

## EXTENDING HOLOMORPHIC MAPPINGS FROM SUBVARIETIES IN STEIN MANIFOLDS

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### 1. Introduction.

We shall say that a complex manifold  $Y$  satisfies the *convex approximation property* (CAP) if every holomorphic map  $f:U \rightarrow Y$  from an open set  $U \subset \mathbb{C}^n$  ( $n \in \mathbb{N}$ ) can be approximated uniformly on any compact convex set  $K \subset U$  by entire maps  $\mathbb{C}^n \rightarrow Y$ . This Runge type approximation property was first introduced in [13] where it was shown that it implies, and hence is equivalent to, the classic *Oka property* of  $Y$  (see corollary 1.3 below; in the definition of CAP we may even restrict to a certain special class of compact convex sets). The main result of this paper is that CAP also implies the universal extendability of holomorphic maps from closed complex subvarieties in Stein manifolds.

**THEOREM 1.1.** — *Assume that  $Y$  is a complex manifold satisfying CAP. For any closed complex subvariety  $X_0$  in a Stein manifold  $X$  and any continuous map  $f_0:X \rightarrow Y$  such that  $f_0|_{X_0}:X_0 \rightarrow Y$  is holomorphic there is a homotopy  $f_t:X \rightarrow Y$  ( $t \in [0, 1]$ ) which is fixed on  $X_0$  such that  $f_1$  is holomorphic on  $X$ . The analogous conclusion holds for sections of any holomorphic fiber bundle with fiber  $Y$  over a Stein manifold.*

We shall say that a complex manifold  $Y$  satisfying the conclusion of theorem 1.1 for all data  $(X, X_0, f_0)$  enjoys the *Oka property with*

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*interpolation*. The conclusion of theorem 1.1 has been proved earlier under any of the following assumptions (from strongest to weakest):

- $Y$  is complex homogeneous (Grauert [18], [19], [20]);
- $Y$  admits a *dominating spray* ([21], [17]); such  $Y$  is said to be *elliptic*;
- $Y$  admits a *finite dominating family of sprays* [9]; such  $Y$  is said to be *subelliptic*.

Each of the above conditions implies CAP (see the h-Runge theorems proved in [9], [15], [21]). The converse implication CAP  $\Rightarrow$  subellipticity is not known in general, and there are cases when CAP is known to hold but the existence of a dominating spray (or a dominating family of sprays) is unclear; see corollary 6.2 below and the examples in [13] and [14].

Our proof shows without any additional work that the homotopy  $\{f_t\}$  in theorem 1.1 can be chosen to remain holomorphic and uniformly close to  $f_0$  on any compact  $\mathcal{H}(X)$ -convex (holomorphically convex) subset  $K \subset X$  on which the initial map  $f_0$  is holomorphic (compare with Theorem 1.4 in [17]). However, using an idea of Lárússon (Theorem 1 in [29]) we show that this addition, as well as the jet interpolation on complex subvarieties, follows automatically (and rather easily) from the Oka property with interpolation.

**PROPOSITION 1.2.** — *Assume that a complex manifold  $Y$  enjoys the Oka property with interpolation (the conclusion of theorem 1.1). Let  $d$  be a distance function on  $Y$  induced by a Riemannian metric. Let  $X$  be a Stein manifold,  $X_0 \subset X$  a closed complex subvariety,  $K$  a compact  $\mathcal{H}(X)$ -convex subset of  $X$ , and  $f_0: X \rightarrow Y$  a continuous map which is holomorphic in an open set containing  $K \cup X_0$ .*

*For every  $s \in \mathbb{N}$  and  $\epsilon > 0$  there are a neighborhood  $U \subset X$  of  $K \cup X_0$  and a homotopy  $f_t: X \rightarrow Y$  ( $t \in [0, 1]$ ) such that for every  $t \in [0, 1]$ ,  $f_t$  is holomorphic in  $U$ , it agrees with  $f_0$  to order  $s$  on  $X_0$ , and  $\sup_{x \in K} d(f_t(x), f_0(x)) < \epsilon$ .*

Proposition 1.2, proved in §5, easily reduces to the case  $Y = \mathbb{C}^r$  where it follows from Cartan's theorems A and B. The analogous result holds for sections of holomorphic fiber bundles over Stein manifolds (remark 5.2).

We shall say that a complex manifold  $Y$  satisfying the conclusion of proposition 1.2 for any  $(X, X_0, K, f_0)$  enjoys the *Oka property with jet interpolation and approximation*. Deleting the interpolation (i.e., taking  $X_0 = \emptyset$ ) we get the *Oka property with approximation*. Summarizing the

above results we see that these ostensibly different Oka-type properties are equivalent.

**COROLLARY 1.3.** — *The following properties of a complex manifold are equivalent:*

- (a) *the convex approximation property (CAP);*
- (b) *the Oka property with approximation;*
- (c) *the Oka property with interpolation;*
- (d) *the Oka property with jet interpolation and approximation.*

The analogous equivalences holds for the parametric versions of these Oka properties (theorem 6.1). The conditions in corollary 1.3 are implied by ellipticity (the existence of a dominating spray; [21], [15]) and are equivalent to it on Stein manifolds (Proposition 1.3 in [17]).

The equivalence (a) $\Leftrightarrow$ (b) in corollary 1.3 was proved in [13], (a) $\Rightarrow$ (c) is the content of theorem 1.1, (c) $\Rightarrow$ (d) is proposition 1.2, and the remaining implications (d) $\Rightarrow$ (b) and (d) $\Rightarrow$ (c) are consequences of definitions. For (d) $\Rightarrow$ (c) we must observe that a holomorphic map  $X_0 \rightarrow Y$  from a Stein subvariety  $X_0$  of a complex manifold  $X$  to any complex manifold  $Y$  admits a holomorphic extension to an open neighborhood of  $X_0$  in  $X$  (Proposition 1.2 in [17]).

In §2 we prove an extension of a theorem of Siu [33] on the existence of open Stein neighborhoods of Stein subvarieties. This is used to obtain an approximation-interpolation theorem (§3) which is needed in the proof of theorem 1.1 (§4). We hope that both these results will be of independent interest. Proposition 1.2 is proved in §5. In §6 we discuss the analogous equivalences for the parametric Oka properties.

