ $(1 \leq j \leq j_0)$, the parameter of the complex structure on $C_{x,j}$ is locally a holomorphic function of x. Hence the universal covering maps $\mathbb{C} \to C_{x,j}$ can be chosen to be locally holomorphic in x, and these maps give sprays on $Z|_U$ over small open sets $U \subset \tilde{X}$. ♠

Example 2. This is an explicit example of a fibration of type (a) in Corollary 1.2. Let S be a compact complex surface in $\mathbb{CP}^N$. Choose a point $p \in \mathbb{CP}^N \backslash S$ and let $X \simeq \mathbb{CP}^{N-1}$ denote the set of all complex hyperplanes $\lambda \subset \mathbb{CP}^N$ passing through p. Set $Z = \{(\lambda, z) : \lambda \in X, z \in \lambda \cap S\}$ and denote by $h : Z \to X$ the projection $h(\lambda, z) = \lambda$. The ramification locus br_h is the set of points $(\lambda, z) \in Z$ such that the intersection of λ with S is non-transverse at z. Since h is proper, its projection $X_0 = h(\mathrm{br}_h) \subset X$ is a closed complex subvariety of X. For $\lambda \in X \backslash X_0$ the fiber $h^{-1}(\lambda) = \lambda \cap S$ is a union of finitely compact Riemann surfaces whose genus g is independent of λ. If $g = 0$ then S is called a ruled surface, and if $g = 1$ then S is an elliptic surface (see [BV]). In each of these two cases Theorem 1.1 holds over any Stein domain in X, but it fails when $g \geq 2$ because of the hyperbolicity. ♠

Our next result extends Theorem 2.1 in [F2] to ramified maps h.

1.3 Theorem (The Oka principle for liftings). *Let $h : Z \to X$ and $f : Y \to X$ be holomorphic maps of reduced complex spaces. Assume that $X_0 \subset X$ is a closed complex subvariety containing $f(Y_{\text{sing}}) \cup h(\text{br } h)$ and $g_0 : Y \to Z$ is a continuous map which is holomorphic in an open set containing $Y_0 := f^{-1}(X_0)$ and satisfies $h \circ g_0 = f$. If Y is Stein and h is an elliptic submersion over an open set in X containing $f(Y) \backslash X_0$ then for each $k \in \mathbb{N}$ there exists a homotopy of continuous maps $g_t : Y \to Z$ such that for each $t \in [0, 1]$ we have $h \circ g_t = f$, g_t and g_0 are tangent to order k along Y_0, and g_1 is holomorphic on Y. If in addition g_0 is holomorphic in a neighborhood of a compact holomorphically convex subset $K \subset\subset Y$ then the homotopy $\{g_t\}$ can be chosen such that, in addition to the above, each g_t approximates g_0 uniformly on K.*

The following diagram illustrates Theorem 1.3:

$$
\begin{array}{ccc}
 & & Z \\
 & \nearrow^{g} & \downarrow^{h} \\
Y & \xrightarrow[f]{} & X
\end{array}
$$
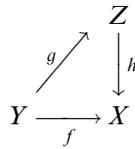
A map g for which this diagram commutes is called a *lifting* of f, and the result is that (under the stated conditions) *a continuous lifting can be homotopically deformed to a holomorphic lifting.* The spaces Z and X in Theorem 1.3 need not be Stein. Theorem 1.3 is proved in section 3.

We can apply Theorem 1.3 to the construction of entire maps on vector bundles whose images avoid certain complex subvarieties. Let $h : E \to X$ be a holomorphic vector bundle of rank q over a Stein manifold X. For each $x \in X$ we denote by $\hat{E}_x \cong \mathbb{CP}^q$ the compactification of the fiber $E_x \cong \mathbb{C}^q$ by the hyperplane at infinity $\Lambda_x \cong \mathbb{CP}^{q-1}$. The resulting fiber bundle $\hat{h} : \hat{E} \to X$ is again holomorphic since the transition maps for E, which are $\mathbb{C}$-linear automorphisms of fibers E_x, extend to projective linear automorphisms of $\hat{E}_x$.

1.4 Corollary. *Let $h : E \to X$ be a holomorphic vector bundle of rank q over a Stein manifold X and let $\hat{E} \to X$ be the associated bundle with fiber $\mathbb{CP}^q$ as above. Let Σ be a closed complex subvariety of $\hat{E}$ whose fiber Σ_x has complex codimension at least two in $\hat{E}_x$ and satisfies $0_x \notin \Sigma_x$ for each $x \in X$. Then for every $k \in \mathbb{N}$ there exists a fiber-preserving holomorphic map $F : E \to E \backslash \Sigma$ which is tangent to the identity to order k along the zero section of E.*

The conclusion of Corollary 1.4 can also be stated as follows: *There exists a family of entire mappings F_x on the fibers $E_x \cong \mathbb{C}^q$, depending holomorphically on the parameter $x \in X$, such that F_x is tangent to the identity at the origin $0_x \in E_x$ and its image $F_x(E_x)$ misses the subvariety $\Sigma_x \subset E_x$ for each $x \in X$.* Of course the point 0_x can be replaced by $g(x)$ where $g : X \to E \backslash \Sigma$ is a holomorphic section. Each fiber Σ_x is projective-algebraic by Chow's theorem. The result is false in general if Σ_x has codimension one in E_x (since its complement may be Kobayashi hyperbolic).

Proof of Corollary 1.4. Let $Z = E \backslash \Sigma$. The hypothesis on Σ implies that the restricted submersion $h|_Z : Z \to X$ is elliptic (see Corollary 1.8 in [FP2]). Take $Y = E$ (which is Stein) and let Y_0 denote the zero section of E. By hypothesis we have $Y_0 \cap \Sigma = \emptyset$. Choose a continuous fiber preserving map $g_0 : E \to Z$ which equals the identity in an open set $U \subset E$ containing Y_0 (such g_0 can be obtained by contracting each fiber E_x to a neighborhood of 0_x which does not intersect Σ_x). Since $h|_Z \circ g_0 = h$, g_0 is a continuous lifting of the holomorphic map $h : Y = E \to X$ with respect to the submersion $h|_Z : Z \to X$. By Theorem 1.3 we obtain a homotopy of liftings $g_t : Y \to Z$ from g_0 to a holomorphic lifting $g_1 : Y \to Z$ such that the homotopy is fixed to order k along Y_0. The map $F = g_1$ satisfies Corollary 1.4. ♠

Open problem. Is it possible to choose $F : E \to E \backslash \Sigma$ in Corollary 1.4 to be *injective* (i.e., such that F_x is a Fatou-Bieberbach map on $E_x \simeq \mathbb{C}^q$ for each $x \in X$)? By Proposition 1.4 in [F1] the answer is affirmative when X consists of a single point.

