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Gunning-Narasimhan's theorem with a growth condition (with F.Forstnerič)

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はじめに ----- 解析接続におけるホモトピー原理

Cauchyの留数解析を発端とする一変数の複素解析においては、Mittag-Lefflerの定理、Weierstrassの乗積定理、およびRungeの近似定理を柱とする基礎理論がある。この理論では、解析関数は一つの関数要素を曲線に沿って可能な限り解析接続してできると考える。Weierstrassによるこの観点から、極や零点を与えて関数を作る問題の解として、ガンマ関数をはじめとする種々の特殊関数や補間公式などが、見通しよく理解できるようになったのだった。ちなみにPoincaréはWeierstrassの業績中で乗積定理を最も高く評価したそうである(cf.[R])。

解析接続を実行するにあたっては、与えられた関数要素の集合を、隣同士の食い違いをなくすように補正して、全体をつなげる必要が生ずる。この種の問題には加法的なものと乗法的なものがあり、それぞれCousinの第１問題、第２問題と呼ばれている。今日の層係数コホモロジー論はこれらに端を発している。

数学の他の理論と同様、解析接続の理論にも一意性定理と存在定理がある。一意性定理であるmonodromy定理(一価性定理)とは、一つの関数要素から解析接続によって生ずる関数要素は、接続路である曲線の、端点を止めたホモトピー類にのみ依存するというものである。これは一致の定理の自明な帰結であるとはいえ、被覆空間の導入を決定づけた重要な原理である。層の言葉でいうなら、monodromy定理は構造層のハウスドルフ性によっている。

存在定理だが、Mittag-Lefflerの定理とWeierstrassの乗積定理は、上記のようなコホモロジー的な意味での存在定理といえるだろうが、「ホモトピー的」ではない。Rungeの近似定理は広い意味ではホモトピー的な接続原理といえるかもしれない。Riemannの写像定理がそうだといえなくもないが、これは解析接続の理論からは遠いところにある。いずれにせよ、一変数の世界ではホモトピー的な存在原理は見えにくい。

ところが多変数の場合は事情が一変し、岡潔の第３論文[O]におけるCousinの第２問題の解こそ、真にホモトピー原理の名に値する。岡の解の内容は、正則領域D内の余次元１の解析集合Xが一つの正則関数の零点集合であるための必要十

この原稿は掲載決定済みの論文[Fn-Oh]に沿って本研究集会用に書いたものを元にして、研究集会「リーマン面・不連続群」(2012/01/07-09)における講演にあたって加筆修正し、さらに付録として未発表の原稿を抄録したものである。

分条件を与えるもので、それはXが「掃散可能」(balayable)かどうか、すなわちDの一点コンパクト化において、Xがホモトピー的に自明かどうかによるものであった。

岡の結果を敷衍しながらStein空間が導入され、一変数関数論の基礎的な定理の多変数版が、Stein空間上の層係数コホモロジー理論としてまとめられていった(cf.[G-R])。「岡の原理」(principe d'Oka)という呼び名は、その過程で上記の定理のコホモロジー完全列を用いた定式化の際に現れた(cf.[Se])。しかしながら、この層係数コホモロジー理論を今日の目で見る時、それは岡が開いた世界のほんの入口付近だけを照らしているように思える。

実際、この岡の原理の一般化にあたるものとしてStein空間上の正則ベクトル束に対するホモトピー原理があるが(Grauert理論[G-1, 2, 3])、その本質はRunge型近似定理によるコホモロジー類の接続であり、代数的なダイアグラム・チェイスの世界から大きくはみ出している。GromovやEliashbergらによるStein多様体上のホモトピー原理も、コホモロジー理論からはみ出した岡の原理の一例である(cf. [Fn-2])。

多変数関数論のこのような展開は開リーマン面の理論にも影響を与え、Runge型近似定理により開リーマン面のStein性が確立された(cf. [B-S])。これより特に、開リーマン面上のすべての正則ベクトル束は自明束であることが従う。正則ベクトル束の自明性を大域的正則枠の存在と解すれば、この命題は開リーマン面上の一つのホモトピー的存在定理である。(多変数的視点に立って初めてよく見えるようになったものは、これだけではない。)

