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Stein Domains in Complex Surfaces

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ABSTRACT. Let S be a closed connected real surface and $\pi : S \to X$ a smooth embedding or immersion of S into a complex surface X. We denote by $I(\pi)$ (resp. by $I_\pm(\pi)$ if S is oriented) the number of complex points of $\pi(S) \subset X$ counted with algebraic multiplicities. Assuming that $I(\pi) \leq 0$ (resp. $I_\pm(\pi) \leq 0$ if S is oriented) we prove that π can be C^0 approximated by an isotopic immersion $\pi_1 : S \to X$ whose image has a basis of open Stein neighborhoods in X which are homotopy equivalent to $\pi_1(S)$. We obtain precise results for surfaces in $\mathbb{CP}^2$ and find an immersed symplectic sphere in $\mathbb{CP}^2$ with a Stein neighborhood.

Introduction

A complex manifold Ω is called *Stein* if it is biholomorphic to a closed complex submanifold of a Euclidean space $\mathbb{C}^N$. Stein manifolds are of special interest in complex analysis due to their rich function theory. Since every n-dimensional Stein manifold is homotopy equivalent to a CW-complex of real dimension at most n [1], it is natural to ask which such complexes can be realized by homologically nontrivial Stein domains in a given complex manifold.

In this article we consider this problem for CW complexes which are represented by smooth embeddings (or immersions) of closed connected real surfaces S in complex surfaces X. Let $\pi : S \to X$ be such an immersion. An open neighborhood $\Omega \subset X$ of $\pi(S)$ is called *regular* if it has a strong deformation retraction onto $\pi(S)$ (and hence Ω is homotopy equivalent to $\pi(S)$). In [10] we proved that every closed real surface S (not necessarily orientable) different from the two-sphere admits a smooth embedding in $\mathbb{C}^2$ with a basis of regular Stein neighborhoods. Here we consider more general complex surfaces with emphasis on the complex projective plane $X = \mathbb{CP}^2$.

Denote by $I(\pi) \in Z$ the number of complex points of π counted with algebraic multiplicities (Section 1). If S is oriented we denote by $I_\pm(\pi)$ the number if positive (resp. negative) complex points. These numbers are sometimes called *Lai indices* after [23], although they appeared already in [4] and [2]. Our main result is that, if $I_\pm(\pi) \leq 0$ (resp. $I(\pi) \leq 0$ when S is unorientable) then π can be C^0-approximated by a regularly homotopic immersion $\pi_1 : S \to X$ whose image $S_1 = \pi_1(S) \subset X$ has a basis of Stein neighborhoods in X homotopically equivalent to S_1 (Theorem 1.1). These neighborhoods are sublevel sets of a smooth nonnegative weakly plurisubharmonic function in a neighborhood of S_1 which vanishes quadratically on S_1 and has

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no other critical points nearby (Theorem 2.2). We obtain precise results on the existence of such embeddings in the projective plane $\mathbb{CP}^2$ (Proposition 1.7, and Theorems 1.9 and 4.3).

For oriented embedded real surfaces $S \subset X$ a converse to Theorem 1.1 is provided by gauge theory. It seems that the connection was first observed by Lisca and Matić [24, Theorem 5.2] and S. Nemirovski [27, Theorem 9]. The condition $I_\pm(S) \leq 0$ is equivalent to the *generalized adjunction inequality*

$$g(S) \geq 1 + \frac{1}{2}\left(S^2 + |c_1(X)\cdot S|\right) , \tag{*}$$

where $g(S)$ is the genus of S, $S^2 = [S]^2 \in Z$ is the self-intersection number of $[S] \in H_2(X; Z)$ in X and $c_1(X) = c_1(\Lambda^2 TX) \in H^2(X; Z)$ is the first Chern class of X (Section 1). The inequality (*) has been proved in many interesting cases using gauge theory methods. The recent Seiberg–Witten approach caused a major revolution in this field and the literature is still growing rapidly. In the Appendix we summarize the results on (*) which are most relevant to us. In particular, (*) *holds if X is a Stein surface and $S \subset X$ is any oriented, connected, embedded real surface except a homologically trivial two-sphere* (Theorem II in the Appendix). Hence the condition $I_\pm(S) \leq 0$ is also necessary for the existence of a Stein neighborhood of an oriented, embedded, homologically nontrivial surface. Nemirovski used this to solve the Vitushkin conjecture [27, Theorem 15] to the effect that there are no nonconstant holomorphic functions in any neighborhood of an embedded homologically nontrivial sphere in $\mathbb{CP}^2$. The corresponding generalized adjunction inequality for immersed oriented surfaces with simple double points is

$$g(S) + \delta_+ \geq 1 + \frac{1}{2}\left(\pi(S)^2 + |c_1(X)\cdot \pi(S)|\right) , \tag{**}$$

where δ_+ denotes the number of positive double points of π. This holds if $\pi(S)$ has a Stein neighborhood $\Omega \subset X$ and $[\pi(S)] \neq 0 \in H_2(\Omega; Z)$ (Theorem III in the Appendix). For $X = \mathbb{CP}^2$ we find embeddings and immersions which realize the lower bounds in (*) resp. (**) (Theorem 4.3).

Our examples, together with a symplectic approximation theorem of Gromov, give an immersed symplectic sphere $\pi : S \to \mathbb{CP}^2$ with a Stein neighborhood (Corollary 4.4). A theorem of Ivashkovich and Shevchishin [18] shows that such π must have some negative double points since otherwise the envelope of meromorphy of any neighborhood of $\pi(S)$ would contain a rational curve which would contradict the existence of a Stein neighborhood. It is not clear what is the minimal number of simple double points of immersed symplectic spheres in $\mathbb{CP}^2$ which admit a Stein neighborhood.

One can find Stein domains of more general homotopy type in a given complex surface by attaching handles as in [6, 12] and using the Kirby calculus [13]. We shall not pursue these matters here.

1. Regular Stein neighborhoods of embedded surfaces

Let S be a closed (= compact without boundary) connected real surface smoothly embedded in a complex surface X (= a complex two dimensional manifold). A point $p \in S$ is said to be *complex* if the tangent plane T_pS is a complex line in T_pX; the other points are called *totally real.* In suitable local holomorphic coordinates (z, w) on X near a complex point $p \in S$ the surface S is the graph $w = f(z)$ of a smooth function over a domain in $\mathbb{C}$, and $(z, f(z)) \in S$ is a complex point of S if and only if $\frac{\partial f}{\partial \bar z}(z) = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})(z) = 0$. If $p = 0$ is an isolated complex point of S then for sufficiently small $\delta > 0$ we have $\frac{\partial f}{\partial \bar z}(z) \neq 0$ for $0 < |z| \leq \delta$, and we define the *index* $I(p; S)$ as the winding number of $\frac{\partial f}{\partial \bar z}(z)$ on the circle $|z| = \delta$. The index is independent of the

choice of local coordinates and is related to the *Maslov index* [9, 10]. A totally real point has index zero.

For a generic choice of the embedding $S \subset X$ there are at most finitely many complex points in S and one defines the index of the embedding by $I(S) = \sum_{p \in S} I(p; S)$. If S is oriented we divide its complex points into *positive* resp. *negative,* depending on whether the orientation on T_pS induced by the complex structure on X agrees or disagrees with the given orientation on S. In this case one defines $I_\pm(S) \in Z$ as the sum of indices over all positive (resp. negative) complex points of S. These notions clearly extend to immersions $\pi : S \to X$ in which case we write $I(\pi) = \sum_{p \in S} I(p; \pi)$ and similarly for $I_\pm(\pi)$.

Definition. A family of open sets $\{\Omega_\epsilon : \epsilon \in (0, \epsilon_0)\}$ in X is said to be a *regular basis of neighborhoods of* $S \subset X$ if for each $\epsilon \in (0, \epsilon_0)$ we have $\Omega_\epsilon = \cup_{t<\epsilon} \Omega_t$, $\overline{\Omega}_\epsilon = \cap_{t>\epsilon} \Omega_t$, and $S = \bigcap_{0<t<\epsilon_0} \Omega_t$ is a strong deformation retraction of Ω_ϵ. In particular, every Ω_ϵ is homotopy equivalent to S.

The following is our main result.