In the remainder of this section we explain the organization of the paper. In section 2 we prove Theorem 1.1 for single-valued sections of ramified maps (this is stated separately as Theorem 2.1). The main point is that the proof of Oka's principle for elliptic submersions, given in [FP2] and [FP3], extends to ramified maps provided that one can construct a local spray around the graph of any holomorphic section of h over a holomorphically convex subset of X, such that this local spray is dominating outside of the branch locus of h. In [FP2, FP3] we used the fact that the vertical tangent space $VT(Z) = \ker dh$ of a holomorphic submersion $h : Z \to X$ is a holomorphic vector bundle over Z and hence is generated over any open Stein subset $\Omega \subset Z$ by finitely many sections (vertical holomorphic vector fields). The composition of local flows of these sections is a local spray on Ω. When h has ramification points, $VT(Z)$ is no longer a vector bundle but merely a *linear space* over Z [Fi]. Nevertheless, germs of holomorphic sections of this space form a coherent analytic sheaf over Z which is locally free over $Z \backslash \mathrm{br}_h$ (Proposition 2.2) which enables us to complete the proof.

In section 3 we reduce Theorem 1.3 to Theorem 2.1. We associate to the map $f : Y \to X$ the pull-back $\hat{h} : \hat{Z} \to Y$ of $h : Z \to X$ so that sections of $\hat{h}$ over Y are in one-to-one correspondence with maps $g : Y \to Z$ satisfying $hg = f$ (i.e., with liftings of f). Furthermore, if $U \subset X$ is an open subset of X and $V = f^{-1}(U) \subset Y$ then any h-spray associated to $h^{-1}(U) \to U$ pulls back to an $\hat{h}$-spray associated to $\hat{h}^{-1}(V) \to V$. Hence Theorem 1.3 follows from Theorem 2.1 applied to sections of $\hat{h}$.

In section 4 we collect some basic results on multivalued sections which are used in the proof of Theorem 1.1 for $d > 1$.

In section 5 we deduce the general case of Theorem 1.1 from Theorem 1.3. Given a d-valued section $F : X \to Z_{\mathrm{sym}}^d$ of $h : Z \to X$ as in Theorem 1.1, we construct a normal Stein space Y and a continuous map $g_0 : Y \to Z$ satisfying the following:

– the composition $f = hg_0 : Y \to X$ is a d-sheeted analytic cover onto X which is unramified over $X \backslash X_0$,

– g_0 maps the fiber $Y_x := f^{-1}(x)$ onto $F(x)$ for each $x \in X$, and

– g_0 is holomorphic in $f^{-1}(U_0)$ if F is holomorphic in $U_0 \subset X$.

The space Y is the *normalized graph* of F in which the multiple points over $X \backslash X_0$ have been separated. Since $f : Y \to X$ is a proper finite map onto a Stein space X, Y is also Stein. The inverse $f^{-1} : X \to Y_{\text{sym}}^d$ is a holomorphic d-valued section of f. The map $g_0 : Y \to Z$ is a continuous lifting of $f = h \circ g_0 : Y \to X$ with respect $h : Z \to X$. Theorem 1.3 provides a homotopy of liftings $g_t : Y \to Z$ of f connecting g_0 to a holomorphic lifting g_1. Then $F_t = g_t \circ f^{-1} : X \to Z_{\text{sym}}^d$ is a homotopy of d-valued sections of $h : Z \to X$ satisfying Theorem 1.1.

2 Proof of Theorem 1.1 for single-valued sections

In this section we prove the following version of Theorem 1.1 for single-valued sections. All complex spaces are assumed to be reduced and finite dimensional.

2.1 Theorem. *Let X be a Stein space (possible reducible) and $h : Z \to X$ a holomorphic map of a complex space Z onto X. Assume that X_0 is a closed complex subvariety of X containing $h(\mathrm{br}_h)$ such that h is an elliptic submersion over $X \backslash X_0$. Given a continuous section $F : X \to Z$ which is holomorphic in an open set containing X_0, there is for any $k \in \mathbb{N}$ a homotopy $F_t : X \to Z$ ($t \in [0,1]$) of continuous sections such that $F_0 = F$, for each $t \in [0,1]$ the section F_t is holomorphic in a neighborhood of X_0 and tangent to F_0 to order k along X_0, and F_1 is holomorphic on X. If F is holomorphic in a neighborhood of $K \cup X_0$ for some compact, holomorphically convex subset K of X then we can choose F_t to be holomorphic in a neighborhood of $K \cup X_0$ and to approximate $F = F_0$ uniformly on K.*

When Z and X are complex manifolds and h is a surjective submersion (i.e., $\mathrm{br}_h = \emptyset$), Theorem 2.1 is a special case of Theorem 1.4 in [FP3]. When $X_0 = \emptyset$ it is included in [Gro, 4.5 Main Theorem] and in [FP2, Theorem 1.5]. The presence of ramification points of h requires a refinement of the proof, related to the construction of local sprays in neighborhoods of graphs of holomorphic sections over Stein sets in X, which we now describe.

2.2 Proposition (Existence of local sprays). *Let $h : Z \to X$ be a holomorphic map of reduced complex spaces. For any open Stein subset Ω of Z there exist an integer $N \in \mathbb{N}$, an open set $V \subset \Omega \times \mathbb{C}^N$ containing $\Omega \times \{0\}$, and a holomorphic map $s : V \to Z$ satisfying the following:*

(i) $s(z,0) = z$ *for* $z \in \Omega$,

(ii) $h(s(z,t)) = h(z)$ *for* $(z,t) \in V$,

(iii) $s(z,t) = z$ *when* $(z,t) \in V$ *and* $z \in \mathrm{br}_h$,

(iv) *for each* $z \in \Omega \backslash \mathrm{br}_h$ *the derivative at* $t = 0 \in \mathbb{C}^N$ *of* $t \to s(z,t) \in Z$ *maps* $T_0 \mathbb{C}^N$ *surjectively onto* $VT_z Z := \ker dh_z$.

A map s satisfying Proposition 2.2 is called a *local spray* for h over Ω. Indeed s satisfies all properties of a spray except that it is not defined globally on $\Omega \times \mathbb{C}^N$ and the domination property (iv) only holds in the complement of the ramification locus br_h. When h is a submersion of complex manifolds, Proposition 2.2 coincides with Lemma 5.3 in [FP1].