## 2. An extension of a theorem of Siu.

All complex spaces in this paper are assumed to be reduced and paracompact. We denote by  $\mathcal{H}(X)$  the algebra of all holomorphic functions on a complex space  $X$ , endowed with the compact-open topology. A function (or a map) is said to be *holomorphic on a compact set*  $K$  in  $X$  if it is holomorphic in an open set  $U \subset X$  containing  $K$ ; a homotopy  $\{f_t\}$  is holomorphic on  $K$  if there is a neighborhood  $U$  of  $K$  independent of  $t$  such that every  $f_t$  is holomorphic on  $U$ . A compact subset  $K$  in a Stein

space  $X$  is  $\mathcal{H}(X)$ -convex if for any  $p \in X \setminus K$  there exists  $f \in \mathcal{H}(X)$  with  $|f(p)| > \sup_K |f|$ . If  $K$  is contained in a closed complex subvariety  $X_0$  of  $X$  then  $K$  is  $\mathcal{H}(X_0)$ -convex if and only if it is  $\mathcal{H}(X)$ -convex. (For the theory of Stein manifolds and Stein spaces we refer to [22] and [25]).

The following result generalizes a theorem of Siu [33] (1976).

**THEOREM 2.1.** — *Let  $X_0$  be a closed Stein subvariety in a complex space  $X$ . Assume that  $K \subset X$  is a compact set which is  $\mathcal{H}(\Omega)$ -convex in some open Stein set  $\Omega \subset X$  containing  $K$  and such that  $K \cap X_0$  is  $\mathcal{H}(X_0)$ -convex. Then  $K \cup X_0$  has a fundamental basis of open Stein neighborhoods in  $X$ .*

Siu's theorem [33] corresponds to the case  $K = \emptyset$ , the conclusion being that every closed Stein subvariety in a complex space admits an open Stein neighborhood. Different proofs of Siu's theorem were given independently by Coltoiu [4] and Demailly [6] in 1990. If  $X$  is Stein and  $K \subset X$  is  $\mathcal{H}(X)$ -convex then  $X_0 \cup K$  admits a basis of Stein neighborhoods which are Runge in  $X$  (Proposition 2.1 in [5]). It seems that under the weaker condition on  $K$  in theorem 2.1 the result is new even when  $X$  is Stein. The necessity of  $\mathcal{H}(X_0)$ -convexity of  $K \cap X_0$  is seen by taking  $X = \mathbb{C}^2$ ,  $X_0 = \mathbb{C} \times \{0\}$ , and  $K = \{(z, w) \in \mathbb{C}^2 : 1 \leq |z| \leq 2, |w| \leq 1\}$ : In this case every Stein neighborhood of  $K \cup X_0$  contains the bidisc  $\{(z, w) : |z| \leq 2, |w| \leq 1\}$ . (I wish to thank N. Øvrelid for this remark).

*Proof.* — We adapt Demailly's proof of Theorem 1 in [6], referring to that paper (or to [32]) for the notion of a strongly plurisubharmonic function on a complex space with singularities. (The proof of Coltoiu [4] is very similar and covers also the more general case when  $X_0$  is a complete pluripolar set). Although we only need the special case of theorem 2.1 where  $X$  is without singularities, the general case does not require any additional effort.

Let  $U \subset X$  be an open set containing  $M := K \cup X_0$ . We shall find an open Stein set  $V$  in  $X$  with  $M \subset V \subset U$ .

By the assumption  $K$  is  $\mathcal{H}(\Omega)$ -convex in an open Stein set  $\Omega \subset X$ . Hence  $K$  has a basis of open Stein neighborhoods, and replacing  $\Omega$  by one of them we may assume that  $\Omega \subset U$ .

Since  $K_0 := K \cap X_0$  is assumed to be  $\mathcal{H}(X_0)$ -convex, it has a compact  $\mathcal{H}(X_0)$ -convex neighborhood  $K'_0$  in  $X_0$  which is contained in  $\Omega$ . Choose a compact neighborhood  $K'$  of  $K$  such that  $K' \subset \Omega$  and  $K' \cap X_0 = K'_0$ .

Since  $K$  is  $\mathcal{H}(\Omega)$ -convex, there is a smooth strongly plurisubharmonic function  $\rho_0$  on  $\Omega$  such that  $\rho_0 < 0$  on  $K$  and  $\rho_0 > 1$  on  $\Omega \setminus K'$  (Theorem 5.1.5 in [25], p. 117). Set  $U_c = \{x \in \Omega: \rho_0(x) < c\}$ . Fix  $c \in (0, 1/2)$ ; then  $K \subset U_c \subset U_{2c} \subset K'$ .

The restriction  $\rho_0|_{X_0 \cap \Omega}$  is smooth strongly plurisubharmonic. Since the set  $K'_0 = K' \cap X_0$  is assumed to be  $\mathcal{H}(X_0)$ -convex, there is a smooth strongly plurisubharmonic exhaustion function  $\rho'_0: X_0 \rightarrow \mathbb{R}$  which agrees with  $\rho_0$  on  $K'_0$  and satisfies  $\rho'_0 > c$  on  $X_0 \setminus \overline{U}_c$ . (To obtain such  $\rho'_0$ , take a smooth strongly plurisubharmonic exhaustion function  $\tau: X_0 \rightarrow \mathbb{R}$  such that  $\tau < 0$  on  $K'_0$  and  $\tau > 1$  on  $X_0 \setminus \Omega$ ; also choose a smooth convex increasing function  $\xi: \mathbb{R} \rightarrow \mathbb{R}_+$  with  $\xi(t) = 0$  for  $t \leq 0$ , and a smooth function  $\chi: X \rightarrow [0, 1]$  such that  $\chi = 1$  on  $\{x \in X_0: \tau(x) \leq 1/2\}$  and  $\chi = 0$  on  $\{x \in X_0: \tau(x) \geq 1\}$ ; the function  $\rho'_0 = \chi\rho_0 + \xi \circ \tau$  satisfies the stated properties provided that  $\xi(t)$  is chosen to grow sufficiently fast for  $t > 0$ ). Let  $\tilde{\rho}_0: K' \cup X_0 \rightarrow \mathbb{R}$  be defined by the conditions  $\tilde{\rho}_0|_{K'} = \rho_0|_{K'}$  and  $\tilde{\rho}_0|_{X_0} = \rho'_0$ .

Choose a smooth convex increasing function  $h: \mathbb{R} \rightarrow \mathbb{R}$  satisfying  $h(t) \geq t$  for all  $t \in \mathbb{R}$ ,  $h(t) = t$  for  $t \leq c$ , and  $h(t) > t + 1$  for  $t \geq 2c$ . The function  $\rho_1 := h \circ \tilde{\rho}_0$  is smooth strongly plurisubharmonic on  $K' \cup X_0$ ; on the set  $\overline{U}_c = \{\rho_0 \leq c\}$  we have  $\rho_1 = \tilde{\rho}_0 = \rho_0$ , while outside of  $U_{2c}$  we have  $\rho_1 > \tilde{\rho}_0 + 1$ .