Theorem 1.1. *Let S be a closed, connected real surface smoothly embedded in a complex surface X. Assume that either S is oriented and $I_\pm(S) \leq 0$ or that S is unorientable and $I(S) \leq 0$. Then S can be C^0-approximated by an isotopic embedding $S' \subset X$ with a regular basis of smoothly bounded Stein neighborhoods. The analogous result holds for immersions.*

Theorem 1.1 is proved in Section 2. The special case $X = \mathbb{C}^2$ was considered in [10] where it was proved that *every closed real surface except the two-sphere admits an embedding in $\mathbb{C}^2$ with a regular Stein neighborhood basis.*

Before proceeding we recall the *index formulas* expressing the indices $I(\pi)$ (resp. $I_\pm(\pi)$) by invariants of the immersion. Isolated complex points of real surfaces in complex surfaces were first investigated by Chern and Spanier [4]. Bishop [2] classified them into *elliptic* (these have index $+1$), *hyperbolic* (with index -1), and *parabolic* (these are degenerate and may have any index). If $S \subset X$ has e elliptic points and h hyperbolic points then $I(S) = e - h$ (this happens for a generic S). Bishop [2] proved that for every oriented embedded surface $S \subset \mathbb{C}^2$ we have $I(S) = \chi(S) = 2 - 2g(S)$ and $I_\pm(S) = \frac{1}{2} I(S) = 1 - g(S)$. In general we have the following [23, 33, 6]:

$$I(\pi) = \chi(S) + \chi_n(\pi), \quad I_\pm(\pi) = \frac{1}{2}\big(I(\pi) \pm c_1(X) \cdot \pi(S)\big) . \tag{1.1}$$

The first of these formulas also holds if S is unorientable. Here $\chi_n(\pi)$ denotes the Euler number of the normal bundle $\nu_\pi = \pi^*(TX)/TS$ of the immersion π (the self-intersection number of the zero section in ν_π), and $c_1(X) \cdot \pi(S)$ is the value of the first Chern class $c_1(X) = c_1(\Lambda^2 TX) \in H^2(X; Z)$ on the homology class $[\pi(S)] \in H_2(X; Z)$. The normal Euler number $\chi_n(\pi)$ is defined also for unorientable surfaces since any local orientation of S coorients ν_π by the condition that the two orientations add up to the orientation of X by the complex structure, and $\chi_n(\pi)$ is independent of these choices.

Suppose now that S is oriented and embedded in X. In this case its normal bundle ν is diffeomorphic to a neighborhood of S in X and hence $\chi_n(S)$ is the *self-intersection number* S^2 of the homology class $[S] \in H_2(X; Z)$. Since $\chi(S) = 2 - 2g(S)$, (1.1) gives

$$I_\pm(S) = 1 - g(S) + \frac{1}{2}\left(S^2 \pm c_1(X) \cdot S\right) \tag{1.2}$$

and the condition $I_\pm(S) \le 0$ in Theorem 1.1 is equivalent to

$$g(S) \ge 1 + \frac{1}{2}\left(S^2 + |c_1(X)\cdot S|\right) . \tag{1.3}$$

This is known as the *generalized adjunction inequality* and has been the subject of intensive recent research using the Seiberg–Witten approach to gauge theory. In the Appendix we summarize the results on (1.3) which are most interesting to us. In particular, (1.3) holds if X is a Stein surface and S is not a homologically trivial two-sphere in X (Theorem II in the Appendix). Hence the condition $I_\pm(S) \le 0$ is also necessary for the existence of a Stein neighborhood of S in X. (It seems that this connection was first observed in [24] and [27].) This gives

Corollary 1.2. *Let S be a closed, oriented, embedded real surface in a complex surface X. If S is not a null-homologous two-sphere then S is isotopic to an embedded surface in X with a regular Stein neighborhood basis if and only if* (1.3) *holds.*

The adjunction inequality also holds in compact Kähler surfaces with $b_2^+(X) > 1$ (Theorem I in the Appendix). Hence Theorem 1.1 gives the following.

Corollary 1.3. *If X is a compact Kähler surface with $b_2^+(X) > 1$ then every closed, connected, oriented, embedded real surface $S \subset X$ of genus $g(S) > 0$ can be $\mathcal{C}^0$ approximated by an isotopic surface with a regular Stein neighborhood basis. When $g(S) = 0$ the same holds provided that $[S] \ne 0$ and none of the homology classes $\pm[S]$ can be represented by a (possibly reducible) complex curve in X.*

Condition $b_2^+(X) > 1$ in Corollary 1.3 cannot be omitted in general (Example 1.5 below). Consider now oriented surfaces $S \subset X$ without negative complex points. These include complex curves and, more generally, symplectic surfaces with respect to a positive symplectic form on X. (If $J \in \mathrm{End}(TX)$ denotes the almost complex structure on X then a symplectic form ω on X is *J-positive,* or J is *tamed* by ω, if $\omega(v, Jv) > 0$ for any $0 \ne v \in TX$. An immersion $\pi : S \to X$ is ω-symplectic if $\pi^*\omega > 0$ on S.) In this case (1.2) gives

$$0 = 2I_-(S) = \chi(S) + S^2 - c_1(X)\cdot S, \quad I_+(S) = c_1(X)\cdot S = \chi(S) + S^2 . \tag{1.4}$$

Now $I_-(S) = 0$ is equivalent to the *genus formula* $g(S) = 1 + \frac{1}{2}(S^2 - c_1(X)\cdot S)$ (see [17, p. 361]), and $I_+(S) \le 0$ if and only if $c_1(X)\cdot S \le 0$.

Example 1.4. If S is a closed Riemann surface and $p\colon E \to S$ is a holomorphic line bundle then the zero section $S \subset E$ satisfies $I_+(S) = \chi(S) + S^2 = 2 - 2g(S) + c_1(E)$. (Here $c_1(E) \in Z$ is the value of the Chern class of E on the fundamental class $[S]$ of S; this is also called the *degree* of E.) Hence the zero section can be approximated by an isotopic surface with a regular Stein neighborhood basis in E if and only if $c_1(E) \le 2g(S) - 2$.

Example 1.5. Let S be a closed, connected, oriented, embedded surface of degree $d \ge 1$ in $\mathbb{CP}^2$ (i. e., $[S] = d[H] \in H_2(\mathbb{CP}^2; Z) = Z$, where $H \simeq \mathbb{CP}^1$ is the projective line). It is well known that $\Lambda^2(T\mathbb{CP}^2) = \mathcal{O}_{\mathbb{CP}^2}(3)$ (its dual, the canonical bundle of $\mathbb{CP}^2$, is $K = \mathcal{O}_{\mathbb{CP}^2}(-3)$). Hence $c_1(\mathbb{CP}^2)\cdot[S] = c_1(\mathbb{CP}^2)\cdot d[H] = 3d$ and (1.3) is equivalent to

$$g(S) \ge 1 + \frac{1}{2}\left(S^2 + \left|c_1\left(\mathbb{CP}^2\right)\cdot S\right|\right) = 1 + \frac{1}{2}\left(d^2 + 3d\right) = \frac{1}{2}(d+1)(d+2) . \tag{1.5}$$

In particular, we must have $g(S) \ge 3$. If $S \subset \mathbb{CP}^2$ is isotopic to a complex or symplectic curve in $\mathbb{CP}^2$ then (by the genus formula) $g(S) = \frac{1}{2}(d-1)(d-2)$ which is smaller than the right-hand side in (1.5); hence, such S does not admit any Stein neighborhood in $\mathbb{CP}^2$.

Example 1.6. Let $X = \mathbb{CP}^1 \times \mathbb{CP}^1$. For any $p \in \mathbb{CP}^1$ the group $H_2(X; Z) = Z \oplus Z$ is generated by $H_1 = \mathbb{CP}^1 \times \{p\}$ and $H_2 = \{p\} \times \mathbb{CP}^1$. For an oriented embedded surface $S \subset X$ we let $d_j = S \cdot H_j \in Z$ for $j = 1, 2$ denote the intersection numbers with the two lines. The pair $d = (d_1, d_2) \in Z^2$ is called the *bidegree* of S. We have $S^2 = 2d_1d_2$, $c_1(X) \cdot S = 2(d_1 + d_2)$, and hence the generalized adjuntion inequality is

$$g(S) \geq 1 + d_1d_2 + |d_1 + d_2| \,. \tag{1.6}$$

If (1.6) holds then by Theorem 1.1 S is isotopic to a surface with a regular Stein neighborhood basis in X. Comparing (1.6) with the genus formula $g(C) = 1 + d_1d_2 - |d_1 + d_2|$ for complex curves in X (for which $d_1, d_2 \geq 0$ and $d_1 + d_2 > 0$) we see that none of them is isotopic to a surface with a Stein neighborhood.

Using (1.5) and elementary complex analysis S. Nemirovski proved that there exist no nonconstant holomorphic functions in any neighborhood of such S [27, Theorem 10], thus solving the *Vitushkin conjecture* to the effect that there are no nonconstant holomorphic functions in any neighborhood of an embedded sphere $S \subset \mathbb{CP}^2$ with $[S] \neq 0$. We show that (1.5) is sharp:

Proposition 1.7. *For any $d \geq 1$ and $g \geq \frac{1}{2}(d+1)(d+2)$ there is an embedded oriented surface $S \subset \mathbb{CP}^2$ of genus g and degree d with a regular Stein neighborhood basis.*

For immersions in $\mathbb{CP}^2$ see Theorem 4.3 below. Proposition 1.7 is proved in Section 3 and is a special case of the following.