Proof of Proposition 2.2. Our reference are Chapters 1 and 2 in [Fi]. Recall that the *tangent space* of a complex space Z is a linear space $\pi : TZ \to Z$ (a complex space with linear fibers over Z) obtained as follows. Fix a point $z_0 \in Z$ and represent an open neighborhood $Z_0 \subset Z$ of z_0 as a closed complex subspace of an open subset W of $\mathbb{C}^m$, defined by a sheaf of ideals $\mathscr{I} \subset \mathscr{O}_W$ which is generated by holomorphic functions $f_1, \ldots, f_r \in \mathscr{O}(W)$. Thus the structure sheaf $\mathscr{O}_{Z_0}$ is isomorphic to the quotient $\mathscr{O}_W / \mathscr{I}$ restricted to $\{f = 0\} = Z_0$. If $(w_1, \ldots, w_m, \xi_1, \ldots, \xi_m)$ are coordinates in $W \times \mathbb{C}^m$ then $TZ_0 = TZ|_{Z_0}$ is the closed complex subspace of $W \times \mathbb{C}^m$ generated by the functions

$$(2.1) \quad f_1, \ldots, f_r \quad \text{and} \quad \frac{\partial f_i}{\partial w_1} \xi_1 + \cdots + \frac{\partial f_i}{\partial w_m} \xi_m \quad \text{for } i = 1, \ldots, r.$$

The projection $TZ_0 \to Z_0$ is induced by $W \times \mathbb{C}^m \to W$, $(w, \xi) \to w$. Different local representations of Z in $\mathbb{C}^m$ give isomorphic representations of the tangent space. Over Z_{reg} the space TZ is the usual tangent bundle of Z.

Suppose furthermore that $h : Z \to X$ is a holomorphic map of complex spaces. The *vertical tangent space* $\pi : VT(Z) \to X$ with respect to h (also called in [Fi] 'the tangent space of Z over X' and denoted $T(Z/X)$) is a linear space over Z (a subspace of TZ) with the following local description. Fix a point $z_0 \in Z$ and let $x_0 = h(z_0) \in X$. Let $Z_0 \subset W \subset \mathbb{C}^m$ be a local representation of a neighborhood of z_0 as above and let $X_0 \subset W' \subset \mathbb{C}^n$ be a local representation of a neighborhood $X_0 \subset X$ of x_0. We may choose these neighborhoods such that $h(Z_0) \subset X_0$ and the restriction $h : Z_0 \to X_0$ extends to a holomorphic map $H = (H_1, \ldots, H_n) : W \to \mathbb{C}^n$. One takes $T(Z/X)|_{Z_0}$ to be the closed complex subspace of $W \times \mathbb{C}^m$ generated by the functions (2.1) together with

$$(2.2) \quad \frac{\partial H_i}{\partial w_1} \xi_1 + \cdots + \frac{\partial H_i}{\partial w_m} \xi_m \quad \text{for } i = 1, \ldots, n.$$

Again the result is independent of the local representations up to isomorphism and the complex space obtained in this way coincides with the usual vertical tangent bundle over $Z \backslash \mathrm{br}_h$. The spaces TZ and $VT(Z)$ need not be reduced even if Z is, but this will not be important for our purposes.

We denote by $\mathscr{T}_Z$ (resp. $\mathscr{VT}_Z$) the sheaf of germs of holomorphic sections of TZ (resp. of $VT(Z)$). These are $\mathscr{O}_Z$-analytic sheaves which are free over $Z \backslash \mathrm{br}_h$. Sections of $\mathscr{T}_Z$ are called *vector fields* on Z and sections of $\mathscr{VT}_Z$ are called *vertical vector fields*.

2.3 Lemma. *The $\mathscr{O}_Z$-analytic sheaves $\mathscr{T}_Z$ and $\mathscr{VT}_Z$ are coherent.*

Proof. Indeed the sheaf $\mathscr{L}$ of holomorphic sections of any linear space $L \to Z$ over a complex space Z is a coherent $\mathscr{O}_Z$-analytic sheaf [Fi, p. 53, Corollary]. In the present case this can be seen directly as follows. Using a local representation for $VT(Z)|_{Z_0}$ as above, the sheaf $\mathscr{VT}_{Z_0}$ consists of germs of holomorphic maps $\xi = (\xi_1, \ldots, \xi_m) : Z_0 \to \mathbb{C}^m$ satisfying

$$\sum_{k=1}^{m} \frac{\partial f_i(w)}{\partial w_k} \xi_k(w) = 0 \quad (1 \le i \le r); \qquad \sum_{k=1}^{m} \frac{\partial H_j(w)}{\partial w_k} \xi_k(w) = 0 \quad (1 \le j \le n).$$

To get $\mathscr{T}_Z$ we only take the first set of equations. Thus both sheaves are locally sheaves of relations and hence coherent [GR, p. 131].   ♠

We continue with the proof of Proposition 2.2. Let $\mathscr{J} \subset \mathscr{O}_Z$ denote the (coherent) sheaf of ideals of the ramification locus br_h. For any $k \in \mathbb{N}$ the sheaf $\mathscr{S}_k = \mathscr{J}^k \cdot \mathscr{VT}_Z$ (the product of k copies of $\mathscr{J}$ with $\mathscr{VT}_Z$) is also coherent analytic. Fix k and write $\mathscr{S} = \mathscr{S}_k$. Let Ω be a Stein open subset of Z. We claim that there exist finitely many sections $X_1, \ldots, X_N$ of $\mathscr{S}$ over Ω which generate $\mathscr{S}$ at each point of $\Omega \backslash \mathrm{br}_h$. This can be seen by a standard argument as follows.

Since Ω is Stein, Cartan's Theorem A gives for each point $z \in \Omega$ finitely many sections of $\mathscr{S}$ over Ω which generate $\mathscr{S}$ at z. We can choose a point a_j in each connected component Ω_j of $\Omega \backslash \mathrm{br}_h$ such that the sequence $\{a_j\}$ is discrete in Ω. Since $\mathscr{S}$ is a free sheaf over the complex manifold $\Omega \backslash \mathrm{br}_h$, a simple (and well known) extension of Cartan's Theorem A gives finitely many sections of $\mathscr{S}$ over Ω which generate $\mathscr{S}$ at each point a_j. The exceptional set Z_1, consisting of points in Ω at which these sections fail to generate $\mathscr{S}$, is a complex subspace of Ω satisfying $\dim Z_1 \cap \Omega_j < \dim \Omega_j$ for each j. For each index j for which $Z_1 \cap \Omega_j \ne \emptyset$ we now choose a point $b_j \in Z_1 \cap \Omega_j$ such that $\{b_j\}$ is discrete in Ω. By Cartan's Theorem A there exist finitely many sections of $\mathscr{S}$ over Ω which, together with the sections chosen in the first step, generate $\mathscr{S}$ at each b_j. The exceptional set $Z_2 \subset \Omega$ at which all these sections fail to generate $\mathscr{S}$ now satisfies $\dim Z_2 \cap \Omega_j \le \dim \Omega_j - 2$ for each j.