By Theorem 4 in [6], applied to  $(\rho_1 - 1)|_{X_0}$ , there exists a smooth strongly plurisubharmonic function  $\rho_2$  in an open neighborhood of  $X_0$ , satisfying

$$\rho_1(x) - 1 < \rho_2(x) < \rho_1(x), \quad x \in X_0.$$

On a small neighborhood of  $\overline{U}_c \cap X_0 = \{x \in X_0: \tilde{\rho}_0(x) \leq c\}$  we have  $\rho_2 < \rho_1 = \tilde{\rho}_0$ , while on  $X_0 \setminus U_{2c}$  we have  $\rho_2 > \rho_1 - 1 > \tilde{\rho}_0$ . It follows that the function

$$\rho = \max\{\tilde{\rho}_0, \rho_2\}$$

is well defined and strongly plurisubharmonic in an open set  $W \subset X$  satisfying  $\overline{U}_c \cup X_0 \subset W \subset U$ . (To see this, observe that the union of the domains of  $\tilde{\rho}_0$  and  $\rho_2$  contains a neighborhood of  $\overline{U}_c \cup X_0$ , and before running out of the domain of one of these two functions, the second function is the larger one and hence takes over). After shrinking  $W$  around the set  $\overline{U}_c \cup X_0$  the function  $\rho$  satisfies the following properties:

- (i)  $\rho = \tilde{\rho}_0 = \rho_0$  on  $\overline{U}_c$  (hence  $\rho < 0$  on  $K$ ),
- (ii)  $\rho > c$  on  $W \setminus \overline{U}_c$ , and

(iii)  $\rho = \rho_2$  on  $W \setminus U_{2c}$ .

Using the smooth version of the maximum operation as in [6] we may also insure that  $\rho$  is smooth. After a further shrinking of  $W$  we may assume that  $\rho$  is a strongly plurisubharmonic exhaustion function on  $\overline{W} \supset \overline{U}_c \cup X_0$  satisfying  $\rho > c$  on  $bW$ . (However,  $\rho$  is not an exhaustion function on  $W$ ). The set  $L = \{x \in W : \rho(x) \leq 0\}$  is then compact and contains  $K$  in its interior.

By Lemma 5 in [6] (see also [31]) there is a smooth function  $v$  on  $X \setminus X_0$  with a logarithmic pole on  $X_0 = \{v = -\infty\}$  whose Levi form  $i\partial\bar{\partial}v$  is bounded on every compact subset of  $X$  (such  $v$  is said to be *almost plurisubharmonic*; this notion is defined on a complex space by considering local ambient extensions as in [32]). By subtracting a constant from  $v$  we may assume that  $v < 0$  on  $K$ .

Let  $g: \mathbb{R} \rightarrow \mathbb{R}$  be a convex increasing function with  $g(t) = t$  for  $t \leq 0$ . For a small  $\epsilon > 0$  (to be specified below) we set

$$\tilde{\rho} = \epsilon v + g \circ \rho, \quad V = \{x \in W : \tilde{\rho}(x) < 0\}.$$

Clearly  $\tilde{\rho}|_{X_0} = -\infty$  and hence  $X_0 \subset V$ . Furthermore, both summands are negative on  $K$  and hence  $K \subset V$ , so  $V$  is an open neighborhood of  $K \cup X_0$  contained in  $W$ .

To complete the proof we show that  $V$  is Stein for a suitable choice of  $\epsilon$  and  $g$ . On  $L = \{\rho \leq 0\}$  we have  $g \circ \rho = \rho$  which is strongly plurisubharmonic; since the Levi form of  $v$  is bounded on  $L$ ,  $\tilde{\rho}$  is strongly plurisubharmonic on  $L$  for a sufficiently small  $\epsilon > 0$ . Fix such  $\epsilon$ . By choosing  $g$  to grow sufficiently fast on  $(0, +\infty)$  we can insure that  $\tilde{\rho}$  is strongly plurisubharmonic on  $\overline{W}$  (since the positive Levi form of  $g \circ \rho$  can be made sufficiently large in order to compensate the bounded negative part of the Levi form of  $\epsilon v$  on each compact in  $\overline{W}$ ). Furthermore, a sufficiently fast growth of  $g$  insures that  $\tilde{\rho}|_{bW} > 0$  and hence  $\overline{V} \subset W$ .

Let  $\tau: (-\infty, 0) \rightarrow \mathbb{R}$  be a smooth convex increasing function such that  $\tau(t) = 0$  for  $t \leq -3$  and  $\tau(t) = -1/t$  for  $t \in (-1, 0)$ . Then  $\psi = \rho + \tau \circ \tilde{\rho}$  is a strongly plurisubharmonic exhaustion function on  $V$ , and hence  $V$  is Stein by Narasimhan's theorem [30].  $\square$

*Remark 2.2.* — The proof of theorem 2.1 applies also if  $X_0$  is a closed Stein subvariety of  $X \setminus L$  for some compact subset  $L \subset \text{Int}K$ , provided that the set  $K \cap X_0$  (which is no longer compact) is  $\mathcal{H}(X_0)$ -convex and  $K$  is  $\mathcal{H}(\Omega)$ -convex in an open Stein neighborhood  $\Omega \subset X$ . For

example, if  $X_0$  is a complex curve in  $X$  (possibly with singularities and with some boundary components in  $K$ ) such that  $X_0 \setminus K$  does not contain any irreducible component with compact  $X_0$ -closure then  $K \cup X_0$  admits a basis of Stein neighborhoods in  $X$ . An example is obtained by attaching to the solid torus

$$K = \{(z, w) \in \mathbb{C}^2 : 1 \leq |z| \leq 2, |w| \leq 1\}$$

finitely many punctured discs  $\Delta_j^* = \{(z, b_j) : |z| < \frac{3}{2}, z \neq a_j\}$  where  $|a_j| < 1$ ,  $|b_j| < 1$  and the numbers  $b_j$  are distinct — the set  $K \cup (\cup_j \Delta_j^*)$  has a basis of Stein neighborhoods (apply theorem 2.1 with  $X = \mathbb{C}^2 \setminus \cup_j \{(a_j, b_j)\}$ ). On the other hand, attaching to  $K$  a non-punctured disc one obtains a Hartogs figure without a basis of Stein neighborhoods.

### 3. An approximation–interpolation theorem.

We shall need the following result whose proof relies on theorem 2.1.