ところで一口に開リーマン面といってもその種類は多様である。大まかにはグリーン関数のあるなしで分かれ、グリーン関数を持たないものはEvansポテンシャルが手がかりになる。さらにその一部として代数曲線の構造を持つものがあり、その上の代数的なベクトル束は一般に代数的な自明束ではない。例えば、一点穴あきトーラスはC^2内の代数曲線 $y^2 = 4x^3 - g_2x - g_3$ $(g_2^3 - 27g_3^2 \neq 0)$と同型だが、このトーラス上には、位数が1の零点を1個だけ持ち、穴で極を持つ正則関数は存在しない。つまり一点穴あきトーラス上では、次数が1の正因子に付随する直線束は、代数的には非自明である。

しかしながらCornalba-Griffiths[C-G]が指摘したように、有限型の開リーマン面上の代数的ベクトル束は、有限位数のベクトル束としては自明である。このことから、C^n 内の代数曲線として埋め込まれた開リーマン面は、ちょうど $n-1$ 個の有限位数の整関数の共通零点となることが[C-G]で予想され、[F-Oh]において証明された。このように、有限位数の解析関数の世界でもホモトピー原理は働いている。

最近、有限型の開リーマン面は有限位数の関数によりC上の不分岐被覆として

実現できることが判明した(cf.[Fn-Oh])。これは有名なGunning-Narasimhanの定理(cf. [G-N])を、上に述べた見方で精密化したものである。Gunning-Narasimhanの定理は任意の開リーマン面がC上の不分岐リーマン領域であることを言っているが、これはRiemannの写像定理になぞらえていうなら「複葉写像の基本定理」とでも呼べそうなもので、その証明には類例がなく、意外な方法で解析接続と等角写像論をつないでいる。ちなみに数学辞典第４版の「リーマン面上の解析学」の項目には、この定理やその楠・斉之内[K-S]による一般化に続けて等角写像論における柴の定理や及川の定理が紹介してあり、興味深い。

以下では[Fn-Oh]の概要を紹介し、関連するいくつかの話題を述べ、最後に境界つきリーマン面について二三の予想を述べたい。

§1．開リーマン面上の岡の原理とその精密化

Gunning-Narasimhanの定理によれば、任意の開リーマン面はCの構造層の一つの連結成分と等角同値だが、与えられた開リーマン面をこのようにC上の不分岐リーマン被覆として実現する仕方は様々なので、その中から標準的なものを見つけ出すことは一つの課題と考えてよいだろう。

例えばそのリーマン面がCであるとき、これをC上の不分岐被覆として実現する写像を $f(z)$ とすれば、$\log f'(z)$ は一つの整関数である。この対応によって恒等写像 $f(z) = z$ は0に、指数写像 $f(z) = \exp z$ は z に対応づけられる。より一般に関数 $\log f'(z)$ が多項式になるような $f(z)$ を決定することは容易であるが、有限位数の整関数のHadamard分解(cf. [A])と位数の微分不変性（Whittaker [W]）により、この写像のクラスは有限位数の不分岐被覆写像として特徴づけられる(有限位数≠有限枚数に注意)。整関数 $g(z)$ に対し$M(r) = \max \{ |g(z)| ; |z| = r \}$とおいたとき、$r \longrightarrow \infty$ のときの $\log \log M(r) / \log r$ の上極限を $g(z)$ の位数というのであった。

この特徴付けを念頭に置けば、Cだけでなく一般の穴あきリーマン面に対しても、それをC上の不分岐被覆として実現する有限位数の正則関数を決定する問題が意味を持つように思われる。

この問題に関し、[Fn-Oh]では閉リーマン面から一点を除いたものが有限位数の写像によってC上の不分岐被覆として実現できることを示した。Gunning-Narasimhanの方法の中身は、開リーマン面上の岡の原理とMergelyanの近似定理の応用であるが、これらを増大度の評価つきでやった[Fn-Oh]の内容を紹介するため、その準備としてまず岡の原理について復習する。

今日「岡の原理」として知られているものは、「Stein空間を定義域とする種々の写像空間はおおむねホモトピー同値である」という大まかな指針であり、その具体的内容は多岐に渡る（cf. [G-R], [F-2]). 一方、J.-P. Serre [Se] によって初めて岡の原理の名で呼ばれた命題は次のものであった。

