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A contractible Levi-flat hypersurface which is a determining set for pluriharmonic functions

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Abstract. We find a real analytic Levi-flat hypersurface in $\mathbf{C}^2$ containing a bounded contractible domain which is a determining set for pluriharmonic functions.

1. The main result

A real hypersurface M in an n-dimensional complex manifold is *Levi-flat* if it is foliated by complex manifolds of dimension $n-1$; this *Levi foliation* is as smooth as M itself according to Barrett and Fornæss [2]. If M is real analytic, it is locally near every point defined by a *pluriharmonic function* v satisfying $dd^c v = 2i\partial\bar{\partial}v = 0$. One might expect that an oriented real analytic Levi-flat hypersurface admits a pluriharmonic defining function on any topologically simple relatively compact domain, perhaps under an additional analytic assumption such as the existence of a fundamental system of Stein neighborhoods (see e.g. Theorem 2 in [10], p. 298). Here we show that, on the contrary, even a most simple domain in a real analytic Levi-flat hypersurface may be a *determining set for pluriharmonic functions.*

Theorem 1.1. *There exist an ellipsoid $B\subset\mathbf{C}^2$ and a real analytic Levi-flat hypersurface $M\subset\mathbf{C}^2$ intersecting the boundary bB transversely such that the Levi foliation of M has trivial holonomy and $A=M\cap B$ satisfies the following conditions*:

(i) *$\bar{A}$ is diffeomorphic to the three-ball and admits a Stein neighborhood basis.*

(ii) *Any real analytic function on A which is constant on Levi leaves is constant.*

(iii) *Any pluriharmonic function in a connected open neighborhood of A in $\mathbf{C}^2$ which vanishes on A is identically zero.*

The Levi foliation of M in our proof is a *simple foliation* ([6], p. 79) whose leaves are complex discs. Likely one can also obtain a similar example in the ball

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of $\mathbf{C}^2$. On the other hand, for any compact subset A in a real analytic simply connected Levi-flat hypersurface M there is a *smooth* defining function v for M such that $dd^c v$ is *flat* on A; this suffices for the construction of Stein neighborhood basis of certain compact subsets of M [4].

We mention that D. Barrett gave an example of a *compact* real analytic Levi-flat hypersurface with trivial holonomy and without a global pluriharmonic defining function (Theorem 3 in [1]). His example is the quotient of $S^1\times\mathbf{C}^*$ by $(\theta,z)\mapsto(\phi(\theta),2z)$, where ϕ is a real analytic diffeomorphism of the circle S^1 which is topologically but not diffeomorphically conjugate to a rotation.

2. A real analytic foliation of $\mathbf{R}^2$ without analytic first integrals

Our construction of the hypersurface M in Theorem 1.1 is based on the following result.

Proposition 2.1. *Let D be the open unit disc in $\mathbf{R}^2$. There exists a real analytic foliation $\mathcal{F}$ of $\mathbf{R}^2$ by closed lines such that any real analytic function on D which is constant on every leaf of the restricted foliation $\mathcal{F}|_D$ is constant.*

Remark 2.2. While we cannot exclude the possibility that an example of this kind is contained in the vast literature on the subject, we could not find a precise reference in some of the standard sources concerning foliations of the plane ([3], [5], [6], [7] and [8]). It is known that every smooth foliation of $\mathbf{R}^2$ by lines has a global continuous first integral but in general not one of class $\mathcal{C}^1$, not even in the analytic case (Wazewsky [11]); however, there exists a smooth first integral without critical points on any relatively compact subset (Kamke [9]).

Proof. Let (x,y) be coordinates on $\mathbf{R}^2$. Define subsets $E_1,E_2\subset\mathbf{R}^2$ by

$$E_1=\{(x,y)\in\mathbf{R}^2:x<-1 \text{ or } y>0\} \quad\text{and}\quad E_2=\{(x,y)\in\mathbf{R}^2:x>1 \text{ or } y>0\}.$$

Let $\mathcal{F}_j$ denote the restriction of the foliation $\{(x,y):y=c\}_{c\in\mathbf{R}}$ to E_j, $j=1,2$. Let ψ be a real analytic orientation preserving diffeomorphism of the half-line $(0,+\infty)$, so $\lim_{t\downarrow 0}\psi(t)=0$. (We do not require that ψ extends analytically to a neighborhood of 0.) Then $\phi(x,y)=(x,\psi(y))$ is a real analytic diffeomorphism of the upper half-plane $E_{1,2}=E_1\cap E_2=\{(x,y)\in\mathbf{R}^2:y>0\}$ onto itself which maps every leaf of $\mathcal{F}_1|_{E_{1,2}}$ to a leaf of $\mathcal{F}_2|_{E_{1,2}}$. Let E be the quotient of the topological (disjoint) sum $E_1\sqcup E_2$ obtained by identifying a point $(x,y)\in E_1$ belonging to $E_{1,2}$ with the point $\phi(x,y)\in E_2$. The foliations $\mathcal{F}_j$, $j=1,2$ amalgamate into a real analytic foliation $\mathcal{F}$ on E.