**THEOREM 3.1.** — *Assume that  $X$  is a Stein manifold,  $X_0 \subset X$  a closed complex subvariety,  $K \subset X$  a compact  $\mathcal{H}(X)$ -convex set and  $U \subset X$  an open set containing  $K$ . Let  $Y$  be a complex manifold with a distance function  $d$  induced by a Riemannian metric. If  $f: U \cup X_0 \rightarrow Y$  is a map whose restrictions  $f|_U$  and  $f|_{X_0}$  are holomorphic then for every  $\epsilon > 0$  there exist an open set  $V \subset X$  containing  $K \cup X_0$  and a holomorphic map  $\tilde{f}: V \rightarrow Y$  such that  $\tilde{f}|_{X_0} = f|_{X_0}$  and  $\sup_{x \in K} d(\tilde{f}(x), f(x)) < \epsilon$ . The analogous result holds for sections of a holomorphic submersion  $Z \rightarrow X$ .*

*Remark 3.2.* — Theorem 3.1 holds without any condition whatsoever on the target manifold  $Y$ ; however, the domain of  $\tilde{f}$  will in general depend on  $\epsilon$ . Our proof gives the analogous result for families of maps depending continuously on a parameter in a compact Hausdorff space, except that the domains of the approximating maps  $\tilde{f}$  must be restricted to a fixed (but arbitrary large) compact in  $X$ ; this suffices for the application to the parametric analogue of theorem 1.1.

*Proof.* — In the case  $Y = \mathbb{C}$  the theorem is well known and follows from Cartan's theorems A and B: The function  $f|_{X_0}: X_0 \rightarrow \mathbb{C}$  admits a holomorphic extension  $\phi: X \rightarrow \mathbb{C}$ ; on  $U$  (which we may assume to be Stein and relatively compact in  $X$ ) we can write  $f = \phi + \sum_{j=1}^m g_j h_j$  where the functions  $h_j \in \mathcal{H}(X)$  vanish on  $X_0$  and generate the ideal sheaf of  $X_0$  over

$U$ , and  $g_j \in \mathcal{H}(U)$ ; approximating  $g_j$  uniformly on  $K$  by  $\tilde{g}_j \in \mathcal{H}(X)$  and taking  $\tilde{f} = \phi + \sum \tilde{g}_j h_j$  completes the proof.

To prove the general case we take  $Z = X \times Y$  and let  $\pi: Z \rightarrow X$  be the projection  $(x, z) \rightarrow x$ . Write  $F(x) = (x, f(x))$ . The set  $Z_0 = \{F(x) \in Z: x \in X_0\}$  is a closed complex subvariety of  $Z$  which is biholomorphic to  $X_0$  (via  $F$ ) and hence is Stein. We may assume that the open set  $U \subset X$  in theorem 3.1 is Stein and  $K$  is  $\mathcal{H}(U)$ -convex. The set  $\tilde{U} = \{F(x): x \in U\}$  is a closed Stein submanifold of  $Z|_U = \pi^{-1}(U)$  and hence it has an open Stein neighborhood  $\Omega \subset Z|_U$ .

Since  $K$  is  $\mathcal{H}(U)$ -convex, the set  $\tilde{K} = \{F(x): x \in K\}$  is  $\mathcal{H}(\tilde{U})$ -convex and hence  $\mathcal{H}(\Omega)$ -convex. Furthermore,  $\tilde{K} \cap Z_0 = \{F(x): x \in K \cap X_0\}$  is  $\mathcal{H}(Z_0)$ -convex since  $K \cap X_0$  is  $\mathcal{H}(X_0)$ -convex. Theorem 2.1 now shows that  $\tilde{K} \cup Z_0$  has an open Stein neighborhood  $W \subset Z$ . Choose a proper holomorphic embedding  $\phi: W \rightarrow \mathbb{C}^r$ . By the special case considered above there is a holomorphic map  $G: X \rightarrow \mathbb{C}^r$  such that  $G|_{X_0} = \phi \circ F|_{X_0}$  and  $G$  approximates  $\phi \circ F$  on a compact neighborhood of  $K$  in  $X$ . Let  $\iota$  denote a holomorphic retraction of a neighborhood of  $\phi(W)$  in  $\mathbb{C}^r$  onto  $\phi(W)$  (Docquier and Grauert [7]). The map  $\tilde{F} = \phi^{-1} \circ \iota \circ G$  (with values in  $Z$ ) is then holomorphic in an open neighborhood  $V \subset X$  of  $K \cup X_0$ , it approximates  $F$  uniformly on  $K$  and satisfies  $\tilde{F}|_{X_0} = F|_{X_0}$ . Writing  $\tilde{F}(x) = (a(x), \tilde{f}(x)) \in X \times Y$  we see that the second component  $\tilde{f}$  satisfies the conclusion of theorem 2.1.

Our proof holds for sections of any holomorphic submersion  $\pi: Z \rightarrow X$ ; when  $Z$  does not have a product structure, the point  $\tilde{F}(x) \in Z$  must be projected back to the fiber  $Z_x = \pi^{-1}(x)$  in order to obtain a bona fide section of  $\pi: Z \rightarrow X$ . This is accomplished by applying to  $\tilde{F}(x)$  a holomorphic retraction onto  $Z_x$  which depends holomorphically on the base point  $x \in V$  (Lemma 3.4 in [12]).  $\square$

#### 4. Proof of theorem 1.1.

The proof which we present requires minimal improvements of the existing tools, thanks to the new theorem 3.1 which provides a local holomorphic extension of a map constructed in an inductive step. Another possibility would be to adapt the proof of Theorem 1.4 in [17] (via Lemma 3.3 and Proposition 4.2 in [17]) to our current situation, thus proving directly that CAP implies the Oka property with jet interpolation

and approximation (the implication (a) $\Rightarrow$ (d) in corollary 1.3). However, since (c) $\Rightarrow$ (d) in corollary 1.3 is quite easy, the latter approach (which seems to require more substantial modifications) appears less attractive.

The main step is provided by the following.

**PROPOSITION 4.1.** — *Let  $K, X_0 \subset X, Y$  and  $d$  be as in theorem 3.1. Assume that  $f_0 : X \rightarrow Y$  is continuous map whose restrictions  $f_0|_K$  and  $f_0|_{X_0}$  are holomorphic. Given a compact  $\mathcal{H}(X)$ -convex set  $L \subset X$  containing  $K$  and a number  $\epsilon > 0$ , there is a homotopy of continuous maps  $f_s : X \rightarrow Y$  satisfying the following for every  $s \in [0, 1]$ :*

- (i)  $f_s|_{X_0} = f_0|_{X_0}$ ,
- (ii)  $f_s$  is holomorphic on  $K$  and satisfies  $\sup_{x \in K} d(f_s(x), f_0(x)) < \epsilon$ ,  
and
- (iii)  $f_1$  is holomorphic on  $L$ .