Serreの岡の原理　任意のStein多様体Mに対し、層係数コホモロジー群の同型 $H^1(M,\mathcal{O}^*) \cong H^2(M,\mathbf{Z})$ が成立する。

ただし$\mathcal{O}^*$はMの構造層$\mathcal{O}$の乗法的可逆元からなる乗法群の層、$\mathbf{Z}$は階数１の自由アーベル群の層を表し、同型はM上の層の短完全列

(1)　$$0 \longrightarrow 2\pi\sqrt{-1}\mathbf{Z} \xrightarrow{\iota} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$　（ι は入射でexpは指数写像）

から導かれる連結準同型についてである。

この命題は岡潔(1901-1978)の第３論文で述べられた命題、「正則領域上では、余次元が１の解析的集合が正則関数の零点集合であるためには掃散可能(balayable)であることが必要かつ十分である」の一般化である。Stein多様体の概念は、岡のこの結果を多様体上へと一般化するため、K. Stein [S]により正則領域を特徴づける性質を用いて導入されたものだが、それに先立って[B-S]では開リーマン面の場合が調べられ、一変数関数論で基本的なMittag-Lefflerの定理、Weierstrassの乗積定理およびRungeの近似定理がすべて開リーマン面上で成立することが示されていた。とくに開リーマン面R上では任意に与えられた零点分布を持つ正則関数が存在するが、このことをR上の正則直線束の自明性つまり$H^1(R,\mathcal{O}^*)=1$の帰結であるとみなせば、自然にSerreの岡の原理へと到達する。

さて、以下ではRが穴あきリーマン面の場合、すなわち閉リーマン面XとXの有限個の点$p_1, \ldots, p_k$ があって$R = X-\{p_1, \ldots, p_k\}$ となっているときを考える。[Fn-Oh]の主問題は、Rから$\mathbf{C}$への正則写像 f で臨界点を持たないものをうまくとって、有限位数の範囲に収めることができるかというものである。ただしR上の関数 f が有限位数であるとは、各点 p_j のまわりの局所座標 z_j に関し、z_j を用いて f を $f_j(z_j)$ で表したとき、ある定数mが存在して

(2)　$$|f_j(z_j)| \le \exp(1/|z_j|^m) \qquad (0<|z_j|<<1)$$

がすべての j に対して成り立つようにできるという条件である。このようなmの

下限が前述の意味の位数の自然な拡張になっていることは明白であろう。

臨界点を持たない有限位数の関数を作るためには、零点を持たない有限位数の完全微分を作ればよい (X上の直線束のR上の断面についても有限位数の定義は上と同様)。それには[G-N]にならい、まず完全ではないかもしれないが零点を持たない有限位数の正則微分ωを作った上で、X上の有理型関数ϕでR内に極を持たないものを、$\omega\exp\phi$が完全微分になるように作る。このとき f は$\omega\exp\phi$の原始関数として定まるので、その位数はωの位数と$\exp\phi$の位数（＝ϕの極の位数の最大値）の大きい方を超えない。ωの存在はCornalba-Griffiths[C-G]による次の一般的な定理に含まれる。

定理 I. X 上の任意の正則直線束 L と穴あきリーマン面 $R\subset X$ に対し、 R 上のL の正則断面で、有限位数であり、かつ零点を持たないものが存在する。

証明. $R = X-A$ ($\#A = n < \infty$)とし、Aの近傍Uを、$U-A$ の各連結成分が 2 重連結であるようにとる。Grauertの定理より$L|R$ (LのR上への制限)と$L|U$は自明束だから、Lは$H^0(U-A,\mathcal{O}^*)$のある要素σを変換関数とする直線束に同値である。U上のAの定義関数をwとし、Uの連結成分を$U_1, \ldots, U_n$とし、$w_j = w|U_j$とおく。するとある整数の組$(m_1, \ldots, m_n)$がとれて、$\log(\sigma\cdot w_j^{m_j})$は$U_j-A$内で一価の分枝$\tau_j$ をもつようにできる。Aの近傍VをU内で相対コンパクトになるようにとると、$X-V$ 上の正則関数 τ と Aで高々極をもつ $U-A$ 上の正則関数 τ' が存在し、U_j-V 上で $\tau_j = \tau - \tau'$ となる(小平の消滅定理またはRiemann-Rochの公式の応用)。するとU_j-V 上で $\exp\tau = \sigma\cdot w_j^{m_j}\exp\tau'$ となるから、$\exp\tau$ は $L|R$ の有限位数の断面へと解析接続可能である。 □