Theorem 1.8. *Let $S \subset X$ be a closed, connected, real surface smoothly embedded in a complex surface X.*

(a) *If S is oriented and $k = \max\{I_+(S), I_-(S), 0\}$ then S is homologous to an embedded surface $S' \subset X$ of genus $g(S') = g(S) + k$ with a regular Stein neighborhood basis.*

(b) *If S is unorientable and k is the smallest integer with $3k \geq \max\{I(S), 0\}$ then S is Z_2-homologous in X to an embedded unorientable surface $S' \subset X$ of genus $g(S') = g(S)+k$ with a regular Stein neighborhood basis.*

The next result shows that there are no genus restrictions for the existence of Stein neighborhoods of embedded unorientable surfaces in $\mathbb{CP}^2$.

Theorem 1.9. *Every closed unorientable real surface embeds in $\mathbb{CP}^2$ with a regular basis of Stein neighborhoods intersecting every projective line.*

Remark. By Theorem 1.8 in [10] every closed unorientable surface S also embeds in $\mathbb{C}^2$ with a regular Stein neighborhood basis. The set of possible indices $I(\pi)$ of embeddings $\pi : S \hookrightarrow \mathbb{C}^2$ equals $\{3\chi - 4, 3\chi, 3\chi + 4, \ldots, 4 - \chi\}$ where $\chi = 2 - g(S)$ [10, p. 358]. This set contains both positive and negative numbers for any value of χ.

Further results on Stein neighborhoods of immersed surfaces are given in Section 4. We mention a couple of related open problems.

Problem 1.10. *Does there exist a Stein domain $\Omega \subset \mathbb{C}^2$ which is homotopically equivalent to the two-sphere ?* This question, which was raised in [10], is apparently still open. It is known that any embedded two-sphere $S \subset \Omega$ must be null-homologous in Ω [27, Theorem 15] and hence $[S]$

cannot generate $H_2(\mathbb{CP}^2; Z)$. There exists a precise description of the envelope of holomorphy of smooth two-spheres contained in closed strongly pseudoconvex hypersurfaces in $\mathbb{C}^2$ [3, 22].

Problem 1.11. *Let $S \subset X$ be an embedded real surface with isolated complex points which admits a regular basis of Stein neighborhoods. Is $I(p; S) \leq 0$ for every $p \in S$?* Clearly S cannot have elliptic complex points due to Bishop discs [2, 20]. Further results on the existence of families of small analytic discs at certain complex points of higher index were obtained by Wiegerinck [35] and Jöricke [19]. The problem seems open in general.

2. Construction of regular Stein neighborhoods

Theorem 2.1 below gives the existence of certain special immersions regularly homotopic to a given initial immersion $S \to X$, and Theorem 2.2 provides regular Stein neighborhood basis of special immersed surfaces.

Theorem 2.1. *Any immersion $\pi_0 : S \to X$ of a closed real surface S into a complex surface X can be C^0-approximated by a regularly homotopic smooth immersion $\pi : S \to X$ satisfying the following properties:*

(a) *At every complex point $p \in S$ of π there are open neighborhoods $p \in U \subset S$, $q = \pi(p) \in V \subset X$ and local holomorphic coordinates (z, w) on V such that $z(q) = w(q) = 0$ and $\pi(U) \subset V$ is given either by $w = z\bar{z}$ (a special elliptic point) or by $w = \bar{z}^2$ (a special hyperbolic point). If $I_\pm(\pi_0) \leq 0$ (resp. $I(\pi_0) \leq 0$ if S is unorientable) then we can choose π as above to be without elliptic points.*

(b) *The immersion π only has transverse double points (and no multiple points), and in a neighborhood of each double point there exist local holomorphic coordinates $z = x + iy$, $w = u + iv$ on X such that $\pi(S)$ is given locally by $\{y = 0,\ v = 0\} \cup \{x = 0,\ u = 0\} = (\mathbb{R}^2 \times i\{0\}^2) \cup (\{0\}^2 \times i\mathbb{R}^2)$.*

(c) *Any embedding $S \hookrightarrow X$ can be C^0-approximated by an isotopic embedding satisfying (a).*

Theorem 2.1 is proved by cancellation of pairs of complex points due to Eliashberg and Harlamov (see the reference in [7]). Their article is not available in a standard source and we refer instead to Theorem 1.1 in [10] where all details can be found. (We apologize for the incorrect reference [24] in [10].) The result falls within the scope of Gromov's h-principle (Section 2.4.5 in [15]). We recall the main steps.

First we modify the immersion π_0 near each double point so that the new immersion satisfies property (b) (this is completely elementary and can be accomplished by a modification supported in small neighborhoods of the double points). Suppose now that $p, q \in S$ are distinct isolated complex points of π_0 with $I(p; \pi_0) + I(q; \pi_0) = 0$. If S is oriented we assume that either both points are positive or both are negative. Choose a simple smooth arc $\gamma \subset S$ with endpoints p and q which does not contain any other complex point or double point of π_0. There exist holomorphic coordinates in an open set $U \supset \pi_0(\gamma)$ in X which embed U onto a domain in $\mathbb{C}^2$ such that $\pi_0(S) \cap U$ is mapped onto a graph $w = f(z)$ over a disc $D \subset \mathbb{C}$ (see Section 5 in [10] for the details). The winding number of $\frac{\partial}{\partial \bar{z}} f$ around ∂D equals $I(p; \pi_0) + I(q; \pi_0)$ which is assumed to be zero. Such f can be uniformly approximated by a smooth function g on D which equals f near ∂D and satisfies $\frac{\partial}{\partial \bar{z}} g \neq 0$ on D (Lemma 4.1 in [10] or Lemma 1 in [27, p. 735]). This gives a C^0-approximation of π_0 by a regularly homotopic immersion which equals π_0 outside a small neighborhood $V \supset \gamma$ in S and is totally real on V.

Using this procedure repeatedly one obtains an immersion $\pi : S \to X$ with $I_\pm(\pi) = I_\pm(\pi_0)$ such that each orientation class of S contains only elliptic or only hyperbolic points of π, depending on the sign of $I_\pm(\pi)$. If $I_\pm(\pi) \leq 0$ then all complex points of π are hyperbolic. It is completely elementary to modify each elliptic resp. hyperbolic point to a complex point of special type as in Theorem 2.1 (a). If π_0 is an embedding, these deformations are carried out by an isotopy of embeddings.

Theorem 2.2. *If $\pi : S \to X$ is an immersion satisfying Theorem 2.1 and containing only special hyperbolic complex points then there are an open set $\Omega \subset X$ containing $M = \pi(S)$ and a smooth plurisubharmonic function $\rho : \Omega \to [0, 1]$ satisfying $M = \{x \in \Omega : \rho(x) = 0\}$, $d\rho \neq 0$ on $\Omega \backslash M$, and such that for every $\epsilon \in (0, 1)$ the set $\Omega_\epsilon = \{x \in \Omega : \rho(x) < \epsilon\}$ is a smoothly bounded Stein domain which admits a strong deformation retraction onto M.*

Remarks: 1. An embedded real surface in a complex surface is locally holomorphically convex in a neighborhood of any hyperbolic complex point [11]. From this it is easy to show that a surface $S \subset X$ with only hyperbolic complex points has a basis of Stein neighborhoods (see [10] and [27, Theorem 4]). However, at the time of this writing, *it is not known whether every such S admits small Stein neighborhoods homotopically equivalent to S.* The problem is to find a 'good' plurisubharmonic function near every hyperbolic complex point whose critical points do not accumulate on S. (One must deal with a similar problem near each double point.) This is why we restrict our attention to *special hyperbolic complex points* and *special double points.*

2. After the completion of this article M. Slapar informed the author that he had recently obtained regular Stein neighborhoods in the presence of arbitrary hyperbolic complex points (personal communication, August 2002).

Proof of Theorem 2.2. We first define ρ near complex and double points. Let $p_1, \ldots, p_m \in M = \pi(S) \subset X$ be the complex points and $q_1, \ldots, q_k \in M$ the double points of π. By hypothesis there exist a neighborhood $V_j \subset X$ of p_j and holomorphic coordinates $\phi_j(p) = (z(p), w(p))$ on V_j such that $z(p_j) = w(p_j) = 0$, $\phi_j(V_j) = r_j\Delta \times \Delta \subset \mathbb{C}^2$ for some $r_j \in (0, 1)$ (where $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$), and $M \cap V_j = \{w = \bar{z}^2\}$. The nonnegative function

$$\rho(z, w) = \left|w - \bar{z}^2\right|^2 = |w|^2 + |z|^4 - 2\Re\left(wz^2\right)$$

has all the required properties in V_j. Indeed it is plurisubharmonic (since $\Re(wz^2)$ is pluriharmonic), strongly plurisubharmonic outside the complex disc $\Lambda_j = \{p \in V_j : z(p) = 0\}$, and $d\rho$ vanishes precisely on $M \cap V_j = \{w = \bar{z}^2\}$.