Continuing this way we obtain in finitely many steps sections $X_1, \ldots, X_N$ of the sheaf $\mathscr{S}$ over Ω which generate $\mathscr{S}$ at each point of $\Omega \backslash \mathrm{br}_h$. (However, the minimal number of generators of $\mathscr{S}$ at ramification points $z \in \mathrm{br}_h$ need not be bounded from above and hence there need not exist finitely many sections generating $\mathscr{S}$ over Ω.) By construction $X_1, \ldots, X_N$ are holomorphic vector fields on Ω which are vertical with respect to h, they vanish to order k along br_h, and they generate the vertical tangent space $VT_z Z = \ker dh_z$ at each point $z \in \Omega \backslash \mathrm{br}_h$. Let ϕ_t^j denote the flow of X_j for complex time t. The map

$$s(z, t_1, \ldots, t_N) = \phi_{t_1}^1 \circ \cdots \circ \phi_{t_N}^N(z),$$

which is defined and holomorphic in an open neighborhood of $\Omega \times \{0\}^N$ in $\Omega \times \mathbb{C}^N$ and takes values in Z, satisfies Proposition 2.2. Indeed, its partial derivative on t_j at

$t = 0$ equals $X_j(z)$, and since these vectors generate $VT_z(Z)$ for $z \in \Omega \backslash \mathrm{br}_h$, s satisfies (iv). The properties (i)–(iii) are clear. ♠

Proof of Theorem 2.1. Proposition 2.2 enables us to prove Theorem 2.1 by following step by step the proof of Theorem 1.4 in [FP3]. We shall point out those places in the proof where a change or remark is needed.

The reader should first look at Theorem 5.2 in [FP3] (and Theorem 5.1 in [FP1]). The situation is the following (we describe the basic case without parameters). We are given a Cartan pair (A, B) in X, where the set A contains a neighborhood of the subvariety $X_0 \subset X$ and where h admits a dominating spray over a neighborhood of $B \subset X \backslash X_0$, and holomorphic sections a, b of h defined over a neighborhood of A resp. of B such that a and b are uniformly close to each other over a neighborhood of $C = A \cap B$. We wish to patch a and b into a single holomorphic section $\tilde{a}$ over a neighborhood of $A \cup B$ which is uniformly close to a over A.

The problem is reduced to the model situation given by Proposition 5.2 in [FP1] (when $X_0 = \emptyset$) or by Proposition 4.2 in [FP3] (in the general case). The model situation also applies in the present case without any changes. The reduction is accomplished by Lemmas 5.3 and 5.4 in [FP1]. Denote by $B^n(\varepsilon) \subset \mathbb{C}^n$ the open ball of radius ε and center at the origin. These lemmas show how to construct the following:

– a local h-spray $s_1 : U_A \times B^n(\varepsilon) \to Z$ over a Stein neighborhood $U_A \subset Z$ of $a(A)$ such that s_1 is dominating over a neighborhood of $a(C)$,

– a global spray $s_2 : U_B \times \mathbb{C}^n \to Z$ over a Stein neighborhood $U_B \subset Z$ of $b(B)$ such that s_2 is dominating over a neighborhood of $b(C)$, and

– an injective fiber preserving holomorphic map $\psi : \tilde{C} \times B^n(\varepsilon) \to \tilde{C} \times \mathbb{C}^n$ (where $\tilde{C} \subset X \backslash X_0$ is an open neighborhood of $C = A \cap B$) such that

$$s_2(b(x), \psi(x, t)) = s_1(a(x), t) \quad (x \in \tilde{C}, \ t \in B^n(\varepsilon)).$$

The local spray s_1 is obtained from the spray s granted by Proposition 2.2 above (the corresponding spray in [FP1] was denoted $\tilde{s}$). The main point to observe with respect to the exposition in [FP1] is that the construction of s_1 only requires the domination property of s over a neighborhood of $a(C)$ (and not over a neighborhood of $a(A)$). Since the spray s furnished by Proposition 2.2 is dominating outside of br_h and since $h(\mathrm{br}_h) \cap C = \emptyset$, s is dominating over a neighborhood of $a(C)$ as required. The spray s_2 is obtained from the global dominating spray over a neighborhood of B which exists by assumption.

In order to get the transition map ψ as above we must insure in addition that the kernels of ds_1 and ds_2 along the zero section are isomorphic (as holomorphic vector bundles) over a neighborhood $U_C \subset Z$ of $b(C)$ (which is chosen such that $U_C \subset U_A \cap U_B$). The details of this construction are given by Lemmas 5.3 and 5.4 in [FP1] (where s_1 and s_2 are constructed such that the above kernels are even close to each other and hence isomorphic).

Since X may have singularities contained in the subvariety X_0, a remark is in order regarding the proof of Proposition 4.2 in [FP3] (the attaching lemma in the model situation). In the present case the patching is performed over open sets in $X \backslash X_0 \subset X_{\mathrm{reg}}$. The $\bar{\partial}$-problems which arise in this patching have compact support contained in $X \backslash X_0 \subset X_{\mathrm{reg}}$. Such $\bar{\partial}$-problems can be solved by transporting them to $\mathbb{C}^N$ via a holomorphic map $g : X \to \mathbb{C}^N$ which is a homeomorphism of X onto a closed complex subvariety $\tilde{X} \subset \mathbb{C}^N$ and which is biholomorphic on X_{reg}. (Compare with section 7 in [FP3].)

We now proceed to section 6 of [FP3] where Theorem 1.4 of that paper is proved. The crucial step is furnished by Proposition 6.1 in [FP3]. To see that its proof remains valid in our current situation we observe that the sets $A_0, A_1, \ldots, A_n \subset X$, which are chosen at the beginning of the proof of Proposition 6.1 in [FP3], are such that A_0 contains a neighborhood of X_0 while the sets $A_1, \ldots, A_n$ do not intersect X_0. Since h is a submersion of complex manifolds over $X \backslash X_0$, the techniques developed in [FP2, FP3] for holomorphic submersions onto Stein manifolds can be applied whenever the first set A_0 is not involved. Using those techniques we can patch any collection of holomorphic sections $a_j : \tilde{A}_j \to Z$ $(1 \le j \le n)$, where $\tilde{A}_j \subset X \backslash X_0$ is a small open neighborhood of A_j over which h admits a spray, into a single holomorphic section b over a neighborhood of $A^n = A_1 \cup A_2 \cup \cdots \cup A_n$, provided that the sections a_j belong to a holomorphic complex. (We are referring to the transformation of a holomorphic complex associated to the Cartan string $(A_1, \ldots, A_n)$ into a holomorphic section over their union A^n; the details of this procedure are explained in [FP2, Proposition 5.1].) The same procedure also gives a homotopy of holomorphic sections over a neighborhood of $A_0 \cap A^n$ connecting a and b.

