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Complex Analysis

Oka maps

Les applications d'Oka

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ABSTRACT

We prove that for a holomorphic submersion of reduced complex spaces, the basic Oka property implies the parametric Oka property. It follows that a stratified subelliptic submersion, or a stratified fiber bundle whose fibers are Oka manifolds, enjoys the parametric Oka property.

RÉSUMÉ

Nous prouvons que, pour une submersion holomorphe des espaces complexes réduits, la propriété d'Oka simple implique la propriété d'Oka paramétrique. En particulier, toute submersion sous-elliptique stratifié possède la propriété d'Oka paramétrique.

1. Oka properties of holomorphic maps

Let E and B be reduced complex spaces. A holomorphic map $\pi : E \to B$ is said to enjoy the *Basic Oka Property* (BOP) if, given a holomorphic map $f : X \to B$ from a reduced Stein space X and a continuous map $F_0 : X \to E$ satisfying $\pi \circ F_0 = f$ (a *lifting* of f) such that F_0 is holomorphic on a closed complex subvariety X' of X and in a neighborhood of a compact $\mathcal{O}(X)$-convex subset K of X, there is a homotopy of liftings $F_t : X \to E$ $(t \in [0,1])$ of f to a holomorphic lifting F_1 such that for every $t \in [0,1]$, F_t is holomorphic in a neighborhood of K (independent of t), $\sup_{x\in K} \operatorname{dist}(F_t(x), F_0(x)) < \epsilon$, and $F_t|_{X'} = F_0|_{X'}$ (the homotopy is fixed on X').

By definition, a complex manifold Y enjoys BOP if and only if the trivial map $Y \to$ *point* does. This is equivalent to several other properties, from the simplest *Convex Approximation Property* (CAP) to the *Parametric Oka Property* (POP) concerning compact families of maps from reduced Stein spaces to Y [2]. A complex manifold enjoying these equivalent properties is called an *Oka manifold* [2,11]; these are precisely the *fibrant complex manifolds* in Lárusson's model category [9]. Here we prove that BOP $\Rightarrow$ POP also holds for holomorphic submersions. (The submersion condition corresponds to requiring smoothness as part of the definition of a variety being Oka. The singular case is rather problematic.)

Theorem 1.1. *For every holomorphic submersion $\pi : E \to B$ of reduced complex spaces, the basic Oka property implies the parametric Oka property.*

Recall [9] that a holomorphic map $\pi : E \to B$ enjoys the *Parametric Oka Property* (POP) if for any triple (X, X', K) as above and for any pair $P_0 \subset P$ of compact subsets in an Euclidean space $\mathbb{R}^m$ the following holds. Given a continuous

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map $f: P \times X \to B$ that is *X-holomorphic* (that is, $f(p, \cdot): X \to B$ is holomorphic for every $p \in P$) and a continuous map $F_0: P \times X \to E$ such that (a) $\pi \circ F_0 = f$, (b) $F_0(p, \cdot)$ is holomorphic on X for all $p \in P_0$ and is holomorphic on $K \cup X'$ for all $p \in P$, there exists for every $\epsilon > 0$ a homotopy of continuous liftings $F_t: P \times X \to E$ of f to an X-holomorphic lifting F_1 such that the following hold for all $t \in [0, 1]$:

(i) $F_t = F_0$ on $(P_0 \times X) \cup (P \times X')$, and
(ii) F_t is X-holomorphic on K and $\sup_{p \in P,\, x \in K} \operatorname{dist}(F_t(p, x), F_0(p, x)) < \epsilon$.

A stratified subelliptic holomorphic submersion, or a stratified fiber bundle with Oka fibers, enjoys BOP [3,4]. Hence Theorem 1.1 implies:

Corollary 1.2.

(i) *Every stratified subelliptic submersion enjoys POP.*
(ii) *Every stratified holomorphic fiber bundle with Oka fibers enjoys POP.*

If $\pi: E \to B$ enjoys the Oka property then by considering liftings of constant maps $X \to b \in B$ we see that every fiber $E_b = \pi^{-1}(b)$ is an Oka manifold. For stratified fiber bundles the converse holds by Corollary 1.2.