Each double point $q_j \in M$ of π has a neighborhood $W_j \subset X$ and holomorphic coordinates $\psi_j = (z, w) = (x + iy, u + iv)$ on W_j such that in these coordinates $q_j = 0$ and $M \cap W_j = \{y = 0,\ v = 0\} \cup \{x = 0,\ u = 0\}$. Set

$$\rho(x + iy, u + iv) = \left(x^2 + u^2\right)\left(y^2 + v^2\right).$$

Its differential

$$d\rho = 2\left(x\left(y^2 + v^2\right), y\left(x^2 + u^2\right), u\left(y^2 + v^2\right), v\left(x^2 + u^2\right)\right)$$

vanishes precisely on $\{\rho = 0\} = M \cap W_j$. We have

$$\rho_{z\bar{z}} = \rho_{w\bar{w}} = \frac{1}{2}\left(x^2 + y^2 + u^2 + v^2\right), \quad \rho_{z\bar{w}} = i(xv - yu) = \overline{\rho_{w\bar{z}}}.$$

Since $\rho_{z\bar{z}} > 0$ except at the origin, its complex Hessian $H_\rho(z,w)$ has at least one positive eigenvalue there. By Cauchy–Schwarz we have

$$\begin{aligned} 4\det H_\rho(z,w) &= \left(x^2+y^2+u^2+v^2\right)^2 - 4(xv-yu)^2 \\ &\geq \left(x^2+y^2+u^2+v^2\right)^2 - 4\left(x^2+y^2\right)\left(u^2+v^2\right) \\ &= |z|^4+|w|^4-2|z|^2|w|^2 \\ &= \left(|z|^2-|w|^2\right)^2 . \end{aligned}$$

Thus $\det H_\rho(z,w) \geq 0$ which shows that both eigenvalues are nonnegative and hence ρ is plurisubharmonic. The equality $\det H_\rho(z,w) = 0$ holds precisely when $(x,y) = \lambda(v,-u)$ for some $\lambda \in \mathbb{R}$ and $|z|^2 = |w|^2$, and this gives $w = \pm iz$. Let $L_j = \{p \in W_j : w(p) = \pm iz(p)\}$. Thus ρ satisfies all required properties on W_j and is strongly plurisubharmonic on $W_j \backslash L_j$.

A function ρ with the required properties has been defined on neighborhoods $p_j \in V_j$ and $q_j \in W_j$. We now extend ρ to a smooth nonnegative function in a neighborhood $V \supset M$ such that the extension vanishes precisely on M and its real Hessian is nondegenerate in any normal direction to M at all points of $M_0 = M\backslash\{p_j, q_j\}$. More precisely, if we denote by $\nu \to M_0$ the normal bundle to M_0 in X and realize it as a subbundle of $TX|_{M_0}$ such that $TX|_{M_0} = TM_0 \oplus \nu$, we require that the second-order derivatives of ρ in the fiber directions ν_x at any $x \in M_0$ give a nondegenerate quadratic form on ν_x (which is necessarily positive definite since ρ has a local minimum at $0_x \in \nu_x$). One can obtain such an extension by taking a suitable second order jet along M with the required properties and applying Whitney's theorem to find a function which matches this jet.

A more explicit construction is the following. Choose a Riemannian metric $h(x;\xi,\eta)$ on the normal bundle $\nu = \{(x,\xi) : x \in M_0, \xi \in \nu_x\}$. The function $\widetilde{\rho}(x,\xi) = h(x;\xi,\xi) \in \mathbb{R}_+$ has suitable properties on ν (with M_0 corresponding to the zero section of ν). Let ψ be a diffeomorphism of ν onto a tubular neighborhood $U \subset X$ of M_0 such that $\psi(x,0) = x$ and $d\psi(x,0)$ is the identity for each $x \in M_0$. Then $\rho_0 = \psi \circ \widetilde{\rho} \circ \psi^{-1} : U \to \mathbb{R}_+$ satisfies the required properties near M_0. Within each coordinate neighborhood V_j and W_j as above we patch ρ_0 with the previously chosen function $\rho = \rho_j$ on this neighborhood by taking $\rho = \chi_j\rho_j + (1-\chi_j)\rho_0$ on V_j, where χ_j is a smooth cut-off function which is supported in V_j (resp. in W_j) and equals one in a smaller neighborhood of p_j resp. q_j. At points $x \in M_0 \cap V_j$ the real Hessian of ρ equals

$$H_\rho^{\mathbb{R}}(x) = \chi_j(x)H_{\rho_j}^{\mathbb{R}}(x) + (1-\chi_j(x))H_{\rho_0}^{\mathbb{R}}(x) .$$

This is nonnegative on T_xX and strongly positive on ν_x (since it is a convex combination of two forms with these properties).

We claim that the function ρ obtained in this way is plurisubharmonic in a neighborhood of M, it has no critical points near M (except on M), and the sublevel sets $\{\rho < \epsilon\}$ for sufficiently small $\epsilon > 0$ are Stein. We first show plurisubharmonicity. By construction ρ is such near each p_j and q_j. Let $p \in M_0 = M\backslash\{p_j, q_j\}$. Since M_0 is totally real, there exist local holomorphic coordinates $(z,w) = (x+iy, u+iv)$ near p such that in these coordinates $p = 0$ and $T_pM_0 = \mathbb{R}^2 \oplus i\{0\}^2 = \{y=0,\ v=0\}$. Since $\rho = 0$ and $d\rho = 0$ on M_0, those second-order derivatives of ρ at 0 which contain at least one differentiation on x or u vanish at 0 and hence

$$4\rho_{z\bar{z}}(0) = \rho_{yy}(0), \quad 4\rho_{w\overline{w}}(0) = \rho_{vv}(0), \quad 4\rho_{z\overline{w}}(0) = \rho_{yv}(0) .$$

Thus the complex Hessian of ρ at 0 equals one quarter of the real Hessian of the function $(y,v) \to \rho(0+iy, 0+iv)$ at $y = 0, v = 0$. By construction this is positive definite and

hence ρ is strongly plurisubharmonic at every $p \in M_0$. Thus ρ is plurisubharmonic in a small neighborhood $\Omega \supset M$ and strongly plurisubharmonic in $\Omega' = \Omega \backslash \big((\cup \Lambda_j) \cup (\cup L_j)\big)$. We may choose $\Omega = \{\rho < \epsilon_0\}$ for some $\epsilon_0 > 0$.

Next we show that $d\rho \neq 0$ on $\Omega \backslash M$ if $\epsilon_0 > 0$ (and hence Ω) are chosen sufficiently small. By construction this holds in small neighborhoods of the points p_j and q_j. Over $M_0 = M \backslash \{p_j, q_j\}$ we consider the conjugate function $\widetilde{\rho} = \psi^{-1} \circ \rho \circ \psi$ on the normal bundle $\nu \to M_0$. Suppose that $d\widetilde{\rho}(x, \xi) \cdot \xi = 0$ for some $(x, \xi) \in \nu$ with $x \in M_0$ and $\xi \neq 0$. Consider the function $t \in \mathbb{R} \to \widetilde{\rho}(x, t\xi)$. By hypothesis its derivative vanishes at $t = 0$ and $t = 1$ and hence its second derivative vanishes at some $t_0 \in (0, 1)$. This means that the Hessian of $\widetilde{\rho}(x, \cdot)$ vanishes at the fiber point $t_0\xi \in \nu_x$ in the direction of the vector $\xi \in \nu_x$. We have seen that this does not happen in a sufficiently small neighborhood of the zero section of ν which establishes our claim.

Thus for any $\epsilon \in (0, \epsilon_0)$ the set $\Omega_\epsilon = \{x \in \Omega \colon \rho(x) < \epsilon\}$ is a relatively compact smoothly bounded (weakly) pseudoconvex domain in X. We obtain a strong deformation retraction of Ω_ϵ onto M by integrating the flow of the negative gradient of ρ from $t = 0$ to $t = +\infty$ and rescaling the time interval to $[0, 1]$. (The gradient of ρ is defined by the equation $\nabla\rho \rfloor h = d\rho$ where h is a Riemannian metric on X and $\rfloor$ denotes the contraction.)

It remains to show that Ω_ϵ is Stein for each $\epsilon \in (0, \epsilon_0)$. Let $h_\epsilon \colon (-\infty, \epsilon) \to \mathbb{R}$ be an increasing strongly convex function with $\lim_{t \to \epsilon} h(t) = +\infty$. If there exists a strongly plurisubharmonic function ψ in a neighborhood of $\overline{\Omega}_\epsilon$ in X (this is the case for instance if X is Stein) then $\psi + h_\epsilon \circ \rho$ is a strongly plurisubharmonic exhaustion function on Ω_ϵ and hence Ω_ϵ is Stein according to [14].