It remains to patch a and b into a single holomorphic section over a neighborhood of $A_0 \cup A^n$. This is accomplished as in [FP3] by combining the homotopy version of the Oka-Weil approximation theorem (see e.g. Theorem 2.1 in [FP3]) with Theorem 5.2 in [FP3] which holds in the current situation as explained above. The proof of Theorem 2.1 can now be concluded by the globalization procedure given in [FP3] (proof of Theorem 1.4).

3 Proof of Theorem 1.3

In this section we reduce Theorem 1.3 to Theorem 2.1. Let $h : Z \to X$ and $f : Y \to X$ be as in Theorem 1.3. Set

$$(3.1) \quad \hat{Z} = \{(y, z) : y \in Y, z \in Z, f(y) = h(z)\},$$

$$\hat{h}(y, z) = y \in Y, \quad \sigma(y, z) = z \in Z.$$

Clearly $\hat{Z}$ is a closed complex subspace of $Y \times Z$, the maps $\hat{h} : \hat{Z} \to Y$ and $\sigma : \hat{Z} \to Z$ are holomorphic, and we have $f\hat{h} = h\sigma$.

By assumption the set $Y_0 = f^{-1}(X_0) \subset Y$ contains the singular locus Y_{sing}. For each $y \in Y \backslash Y_0$ we have $f(y) \in X \backslash X_0$ and hence h is a submersion over an open neigh-

borhood $U \subset X \backslash X_0$ of $f(y)$. Setting $V = f^{-1}(U)$ it follows that $\hat{h} : \hat{h}^{-1}(V) \to V$ is a submersion. Thus $\hat{h}$ is a surjective submersion over $Y \backslash Y_0$. For any section $\hat{g} : Y \to \hat{Z}$ of $\hat{h} : \hat{Z} \to Y$ the map $g = \sigma \hat{g} : Y \to Z$ is a lifting of f with respect to h:

$$hg = h(\sigma \hat{g}) = (h\sigma)\hat{g} = (f\hat{h})\hat{g} = f(\hat{h}\hat{g}) = f.$$

Moreover, any lifting g of f is of this form: from $h(g(y)) = f(y)$ ($y \in Y$) it follows that the point $\hat{g}(y) := (y, g(y)) \in Y \times Z$ belongs to the subset $\hat{Z} \subset Y \times Z$ (3.1) and hence $\hat{g} : Y \to \hat{Z}$ is a section of $\hat{h}$. Furthermore, $\sigma(\hat{g}(y)) = \sigma(y, g(y)) = g(y)$ whence g is obtained from the section $\hat{g} : Y \to \hat{Z}$. Therefore Theorem 1.3 follows immediately from Theorem 2.1 and the following lemma.

3.1 Lemma (Pull-back sprays). *Let $f : Y \to X$ and $h : Z \to X$ be holomorphic maps. Assume that $U \subset X$ is an open set such that $h : Z|_U = h^{-1}(U) \to U$ is a submersion which admits a spray. Then the map $\hat{h} : \hat{Z} \to Y$ defined by (3.1) is a submersion with spray over $V = f^{-1}(U) \subset Y$.*

Proof. Let (E, p, s) be a spray associated to the submersion $h : Z|_U \to U$ (Definition 1). Set $V = f^{-1}(U) \subset Y$ and observe that σ maps $\hat{Z}|_V = \hat{h}^{-1}(V)$ to $Z|_U$. Let $\hat{p} : \hat{E} \to \hat{Z}|_V$ denote the pull-back of the holomorphic vector bundle $p : E \to Z|_U$ by the map $\sigma : \hat{Z}|_V \to Z|_U$. Explicitly, we have

$$\hat{E} = \{(\hat{z}, e) : \hat{z} \in \hat{Z}|_V, e \in E; \sigma(\hat{z}) = p(e)\}$$

$$= \{(y, z, e) : y \in V, z \in Z, e \in E; f(y) = h(z), p(e) = z\};$$

$$\hat{p}(\hat{z}, e) = \hat{z}.$$

Consider the map $\hat{s} : \hat{E} \to \hat{Z}|_V$, $s(y, z, e) = (y, s(e))$. We claim that $(\hat{E}, \hat{p}, \hat{s})$ is a spray associated to the submersion $\hat{h} : \hat{Z}|_V \to V$. We first check that $\hat{s}$ is well defined. If $(y, z, e) \in \hat{E}$ then $p(e) = z$ and $h(z) = f(y)$. Since s is a spray for h, we have $h(s(e)) = h(z) = f(y)$ which shows that the point $\hat{s}(y, z, e) = (y, s(e)) \in Y \times Z$ belongs to the fiber $\hat{Z}_y$. This verifies property (i) in Definition 1. Clearly $\hat{s}(y, z, 0_{(y,z)}) = (y, s(0_z)) = (y, z)$ which verifies property (ii) in Definition 1. It is also immediate that $\hat{s}$ satisfies property (iii) provided that s does since the vertical derivatives of the two maps coincide under the identification $\hat{Z}_y \cong Z_{f(y)}$ and $\hat{E}_{(y,z)} \cong E_z$. This proves Lemma 3.1.

4 Multivalued sections and analytic covers

In this section we recall some well known results on symmetric products and multivalued sections which will be used in the proof of Theorem 1.1. Our reference is Appendix V in [W].

Denote by Z^d the d-fold Cartesian power of a set Z. The group Π_d of all permutations on d elements acts on Z^d by permuting the entries, and we denote this action by

ρ. The quotient space is called the *d-fold symmetric power* of Z and is denoted Z_{sym}^d. For $z = (z_1, \ldots, z_d) \in Z^d$ we write $\pi(z) = [z] = [z_1, \ldots, z_d] \in Z_{\text{sym}}^d$. A *d*-valued map from X to Z is a map $F : X \to Z_{\text{sym}}^d$. The number d is called the *degree* of F and denoted $\deg F$.

Assume from now on that X and Z are reduced complex spaces. Then $\pi : Z^d \to Z_{\text{sym}}^d$ induces a natural (quotient) complex structure on Z_{sym}^d such that holomorphic functions on Z_{sym}^d correspond to ρ-invariant holomorphic functions on Z^d. In particular, if $F = [f_1, \ldots, f_d] : X \to Z_{\text{sym}}^d$ is a holomorphic map and if P is a ρ-invariant holomorphic function on Z^d then $P(f_1, \ldots, f_d)$ is a well defined holomorphic function on X.

