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## Most real analytic Cauchy-Riemann manifolds are nonalgebraizable

Received: 28 June 2004 / Published online: 5 November 2004

**Abstract.** We give a simple argument to the effect that most germs of generic real analytic Cauchy-Riemann manifolds of positive CR dimension are not holomorphically embeddable into a generic real algebraic CR manifold of the same real codimension in a finite dimensional space. In particular, most such germs are not holomorphically equivalent to a germ of a generic real algebraic CR manifold.

### Introduction

A smooth real submanifold  $M \subset \mathbf{C}^n$  in a complex Euclidean space is said to be a *generic Cauchy-Riemann (CR) submanifold* of CR dimension  $m$  and codimension  $d$  ( $m + d = n$ ) if it is locally near every point  $x \in M$  defined by  $d$  real equations  $\rho_1 = 0, \dots, \rho_d = 0$  satisfying  $\partial\rho_1 \wedge \dots \wedge \partial\rho_d \neq 0$ . (Here  $\partial\rho = \sum_{j=1}^d \frac{\partial\rho}{\partial z_j} dz_j$  and  $\wedge = \wedge_C$ .) A germ  $(M, x)$  is *real analytic* (respectively *real algebraic*) if it is defined locally near  $x$  by real analytic (resp. real algebraic) functions. Germs  $(M, x), (M', x')$  are *holomorphically equivalent* if there exists a biholomorphic map  $f: U \rightarrow U'$  from a neighborhood  $U$  of  $x$  onto a neighborhood  $U'$  of  $x'$  with  $f(x) = x'$  and  $f(M \cap U) = M' \cap U'$ .

Beginning with Ebenfelt [4] (1996) several authors have given examples of analytic CR manifolds which are not locally holomorphically equivalent to an algebraic one (Baouendi, Ebenfelt and Rothschild ([1], 9.11.4), ([2], 7.2); Huang, Ji and Yau [8]). S. Ji studied the algebraization problem for real analytic strongly pseudoconvex hypersurfaces and established the propagation of algebraization for hypersurfaces with maximal Cartan-Chern-Moser rank [9], [10]. Gaussier and Merker studied the problem for a class of tuboids [6].

While it seems rather difficult to decide whether a specific analytic CR manifold is locally holomorphically equivalent to an algebraic one, the phenomenon itself is not at all surprising. The purpose of this note is to give a very simple argument to the effect that *most germs of generic real analytic CR manifolds  $M \subset \mathbf{C}^n$  of positive CR dimension are not holomorphically embeddable into any generic real algebraic CR manifold  $M' \subset \mathbf{C}^{n'}$  of the same codimension as  $M$* ; in particular,

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*Mathematics Subject Classification (2000): Primary 32V20, 32V30*

Supported in part by Research Program P1-0291, Republic of Slovenia

they are not holomorphically equivalent to a germ of a real algebraic CR manifold in  $\mathbf{C}^n$ . More precisely, the embeddable ones form a set of the first category in a suitable Baire space (theorem 1.2). The same conclusion holds for embeddings into any countable union of finite dimensional families of CR manifolds of the same codimension.

Our proof employs an argument from [5] (which essentially goes back to Poincaré [12]) where it was proved that most germs of real analytic strongly pseudoconvex hypersurfaces in  $\mathbf{C}^n$  for  $n > 1$  are not holomorphically embeddable into any sphere  $\sum_{j=1}^N |z_j|^2 = 1$  (Theorem 2.2 in [5]). For embeddings into infinite dimensional spheres see Lempert [11].

## 1. The main result

Every germ of analytic CR manifold in  $\mathbf{C}^n$  of CR dimension  $m$  and codimension  $d$ , with  $m + d = n$ , is holomorphically equivalent to one of the form

$$M = \{v_j = r_j(x, y, u) : j = 1, \dots, d\} \quad (1)$$

where  $z = x + iy \in \mathbf{C}^m$ ,  $w = u + iv \in \mathbf{C}^d$  and  $r = (r_1, \dots, r_d)$  is an  $\mathbf{R}^d$ -valued convergent power series without constant and linear terms. Let  $\mathcal{R}$  denote the space of all formal power series