In general a weakly pseudoconvex domain in a non-Stein manifold need not be Stein. In our situation we proceed as follows. Recall that the Levi form of ρ and hence of $\rho_\epsilon = h_\epsilon \circ \rho$ is degenerate only on the complex curves $\Lambda_j \subset \Omega$ and $L_j \subset \Omega$. These curves intersect M transversely at the points p_j resp. q_j. We take $\phi = \rho_\epsilon + \delta\tau$ where $\delta > 0$ and τ is a smooth function in a neighborhood of $\overline{\Omega}_\epsilon$ which is strongly plurisubharmonic on the curves Λ_j and L_j and which vanishes outside the coordinate neighborhood V_j of p_j (resp. q_j). Since the complex Hessian of ρ_ϵ is bounded away from zero on the set in Ω_ϵ where τ fails to be plurisubharmonic, ϕ is a strongly plurisubharmonic exhaustion function on Ω_ϵ provided that $\delta > 0$ is chosen sufficiently small.

We take τ to be given in local coordinates (z, w) near p_j resp. q_j by $\tau(z, w) = \chi(z, w)(|z|^2 + |w|^2)$ where χ is a suitable cut-off function. At p_j we have $\Lambda_j = \{z = 0\}$, the coordinate neighborhood $V_j \subset X$ of p_j is mapped onto $\{|z| < r,\ |w| < 1\} \subset \mathbb{C}^2$, and a suitable cut-off function is $\chi(|z|)$ where $\chi(t) = 1$ for $t \leq r/2$ and $\chi(t) = 0$ for $t \geq 3r/4$. At q_j we may assume that the coordinate neighborhood $W_j \subset X$ is mapped onto $\{|z| < 1,\ |w| < 1\} \subset \mathbb{C}^2$ and in this case a suitable cut-off function is $\chi(|z|^2 + |w|^2)$ where $\chi(t) = 1$ for $t \leq 1/4$ and $\chi(t) = 0$ for $t \geq 3/4$. In both cases the support of the differential $d\chi$ intersects Ω_ϵ for all sufficiently small $\epsilon > 0$ in a set whose closure (in X) is compact and does not meet any of the curves Λ_j and L_j. On this set the eigenvalues of the Levi form of $\rho_\epsilon = h_\epsilon \circ \rho$ are bounded away from zero and hence for $\delta > 0$ sufficiently small $\rho_\epsilon + \delta\tau$ is a strongly plurisubharmonic exhaustion function in Ω_ϵ. Hence Ω_ϵ is Stein by Grauert's theorem [14].

Note that we can perturb each domain Ω_ϵ constructed above to a strongly pseudoconvex domain in X by adding to ρ a small function which is strongly plurisubharmonic on the sets $\Lambda_j \cap b\Omega_\epsilon$ and $L_j \cap b\Omega_\epsilon$. □

Proof of Theorem 1.1. Let $S \subset X$ be an embedded surface with $I_\pm(S) \leq 0$ (resp. $I(S) \leq 0$ if S is unorientable). Let $S' \subset X$ be an isotopic embedding satisfying the conclusion of Theorem 2.1.

Theorem 2.2 then gives a regular basis of Stein neighborhoods of S'. The analogous conclusions hold for immersions. □

3. Connected sums of embedded and immersed surfaces

Let X_1 and X_2 be smooth n-manifolds. Choose embedded n-dimensional discs $D_j \subset X_j$ for $j = 1, 2$ and let $\phi\colon \partial D_1 \to \partial D_2$ be a smooth diffeomorphism which is orientation reversing if the two manifolds are oriented. The identification space $(X_1 \setminus \operatorname{int} D_1) \cup_\phi (X_2 \setminus \operatorname{int} D_2)$ can be given the structure of a smooth manifold called the *connected sum* $X_1 \# X_2$ of X_1 and X_2 (see [13, p. 20]). The smoothing process can be visualized by connecting the two pieces $X_j \setminus \operatorname{int} D_j$ by a cylinder $[0, 1] \times S^{n-1}$ glued along its boundary spheres $(\{0\} \times S^{n-1}) \cup (\{1\} \times S^{n-1})$ to the spheres ∂D_1 resp. ∂D_2. If $X_1 = X_2 = X$ and we wish to use as ϕ the identity map on $\partial D \subset X$, we must reverse the orientation on one of the copies of X; in this case we write $X \# \overline{X}$. For example, if X is a complex n-manifold then $X \# \overline{\mathbb{CP}^n}$ is diffeomorphic to the complex manifold obtained by blowing up a point in X. The connected sum operation extends to several terms and we shall write $X_1 \# k X_2$ for the connected sum of X_1 with k copies of X_2.

Suppose now that S_1, $S_2 \subset X$ are embedded or immersed real surfaces in general position in a complex surface X. We can realize their connected sum $S_1 \# S_2$ as an immersed surface in X with the same number of double points and with two additional hyperbolic complex points, one positive and one negative if the surfaces are oriented, which are contained in the attaching cylinder $\Sigma \simeq [0, 1] \times S^1$ (see Section 3 in [10] for the details). The two removed discs $D_j \subset S_j$ are chosen totally real. Thus we have $I(S_1 \# S_2) = I(S_1) + I(S_2) - 2$, and if both surfaces are oriented we also have

$$I_+(S_1 \# S_2) = I_+(S_1) + I_+(S_2) - 1, \quad I_-(S_1 \# S_2) = I_-(S_1) + I_-(S_2) - 1 .$$

Furthermore, $g(S_1 \# S_2) = g(S_1) + g(S_2)$ and $[S_1 \# S_2] = [S_1] + [S_2] \in H_2(X; Z)$. If S_1 and S_2 are disjoint embedded surfaces in X then $S_1 \# S_2$ can also be realized as an embedded surface.

Proof of Theorem 1.8. Let $T \subset X$ be an embedded null-homologous totally real torus (take a small torus in a coordinate chart). For any embedded oriented surface $S \subset X$ we have

$$g(S \# T) = g(S) + 1, \quad I_\pm(S \# T) = I_\pm(S) - 1, \quad [S] = [S \# T] \in H_2(X; Z) .$$

Attaching $k = \max\{I_+(S), I_-(S), 0\}$ torus handles we obtain an embedded surface $S \# kT \subset X$ with $g(S \# kT) = g(S) + k$ and $I_\pm(S \# kT) = I_\pm(S) - k \le 0$. Part (a) now follows from Theorem 1.1. The same argument applies to immersed surfaces.

Suppose now that $S \subset X$ is an embedded (or immersed) unorientable surface. Let $M \subset X$ be a copy of the real projective plane $\mathbb{RP}^2$ embedded in a coordinate chart in X with $I(M) = -1$ [10, p. 367]. Then $g(S \# M) = g(S) + 1$ and $I(S \# M) = I(S) - 3$. Thus $g(S \# kM) = g(S) + k$, $I(S \# kM) = I(S) - 3k \le 0$, and hence (b) follows from Theorem 1.1. We can use other types of handles such as a totally real torus T or a Klein bottle K (for an explicit totally real Klein bottle in $\mathbb{C}^2$ see [29]). The surfaces $S \# T$ and $S \# K$ are diffeomorphic and satisfy $I(S \# T) = I(S) - 2$ and $g(S \# T) = g(S) + 2$; hence these handles are less effective in lowering the index than $\mathbb{RP}^2$. □

Proof of Proposition 1.7. Let $C \subset \mathbb{CP}^2$ be a smooth complex curve of degree d. By (1.4) we have $I_-(C) = 0$, $I_+(C) = 3d$, $g(C) = g_{\mathbb{C}}(d) = \frac{1}{2}(d-1)(d-2)$. Attaching $k \ge 3d$ torus handles we obtain an embedded surface $S = C \# kT \subset \mathbb{CP}^2$ homologous to C, with $I_\pm(S) \le 0$ and

$$g(S) = g(C) + k = \frac{1}{2}(d-1)(d-2) + k \ge \frac{1}{2}(d+1)(d+2) .$$

The result now follows from Theorem 1.1. Note that, by the Thom conjecture proved in [21], a smooth complex curve of degree d in $\mathbb{CP}^2$ has the smallest genus $g_{\mathbb{C}}(d) = \frac{1}{2}(d-1)(d-2)$ among all smooth oriented real surfaces of degree d in $\mathbb{CP}^2$. For the symplectic case see [28] and [26]. □

Proof of Theorem 1.9. Let $\mathbb{RP}^2 \simeq M \hookrightarrow \mathbb{C}^2 \subset \mathbb{CP}^2$ be an embedding with index $I(M) = -1$ mentioned above [10, p. 367]. Let $S = \{[x:y:z] \in \mathbb{CP}^2 : x, y, z \in \mathbb{R}\} \simeq \mathbb{RP}^2$. Clearly S is totally real and intersects every projective line in $\mathbb{CP}^2$; in fact their (mod 2) intersection number equals one. The connected sum $S\#kM \subset \mathbb{CP}^2$ is an embedded unoriented surface with genus $1+k$ and index $-3k \leq 0$, and hence Theorem 1.1 applies. □

4. Regular Stein neighborhoods of immersed surfaces

In this section S denotes a closed connected oriented real surface of genus $g(S)$. We shall consider immersions $\pi : S \to X$ into a complex surface X with simple (transverse) double points and with no multiple points. Furthermore we shall assume that both tangent planes at any double point are totally real. At each double point π has self-intersection index ± 1 which is independent of the choice of the orientation on S. Double points with index $+1$ will be called *positive* and those with index -1 will be called *negative*. If π has δ_+ positive and δ_- negative double points then $\delta(\pi) = \delta_+ - \delta_-$ is the (geometrical) *self-intersection index* of π which only depends on its regular homotopy class.