We recall some natural operations on symmetric products.

1. If Z is a complex subspace of another complex space S then Z_{sym}^d is in a natural way a subspace of S_{sym}^d, and any map $X \to S_{\text{sym}}^d$ whose image belongs to Z_{sym}^d may also be considered as a map $X \to Z_{\text{sym}}^d$. More generally, any holomorphic map $g : Z \to S$ induces a holomorphic map $\hat{g} : Z_{\text{sym}}^d \to S_{\text{sym}}^d$.

2. For any pair of integers $d, k \in \mathbb{N}$ we have a natural holomorphic map $\tau : Z_{\text{sym}}^d \times Z_{\text{sym}}^k \to Z_{\text{sym}}^{d+k}$ induced by the identification $Z^d \times Z^k = Z^{d+k}$.

Given a pair of maps $F_1 : X \to Z_{\text{sym}}^d$ and $F_2 : X \to Z_{\text{sym}}^k$, we denote

$$F_1 \oplus F_2 = \tau(F_1, F_2) : X \to Z_{\text{sym}}^{d+k}.$$

The direct sum generalizes to several terms and we write $F = \bigoplus_j m_j F_j$, where the F_j's are multivalued maps of X to Z and $m_j \in \mathbb{N}$. Clearly we have $\deg F = \sum_j m_j \deg F_j$. A map $F : X \to Z_{\text{sym}}^d$ is called *irreducible* if it cannot be decomposed as a direct sum of multivalued maps of smaller degrees. We recall the notation

$$\mu_F = \max\{\#F(x) : x \in X\}, \quad \delta_F = \{x \in X : \#F(x) < \mu_F\},$$

where $\#F(x)$ denotes the number of distinct points in $F(x)$.

4.1 Proposition. *Assume that $F : X \to Z_{\text{sym}}^d$ is a continuous (resp. holomorphic) map such that δ_F is nowhere dense in X and $X \backslash \delta_F$ is pathwise connected and locally pathwise connected. Then F has a decomposition $F = \bigoplus m_j F_j$ where the continuous (resp. holomorphic) maps $F_j : X \to Z_{\text{sym}}^{d_j}$ are irreducible and the decomposition is unique up to the order of terms. Furthermore $\operatorname{br} F_j \subset \operatorname{br} F$ and $\delta_{F_j} \subset \delta_F$ for each j. Such a decomposition exists in particular if X is an irreducible n-dimensional complex space and $\mathcal{H}^{2n-1}(\delta_F) = 0$.*

Proposition 4.1 is proved in [W, Appendix V]. The idea of the proof is as follows. Let $\tilde{V}$ be the graph of F over $\tilde{X} = X \backslash \delta_F$, i.e., $\tilde{V}$ consists of all points in the fibers $F(x)$ for $x \in \tilde{X}$. Since $\tilde{X}$ is connected and $\#F(x) = \mu_F$ for all $x \in \tilde{X}$, $\tilde{V}$ is a union of finitely many connected componets $\tilde{V}_j$ such that $h : \tilde{V}_j \to \tilde{X}$ is a finite unramified covering projection onto $\tilde{X}$, say with d_j sheets, and F has constant multiplicity $m_j \in \mathbb{N}$ along

$\tilde{V}_j$. Let V_j denote the closure of $\tilde{V}_j$ in Z. One can then show that V_j is the graph of a d_j-valued section F_j of h and $F = \bigoplus m_j F_j$. ♠

Assume now that $h : Z \to X$ is a surjective holomorphic map of reduced complex spaces. A map $F = [f_1, \ldots, f_d] : X \to Z^d_{\text{sym}}$ such that $f_j(x) \in Z_x = h^{-1}(x)$ for all $x \in X$ and all j is called a *d-valued section* of h. The direct sum operation and Proposition 4.1 clearly extend to multivalued sections.

We shall assume from now on that $F : X \to Z^d_{\text{sym}}$ is a d-valued section of $h : Z \to X$ which satisfies the hypothesis of Proposition 4.1. If F is irreducible, we define its *graph* $V(F) \subset Z$ by

$$V(F) = \{z \in Z : z \in F(x) \text{ for some } x \in X\}.$$

If $F = \bigoplus m_j F_j$ with F_j irreducible for all j, we let $V(F) = \sum m_j V(F_j)$ be the disjoint union of m_j copies of $V(F_j)$ for each j. If we consider $V(F)$ as a *multiplicity subset* of Z then $h : V(F) \to X$ is a proper continuous map onto X which is d-sheeted over $X \setminus \delta_F$.

Recall that an *analytic chain* in Z is a formal locally finite combination $V = \sum m_j V_j$ of closed complex subvarieties $V_j \subset Z$ with integer coefficients. If $m_j \geq 0$ for all j then V is called *effective*, and if $\dim V_j = n$ for all j then V is said to be purely *n*-dimensional. In the sense of currents we have $[V] = \sum_j m_j [V_j]$. The following proposition shows that over an irreducble base X there is a bijective correspondence between multivalued holomorphic sections of $h : Z \to X$ and analytic chains $V \subset Z$ such that $h|_V : V \to X$ is an analytic cover.

4.2 Proposition. *Let $h : Z \to X$ be a holomorphic map onto an irreducible n-dimensional complex space X. If $F : X \to Z^d_{\text{sym}}$ is a holomorphic d-valued section of h then its graph $V(F)$ is a chain in Z such that $h : V(F) \to X$ is a d-sheeted analytic cover. Conversely, any effective n-chain $V \subset Z$ such that $h : V \to X$ is a d-sheeted analytic cover is the graph of a holomorphic d-valued section F of h. Furthermore, the decomposition of F into irreducible components given by Proposition 4.1 correponds to the decomposition of its graph $V(F)$ into irreducible complex subvarieties of Z. In particular, if $h : V \to X$ is a d-sheeted analytic cover then $h^{-1} : X \to V^d_{\text{sym}}$ is a d-valued holomorphic section of h.*

Proof. We include a proof only for the sake of completness since arguments of this kind are well known and we certainly do not claim any originality here (see e.g. [W]).