$$r(x, y, u) = \sum_{\alpha, \beta \in \mathbf{Z}_+^m, \gamma \in \mathbf{Z}_+^d} c_{\alpha, \beta, \gamma} x^\alpha y^\beta u^\gamma \quad (c_{\alpha, \beta, \gamma} \in \mathbf{R}^d)$$

without constant and linear terms in  $2m + d$  real variables  $(x, y, u)$ . (One could put  $M$  in a Chern-Moser normal form [3], although this will not be necessary for our purposes.) We shall identify  $r \in \mathcal{R}$  with the (formal) germ at  $0 \in \mathbf{C}^n$  of the CR manifold (1).  $\mathcal{R}$  is a Fréchet space in the topology induced by the seminorms  $\|r\|_{\alpha, \beta, \gamma} = |c_{\alpha, \beta, \gamma}|$  for all multiindices  $\alpha, \beta \in \mathbf{Z}_+^m, \gamma \in \mathbf{Z}_+^d$ . The convergent power series, representing germs of real analytic CR manifolds, form a union  $\cup_{t>0} \mathcal{R}^t \subset \mathcal{R}$  of Banach spaces (in fact, Banach algebras)

$$\mathcal{R}^t = \{r \in \mathcal{R} : \|r\|_t = \sum |c_{\alpha, \beta, \gamma}| \cdot t^{|\alpha|+|\beta|+|\gamma|} < +\infty\} \quad (2)$$

with the norm  $\|r\|_t$  ([7], p. 15). For  $r \in \mathcal{R}$  and  $k \in \mathbf{N}$  we denote by  $r_k$  its the truncation (Taylor polynomial) of order  $k$ . Let  $\mathcal{R}_k$  be the (finite dimensional real) vector space of all such truncations.

**Definition 1.1.** A manifold  $M$  (1) is embeddable into an algebraic model if there exists a real algebraic CR manifold  $M' \subset \mathbf{C}^{n'} (n' \geq n)$  of real codimension  $d$  and a holomorphic embedding  $F = (f_1, \dots, f_{n'}) : U \rightarrow \mathbf{C}^{n'}$ , defined in an open neighborhood  $U \subset \mathbf{C}^n$  of 0, such that  $F$  is transverse to  $M'$  at 0 and  $F(M \cap U) = M' \cap F(U)$ .

One defines *formal holomorphic embeddability* of a jet (1) into a similar jet  $M' \subset \mathbf{C}^{n'}$  defined by  $v' = \rho(x', y', u')$  by requiring that the composition  $\rho \circ F$  is formally holomorphically equivalent to the jet (1) (see [5]).

**Theorem 1.2.** Let  $t > 0$  and  $m \geq 1$ . The set of all  $r \in \mathcal{R}^t$  for which the germ at 0 of the real analytic CR manifold  $M = \{v = r(x, y, u)\}$  (1) of  $\text{CRdim}M = m$  is holomorphically embeddable into an algebraic model is of the first category in the Banach algebra  $\mathcal{R}^t$ . The same holds for the set of germs in  $\mathcal{R}^t$  or in  $\mathcal{R}$  which are formally holomorphically embeddable into an algebraic model.

*Proof.* Fix the dimension  $n' = m' + d \geq n = m + d$  of the target space and denote the variables by  $(z', w')$ , with  $z' = x' + iy' \in \mathbf{C}^{m'}$  and  $w' = u' + iv' \in \mathbf{C}^d$ . Every germ at 0 of a generic algebraic CR manifold in  $\mathbf{C}^{n'}$  of CR dimension  $m'$  is linearly equivalent to one of the form

$$A: \rho(z', \bar{z}', w', \bar{w}') = \Im w' + \tilde{\rho}(z', \bar{z}', w', \bar{w}') = 0 \quad (3)$$

where  $\rho = (\rho_1, \dots, \rho_d)$  is a  $d$ -tuple of real-valued polynomials and  $\tilde{\rho} = O(2)$  (i.e., it only contains terms of order  $\geq 2$ ). We fix a germ at  $0 \in \mathbf{C}^n$  of an analytic CR manifold  $M$  of the form (1) and ask whether there exists a germ of a holomorphic embedding  $F = (f, g): (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^{n'}, 0)$ , with  $f = (f_1, \dots, f_{m'})$  and  $g = (g_1, \dots, g_d)$ , such that the equation