Since  $X$  is exhausted by a sequence of compact  $\mathcal{H}(X)$ -convex subsets  $K = K_0 \subset K_1 \subset K_2 \subset \dots$ , theorem 1.1 follows from proposition 4.1 by an obvious induction.

The remainder of this section is devoted to the proof of proposition 4.1. We may assume that  $L = \{x \in X : \rho(x) \leq 0\}$  where  $\rho : X \rightarrow \mathbb{R}$  is a smooth strongly plurisubharmonic exhaustion function on  $X$ ,  $\rho|_K < 0$ , and 0 is a regular value of  $\rho$ . By theorem 3.1 we may assume that the map  $f_0$  in proposition 4.1 is holomorphic in an open set  $U \subset X$  containing  $K \cup X_0$ .

A homotopy satisfying proposition 4.1 is obtained by the *bumping method* introduced by Grauert in the 1960's to solve  $\bar{\partial}$ -problems [23]. To our knowledge this method was first used in the Oka-Grauert theory by Henkin and Leiterer in an unpublished preprint (1986) on which the paper [24] is based. (In these papers the target manifold  $Y$  was assumed to be complex homogeneous). Subsequently it has been used in most recent works on the subject. The interpolation requirement presents additional difficulties. It seems that the first such result (besides those of Grauert) is Theorem 1.7 in [16]; its proof depends on a geometric construction (Lemma 8.4 in [16]) which cannot be used directly in our situation since it does not insure convexity of the bumps.

The general outline of the bumping method suitable to our current needs can be found in §6 of [10] and in §4 of [12]. The geometric situation considered here is precisely the same as in §6.5 of [10] where it was used in the construction of holomorphic submersions  $X \rightarrow \mathbb{C}^q$  with interpolation

on a subvariety  $X_0 \subset X$ . To avoid repetition as much as possible we shall outline the proof (with necessary modifications) and refer to §6 of [10] and to [13] for further details where appropriate.

We recall the geometric setup in [10] (p. 179); our current notation is harmonized with that in [10]. The compact set  $K' := (K \cup X_0) \cap \{\rho \leq 1\} \subset U$  is  $\mathcal{H}(X)$ -convex; hence there exists a smooth strongly plurisubharmonic exhaustion function  $\tau: X \rightarrow \mathbb{R}$  such that  $\tau < 0$  on  $K'$  and  $\tau > 0$  on  $X \setminus U$ . We may assume that 0 is a regular value of  $\tau$  and the hypersurfaces  $\{\rho = 0\} = bL$  and  $\{\tau = 0\}$  intersect transversely along the real codimension two submanifold  $S = \{\rho = 0\} \cap \{\tau = 0\}$ . Hence  $D_0 := \{\tau \leq 0\} \subset U$  is a smoothly bounded strongly pseudoconvex domain. Set  $\rho_t = \tau + t(\rho - \tau) = (1-t)\tau + t\rho$  and

$$D_t = \{\rho_t \leq 0\} = \{\tau \leq t(\tau - \rho)\}, \quad t \in [0, 1].$$

We have  $D_0 = \{\tau \leq 0\}$ ,  $D_1 = \{\rho \leq 0\} = L$  and  $D_0 \cap D_1 \subset D_t$  for all  $t \in [0, 1]$ . Let

$$\Omega = \{\rho < 0, \tau > 0\} \subset D_1 \setminus D_0 \quad \text{and} \quad \Omega' = \{\rho > 0, \tau < 0\} \subset D_0 \setminus D_1.$$

As  $t$  increases from 0 to 1,  $D_t \cap L$  increases to  $D_1 = L$  while  $D_t \setminus L \subset D_0$  decrease to  $\emptyset$ . All hypersurfaces  $\{\rho_t = 0\} = bD_t$  intersect along  $S = \{\rho = 0\} \cap \{\tau = 0\}$ . Since  $d\rho_t = (1-t)d\tau + td\rho$  and the differentials  $d\tau, d\rho$  are linearly independent along  $S$ ,  $bD_t$  is smooth near  $S$ . Finally,  $\rho_t$  is strongly plurisubharmonic and hence  $D_t$  is strongly pseudoconvex at every smooth boundary point for every  $t \in [0, 1]$ .

We find a homotopy satisfying proposition 4.1 by a sequence of finitely many localized steps. Fix two nearby values of the parameter, say  $0 \leq t_0 < t_1 \leq 1$ . Consider first the case that all boundaries  $bD_t$  for  $t \in [t_0, t_1]$  are smooth. Then  $D_{t_1}$  is obtained from  $D_{t_0}$  by attaching finitely many small convex bumps of the type described in [10] (disjoint from  $X_0$ ) and intersecting the union with the set  $D_{t_1}$ . Here ‘small convex’ refers to a suitable local holomorphic coordinate system on  $X$ . At every step we have a map  $X \rightarrow Y$  which is holomorphic on a certain compact set  $A$  obtained from  $D_{t_0}$  by the previous attachments of bumps; the goal of the step is to obtain a new map which is holomorphic on  $A \cup B$  (where  $B$  is the next convex bump), approximates the given map uniformly on  $A$  and agrees with it on  $X_0$ . The pair  $(A, B)$  is chosen such that  $C = A \cap B$  is convex in some local holomorphic coordinates in a neighborhood of  $B$  in  $X$ ,  $A \cup B$  admits a Stein neighborhood basis in  $X$ ,  $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$ , and  $B \cap X_0 = \emptyset$ . (To prove an effective version of theorem 1.1 in which the CAP axiom is used only for maps from Euclidean spaces of dimension  $\leq \dim X + \dim Y$

to  $Y$ , we must assume in addition that  $C$  is Runge in  $A$ ). The solution of this local problem is obtained in three steps as in §3 of [13].

*Step 1:* We denote by  $f_0: X \rightarrow Y$  a map which is holomorphic on the set  $A$  obtained in the earlier steps of the process. Write  $F_0(x) = (x, f_0(x)) \in X \times Y$ . We find a small ball  $0 \in D \subset \mathbb{C}^p$  ( $p = \dim Y$ ) and a holomorphic map  $f: A \times D \rightarrow Y$  such that  $f(\cdot, 0) = f_0$ ,  $f(x, t) = f_0(x)$  for all  $x \in X_0 \cap A$  and  $t \in D$ , and the partial differential  $\partial_t f(x, t): T_t \mathbb{C}^p \rightarrow T_{f(x,t)} Y$  in the  $t$  variable is surjective for  $x$  in a neighborhood of  $C = A \cap B$  and  $t \in D$  (Lemma 3.2 in [13]). Such  $f$  is found by choosing holomorphic vector fields  $\xi_1, \dots, \xi_p$  in a Stein neighborhood of  $F_0(A)$  in  $Z = X \times Y$  which are tangent the fibers  $Y$ , they vanish on the intersection of their domains with  $X_0 \times Y$  and span the vertical tangent bundle over a neighborhood of  $F_0(C)$ . The flow  $\theta_t^j$  of  $\xi_j$  exists for sufficiently small  $t \in \mathbb{C}$ , and the map

$$F(x, t_1, \dots, t_p) = (x, f(x, t)) = \theta_{t_1}^1 \circ \dots \circ \theta_{t_p}^p \circ F_0(x) \in X \times Y$$

satisfies the stated requirements.