First we prove the converse part. Thus, let $V = \sum_{j=1}^{k} m_j V_j$ be an effective purely *n*-dimensional analytic chain in Z such that $h : V_j \to X$ is a d_j-sheeted analytic cover for each $j = 1, \ldots, k$. For any $x \in X$ we denote the points in $V_j \cap h^{-1}(x)$ by $f_{j,m}(x)$, $1 \leq m \leq d_j$, listed according to algebraic multiplicities. Then $x \in X \to F_j(x) = [f_{j,1}(x), \ldots, f_{j,d_j}(x)] \in Z^{d_j}_{\text{sym}}$ is a d_j-valued section of $h : Z \to X$. It remains to prove that F_j is holomorphic. Let $z_0 \in F_j(x_0)$ for some $x_0 \in X$. Let $I = \{i_1, \ldots, i_s\}$ be precisely those indices $i \in \{1, \ldots, d_j\}$ for which $f_{j,i}(x_0) = z_0$. Let g be a germ of a symmetric holomorphic function on Z^s at the point $(z_0, \ldots, z_0) \in Z^s$ (i.e., g is invariant

under the action of the permutation group on Z^s). By the elementary properties of analytic covers the function $x \to g(f_{j,i_1}(x), \ldots, f_{j,i_s}(x))$ is well defined and holomorphic for x near x_0. This means that the local branches of F_j which at $x = x_0$ pass through z_0 are local holomorphic multivalued sections. Hence F_j is holomorphic as claimed.

Assume now that $F : X \to Z_{\mathrm{sym}}^d$ is a holomorphic d-valued section of h. Denote by $\Gamma(F)$ the subset of Z with fibers $F(x)$, $x \in X$, where the points in $F(x)$ are considered without multiplicities. First we show that $\Gamma(F)$ is a closed complex subvariety of Z and $h : \Gamma(F) \to X$ is an analytic cover.

Fix a point $x_0 \in X$ and let $z_1, \ldots, z_k$ denote the distinct points in $F(x_0)$, where z_j appears with multiplicity d_j (hence $\sum_{j=1}^k d_j = d$). Choose pairwise disjoint open neighborhoods $U_j \subset Z$ of z_j for $j = 1, \ldots, k$. By continuity of F there is a neighborhood U of x_0 in X such that for each $x \in U_0$ we have $F(x) \subset \bigcup_{j=1}^k U_j$ and U_j contains precisely d_j of the points in the d-tuple $F(x) = [f_1(x), \ldots, f_d(x)]$. These points define a d_j-valued section $F_j : U \to Z|_U$ of $h : Z|_U \to U$ with values in U_j. It suffices to consider the maps F_j separately.

Thus we may assume that $F(x_0)$ consists of a single point z_0 of multiplicity d, $\tilde{U}$ is an open neighborhood of z_0 in Z embedded as a closed complex subvariety in some open set in $\mathbb{C}^N$, and $U \subset X$ is an open neighborhood of x_0 such that $F(x) \subset \tilde{U}$ for all $x \in U$. Write $F = [f_1, \ldots, f_d]$ where $f_j(x)$ is thought of as a point in $\mathbb{C}^N$ for each $x \in U$ and each j. Denote by $z = (z_1, \ldots, z_N)$ the coordinates on $\mathbb{C}^N$. Let $W = \{W_1, \ldots, W_d\}$ be any d-tuple of points $W_k \in \mathbb{C}^N$. There exist finitely many holomorphic polynomials of the form

$$\Phi_j(z, W) = \sum_\alpha P_{j,\alpha}(W) z^\alpha, \quad z \in \mathbb{C}^N, \quad W \in (\mathbb{C}^N)^d, \quad 1 \le j \le j_0$$

such that (i) the coefficients $P_{j,\alpha} : (\mathbb{C}^N)^d \to \mathbb{C}$ are symmetric polynomials in the components $W_k \in \mathbb{C}^N$ of W (i.e., they are invariant under permutations of the W_k's), and (ii) a point $w \in \mathbb{C}^N$ belongs to the d-tuple W if and only if $\Phi_j(w, W) = 0$ for all $j = 1, \ldots, j_0$. (Any such collection of polynomials is called a set of *canonical defining functions* for d-tuples in $\mathbb{C}^N$ (see [W]). Set $W = F(x)$ for some $x \in U$. Then $x \to P_{j,\alpha}(F(x))$ is a well-defined holomorphic function on U and we have

$$\Gamma(F|_U) = \{(x, z) \in U \times \mathbb{C}^N : \Phi_j(z, F(x)) = 0, 1 \le j \le j_0\}$$

(with obvious identifications). This is a closed complex subvariety of $U \times \mathbb{C}^N$ contained in $U \times \tilde{U} \subset U \times Z$.

This shows that $\Gamma(F)$ is a closed complex subvariety of Z and $h : \Gamma(F) \to X$ is an analytic cover onto X. Let $\Gamma(F) = \bigcup_{j=1}^k V_j$ be its decomposition into irreducible components. Then $h : V_j \to X$ is an analytic cover for each $j = 1, \ldots, k$. Since V_j is irreducible, F has constant multiplicity m_j along V_j and $V_j \backslash h^{-1}(\delta_F \cup X_0)$ is a connected complex manifold. Hence V_j equals the graph $V(F_j)$ of an irreducible component F_j

of F granted by Proposition 4.1. By the converse part proved above F_j is holomorphic, and we have $V(F) = \sum_{j=1}^{k} m_j V(F_j)$ in the sense of chains. This proves Proposition 4.2.   ♠

A suitable version of Proposition 4.2 remains valid over reducible spaces X. In such case we must assume in addition that $X \backslash \delta_F$ is pathwise connected and locally pathwise connected to insure that Proposition 4.1 applies (this is always the case if X is irreducible).

The following lemma shows that local multivalued sections of h exist at any point where h has maximal rank.

4.3 Lemma. *Let $h : Z \to X$ be a holomorphic map. Suppose that Z is locally irreducible at $z_0 \in Z$, X is locally irreducible at $x_0 := h(z_0) \in X$ and*

$$\dim_{z_0} h^{-1}(x_0) = \dim_{z_0} Z - \dim_{x_0} X.$$

Then there exist an integer $d \in \mathbb{N}$ and a local holomorphic d-valued section F of h in a neighborhood of x_0 such that $F(x_0) = [z_0, \ldots, z_0]$.

The number $k := \dim_{z_0} h^{-1}(x_0)$ is called the *corank* of h at z_0 and $\dim_{z_0} Z - k$ is the *rank* of h at z_0. Clearly the rank cannot exceed $\dim_{x_0} X$, and the hypothesis in the lemma is that h has maximal rank at z_0.