By construction E is a real analytic manifold homeomorphic to $\mathbf{R}^2$, and hence there exists a real analytic diffeomorphism of E onto $\mathbf{R}^2$. (This follows in particular from the classification theorem for simply connected Riemann surfaces.) We identify E with $\mathbf{R}^2$ and denote the resulting real analytic foliation of $\mathbf{R}^2$ by $\mathcal{F}=\mathcal{F}_\psi$. Let $\pi\colon \mathbf{R}^2\to Q=\mathbf{R}^2/\mathcal{F}$ denote the projection onto the space of leaves. Q admits the structure of a non-Hausdorff real analytic manifold such that π is a real analytic submersion. (The real analytic structure on Q is obtained by declaring the restriction of π to any local analytic transversal l to $\mathcal{F}$ to be a diffeomorphism of l onto the open set $\pi(l)\subset Q$. For the details see [7] and [8].) In our case Q is the quotient of the topological sum $\mathbf{R}_1\sqcup\mathbf{R}_2$ of two copies of the real axis obtained by identifying a point $t>0$ in $\mathbf{R}_1$ with the point $\psi(t)\in\mathbf{R}_2$ (no identifications are made for points $t\le 0$). The only pair of branch points in Q (i.e., points without a pair of disjoint neighborhoods) are those corresponding to $0\in\mathbf{R}_1$ and $0\in\mathbf{R}_2$.

Lemma 2.3. *If ψ is flat at origin (i.e. $\lim_{t\downarrow 0}\psi^{(k)}(t)=0$ for $k\in\mathbf{N}$) then every real analytic function on $\mathbf{R}^2$ which is constant on every leaf of $\mathcal{F}_\psi$ is constant.*

Proof. A real analytic function f on $\mathbf{R}^2$ which is constant on the leaves of the foliation $\mathcal{F}_\psi$ is of the form $f=h\circ\pi$ for some real analytic function $h\colon Q\to\mathbf{R}$, where Q is the space of leaves. From our construction of the foliation it follows that h is given by a pair of real analytic functions $h_j\colon \mathbf{R}\to\mathbf{R}$, $j=1,2$, satisfying $h_1(t)=h_2(\psi(t))$ for $t>0$. As $t\downarrow 0$, the flatness of ψ at 0 implies that the derivative h_1' is flat at 0. Hence h_1, and therefore also h_2, are constant. $\square$

Fix ψ and consider the following pair of subsets of E_1 resp. E_2:

$$\begin{aligned} D_1 &= \{(x,y)\in\mathbf{R}^2 : -3<x<-2 \text{ and } -1<y<2\},\\ D_2 &= \{(x,y)\in\mathbf{R}^2 : 2<x<3 \text{ and } -1<y<\psi(2)\}\\ &\quad \cup\{(x,y)\in\mathbf{R}^2 : -3<x<3 \text{ and } \psi(1)<y<\psi(2)\}. \end{aligned}$$

Let D be the quotient of the disjoint sum $D_1\sqcup D_2$ obtained by identifying any point $(x,y)\in D_1$ such that $1<y<2$ with the point $\phi(x,y)=(x,\psi(y))\in D_2$. Clearly D is a simply connected domain with compact closure in $E\simeq\mathbf{R}^2$, and the space of leaves $Q_D=D/\mathcal{F}$ is a non-Hausdorff manifold with a simple branch at $t=1\in\mathbf{R}_1$ resp. $\psi(1)\in\mathbf{R}_2$.

Lemma 2.4. *If ψ is flat at the origin then every real analytic function f on D which is constant on every leaf of $\mathcal{F}_\psi|_D$ is constant.*

Proof. As in Lemma 2.3 such an f is of the form $f=h\circ\pi$ for some real analytic function h on $Q_D=D/\mathcal{F}_\psi$. Such an h is given by a pair of real analytic functions $h_1\colon (-1,2)\to\mathbf{R}$ and $h_2\colon (-1,\psi(2))\to\mathbf{R}$ satisfying $h_1(t)=h_2(\psi(t))$ for $1<t<2$. By analyticity this relation persists on the largest interval on which both sides are defined, which is $(0,2)$. By flatness of ψ at 0 we conclude as in Lemma 2.3 that h_1 and h_2 must be constant. $\square$