*Step 2:* We approximate the holomorphic map  $f$  from Step 1 uniformly on a neighborhood of the compact convex set  $C \times D$  by a map  $g$  which is holomorphic in an open neighborhood of  $B \times D$ . This is possible since  $C$  is a compact convex set in  $\mathbb{C}^n$  ( $n = \dim X$ ) with respect to some local holomorphic coordinates in a neighborhood of  $B$  in  $X$ ; hence  $C \times D$  may be considered as a compact convex set in  $\mathbb{C}^{n+p}$  and  $g$  exists by the CAP property of  $Y$ . (In [13], Definition 1.1, we used a more technical version of CAP which required approximability only on special compact convex sets with the purpose of making it easier to verify. However, by the main result of [13] this more restrictive definition implies the Oka property with approximation, hence in particular the approximability of holomorphic maps  $K \rightarrow Y$  on any compact convex set  $K \subset \mathbb{C}^m$  by entire maps  $\mathbb{C}^m \rightarrow Y$ ).

*Step 3:* We ‘glue’  $f$  and  $g$  into a holomorphic map  $f': (A \cup B) \times D \rightarrow Y$  which approximates  $f$  uniformly on  $A \times D$  and agrees with it on  $(X_0 \cap A) \times D$ . (A small shrinking of the domain is necessary). This is done by finding a fiberwise biholomorphic transition map  $\gamma(x, t)$  which is close to  $(x, t) \rightarrow t$  and satisfies  $f(x, t) = g(x, \gamma(x, t))$  on  $C \times D$ , splitting it in the form  $\gamma_x = \gamma(x, \cdot) = \beta_x \circ \alpha_x^{-1}$  ( $x \in C$ ) where  $\alpha$  is holomorphic on  $A \times D$  and  $\beta$  is holomorphic on  $B \times D$  (Lemma 2.1 in [13]), and taking  $f'(x, t) = f(x, \alpha(x, t)) = g(x, \beta(x, t))$ . The map  $f'_0 = f'(\cdot, 0): A \cup B \rightarrow Y$  is holomorphic on  $A \cup B$ , it approximates  $f_0$  uniformly on  $A$  and agrees with  $f_0$  on  $X_0$  (the last observation follows from the construction of  $f$ ).

The construction also gives a homotopy with the required properties from  $f_0$  to  $f'_0$ . (We can also obtain  $f'_0$  by applying Proposition 5.2 in [15] which is available in the parametric case; this can be used in the proof of the parametric analogue of theorem 1.1).

This explains the *noncritical* case in which we are not passing any nonsmooth boundary points of  $bD_t$ . The number of steps needed to reach  $D_{t_1}$  from  $D_{t_0}$  does not depend on the partial solutions obtained in the intermediate steps.

It remains to consider the critical values  $t \in [0, 1]$  for which  $bD_t$  has a nonsmooth point in  $\Omega$ . The defining equation of  $D_t \cap \Omega$  can be written as

$$D_t \cap \Omega = \left\{ x \in \Omega : h(x) \stackrel{\text{def}}{=} \frac{\tau(x)}{\tau(x) - \rho(x)} \leq t \right\},$$

and the equation for critical points  $dh = 0$  is equivalent to

$$(\tau - \rho)d\tau - \tau(d\tau - d\rho) = \tau d\rho - \rho d\tau = 0.$$

A generic choice of  $\rho$  and  $\tau$  insures that there are at most finitely many solutions in  $\Omega$  and no solution on  $b\Omega$ ; furthermore, these solutions lie on distinct level sets of  $h$ . At each critical point of  $h$  the complex Hessians satisfy  $(\tau - \rho)^2 H_h = \tau H_\rho - \rho H_\tau$ . Since  $\tau > 0$  and  $-\rho > 0$  on  $\Omega$ , we conclude that  $H_h > 0$  at such points, i.e.,  $h$  is strongly plurisubharmonic near its critical point in  $\Omega$ . (See [10] for more details).

A method for passing the isolated critical points of  $h$  was explained in §6.2–§6.4 of [10] for submersions  $X \rightarrow \mathbb{C}^q$ . In §6 of [12] this method was adapted to holomorphic maps to any complex manifold  $Y$ ; we include a brief description. Assume that  $f: X \rightarrow Y$  is holomorphic on a certain sublevel set  $A = D_{t_0}$  of  $h$  just below a critical level  $t_1$ . We attach to  $A$  a totally real handle  $E \subset \Omega \setminus X_0$  containing the critical (nonsmooth) point  $p \in bD_{t_1}$  which describes the change of topology of  $D_t$  as  $t$  passes  $t_1$ . By theorem 3.2 in [12] the map  $f|_{A \cup E}$  can be uniformly approximated by a map which is holomorphic in an open neighborhood of  $A \cup E$  and agrees with  $f$  on  $X_0$ . Finally, by the method explained in §6.4 of [10] we can proceed by the noncritical case (applied with a different strongly plurisubharmonic function constructed especially for this purpose) to a sublevel set  $D_{t_2}$  above the critical level ( $t_2 > t_1$ ). Now we can proceed to the next critical value of  $h$ .

This completes the proof of theorem 1.1 for maps  $X \rightarrow Y$ . The same proof applies to sections of holomorphic fiber bundles with fiber  $Y$  over a Stein manifold  $X$  because the use of CAP can be localized to small subsets in  $X$  over which the bundle is trivial (compare with [13]).

### 5. Proof of proposition 1.2.

Let  $Y$  be a complex manifold satisfying the conclusion of theorem 1.1 for closed complex submanifolds in Stein manifolds. Let  $X$  be a Stein manifold,  $K \subset X$  a compact  $\mathcal{H}(X)$ -convex subset,  $X_0$  a closed complex subvariety of  $X$  and  $f: X \rightarrow Y$  a continuous map which is holomorphic in an open set  $U_0 \subset X$  containing  $K \cup X_0$ .