一つの近道： よく知られた同型

(3) $$H^1(R,\mathbf{C}) \cong H^0_{alg}(R,d\mathcal{O})/dH^0_{alg}(R,\mathcal{O})$$

を用いてωを作ることもできる。ただしH^0 の意味は

$$H^0_{alg}(R,\cdot) = \{ s\in H^0(R,\cdot)\ ;\ s \text{ はX上有理型} \}.$$

実際、$\omega_0(\neq 0)$ をX上の$d\mathcal{O}$の有理型断面で高々$X-R$にのみ極を持つものとし、$\omega_0|R$ の因子を $m_1 q_1 + \cdots + m_\ell q_\ell$ ($m_j\in\mathbf{N}$)とする。
(3)より各 q_j で留数 $-m_j$ を持ち、Rの 1 サイクル上の積分がすべて$2\pi\sqrt{-1}$

の整数倍であるような$H^0_{alg}(R-\{q_1, \ldots, q_\ell\}, d\mathcal{O})$の要素 η が存在するから、$R-\{q_1, \ldots, q_\ell\}$ の点p_0を固定して

$$\omega(p) = \omega_0(p)\exp\int_{p_0}^{p}\eta$$

とおけば、ω_0 の零点がみな消えてしまう。

ついでにStein多様体上のSerre同型

$$(4) \qquad H^k(M,C) \cong H^0(M,d\Omega^{k-1})/dH^0(M,\Omega^{k-1}) \quad (k \geq 1)$$

を思い出しておこう。ただしΩ^k は正則 k 形式の芽の層を表す。

§2. Mergelyanの定理とその応用

Mergelyanの定理(cf. [M])を述べよう。開リーマン面Rに対し、$H^0(R,\mathcal{O})$を単に$\mathcal{O}(R)$で表す。とくにRが穴あきリーマン面のとき、$H^0_{alg}(R,\mathcal{O})$を$\mathcal{O}_{alg}(R)$で表す。Rの部分集合Aに対して$C^0(A)$でA上のC値連続関数の集合を表す。Aがコンパクトなら$C^0(A)$はL^∞ノルムに関してバナッハ環となる。

$$\mathcal{O}(A) = \varinjlim_{U} \mathcal{O}(U)$$

とおく。ここでUはAの近傍全体を動く。$\mathcal{O}(A)$の要素を制限して得られる$C^0(A)$の要素全体の集合を$\mathcal{O}(A)\hat{}$で表す。A° でAの開核を表し$\mathcal{O}(A)\check{}=\{g\in C^0(A) ; g|A^\circ \in\mathcal{O}(A^\circ)\}$ とおく。

Mergelyanの定理(その一)　　開リーマン面Rの任意のコンパクト集合Kに対し、$C^0(K)$における$\mathcal{O}(K)\hat{}$の閉包は$\mathcal{O}(K)\check{}$に等しい。

Mergelyanの定理(その二)　　上のRとKに対し、特にR−Kが相対コンパクトな連結成分を持たなければ、$C^0(K)$における $\{f|K ; f\in\mathcal{O}(R)\}$ の閉包は$\mathcal{O}(K)\check{}$に等しい。

Mergelyanの定理(その三)　上のようにR−K が相対コンパクトな連結成分を持たない場合、さらにRが穴あきリーマン面ならば、$C^0(K)$における $\{ f|K ; f\in\mathcal{O}_{alg}(R)\}$ の閉包は$\mathcal{O}(K)\check{}$に等しい。

証明は[K]または[G]を参照されたい。

Mergelyanの定理(その二)により次は明白である。

命題1. 開リーマン面R上の任意の正則微分ωと有限個の1サイクル$\gamma_1, ... \gamma_n$、および$\varepsilon > 0$ に対し、$\mathcal{O}(R)$の要素Fが存在して

$$\left| \int_{\gamma_k} e^F \omega \right| < \varepsilon \quad (1 \leq {}^\forall k \leq n)$$

となる。

命題1においてRが穴あきリーマン面ならば、Mergelyanの定理(その三)より$F\in\mathcal{O}_{alg}(R)$ としてよいことに注意しよう。