Question: *Does every holomorphic submersion with Oka fibers enjoys the Oka property?*

A holomorphic map is said to be an *Oka map* if it is a topological (Serre) fibration and it enjoys POP. Such maps are *intermediate fibrations* in Lárusson's model category [9,10]. Corollary 1.2 implies:

Corollary 1.3.

(i) *Every holomorphic fiber bundle projection with Oka fiber is an Oka map.*
(ii) *A stratified subelliptic submersion, or a stratified holomorphic fiber bundle with Oka fibers, is an Oka map if and only if it is a Serre fibration.*

Corollary 1.2(i) and the proof by Ivarsson and Kutzschebauch [8] give the following solution of the parametric Gromov–Vaserstein problem [7,12].

Theorem 1.4. *Assume that X is a finite-dimensional reduced Stein space, P is a compact subset of $\mathbb{R}^m$, and $f: P \times X \to \mathrm{SL}_n(\mathbb{C})$ is a null-homotopic X-holomorphic mapping. Then there exist a natural number N and X-holomorphic mappings $G_1, \ldots, G_N: P \times X \to \mathbb{C}^{n(n-1)/2}$ such that*

$$f(p, x) = \begin{pmatrix} 1 & 0 \\ G_1(p, x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(p, x) \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_N(p, x) \\ 0 & 1 \end{pmatrix}$$

is a product of upper and lower diagonal unipotent matrices.

2. Reduction of Theorem 1.1 to an approximation property

Assume that $\pi: E \to B$ enjoys BOP and that $(X, X', K, P, P_0, f, F_0)$ are as in the definition of POP, with $P_0 \subset P \subset \mathbb{R}^m \subset \mathbb{C}^m$. Set

$$Z = \mathbb{C}^m \times X \times E, \qquad Z_0 = \mathbb{C}^m \times X \times B, \qquad \psi = (\mathrm{id}_{\mathbb{C}^m \times X}) \times \pi: Z \to Z_0. \tag{1}$$

Observe that ψ enjoys BOP (resp. POP) if and only if π does. To the map $f: P \times X \to B$ we associate the X-holomorphic section

$$g: P \times X \to Z_0, \qquad g(p, x) = \big(p, x, f(p, x)\big) \quad (p \in P,\ x \in X), \tag{2}$$

and to the π-lifting $F_0: P \times X \to E$ of f we associate the section

$$G_0: P \times X \to Z, \qquad G_0(p, x) = \big(p, x, F_0(p, x)\big) \quad (p \in P,\ x \in X). \tag{3}$$

Then $\psi \circ G_0 = g$, G_0 is X-holomorphic over $K \cup X'$, and $G_0|_{P_0 \times X}$ is X-holomorphic. We must find a homotopy $G_t: P \times X \to Z$ ($t \in [0, 1]$) such that $\psi \circ G_t = g$ for all $t \in [0, 1]$, G_1 is X-holomorphic, and for all $t \in [0, 1]$ the map G_t has the same properties as G_0, G_t is uniformly close to G_0 on $K \times P$, and $G_t = G_0$ on $(P_0 \times X) \cup (P \times X')$. Set

$$Q = [0, 1] \times P, \qquad Q_0 = \big(\{0\} \times P\big) \cup \big([0, 1] \times P_0\big).$$

The following result is the key to the proof of Theorem 1.1.

Proposition 2.1. *If the submersion $\psi : Z \to Z_0$ (1) enjoys the basic Oka property, then it also enjoys the following Parametric Homotopy Approximation Property (PHAP): Let $K \subset L$ be compact $\mathcal{O}(X)$-convex subsets and let $U \supset K$, $V \supset L$ be open neighborhoods in X. Assume that $g : P \times V \to Z_0$ is an X-holomorphic section of the form (2) and $G_t : P \times V \to Z$ $(t \in [0, 1])$ is a homotopy of sections (3) satisfying*

(a) $\psi \circ G_t = g$ *for* $t \in [0, 1]$,
(b) $G_t(p, \cdot)$ *is holomorphic on* U *for* $(t, p) \in Q$, *and*
(c) $G_t(p, \cdot) = G_0(p, \cdot)$ *for* $(t, p) \in Q_0$, *and these are holomorphic on* V.