Recall that $\chi_n(\pi)$ denotes the normal Euler number of π. It is easily seen that each double point of π contributes ± 2 (the sign depending on its self-intersection index) to the homological self-intersection number of the image $\pi(S)$ in X. This gives $\chi_n(\pi) + 2\delta(\pi) = \pi(S)^2$. From (1.1) we get

$$I_{\pm}(\pi) = 1 - g(S) - \delta(\pi) + \frac{1}{2}\left(\pi(S)^2 \pm c_1(X)\cdot\pi(S)\right) , \tag{4.1}$$

and the condition $I_{\pm}(\pi) \leq 0$ is equivalent to

$$g(S) + \delta_+ \geq 1 + \delta_- + \frac{1}{2}(\pi(S)^2 + |c_1(X)\cdot\pi(S)|) . \tag{4.2}$$

Corollary 4.1. *If (4.2) holds then π is regularly homotopic to an immersion $S \to X$ whose image has a regular Stein neighborhood basis in X. The regular homotopy can be chosen such that it preserves the location (and number) of double points. Conversely, if $\pi(S)$ has a Stein neighborhood $\Omega \subset X$ and if $[\pi(S)] \neq 0$ in $H_2(\Omega; Z)$ then the following generalized adjunction inequality holds:*

$$g(S) + \delta_+ \geq 1 + \frac{1}{2}\left(\pi(S)^2 + |c_1(X)\cdot\pi(S)|\right) . \tag{4.3}$$

Unlike in (4.2), the term δ_- is not present on the right-hand side of (4.3) and hence the two inequalities coincide only when π has no negative double points. If $\delta_- > 0$, there is a 'gray area' between the direct and the converse part of Corollary 4.1. The direct part follows from Theorems 2.1 and 2.2; for the converse part see Theorem III in the Appendix. When π has no negative complex points (for instance if π is complex or symplectic) then $I_-(\pi) = 0$ which gives the *genus formula* (see e. g. [18, p. 576])

$$g(S) = 1 - \delta(\pi) + \frac{1}{2}\left(\pi(S)^2 - c_1(X)\cdot\pi(S)\right) .$$

Theorem 4.2. *Assume that S is an oriented closed surface and $\pi : S \to X$ is an immersion with k double points. If*

$$g(S) + k \geq 1 + \frac{1}{2}\left(\pi(S)^2 + |c_1(X)\cdot \pi(S)|\right) \tag{4.4}$$

then every open neighborhood $\Omega \subset X$ of $\pi(S)$ contains an embedded oriented surface $M \subset \Omega$ of genus $g(M) = g(S) + k$ such that $[M] = [\pi(S)] \in H_2(\Omega; Z)$ and M admits a regular basis of Stein neighborhoods.

Proof. At each double point of $\pi(S)$ we replace a pair of small intersecting discs in $\pi(S)$ by an embedded annulus $\Sigma \simeq S^1 \times [0, 1]$. This well known procedure, which amounts to replacing double points by handles, can be done within Ω and it does not change the homology class of the image, but it increases the genus of the immersed surface by k. Here is a precise description.

By Theorem 2.1 we may assume that in local holomorphic coordinates $(z, w) = (x+iy, u+iv)$ on X near a double point the immersed surface is the union of discs in the Lagrangian planes $\Lambda_1 = \{y = 0,\ v = 0\} \subset \mathbb{C}^2$, $\Lambda_2 = \{x = 0,\ u = 0\} \subset \mathbb{C}^2$. Let Λ_1 be oriented by $\partial_x \wedge \partial_u$ and Λ_2 by $\kappa\, \partial_v \wedge \partial_y$ where $\kappa = \pm 1$ is the self-intersection index of the double point.

If $\kappa = +1$ we can use the handle $\Sigma_+ = \{(x + iu)(y - iv) = \epsilon\} \cap B$ for a small $\epsilon \neq 0$, where $B \subset \mathbb{C}^2$ is a small ball around the origin. Outside of B we smoothly patch Σ_+ with $(\Lambda_1 \backslash D_1) \cup (\Lambda_2 \backslash D_2)$ without introducing new complex points (D_j is a disc in Λ_j). A simple calculation shows that Σ_+ is totally real in $\mathbb{C}^2$ for every $\epsilon \neq 0$, and hence this handle does not change the indices $I_\pm$.

If $\kappa = -1$, an appropriate handle is $\Sigma_- = \{(x + iu)(y + iv) = \epsilon\}$ for small $\epsilon \neq 0$. It has four hyperbolic complex points at $x = y = \pm\sqrt{\epsilon/2}$, $v = -u = \pm\sqrt{\epsilon/2}$ (independent choices of signs), two positive and two negative. Hence $I_\pm(\Sigma_-) = -2$ and the indices of the immersion decrease by two.

After replacing all double points by handles in this way we obtain an embedded surface $M \subset \Omega$ with $g(M) = g(S) + k$ and $[M] = [\pi(S)] \in H_2(X; Z)$. From (4.4) it follows that $g(M) \geq 1 + \frac{1}{2}(M^2 + |c_1(X)\cdot M|)$ and hence Theorem 1.1 applies to M. □

Remark: The replacement of double points by handles can also be explained using the following complex model. In local holomorphic coordinates $(z, w) = (x + iy, u + iv)$ we represent the double point by $\{zw = 0\} = L_1 \cup L_2$, where $L_1 = \{w = 0\}$ is oriented by $\partial_x \wedge \partial_y$ and $L_2 = \{z = 0\}$ is oriented by $\kappa\, \partial_u \wedge \partial_v$ where $\kappa = \pm 1$ is the intersection index. An appropriate handle is $zw = \epsilon$ when $\kappa = +1$ and $z\overline{w} = \epsilon$ when $\kappa = -1$.

We now consider immersed surfaces in $\mathbb{CP}^2$. Recall that the degree of $\pi : S \to \mathbb{CP}^2$ is the integer d satisfying $[\pi(S)] = d[H] \in H_2(\mathbb{CP}^2; Z) = Z$ where H is the projective line. Reversing the orientation on S if necessary we may assume $d \geq 0$. The number on the right-hand side of (4.2)–(4.4) equals $1 + \frac{1}{2}(d^2 + 3d) = \frac{1}{2}(d + 1)(d + 2)$.

Theorem 4.3. *Let S_g be a closed oriented surface of genus g and let $\pi : S_g \to \mathbb{CP}^2$ be an immersion of degree $d \geq 1$ with δ_+ positive double points. If $\pi(S_g)$ has a Stein neighborhood in $\mathbb{CP}^2$ then $g + \delta_+ \geq \frac{1}{2}(d+1)(d+2)$. Conversely, for any pair of integers $g, \delta_+ \geq 0$ satisfying the above inequality there exists an immersion $\pi : S_g \to \mathbb{CP}^2$ of degree d with δ_+ positive double points (and no negative double points) whose image $\pi(S_g)$ admits a regular Stein neighborhood basis in $\mathbb{CP}^2$. In particular, there is an immersed sphere in $\mathbb{CP}^2$ in the homology class of the projective line, with three double points, which admits a regular Stein neighborhood basis.*

Proof. The first part follows from Corollary 4.1. We prove the converse part by an explicit construction. Let $\pi_0\colon S_0 \to \mathbb{C}^2$ be Weinstein's Lagrangian (hence totally real) 'figure eight' immersion of the sphere [34]:

$$\pi_0(x, y, u) = \big(x(1+2iu), y(1+2iu)\big), \quad x^2+y^2+u^2 = 1\,.$$

It identifies the points $(0, 0, \pm 1)$ and the corresponding double point at $0 \in \mathbb{C}^2$ has self-intersection index $+1$.

Let $C_1, C_2, \dots, C_d \subset \mathbb{CP}^2$ be projective lines in general position. There are $\frac{1}{2}d(d-1)$ positive intersection points. For each $j \in \{2, \dots, d\}$ we replace one of the intersection points of C_j with $C_1 \cup \cdots \cup C_{j-1}$ by a handle as in the proof of Theorem 4.2. This eliminates $d-1$ double points and changes the given system of lines to an immersed sphere $\rho\colon S_0 \to \mathbb{CP}^2$ of degree d with $I_+(\rho) = 3d$, $I_-(\rho) = 0$ and with $\frac{1}{2}(d-1)(d-2)$ positive double points.