§3. 主定理とその証明

主定理の内容についてはすでに述べたが、ここでそれを再掲し、§1で述べた方針に沿って証明を完成させよう。

主定理　任意の穴あきリーマン面上に、臨界点を持たない有限位数の正則関数が存在する。

証明.　Rとωを§1の通りとする。Rが一点穴あきリーマン面の場合に示せばよい。このときRの一点pを固定し、ホモロジー群$H_1(R,\mathbb{Z})$の基底 e_i (i = 1, 2, . . . , 2g) を、e_iを代表するサイクル γ_i が単純閉曲線であり、γ_i の台 $|\gamma_i|$ の共通部分が {p} であり、しかも$|\gamma_i|$の和集合をΓとしたとき R−Γが連結であるようにとる(例えば[K]を見よ)。

$\omega\exp\phi$が完全になるようなϕの存在は次の手順で示される。

1)　「一つの近道」の方法で、R上に零点なしの有限位数の正則関数 u を作

り、$u\omega$ が各$|\gamma_i|$の近傍U_iで $dw_i \exp h_i$ なる形で書けるようにする。ただしh_iは U_i 上で正則で、w_i は U_i を円環領域 $A_r = \{w \in C ; 1-r<|w|<1+r\}$ $(r > 0)$ へと等角に写す写像とする。

2） これらのw_i に対し、$f_i \in \mathcal{O}_{alg}(R)$ を適当にとって

(5) $$\exp(h_i(p))\int_{\gamma_i} f_j \, dw_i = \delta_{ij} \qquad (\delta_{ij} \text{ はKroneckerのデルタ})$$

が成り立つようにする。このような f_i を作るには、まずΓの近傍で正則な関数で上の積分条件を満たすものを作り (Serre同型)、それらをMergelyanの定理 (その三) により$\mathcal{O}_{alg}(R)$ の元で近似し、その後適当な線形変換を施せば良い。

3） $|\gamma_i|$上で $h_i - h_i(p)$ となるΓ上の連続関数がある。それを h_Γ と書く。Γは連結な補集合を持つから、Mergelyanの定理（その三）より h_Γ は$\mathcal{O}_{alg}(R)$ の元でΓ上一様近似できる。したがって次の主張が正しければ証明が完了する。

主張 次を満たす正数εが存在する：

(6) $$\sup_\Gamma |h - h_\Gamma| < \varepsilon$$

を満たす任意の $h \in \mathcal{O}_{alg}(R)$に対し、

$$\int_{\gamma_j} u\omega \exp\Big(\sum_{i=1}^{2g} \tau_i f_i - h\Big) = 0, \quad j = 1, 2, \ldots, 2g$$

が成立するような$\tau_i \in C$ が存在する。

主張の証明. まず、(6)をみたす h $(= h_\varepsilon)$に対し、$\varepsilon \to 0$ のときΓ上で$u\omega\exp(-h)$ は一様に有界であり、かつ $\int_{\gamma_j} u\omega \exp(-h) \to 0$ $(\forall j)$となることに注意しよう。これと(5)を合わせると、$\tau = (\tau, \ldots, \tau_{2g}) \in C^{2g}$に対し、$\varepsilon \to 0$ かつ $|\tau| \to 0$ のとき

(7) $$|\tau|^{-1} \Big| \tau_j - \int_{\gamma_j} u\omega \exp(-h) \Sigma \tau_i f_i \Big| \longrightarrow 0 \quad (\forall j)$$

である。(6), (7)と $u\omega\exp(-h)$ の一様有界性から $\varepsilon\to 0$ かつ $|\tau|\to 0$ のとき

$$(8)\qquad |\tau|^{-1}\left|\tau - \int_{\gamma_j} u\omega\exp(\Sigma\tau_i f_i - h)\right| \longrightarrow 0 \qquad (\forall j)$$