Let $\epsilon > 0$. After shrinking the neighborhoods $U \supset K$ and $V \supset L$, there exists a homotopy $\widetilde{G}_t : P \times V \to Z$ $(t \in [0, 1])$ of the form (3) such that

(i) $\psi \circ \widetilde{G}_t = g$ *for all* $t \in [0, 1]$,
(ii) *for each* $(t, p) \in Q$ *the map* $\widetilde{G}_t(p, \cdot) : V \to Z$ *is holomorphic and it satisfies* $\sup_{x \in K} \mathrm{dist}(\widetilde{G}_t(p, x), G_t(p, x)) < \epsilon$, *and*
(iii) $\widetilde{G}_t(p, \cdot) = G_t(p, \cdot)$ *for each* $(t, p) \in Q_0$.

Furthermore, there is a homotopy from $\{G_t\}$ to $\{\widetilde{G}_t\}$ consisting of homotopies with the same properties as $\{G_t\}$.

For families of sections of a holomorphic submersion $\pi : Z \to X$ over a Stein space X, PHAP holds if $Z \to X$ admits a fiber-dominating spray over a neighborhood of L [7,5], or a finite fiber-dominating family of sprays [1]. Submersions admitting such sprays over small open subsets of X are called *elliptic*, resp. *subelliptic*. If PHAP holds over small open subsets of X then sections $X \to Z$ satisfy the parametric Oka property (Gromov [7, Theorem 4.5]; the details can be found in [3,5]). The same proof applies in our situation (see [4, Theorem 4.2]) and shows that PHAP implies Theorem 1.1.

3. Proof of Proposition 2.1

Let $h : \mathcal{E} \to Z$ denote the holomorphic vector bundle whose fiber over a point $z \in Z$ equals the tangent space at z to the (smooth) fiber of ψ. The restriction $\mathcal{E}|_\Omega$ to any open Stein domain $\Omega \subset Z$ is a reduced Stein space. By standard techniques we obtain for every such Ω an open Stein neighborhood $W \subset \mathcal{E}|_\Omega$ of the zero section $\Omega \subset \mathcal{E}|_\Omega$, with W Runge in $\mathcal{E}|_\Omega$, and a continuous map $s : \mathcal{E}|_\Omega \to Z$ satisfying $\psi \circ s = \psi \circ h$, such that s is the identity on the zero section, it is holomorphic on W, and it maps the fiber $W_z = \mathcal{E}_z \cap W$ over a point $z \in Z$ biholomorphically onto a neighborhood of the point $z = s(0_z)$ in the fiber $Z_{\psi(z)} = \psi^{-1}(\psi(x))$. Such s is a fiber-dominating spray in the sense of [7], except that it is not globally holomorphic.

By [6, Corollary 2.2] each of the sets

$$S_0 = G_0(P \times L) \subset Z, \qquad \Sigma_t = G_t(P \times K) \subset Z \quad \big(t \in [0, 1]\big)$$

is a Stein compactum in Z. By compactness of $\bigcup_{t \in [0,1]} \Sigma_t$ there exist numbers $0 = t_0 < t_1 < \cdots < t_N = 1$, Stein domains $\Omega_0, \ldots, \Omega_{N-1} \subset Z$ satisfying

$$\bigcup_{t \in [t_j, t_{j+1}]} \Sigma_t \subset \Omega_j \quad (j = 0, 1, \ldots, N - 1), \tag{4}$$

and for every $j = 0, 1, \ldots, N - 1$ there exist an open Stein neighborhood $W_j \subset \mathcal{E}|_{\Omega_j}$ of the zero section Ω_j of $\mathcal{E}|_{\Omega_j}$ such that W_j is Runge in $\mathcal{E}|_{\Omega_j}$ and has convex fibers, a fiber-spray $s_j : \mathcal{E}|_{\Omega_j} \to Z$ as above that is holomorphic on W_j, and a homotopy ξ_t $(t \in [t_j, t_{j+1}])$ of X-holomorphic sections of the restricted bundle $\mathcal{E}|_{G_{t_j}(P \times U)}$, with the range contained in W_j, such that

(i) ξ_{t_j} is the zero section of $\mathcal{E}|_{G_{t_j}(P \times U)}$,
(ii) $\xi_t(p, \cdot)$ is the zero section when $p \in P_0$ and $t \in [t_j, t_{j+1}]$, and
(iii) $s_j \circ \xi_t \circ G_{t_j} = G_t$ on $P \times U$ for all $t \in [t_j, t_{j+1}]$.