The set  $K \cup X_0$  has a basis of open Stein neighborhoods in  $X$  and hence we may assume (by shrinking  $U_0$  if necessary) that  $U_0$  is Stein. Choose a proper holomorphic embedding  $\phi: U_0 \rightarrow \mathbb{C}^r$ ; its graph  $\Sigma = \{(x, \phi(x)): x \in U_0\}$  is a closed complex submanifold of the Stein manifold  $M = X \times \mathbb{C}^r$ . The map  $g_0: M \rightarrow Y$ ,  $g_0(x, z) = f_0(x)$  ( $x \in X$ ,  $z \in \mathbb{C}^r$ ), is continuous on  $M$  and holomorphic on the open set  $U_0 \times \mathbb{C}^r \subset M$  containing  $\Sigma$ .

By the assumption on  $Y$  there is a homotopy  $g_t: M \rightarrow Y$  ( $t \in [0, 1]$ ) which is fixed on the submanifold  $\Sigma$  and such that  $g_1$  is holomorphic on  $M$ . For later purposes we also need that  $g_t$  be holomorphic in an open neighborhood  $V \subset X \times Y$  (independent of  $t \in [0, 1]$ ) of  $\Sigma$ . Since both  $g_0$  and  $g_1$  are holomorphic near  $\Sigma$ , this can be accomplished (without changing  $g_0$  and  $g_1$ ) by a simple modification of the homotopy  $g_t$ , using a strong holomorphic deformation retraction of a neighborhood of  $\Sigma$  in  $M$  onto  $\Sigma$  (such exists by combining Siu's theorem [33], [6] with the Docquier-Grauert theorem [7]; see Corollary 1 in [33]).

To complete the proof we shall need the following (known) result which follows from Cartan's theorems A and B (see lemma 8.1 in [16] for a reduction to the Oka-Weil approximation theorem).

LEMMA 5.1. — *Let  $X$  be a Stein manifold,  $L \subset X$  a compact  $\mathcal{H}(X)$ -convex subset and  $X_0 \subset X$  a closed complex subvariety. Let  $\phi$  be a holomorphic function in an open set containing  $L \cup X_0$ . For every  $s \in \mathbb{N}$  and  $\eta > 0$  there exists  $\varphi \in \mathcal{H}(X)$  such that  $\varphi - \phi$  vanished to order  $s$  on  $X_0$  and  $\sup_{x \in L} |\varphi(x) - \phi(x)| < \eta$ .*

We apply this lemma to the embedding  $\phi: U \rightarrow \mathbb{C}^r$  and a compact set  $L \subset U_0$  with  $K \subset \text{Int } L$ . Denote the resulting map by  $\varphi: X \rightarrow \mathbb{C}^r$ . Consider the homotopy  $f_t: X \rightarrow Y$  defined by  $f_t(x) = g_t(x, \varphi(x))$  for  $x \in X$  and  $t \in [0, 1]$ . At  $t = 0$  we get the initial map  $f_0(x) = g_0(x, \varphi(x))$ , and at  $t = 1$  we get a map  $f_1(x) = g_1(x, \varphi(x))$  which is holomorphic on  $X$ . There is a small open neighborhood  $U \subset U_0$  of  $K \cup X_0$  such that  $(x, \varphi(x)) \in V$  for  $x \in U$ ; since  $g_t$  is holomorphic on  $V$ , we see that  $f_t$  is holomorphic on  $U$ . It

is easily verified that for every  $t \in [0, 1]$ ,  $f_t$  agrees with  $f_0$  to order  $s$  along  $X_0$ , and it approximates  $f_0$  uniformly on  $K$  as well as desired provided that the number  $\eta$  in lemma 5.1 is chosen sufficiently small. Thus  $f_t$  satisfies the conclusion of proposition 1.2.

All steps go through in the parametric case and give the corresponding parametric analogue of proposition 1.2.

*Remark 5.2.* — The analogue of proposition 1.2 holds for sections of fiber bundles with fiber  $Y$  over Stein manifolds: interpolation implies jet interpolation and approximation. To see this, assume that  $\pi: E \rightarrow X$  is such a bundle,  $X_0$  is a closed complex subvariety of  $X$  and  $K$  is a compact  $\mathcal{H}(X)$ -convex subset of  $X$ . Let  $f_0: X \rightarrow E$  be a continuous section which is holomorphic in an open Stein neighborhood  $U \subset X$  of  $K \cup X_0$ . As before we embed  $U$  to  $\mathbb{C}^r$  and denote the graph of the embedding by  $\Sigma \subset M = X \times \mathbb{C}^r$ . Let  $p^*E \rightarrow M$  be the pull-back of  $E$  by the projection  $p: M \rightarrow X$ ; this bundle has the same fiber  $Y$  and there is a natural map  $\theta: p^*E \rightarrow E$  covering  $p$  which is isomorphism on the fibers. The section  $f_0$  pulls back to a section  $h_0: M \rightarrow p^*E$  satisfying  $\theta(h_0(x, y)) = f_0(x)$ . Note that  $h_0$  is holomorphic on  $U' = p^{-1}(U) = U \times \mathbb{C}^r$ . By the hypothesis there is a homotopy  $h_t: M \rightarrow p^*E$  which is fixed on  $\Sigma$  such that  $h_1$  is holomorphic. Choosing  $\varphi: X \rightarrow \mathbb{C}^r$  as before we obtain a homotopy  $f_t(x) = \theta(h_t(x, \varphi(x))) \in E_x$  ( $x \in X$ ) satisfying the desired conclusion.

*Remark 5.3.* — Lárusson proved that for a Stein manifold, the basic Oka property with interpolation implies ellipticity, i.e., existence of a dominating spray (Theorem 2 in [29]). This also follows from proposition 1.2 and the known result that ellipticity of a Stein manifold is implied by second order jet interpolation ([21]; proposition 1.2 in [17]).

## 6. Equivalences between the parametric Oka properties.

In this section we discuss the parametric analogue of corollary 1.3.

Let  $P$  be a compact Hausdorff space (the parameter space) and  $P_0 \subset P$  a closed subset of  $P$  which is a strong deformation retract of some open neighborhood of  $P_0$  in  $P$ . In most applications  $P$  is a polyhedron and  $P_0$  a subpolyhedron; an important special case is  $P_0 = \{0, 1\} \subset [0, 1] = P$ .