となる。したがって ε が十分小さければ、

$$\Phi(\tau) = \int_{\gamma_j} u\omega\exp(\Sigma\tau_i f_i - h)$$

によって定まる写像

$$C^{2g}\ni\tau \longmapsto (\Phi_1(\tau),\ldots,\Phi_{2g}(\tau))\in C^{2g}$$

の像は原点の近傍を含む。これが示すべきことであった。 □

ザックリいえば、線形補正ができれば非線形補正もできるといういつものやり方である。

§4．二三の予想

以上の結果は実質的には[G-N]の方法をなぞったものに過ぎないが、実際になぞってみた結果、次の問題が浮上したように思われる。

予想1． 有限型リーマン面Xに対し、Xの種数 g にのみ依存する整数 m(g) があり、XからCへの正則なはめ込みで位数が m(g) 以下のものが存在する。

穴あきリーマン面上の有限位数正則関数が無数の臨界点を持たなければ、その位数は整数であることに注意しよう(Hadamard分解と位数の微分不変性)。

西村保一郎氏により、Gunning-Narasimhanの定理は開リーマン面の解析族へと（部分的に）一般化されている(cf. [N])。穴あきリーマン面の解析族に対してこれを精密化することは容易であろう。

元々の（増大度の条件なしの）Gunning-Narasimhanの定理にはForstnerič氏

による別証明があり(cf. [Oh])、その方法を一般化することにより[Fn-1]において次が示された。

定理２． n次元Stein多様体は$C^{[\frac{n+1}{2}]}$への正則な沈入(submersion)をもつ。

したがって次が正しいかどうかが気になるところである。

予想２． n次元アファイン代数多様体は$C^{[\frac{n+1}{2}]}$への有限位数の正則な沈入をもつ。

小論で解説した方法を高次元の場合へと一挙に一般化することは難しい。

種数が１以上の一点穴あきリーマン面からCへの正則関数がいたるところ不分岐なら、それは穴を真性特異点として持たざるを得ない。従ってこの写像は無限葉である。一方、Yを種数が有限でコンパクトな境界つきリーマン面とすると、YはCへの有限枚数の不分岐正則写像をもつから、その枚数の下限が定まる。これをm(Y)としたとき、次を予想するのは自然であろう。

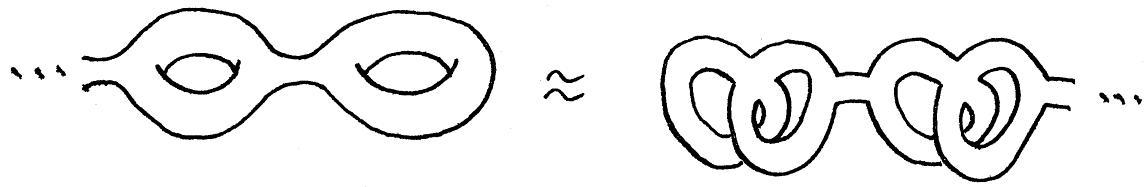
予想３． 任意の自然数 m に対し、$m(Y) \leq m$ をみたす Y はYのモジュライ空間の中で有界集合をなす。

数学辞典第２版のリーマン面の項には「開いた,しかも種数が ∞ のRiemann面は多種多様であって」とあり、第３版ではそこが「開リーマン面は, 種数が ∞ である場合とくに, 極めて複雑である」となり、第４版には対応する文章はない。実際、種数が ∞ のリーマン面は位相的にも複雑である。しかし次の予想はトリビアルかもしれない。

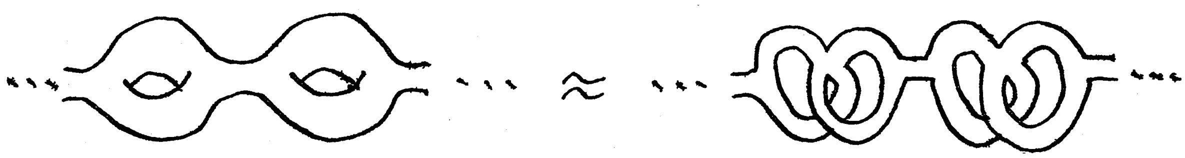
予想４． 任意の開リーマン面はC上の高々２葉のリーマン領域と同相である。

リーマン面の端(end)の個数が２以下の場合にこれを確認したが(次ページの図を参照)、一般の場合は個人的にはわからない。

#end(R)=1



#end(R)=2



さらにもう一つの課題は、ここでの存在定理をふまえて標準的な有限位数不分岐被覆写像を具体的に構成することであろう。たとえば一点穴あきトーラスの場合にどうなるかは興味深いと思われる。

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付録

Topics in complex analysis from the viewpoint of Oka principle*

(岡の原理から見た複素解析の話題)