Let $\mathcal{F}=\mathcal{F}_\psi$ be the foliation of $\mathbf{R}^2$ constructed above with the diffeomorphism $\psi(t)=te^{-1/t}$ of $(0,+\infty)$ (which is flat at 0). Let $D\Subset\mathbf{R}^2$ satisfy the conclusion of Lemma 2.4. Choose a disc containing D; clearly Lemma 2.4 still holds for this disc, and by an affine change of coordinates on $\mathbf{R}^2$ we may assume this to be the unit disc. This completes the proof of Proposition 2.1. □

Remark 2.5. Proposition 2.1 holds for any foliation $\mathcal{F}_\psi$ constructed above for which the diffeomorphism ψ of $(0,+\infty)$ is such that $h\circ\psi$ does not extend as a real analytic function to a neighborhood of 0 for any real analytic function h near 0. An example is t^α for an irrational $\alpha>0$. The foliation of $\mathbf{R}^2$ determined by the algebraic 1-form $\omega=(\alpha-x)(1+x)\,dy-x\,dx$ has the space of leaves $\mathcal{C}^1$-diffeomorphic to the 'simple branch' Q determined by $\psi(t)=t^\alpha$ ([5], p. 120); hence it might be possible to find a disc $D\subset\mathbf{R}^2$ satisfying Proposition 2.1 for this foliation. These examples indicate that a real analytic foliation of $\mathbf{R}^2$ only rarely admits real analytic first integrals on large compact subsets.

3. Proof of Theorem 1.1

Let $\mathcal{F}$ be a real analytic foliation of $\mathbf{R}^2$ furnished by the Proposition 2.1 such that any real analytic function on $D=\{(x_1,x_2):x_1^2+x_2^2<1\}\subset\mathbf{R}^2$ which is constant on the leaves of $\mathcal{F}|_D$ is constant. Denote by (x_1+iy_1,x_2+iy_2) the coordinates on $\mathbf{C}^2$ and identify $\mathbf{R}^2$ with the plane $\{(x_1+iy_1,x_2+iy_2):y_1=0 \text{ and } y_2=0\}\subset\mathbf{C}^2$. Complexifying the leaves of $\mathcal{F}$ we obtain the Levi foliation of a closed real analytic Levi-flat hypersurface M in an open tubular neighborhood $\Omega\subset\mathbf{C}^2$ of $\mathbf{R}^2$. Set $B=\{(x_1+iy_1,x_2+iy_2):x_1^2+x_2^2+c(y_1^2+y_2^2)<1\}$ where $c>0$ is chosen so large that $\overline{B}\subset\Omega$. Note that $B\cap\mathbf{R}^2=D$. A generic choice of c insures that M intersects bB transversely (since transversality holds along $bD\cap M$). Set $A=M\cap B\Subset M$. If B is sufficiently thin (which is the case if c is sufficiently large) then clearly $\bar{A}$ is diffeomorphic to the closed ball in $\mathbf{R}^3$. If a real analytic function $u\in\mathcal{C}^\omega(A)$ is constant on every Levi leaf of A then $u|_D$ is constant on every leaf of $\mathcal{F}|_D$ and hence is constant. Thus A satisfies property (ii) in Theorem 1.1.

The foliation $\mathcal{F}$ of $\mathbf{R}^2$ is transversely orientable and hence admits a transverse real analytic vector field ν. Its complexification is a holomorphic vector field w in a neighborhood of $\mathbf{R}^2$ in $\mathbf{C}^2$ such that iw is transverse to M in a neighborhood of $\overline{B}$, provided that B is chosen sufficiently thin. Moving M off itself to either side by a short time flow of iw in a neighborhood of $\overline{B}$ we obtain thin neighborhoods of $\bar{A}$ with two Levi-flat boundary components; intersecting these with rB for $r>1$ close to 1 gives a fundamental system of Stein neighborhoods of $\bar{A}$.

Suppose that v is a real pluriharmonic function in a connected open neighborhood of A such that $v|_A=0$. For every point $x\in A$ there is an open connected neighborhood $U_x \subset B$ and a pluriharmonic function u_x on U_x, determined up to a real constant, such that u_x+iv is holomorphic on U_x. Since A is contractible, $H^1(A,\mathbf{R})=0$ and hence the collection $\{u_x\}_{x\in A}$ can be assembled into a pluriharmonic function u in a neighborhood of A such that $u+iv$ is holomorphic. Since $v|_A=0$, u is constant on every Levi leaf on A and hence constant by property (ii) of A. Thus v is constant and hence identically zero. This proves Theorem 1.1.

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