$$\rho(f, \bar{f}, g, \bar{g}) = \Im g + \tilde{\rho}(f, \bar{f}, g, \bar{g}) = 0 \quad (4)$$

defines the germ of  $M$  at  $0 \in \mathbf{C}^n$ . (Any local holomorphic change of coordinates of  $(\mathbf{C}^n, 0)$  may be included in  $F$ .) Our normalizations imply

$$\begin{aligned} T_0 M &= \{v = 0\}, & T_0^{\mathbf{C}} M &= T_0 M \cap i T_0 M = \{w = 0\}, \\ T_0 A &= \{v' = 0\}, & T_0^{\mathbf{C}} A &= \{w' = 0\}. \end{aligned}$$

Hence  $g(z, w) = Bw + \tilde{g}(z, w)$  for some  $B \in GL_d(\mathbf{R})$  and  $\tilde{g} = O(2)$ . Insertion into (4) gives

$$\Im(Bw + \tilde{g}(z, w)) + \tilde{\rho}(f, \bar{f}, g, \bar{g}) = 0. \quad (5)$$

Set  $g^*(z, w) = B^{-1}g(z, w) = w + B^{-1}\tilde{g}(z, w)$  and

$$\tilde{\rho}^*(z', \bar{z}', w', \bar{w}') = B^{-1}\rho(z', \bar{z}', Bw', B\bar{w}') = \Im w' + \tilde{\rho}^*(z', \bar{z}', w', \bar{w}').$$

Multiplying (5) on the left by  $B^{-1}$  we see that  $M$  is also defined by

$$\rho^*(f, \bar{f}, g^*, \bar{g}^*) = \Im g^* + \tilde{\rho}^*(f, \bar{f}, g^*, \bar{g}^*) = 0$$

where  $\tilde{\rho}^* = O(2)$ . Thus the germ  $M$  also arise as the preimage of the algebraic CR manifold  $\tilde{A} = \{\rho^* = 0\} \subset \mathbf{C}^{n'}$  by the holomorphic embedding  $F^* = (f, g^*)$ . This shows that it suffices to consider preimages of algebraic manifolds (3) by (formal) holomorphic embeddings

$$F = (f, g), \quad F(0) = 0, \quad g(z, w) = w + \tilde{g}(z, w), \quad \tilde{g} = O(2). \quad (6)$$

The  $F$ -preimage of  $A$  (3) is given by

$$0 = \rho(f, \bar{f}, g, \bar{g}) = \Im g + \tilde{\rho}(f, \bar{f}, g, \bar{g}) = v - r'(x, y, u, v) \quad (7)$$

where  $r'$  is a power series containing only terms of order  $\geq 2$ . To change the equation  $v = r'(x, y, u, v)$  (7) into one of the form (1) one performs the iteration

$$v^0 = 0, \quad v^{j+1} = r'(x, y, u, v^j) \quad (j = 0, 1, \dots).$$

In the convergent (real analytic) case this amounts to solving (7) on  $v$  by the implicit function theorem. The iteration converges also on the formal level, that is, every coefficient of order  $k$  in the power series  $r$  is determined after at most  $k$  iterations and does not change during subsequent iterations.

*Key observation.* If the germ at 0 of the manifold  $v = r(x, y, u)$  (1) is the preimage of the manifold (3) by a (formal) holomorphic map (6) then the coefficient of every monomial of order  $\leq k$  in the series for  $r$  is a polynomial function of the coefficients (and their conjugates) of order  $\leq k$  in the series for  $\rho$  and  $F$ , and it does not depend on the coefficients of order  $> k$  of  $\rho$  or  $F$ .

To see this it suffices to observe that all operations with power series without a constant term which were used in the process have this property (since they only involve conjugation, addition, multiplication, and insertion of one series into another, and each of these operations has the stated property).