Takeo Ohsawa

Introduction In complex analysis, important basic theorems for analytic functions in one variable were mostly established in the 19th century. In the last century, it turned out that they are naturally carried over to several variables on Stein spaces. Stein spaces (cf. §2 for the definition), which are named after K. Stein (1913-2000) from [Stn-2], are thus home grounds for functions. On Stein spaces, there is a loosely formulated principle: What can be done topologically on Stein spaces can also be done holomorphically. This is called Oka principle. A well known theory of H. Grauert [G-2, 3, 4] asserting the equivalence of continuous and holomorphic vector bundles over Stein spaces is a part of it (cf. [G-6]). J-P. Serre [Se-1] coined the word "Oka principle" because Kiyoshi Oka (1901-78) first established in [O-3] that holomorphic line bundles over a Stein space is trivial if and only if so are they topologically, according to the expression of [Se-1, 2]. Oka principle has been realized in many situations. Grauert's theory is a good example. As a categorical equivalence assertion, Oka principle has an analog in algebraic geometry, the equivalence between analytic and algebraic coherent sheaves over projective algebraic varieties (Serre's GAGA principle, cf. [Se-3]). Also, on the analytic side, the L^2 method by J. Kohn [Kn-1, 2] and L. Hörmander [H-1, 2] is viewed as a part of Oka principle, since its spirit is to correct a C^∞ solution to get a holomorphic one by solving an inhomogeneous Cauchy-Riemann equation. Admitting such a broad scope of Oka principle, we would like to restrict ourselves here to overview those results which directly extend a theorem in [O-3]. The goal of the exposition is to review some of the recent works of F. Forstnerič who solved a conjecture of M. Gromov [Gm-2] and introduced the notion of Oka manifolds and Oka maps (cf. [Fn-7, 8, 9]). On the way, special attention will be paid to questions of interpolation and approximation.

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*本文の引用文献中の[Oh]（第55回 ENCOUNTER with MATHEMATICS（2011/2/21~22 於中央大学)のための原稿)から、岡の原理の背景とForstneričの仕事に関係する部分を抜粋したものである。

§1. Oka's theory and its background It is said that Oka was aware of a completely new method of constructing global analytic objects from local ones when he wrote [O-1]. According to Oka's renowned essay "Shunsho-Juwa" (Ten stories in spring nights), his first five articles starting from [O-1] are based on what he saw on a summer day in 1935. Recalling this great moment, Oka once told his friend Yasuo Akizuki (1902-84) that it was as if the whole universe stood before him in one line, according to Shigeo Nakano (1923-97), a student of Akizuki and the author's thesis adviser. Since the paper [O-3] of Oka principle is in the above five, we shall start our exposition from [O-1].

Among other things, the impact of [O-1] is that it brought a new insight into the classical theory of interpolation due to Newton, Lagrange and Weierstrass. Let's recall this good-old theory at first.

Lagrange's formula (written in a lecture note for the undergraduates): Given $n+1$ mutually distinct points $c_0, \dots, c_n \in \mathbf{R}$, put $g(x) = \prod_{k=0}^{n}(x - c_k)$. Then the polynomial of degree $\leq n$ that takes values a_k at c_k is

(1) $$f(x) = \sum_{k=0}^{n} a_k g(x)/(g'(c_k)(x - c_k)).$$

A significance of (1) is that it suggests a way to decompose a function into its building blocks. As a formula of such a spirit, it is naturally extended to solve interpolation problems by infinite series

(2) $$\sum_{k=0}^{\infty} a_k \varphi(x)/(\varphi'(c_k)(x - c_k))$$

for a discrete subset $\{c_k\}_{k=0}^{\infty} \subset \mathbf{R}$, where φ is a function satisfying $\varphi(c_k) = 0$ and $\varphi'(c_k) \neq 0$.

E. Borel (1871-1956) asked for the convergence criterion for (2). An extremely elegant answer to Borel's question was given by the formula

(3) $$f(x) = \sum_{k=-\infty}^{\infty} f(k)\sin(\pi(x-k))/(\pi(x-k)).$$

Here $f(x)$ is any continuous square integrable function on $\mathbf{R}$ whose Fourier transform is supported in the interval $[-1,1]$.