Given a Stein manifold  $X$  and a compact  $\mathcal{H}(X)$ -convex subset  $K$  in  $X$  we consider continuous maps  $f: X \times P \rightarrow Y$  such that for every  $p \in P$  the map  $f^p = f(\cdot, p): X \rightarrow Y$  is holomorphic in an open neighborhood of  $K$  in  $X$  (independent of  $p \in P$ ), and for every  $p \in P_0$  the map  $f^p$  is holomorphic on  $X$ . We say that  $Y$  satisfies the *parametric Oka property with approximation* if for every such data  $(X, K, P, P_0, f)$  there is a homotopy  $f_t: X \times P \rightarrow Y$  ( $t \in [0, 1]$ ) consisting of maps satisfying the same properties as  $f_0 = f$  such that

- (i) the homotopy is fixed on  $P_0$ :  $f_t^p = f_0^p$  when  $p \in P_0$  and  $t \in [0, 1]$ ,
- (ii)  $f_t$  approximates  $f_0$  uniformly on  $K \times P$  for all  $t \in [0, 1]$ , and
- (iii)  $f_1^p: X \rightarrow Y$  is holomorphic for every  $p \in P$ .

We say that  $Y$  satisfies the *parametric convex approximation property* (PCAP) if the above holds for any compact convex set  $K \subset X = \mathbb{C}^n$ ,  $n \in \mathbb{N}$ .

Similarly,  $Y$  satisfies the *parametric Oka property with interpolation* if for any closed complex subvariety  $X_0$  in a Stein manifold  $X$  and continuous map  $f: X \times P \rightarrow Y$  such that  $f^p|_{X_0}$  is holomorphic for all  $p \in P$  and  $f^p$  is holomorphic on  $X$  for  $p \in P_0$  there is a homotopy  $f_t: X \times P \rightarrow Y$  ( $t \in [0, 1]$ ), with  $f_0 = f$ , which satisfies (i) and (iii) above together with the interpolation condition

- (ii')  $f_t^p(x) = f_0^p(x)$  for  $x \in X_0$ ,  $p \in P$  and  $t \in [0, 1]$ .

Similarly one introduces the *parametric Oka property with jet interpolation* on subvarieties  $X_0 \subset X$ . Combining it with approximation on  $\mathcal{H}(X)$ -convex sets one obtains *the parametric Oka property with jet interpolation and approximation* which coincides with Gromov's  $\text{Ell}_\infty$  property ([21], §3.1). By Theorem 1.5 in [16] and Theorem 1.4 in [17] all these properties are implied by ellipticity of  $Y$  (the existence of a dominating spray), and also by subellipticity of  $Y$  [9]. In analogy to the non-parametric case (corollary 1.3) these ostensibly different properties are equivalent.

**THEOREM 6.1.** — *The following properties of a complex manifold are equivalent:*

- (a) *the parametric convex approximation property (PCAP);*
- (b) *the parametric Oka property with approximation;*
- (c) *the parametric Oka property with interpolation;*
- (d) *the parametric Oka property with jet interpolation and approximation, i.e., Gromov's  $\text{Ell}_\infty$  property ([21], §3.1).*

*Proof.* — The logic of proof is the same as in corollary 1.3: (a) $\Leftrightarrow$ (b) is Theorem 5.1 in [13], (a) $\Rightarrow$ (c) is the parametric version of theorem 1.1 in this paper, and (c) $\Rightarrow$ (d) is the parametric version of proposition 1.2. The remaining implications are consequences of definitions and of Proposition 1.2 in [17] concerning the holomorphic extendability of holomorphic maps from a subvariety  $X_0 \subset X$  to an open neighborhood of  $X_0$  in  $X$ . The implication (c) $\Rightarrow$ (b) was first proved by Lárusson [29].

To obtain these implications in the parametric case it suffices to observe that theorem 1.1 and proposition 1.2 remain valid in this more general situation as we have already indicated at the appropriate places in their proofs. The reader who may be interested in further details should consult the proof of the parametric Oka property with approximation for elliptic manifolds in [15] (Theorems 1.4 and 5.5).

The following observation might be helpful concerning the use of theorem 2.1 (the existence of Stein neighborhoods) in the proof of (a) $\Rightarrow$ (c) in theorem 6.1. In the proof of proposition 4.1 (§4) we needed a Stein neighborhood  $U \subset X \times Y$  of the graph of a holomorphic map  $K \cup X_0 \rightarrow Y$ . It may appear that one needs a stronger theorem in the parametric case (a family of Stein neighborhoods depending continuously on the parameter  $p \in P$ ). This is unnecessary because the graphs of maps  $f^p$  over the set  $K' = K \cup (L \cap X_0)$  in the proof of proposition 4.1 belong to the same Stein set  $U$  for  $p \in P$  close to some initial point  $p_0$  for which  $U$  was chosen. Finitely many such Stein open sets in  $X \times Y$  cover the graph of  $f: X \times P \rightarrow Y$  over the compact set  $K' \times P$ ; the solutions in each of these Stein neighborhoods (for an open set of parameters in  $P$ ) are patched together by a continuous partition of unity in the parameter space (see the proof of Theorem 5.5 in [15]).  $\square$

A continuous map satisfying the homotopy lifting property is called a *Serre fibration* [34], p. 8. A holomorphic submersion  $\pi: Y \rightarrow Y_0$  is said to be *subelliptic* if every point  $y_0 \in Y_0$  has an open neighborhood  $U \subset Y_0$  such that  $\pi: \pi^{-1}(U) \rightarrow U$  admits a finite fiber-dominating family of sprays ([9], p. 529); for a holomorphic fiber bundle this is equivalent to subellipticity of the fiber. Theorem 6.1 together with the parametric version of Theorem 1.3 in [13] (see the discussion following Theorem 5.1 in [13]) we obtain the following.

**COROLLARY 6.2.** — *Suppose that  $\pi: Y \rightarrow Y_0$  is a subelliptic submersion which is also a Serre fibration. If  $Y_0$  satisfies any (and hence all) of*

the properties in theorem 6.1 then so does  $Y$ . Conversely, if  $\pi$  is surjective and  $Y$  satisfies any of these properties with a contractible parameter space  $P$  then so does  $Y_0$ .

*A concluding remark.* Our results suggest that CAP (and its parametric analogue) is the most natural Oka-type property to be studied further since it is the simplest to verify, yet equivalent to all other Oka properties. Indeed CAP is just the localization of the *Oka property with approximation* to maps from compact convex sets in Euclidean spaces (see [14] for this point of view). CAP easily follows from ellipticity or subellipticity, and is a natural opposite property to Kobayashi-Eisenman-Brody hyperbolicity [1], [8], [26]. It is also related to dominability by Euclidean spaces, a property that has been extensively studied [2], [3], [27], [28]. It would be interesting to know how big (if any) is the gap between CAP and (sub)ellipticity. Several related questions are mentioned in [11] and [14].

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