Write $g + \delta_+ = l + \frac{1}{2}(d+1)(d+2)$, with $l \geq 0$, and set $k = l + 3d \geq 3d$. Take k pairwise disjoint copies $M_1, \dots, M_k \subset \mathbb{CP}^2 \backslash \rho(S_0)$ of Weinstein's sphere $\pi_0(S_0)$. The connected sum $\rho(S_0)\#M_1\#\cdots\#M_k \subset \mathbb{CP}^2$ is parametrized by a degree d immersion $\pi\colon S_0 \to \mathbb{CP}^2$ with $I_+(\pi) = 3d - k \leq 0$, $I_-(\pi) = -k < 0$, and with $k + \frac{1}{2}(d-1)(d-2) = g + \delta_+$ positive double points. Replacing g of the double points by handles we obtain an immersed surface of genus g with δ_+ positive double points and with non-positive indices $I_\pm$. The result now follows from Theorem 1.1. □

By a theorem of Gromov [15, p. 334, Theorem A] every immersion $\pi : S \to \mathbb{CP}^2$ of degree $d \geq 1$ can be C^0-approximated by a symplectic immersion $\tilde{\pi}\colon S \to \mathbb{CP}^2$, i. e., $\tilde{\pi}^*(\omega_{FS}) > 0$ on S. (If we specify a two-form $\theta > 0$ on S with $\int_S \theta = 1$, we can even choose $\tilde{\pi}$ such that $\tilde{\pi}^*\omega_{FS} = d \cdot \theta$.) Hence Theorem 4.3 implies the following.

Corollary 4.4. *For any $d \geq 1$ there exists a symplectically immersed sphere in $\mathbb{CP}^2$ of degree d with a Stein neighborhood.*

I thank S. Ivashkovich for telling me the question answered in part by Corollary 4.4 and for pointing out the relevance of Gromov's theorem. The results of [18] imply that *there exist no immersed symplectic spheres in $\mathbb{CP}^2$ with a Stein neighborhood basis.*

Corollary 4.4 may seem somewhat surprising since symplectic curves share many properties with complex curves. The main difference (which is essential here) is that complex curves only intersect positively while symplectic curves may also intersect negatively.

The symplectic approximation furnished by Gromov's theorem need not be regularly homotopic to the initial immersion and hence the number of double points may increase. The question remains about the minimal number of double points of immersed symplectic spheres in $\mathbb{CP}^2$ of a given degree $d \geq 1$ which admit Stein neighborhoods. For $d = 1$ the genus formula gives $\delta(\pi) = 1 + \frac{1}{2}(d^2 - 3d) = 0$ which shows that the number of positive double points equals the number of negative double points. By Theorem 4.3 we have $\delta_+ \geq \frac{1}{2}(d+1)(d+2) = 3$ and hence *there are at least 6 double points.*

Question. *Is there an immersed symplectic sphere in $\mathbb{CP}^2$ of degree one (in the homology class of the projective line) with six double points and with a Stein neighborhood?*

Appendix: Generalized adjunction inequalities

We say that a closed, connected, oriented, smoothly embedded real surface S in a complex surface X satisfies the *generalized adjunction inequality* if

$$g(S) \geq 1 + \frac{1}{2}\left(S^2 + |c_1(X)\cdot S|\right) , \qquad (*)$$

where $g(S)$ is the genus of S, S^2 is the self-intersection number of its oriented homology class $[S] \in H_2(X; Z)$ in X, and $c_1(X) = c_1(\Lambda^2 TX) \in H^2(X; Z)$ is the first Chern class of X. The purpose of this Appendix is to collect from the literature those results on (*) which are most relevant to this article. We emphasize that no originality whatsoever is being claimed on our part.

If X is a closed oriented four-manifold then $H^2(X; \mathbb{R})$ is equipped with the intersection pairing induced by $(\alpha, \beta) \to \int_X \alpha \wedge \beta$ where α, β are closed 2-forms on X. This pairing is Poincaré dual to the intersection pairing of homology classes in $H_2(X; Z)$. Let $b_2^+(X)$ denote the dimension of a maximal linear subspace of $H^2(X; \mathbb{R})$ on which this pairing is positive definite. (If X is a compact Kähler surface then $b_2^+(X) > 0$ is an odd positive integer.)

Theorem I ([21, 26, 8, 28]). *Assume that X is a compact Kähler surface with $b_2^+(X) > 1$. Let $S \subset X$ be a smoothly embedded, closed, connected, oriented real surface in X. Then the generalized adjunction inequality (*) holds in each of the following cases:*

(a) $g(S) > 0$ *(i. e., S is not a sphere),*

(b) $g(S) = 0$, $[S] \neq 0$ *in $H_2(X; Z)$, and none of the homology classes $\pm[S]$ can be represented by a (possibly reducible) complex curve in X.*

For homologically trivial surfaces (*) reduces to $g(S) \geq 1$ which only excludes the sphere. For homologically nontrivial spheres (*) requires in particular that $S^2 \leq -2$ which fails for the exceptional sphere of a blown-up point (this has $S^2 = -1$). The inequality (*) fails for any complex (or symplectic) curve $C \subset X = \mathbb{CP}^2$ since $g(C) = \frac{1}{2}(d-1)(d-2)$ by the classical genus formula (where $d > 0$ is the degree of C), but the right hand side of (*) equals $\frac{1}{2}(d+1)(d+2)$. Hence the condition $b_2^+(X) > 1$ cannot be omitted.

Before proceeding we need to recall the notion of the Seiberg–Witten function (see [13, Section 2.4] for a concise exposition and [25] for a more comprehensive one). If X is a simply connected, closed, oriented four-manifold with $b_2^+(X) > 1$, the *Seiberg–Witten function* on X is an integer valued function $SW_X \colon \mathcal{C}_X \to Z$ defined on the set of all characteristic classes $\mathcal{C}_X \subset H^2(X; Z)$ (see [13, p. 51]). One of its main features is that the value $SW_X(K)$ on any $K \in \mathcal{C}_X$ only depends on the smooth structure on X. The story is more complicated if X is not simply connected or if $b_2^+(X) = 1$ and we shall not go into that. A characteristic class K is a *Seiberg–Witten basic class* if $SW_X(K) \neq 0$, and X is of *Seiberg–Witten simple type* if for any such K the associated moduli space of solutions of the perturbed Seiberg–Witten equations is zero dimensional (hence finite).

For surfaces with $S^2 \geq 0$ Theorem I was proved by Kronheimer and Mrowka [21] and Morgan, Szabó and Taubes [26] when X is an oriented closed four-manifold with $b_2^+(X) > 1$, of Seiberg–Witten simple type, and with $c_1(X)$ replaced in (*) by any Seiberg–Witten basic class. Since any compact symplectic four-manifold (hence any compact Kähler surface) with $b_2^+(X) > 1$ is of Seiberg–Witten simple type [32] and satisfies $SW_X(c_1(X)) \neq 0$ [31], Theorem I is a special case of the quoted results. Note that S cannot be a sphere since the existence of an embedded sphere with $S^2 \geq 0$ and $[S] \neq 0$ implies that SW_X vanishes identically [8].

In the case $S^2 < 0$ Theorem I was proved for spheres by Fintushel and Stern [8, Theorem 1.3] and for surfaces of positive genus by Ozsváth and Szabó [28, Corollary 1.7]. The proofs essentially depend on the assumption that X is of Seiberg–Witten simple type, but the Kähler hypothesis can be relaxed as above. The argument goes as follows. If (*) fails for a given $S \subset X$ then $K = c_1(X) + 2\epsilon PD(S) \in H^2(X; Z)$ is a Seiberg–Witten basic class; here $\epsilon = \pm 1$ is the sign of $c_1(X) \cdot S$ and $PD(S)$ is the Poincaré dual of S (Theorem 1.3 in [28]; Remark 1.6 in [28] explains the factor 2 in front of $PD(S)$ as opposed to the notation in Theorem 1.3). Furthermore, if $g(S) > 0$, the corresponding moduli space of solutions of the perturbed Seiberg–Witten equations has positive dimension (see the proof of Corollary 1.7 in [28]) which contradicts the simple type assumption on X. If $g(S) = 0$ then one cannot draw the last conclusion. However, $SW_X(K) \neq 0$ for a $K \in H^2(X; Z)$ implies that the class $K' = \frac{1}{2}(c_1(X) - K)$ is Poincaré dual to a complex curve $C \subset X$ [13, p. 417]. Applying this to $K = c_1(X) + 2\epsilon PD(S)$ for which $K' = \pm PD(S)$ we obtain $\pm[S] = [C]$ as claimed (see [8, Theorem 1.3]).