For a fixed  $n' = m' + d$  we denote by  $\mathcal{H}^{n'}$  the set of all germs of holomorphic maps  $F: (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^{n'}, 0)$  of the form (6). For fixed  $n', v \in \mathbf{N}$  we denote by  $\mathcal{A}^{n', v}$  the set of all algebraic manifolds (3) in  $\mathbf{C}^{n'}$  defined by real polynomials  $\rho = (\rho_1, \dots, \rho_d)$  of degree at most  $v$ . The corresponding spaces of truncations are denoted by a subscript. The above observation amounts to the following.

**Lemma 1.3.** *Given  $k, n', v \in \mathbf{N}$  there exists a polynomial map  $P_k: \mathcal{H}_k^{n'} \times \mathcal{A}_k^{n', v} \rightarrow \mathcal{R}_k$  whose range contains the truncation  $M_k$  of every germ  $M$  (1) which can be formally holomorphically embedded in an algebraic manifold (4) of degree  $\leq v$  in  $\mathbf{C}^{n'}$ .*

We now compare the real dimensions of the source and the target spaces. It is easily seen that  $\dim \mathcal{H}_k^{n'} \approx 2n'(k+1)^n$  and  $\dim \mathcal{A}_k^{n', v} \approx d(v+1)^{n'}$ , so the dimension of the source space is  $\leq C(2n'(k+1)^n + d(v+1)^{n'})$  for some constant  $C < +\infty$  independent of  $k$ . On the other hand,  $\dim \mathcal{R}_k \geq ck^{2m+d}$  for some  $c > 0$  independent of  $k$ . When  $2m+d > n$  (which is the case if and only if  $m > 0$ ) the latter dimension grows faster as  $k \rightarrow +\infty$  (for fixed values of  $n', v$ ). Hence for a sufficiently large  $k$  the image of the polynomial map  $P_k$  is contained in a union of at most countably many proper closed local real analytic subsets of the vector space  $\mathcal{R}_k$ . Since the natural linear projection  $\mathcal{R}^t \rightarrow \mathcal{R}'_k = \mathcal{R}_k$  is surjective, we conclude that the set of all  $r \in \mathcal{R}^t$  for which the germ (1) can be holomorphically embedded in an algebraic model (3) of degree  $\leq v$  in  $\mathbf{C}^{n'}$  is of the first category in the Banach space  $\mathcal{R}^t$ . The same remains true for the countable union of these sets over all  $n', v \in \mathbf{N}$ . This concludes the proof of the theorem.  $\square$

## 2. Remarks and open problems

*Remark 2.1.* Clearly the proof of theorem 1.2 applies to more general families of domains and targets; our goal here was to illustrate a general principle without

aiming at the most general results. Similar observations in a related context have been made recently in [6] (see especially sect. 8).

*Remark 2.2.* I wish to thank P. Ebenfelt (private communication) for pointing out that, at least for hypersurfaces, the result can also be obtained by extending the theorem on non-embeddability of a generic real analytic hypersurface into a sphere (theorem 2.2 in [5]) to show non-embeddability into quadrics of any signature and then applying Webster's result [13] to the effect that any real algebraic hypersurface can be embedded into a quadric in some higher dimensional space. In practice this does not give a shorter proof.

**Problem 2.3.** Let  $d > 1$  and  $t > 0$ . Consider the set of all  $r \in \mathcal{R}^t$  for which the germ at 0 of the CR manifold  $v = r(x, y, u)$  (1) of codimension  $d$  admits a local holomorphic map into some algebraic strongly pseudoconvex *hypersurface*  $M' \subset \mathbf{C}^N$ . Is this set of the first category in  $\mathcal{R}^t$ ?

**Problem 2.4.** What is the answer if one replaces holomorphic embeddings by CR embeddings of certain smoothness class? In particular, does every real analytic strongly pseudoconvex hypersurface in  $\mathbf{C}^n$  ( $n > 1$ ) admit a local CR embedding of class  $C^1$  into an algebraic model? Into a sphere?

**Problem 2.5.** Is there a propagation of holomorphic embeddability into algebraic models, similar to [10], in a suitable class of real analytic (strongly pseudoconvex) CR manifolds?

*Acknowledgements.* I wish to thank Peter Ebenfelt and Alexander Sukhov for their invaluable advice concerning the state of knowledge on the question considered in the paper.

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