(See Fig. 1.) For a given collection (Ω_j, W_j, s_j) the existence of homotopies ξ_t is stable under sufficiently small perturbations of the homotopy G_t.

Consider the homotopy of sections $\{\xi_t\}_{t \in [0, t_1]}$ of $\mathcal{E}|_{G_0(P \times U)}$. By the parametric version of the Oka–Weil theorem we can approximate ξ_t uniformly on $P \times K$ by X-holomorphic sections $\widetilde{\xi}_t$ of $\mathcal{E}|_{G_0(P \times V')}$ for an open set $V' \subset X$ with $L \subset V' \subset V$. Further, we may choose $\widetilde{\xi}_t = \xi_t$ for $t = 0$ and on $P_0 \times V'$. In the sequel the set V' may shrink around L.

By [6, Corollary 2.2] there is an open Stein neighborhood Ω of S_0 in Z such that S_0 is $\mathcal{O}(\Omega)$-convex. Hence $\Sigma_0 = G_0(P \times K) \subset S_0$ is also $\mathcal{O}(\Omega)$-convex, and it follows that $W_0 \cap \mathcal{E}|_{\Sigma_0}$ is exhausted by $\mathcal{O}(\mathcal{E}|_\Omega)$-convex compact sets. Since $\mathcal{E}|_\Omega$ is a reduced Stein space and s_0 extends continuously to $\mathcal{E}|_\Omega$ preserving the property $\psi \circ s_0 = \psi \circ h$, the assumed BOP of ψ implies that s_0 can be approximated on the range of the homotopy $\{\xi_t \colon t \in [0, t_1]\}$ (which is contained in $W_0 \cap \mathcal{E}|_{\Sigma_0}$) by a holomorphic map $\widetilde{s}_0 : \mathcal{E}|_\Omega \to Z$ which equals the identity on the zero section and satisfies $\psi \circ \widetilde{s}_0 = \psi \circ h$. The homotopy

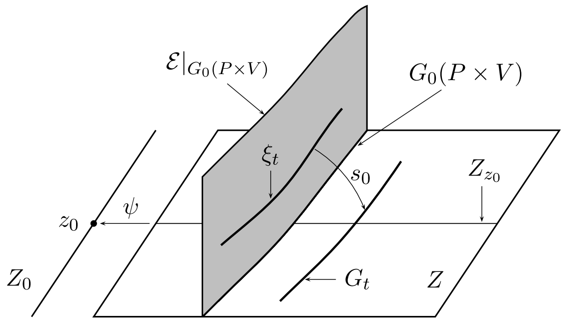

Fig. 1. Lifting sections G_t to the spray bundle $\mathcal{E}|_{G_0(P\times V)}$.

$$\widetilde{G}_t = \widetilde{s}_0 \circ \widetilde{\xi}_t \circ G_0 : P \times V' \to Z \quad \bigl(t \in [0, t_1]\bigr)$$

is fixed over P_0, X-holomorphic on V', $\widetilde{G}_0 = G_0$, and $\widetilde{G}_t$ approximates G_t uniformly on $P \times K$ (also uniformly with respect to $t \in [0, t_1]$). If the approximation is sufficiently close, we obtain a new homotopy $\{G_t\}_{t\in[0,1]}$ that agrees with $\widetilde{G}_t$ for $t \in [0, t_1]$ (hence is X-holomorphic on L), and that agrees with the initial homotopy for $t \in [t_1', 1]$ for some $t_1' > t_1$ close to t_1.

We now repeat the same argument with the parameter interval $[t_1, t_2]$ using G_{t_1} as the initial reference map. This produces a new homotopy that is X-holomorphic on L for all values $t \in [0, t_2]$. After N steps of this kind we obtain a homotopy satisfying the conclusion of Proposition 2.1.

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