Theorem II (Adjunction inequalities in Stein surfaces). *Let X be a Stein surface and let $S \subset X$ be a smoothly embedded, closed, connected, oriented real surface in X. Then the generalized adjunction inequality (*) holds unless S is a homologically trivial sphere.*

Theorem II is stated as Theorem 11.4.7 in [13] and is attributed to Lisca and Matić [24, Theorem 5.2]. (For spheres only the inequality $S^2 \leq -2$ is stated in [13].) The result was stated and proved in [24] only for the case $S^2 \geq 0$, the reason being that the article [24] preceded the work of Ozsváth and Szabó [28] on surfaces with negative self-intersection. Joining [24] and [28] one immediately obtains the general case of Theorem II as follows. (A similar proof is given in [13].)

By Theorem 3.2 and Corollary 3.3 in [24] every relatively compact domain $\Omega \subset\subset X$ in a Stein surface admits a biholomorphic map onto a domain in a compact Kähler surface Y with $b_2^+(Y) > 1$ whose canonical bundle $K_Y = \Lambda^2 T^*Y$ is ample, meaning in particular that $c_1(K_Y)^2 > 0$ [17, p. 365]. (Equivalently, the canonical homology class $[K_Y] = PD(c_1(K_Y)) \in H_2(X; Z)$ satisfies $[K_Y]^2 > 0$.) From $\Lambda^2(TY) \simeq K_Y^* = K_Y^{-1}$ we see that $c_1(Y) = -c_1(K_Y)$ and hence $c_1(Y)^2 > 0$ as well. Such surface is always *minimal* (without complex spheres of self-intersection number -1) and *of general type.* Furthermore, if $S \subset \Omega$ is homologically nontrivial in X then its image is nontrivial in Y. The right-hand side of (*) does not change if we replace $S \subset X$ with its image in Y. The inequality (*) now follows from Theorem I unless S is a sphere with $S^2 < 0$. If (*) fails for such S then by [8] (or by Theorem 1.3 in [28]) $K = c_1(Y) + 2\epsilon PD(S)$ is a Seiberg–Witten basic class, where $\epsilon = \pm 1$ is the sign of $c_1(Y) \cdot S$. Since Y is minimal and of general type, its only Seiber–Witten basic classes are $\pm c_1(Y)$ [36, 32], hence $c_1(Y) + 2\epsilon PD(S) = \pm c_1(Y)$. This gives either $PD(S) = 0$ (a contradiction) or $PD(S) = -\epsilon c_1(Y)$ in which case $S^2 = PD(S)^2 = c_1(Y)^2 > 0$, a contradiction to the assumption $S^2 < 0$.

Remark. Theorem II appears as Theorem 9 in the article of S. Nemirovski [27] without acknowledging the 1997 work of Lisca and Matić [24], even though [24] is included in the bibliography in [27]. (In fact, all relevant references cited above were already available at that time and are included in [27].) Nemirovski gave in [27] a slightly different reduction argument to the Kähler case which nevertheless used essentially the same tools and results. He begins by embedding a domain $\Omega \subset X$ containing the given surface S into a smooth affine algebraic surface $V \subset \mathbb{C}^N$ using a theorem of Stout [30] and Demailly, Lempert and Shiffman [5]. By taking the projective closure of V in $\mathbb{CP}^N$ and desingularizing at infinity one obtains a compact Kähler surface Y containing a biholomorphic copy of S. (So far the argument is the same as in [24], except that Y does not have the additional properties.) Nemirovski then argues that

$S^2 \geq 0 \Rightarrow b_2^+(Y) > 1$ and hence Theorem I applies, while $S^2 < 0$ should imply by [28] that one of the classes $\pm[S]$ is represented by a complex curve, but this cannot happen by the construction of Y. However, the conclusion in Theorem 1.3 of [28] is more complicated when $b_2^+(Y) = 1$ and it was not verified in [27] whether this theorem can indeed be applied as indicated. The argument given above avoids this situation.

For immersed real surfaces in Stein surfaces one has the following adjunction inequality (see Remark 3 in [27, p. 742]).

Theorem III. *Let $\pi : S \to X$ be an immersed closed oriented real surface with simple double points in a Stein surface X. If π has δ_+ positive double points and $[\pi(S)] \neq 0$ in X then*

$$g(S) + \delta_+ \geq 1 + \frac{1}{2}\left(\pi(S)^2 + |c_1(X)\cdot\pi(S)|\right) . \qquad (**)$$

Lacking a precise reference we include a sketch of proof which is intended solely for the non-experts in topology (no originality whatsoever is being claimed). The first step is exactly as in the proof of Theorem II: we replace X by a compact Kähler minimal surface of general type, with $b_2^+(Y) > 1$ and $[K_Y]^2 > 0$, such that $\pi(S)$ is homologically nontrivial in Y. From now on we consider π as an immersion of S into Y. We then replace each positive double point of $\pi(S) \subset Y$ by an embedded handle as in the proof of Theorem 4.2. This increases the genus of the immersed surface by δ_+ but it does not change its homology class in Y, hence the right-hand side of (**) remains unchanged. We denote the new surface again by S.

It remains to desingularize $\pi(S)$ by blowing up Y at each of the remaining (negative) double points. This is described in Section 2.2 of [13] and goes as follows. (I thank S. Ivashkovich for explaining the basic idea to me.) Locally near a negative double point $p \in \pi(S) \subset Y$ we take as the local model for $\pi(S)$ the union $L_1 \cup \overline{L}_2 \subset \mathbb{C}^2$, where $L_1 \neq L_2$ is a pair of complex lines through $p = 0 \in \mathbb{C}^2$ and the bar on L_2 means that the orientation has been reversed (in order to have a negative intersection at 0). Let $\tau : \widetilde{Y} \to Y$ denote the blow-up of Y at p with the exceptional sphere $e = \tau^{-1}(p)$. Denote by $\widetilde{S} = \tau^{-1}(\pi(S)) \subset \widetilde{Y}$ the total transform of $\pi(S)$ and by $S' = \overline{\tau^{-1}(\pi(S)\backslash\{p\})} \subset \widetilde{Y}$ its proper transform. By general properties of blow-ups the right-hand side of (**) does not change if we replace $\pi(S) \subset Y$ by the total transform $\widetilde{S} \subset \widetilde{Y}$. We claim that the same is true if we replace $\widetilde{S}$ by the proper transform $S' \subset \widetilde{Y}$ of $\pi(S)$. This follows immediately from $[S'] = [\widetilde{S}] \in H_2(\widetilde{Y}; Z)$ which can be seen as follows. We obtain $[S']$ by subtracting from $[\widetilde{S}]$ a copy of $[e]$ (to obtain the proper transform along L_1) as well as a copy of $-[e]$ (for the proper transform along $\overline{L}_2$; the minus sign accounts for the reversed orientation). Thus $[e]$ cancels out and we have $[S'] = [\widetilde{S}]$ as claimed. In particular, $S'\cdot e = \widetilde{S}\cdot e = 0$.

Performing blow-up at all negative double points of $\pi(S)$ we get a compact Kähler surface Y_0 with $b_2^+(Y_0) > 1$, along with a natural projection $\tau : Y_0 \to Y$, and a smoothly embedded, oriented, homologically nontrivial real surface $S_0 \subset Y_0$ with $\tau(S_0) = \pi(S) \subset Y$ whose genus $g(S_0)$ equals the genus of the original surface (immersed in X) plus δ_+. By Theorem I the inequality (*) holds for $S_0 \subset Y_0$ (and hence (**) holds for the initial immersion into X) unless S_0 is a sphere with $S_0^2 < 0$ (this happens if we start with an immersed sphere with only negative double points).

In the latter case we proceed as follows. Let $e_j = \tau^{-1}(p_j)$ be the exceptional spheres over the blown-up points and E_j their Poincaré duals. Since (*) fails, we conclude as in the proof of Theorem II (using [8] or [28]) that $c_1(Y_0) + 2\epsilon PD(S_0) \in H^2(Y_0; Z)$ is a Seiberg–Witten basic class for one of the choices of $\epsilon = \pm 1$. By [13, Theorem 2.4.9] the only Seiberg–Witten basic classes of Y_0 are $\pm\tau^* c_1(Y) + \sum \epsilon_j E_j$ where $\epsilon_j = \pm 1$ for every j. Also $c_1(Y_0) = \tau^* c_1(Y) - \sum E_j$. If follows that $PD(S_0) = \pm\tau^* c_1(Y)$ modulo the exceptional classes E_j.

Dualizing gives $[S_0] = \pm\tau^{-1}[K_Y]$ modulo the exceptional spheres $[e_j]$. (Here $[K_Y]$ denotes the Poincaré dual of $c_1(K_Y) = -c_1(Y)$.) Pushing down by $\tau : Y_0 \to Y$ we get $[\pi(S)] = \pm[K_Y]$ and hence $[S_0]^2 = [\pi(S)]^2 = [K_Y]^2 > 0$ in contradiction to the assumption $S_0^2 < 0$.

Problem. *Prove Theorems II and III without using the Seiberg–Witten theory.*

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