Proof. Since $h^{-1}(x_0)$ is a complex subvariety of Z whose dimension at z_0 equals k, there exists a germ of an irreducible complex subvariety $V \subset Z$ at z_0 such that $\dim V + k = \dim_{z_0} Z$ and z_0 is an isolated point of $V \cap h^{-1}(x_0)$. By a standard localization argument we obtain open neighborhoods $U \subset X$ of x_0 and $\tilde{U} \subset Z$ of z_0 such that $h : V \cap \tilde{U} \to U$ is a proper finite map. The rank hypothesis implies $\dim V = \dim_{x_0} X$ and hence $h(V \cap \tilde{U}) = U$ provided that U is irreducible (as we may assume to be the case). By Proposition 4.2 the set $V \cap \tilde{U}$ is the graph of a holomorphic multivalued section of h over U.   ♠

5 Proof of Theorem 1.1

Let $F : X \to Z$ be a d-valued section of $h : Z \to X$ satisfying the hypotheses of Theorem 1.1. Thus F is holomorphic in an open set $U_0 \subset X$ containing a complex subvariety $X_0 \subset X$, unramified over $X \backslash X_0$ and satisfies $\mathscr{H}^{2n-1}(\delta_F) = 0$. Propositions 4.1 and 4.2 show that $F = \bigoplus m_j F_j$ where the F_j are irreducible multivalued sections of h which are holomorphic on U_0, unramified over $X \backslash X_0$ and satisfy $\mathscr{H}^{2n-1}(\delta_{F_j}) = 0$. It suffices to prove the result for each F_j. Hence we may assume that F is irreducible. The heart of the proof is the following lemma.

5.1 Lemma. *There exists a normal complex space Y and a continuous map $g : Y \to Z$ such that, setting $f = h \circ g : Y \to X$ and $Y_x = f^{-1}(x)$ for each $x \in X$, the following hold:*

(a) *g is holomorphic in $f^{-1}(U_0) \subset Y$ and $g(Y_x) = F(x)$ for each $x \in X$,*

(b) *$f : Y \to X$ is a d-sheeted analytic cover which is unramified (a holomorphic covering space) over $X \backslash X_0$.*

The complex space Y is the 'normalized graph' of F in which the self-intersections of the ordinary graph of F over $X \backslash X_0$ (considered as a subset of Z) have been separated.

Proof. Since F is holomorphic over $U_0 \subset X$, its graph $V(F|_{U_0})$ is an effective n-chain in $h^{-1}(U_0) \subset Z$ and $h : V(F|_{U_0}) \to U_0$ is an analytic cover (Proposition 4.2). Let $\iota : V(F|_{U_0}) \to Z$ denote the inclusion map. Let $g'_0 : \tilde{Y} \to V(F|_{U_0})$ be the normalization of $V(F|_{U_0})$ and $g_0 = \iota \circ g'_0 : \tilde{Y} \to Z$. Set $f_0 = h \circ g_0 : \tilde{Y} \to U_0$. Thus $\tilde{Y}$ is a normal complex space and g'_0 is a holomorphic map which is biholomorphic over the regular locus (in particular it is biholomorphic on $f_0^{-1}(U_0 \backslash X_0) \subset \tilde{Y}$). Clearly g_0 and f_0 satisfy Lemma 5.1 over U_0.

We now extend $(\tilde{Y}, g_0)$ as follows. Let $U \subset X \backslash X_0$ be any open set such that $F|_U = \bigoplus_{j=1}^d F_j$ where $F_j : U \to Z$ are continuous single-valued sections of h over U. (Since F is unramified over $X \backslash X_0$, there is a covering of $X \backslash X_0$ by open sets satisfying this property.) From $\mathscr{H}^{2n-1}(\delta_F) = 0$ it follows by Proposition 4.1 that the F_j's are unique up to reordering, and there are no repetitions since $\mu_F = d$. Let $\mathbb{N}_d = \{1, 2, \ldots, d\}$. Set $Y_U = U \times \mathbb{N}_d$ (the disjoint union of d copies of U) and define the map $g_U : Y_U \to Z$ by $g_U(x, j) = F_j(x)$. We introduce a complex structure on Y_U by requiring that the (trivial) d-sheeted projection $f_U = h \circ g_U : Y_U \to U$ is biholomorphic on each sheet $U \times \{j\}$.

If $U' \subset X \backslash X_0$ is another open subset such that $F|_{U'} = \bigoplus_{j=1}^d F'_j$, it follows from $\mathscr{H}^{2n-1}(\delta_F) = 0$ that for each connected component Ω of $U \cap U'$ there is a permutation σ on $\mathbb{N}_p$ such that $F_j(x) = F'_{\sigma(j)}(x)$ for $x \in \Omega$ and $j = 1, \ldots, d$. This defines a transition map

$$\theta_{U, U'} : Y_U|_\Omega \to Y_{U'}|_\Omega, \quad (x, j) \to (x, \sigma(j)).$$

Clearly $\theta_{U, U'}$ is biholomorphic with respect to the complex structures on Y_U and $Y_{U'}$ and satisfies $f_U = f_{U'} \circ \theta_{U, U'}$. Using these transition maps we patch Y_U and $Y_{U'}$ into a complex manifold $Y_{U \cup U'}$ which contains Y_U and $Y_{U'}$ as open subsets such that the maps g_U, $g_{U'}$ coincide on the intersection of their domains to give a continuous map $g_{U \cup U'} : Y_{U \cup U'} \to Z$. Since F is assumed to be unramified over $X \backslash X_0$, we can globalize this construction by covering $X \backslash X_0$ with open sets as above and using the transition maps $\theta_{U, U'}$ to construct a pair (Y, g) with the required properties. ♠

We continue with the proof of Theorem 1.1. Since $f = h \circ g : Y \to X$ is a finite holomorphic map of Y onto the Stein space X, Y is also Stein [LeB]. The inverse $f^{-1} : X \to Y_{\mathrm{sym}}^d$ is a d-valued holomorphic section of $f : Y \to X$ by Proposition 4.2. The map $g : Y \to Z$ is a continuous lifting of $f = h \circ g : Y \to X$ with respect to

$h : Z \to X$, and g is holomorphic on the open set $f^{-1}(U_0) \subset Y$ containing the subvariety $Y_0 = f^{-1}(X_0)$. Note that $Y \backslash Y_0$ is contained in the regular locus of Y. Theorem 1.3 furnishes a homotopy of liftings $g_t : Y \to Z$ ($t \in [0, 1]$) connecting $g_0 = g$ to a holomorphic lifting g_1. In addition, the homotopy can be chosen to be holomorphic near Y_0 and fixed on Y_0. Then $F_t = g_t \circ f^{-1} : X \to Z^d_{\mathrm{sym}}$ for $t \in [0, 1]$ is a homotopy of d-valued sections of $h : Z \to X$ satisfying Theorem 1.1. ♠

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