This formula is often called "Shannon's sampling theorem" because C. E. Shannon (1916-2001) applied it in [Sh] to give a rigorous justification for the Nyquist rate of frequencies in analog-to-digital converters. However, as a theorem of mathematics, it is actually first due to E. T. Whittaker (1873-1956) [W] and K. Ogura (1885-1962) [Og]. See [B-F-H-S-S-S] for the detail.

Anyway, the only point relevant here is that Ogura wrote (3) as

(4) $$f(z) = \frac{\sin \pi z}{\pi} \sum_{k=-\infty}^{\infty} (-1)^k \frac{f(k)}{z-k}$$

putting it into the context of complex function theory as presented in [L], which is a part of the theory of Weierstrass.

Weierstrass's method of interpolation (cf. [Ah], [N]) consists of two parts. Namely, given a domain D in the complex plane **C**, a discrete subset $\Gamma = \{c_k\}_{k=0}^{\infty} \subset D$, and any sequence $a_k \in \mathbf{C}$, one can find a holomorphic function g(z) on D satisfying $g^{-1}(0) = \Gamma$ and $(dg)^{-1}(0) \cap \Gamma = \varnothing$ (Weierstrass's product theorem), and a meromorphic function h(z) on D such that $h^{-1}(\infty) = \Gamma$ and $h(z) - a_k/(g'(c_k)(z-c_k))$ is holomorphic around each c_k (Mittag-Leffler's theorem). Then g(z)h(z) is a holomorphic solution to the interpolation problem with respect to Γ and $\{a_k\}_{k=0}^{\infty}$. The formula (4) matches this manual.

On the other hand, in the earliest days of several complex variables, Poincaré proved that any meromorphic function on $\mathbf{C}^2$ is a quotient of two entire functions that are everywhere coprime to each other. This result is closely related to the project of extending Weierstrass's theory to several variables.

In 1895, P. Cousin (1867-1933) [Cou] systematically studied how to express a meromorphic function as the ratio of two locally coprime holomorphic functions on a domain in $\mathbf{C}^n$, and found a method for special domains by applying the Cauchy's integral formula. Then he posed a question to find meromorphic functions and holomorphic functions on a general domain, say Ω, with given poles and with given zeros, respectively:

(C-I) Given an open covering $\{U_\alpha\}_{\alpha\in A}$ of Ω and meromorphic functions u_α defined on U_α such that $u_\beta - u_\alpha$ are holomorphic on $U_\alpha \cap U_\beta$, find a meromorphic function u on Ω such that $u - u_\alpha$ are holomorphic on U_α.

(C-II) Given $\{U_\alpha\}$ as above and a system of holomorphic functions v_α on U_α such that v_β / v_α are holomorphic on $U_\alpha \cap U_\beta$, find a holomorphic function v on Ω such that v_α/v and v/v_α are holomorphic on U_α.

(C-I) and (C-II) are known as Cousin's first and second problems, respectively (cf. [Ca-2]).

Based on the theory of Poincaré and Cousin, Cartan [Ca-1] solved an interpolation problem on $\mathbf{C}^2$ following the above mentioned recipe of Weierstrass. Up to this point, solvability of the interpolation problem was linked to (C-II).

After the motivating papers of Cartan [Ca-1, 2], Oka [O-1] first solved (C-I) on a reasonably general class of domains, i. e. rationally convex domains in $\mathbf{C}^n$, by introducing a new method for the interpolation problem. In [O-2] he generalized the result by exploiting the work [C-T] of Cartan and P. Thullen (1907-96). The point is that Oka's method does not require a solution of (C-II).

Recall that Ω is called a *domain of holomorphy* if there exists a holomorphic function on Ω which is not extendible analytically across any boundary point of Ω. Cartan and Thullen showed that any domain of holomorphy is *holomorphically convex* in the sense that, for any compact subset $K \subset \Omega$, the set

$$\{w \in \Omega \mid |f(w)| \leq \sup_K |f| \text{ holds for any holomorphic function } f \text{ on } \Omega\}$$

is compact, where $\sup_K |f|$ denotes the supremum of $|f|$ on K.

In short, Oka proved that (C-I) is solvable on domains of holmorphy by embedding them into higher dimensional polydiscs $\mathbf{D}^{m+n}(m \geq 1)$. Here we put $\mathbf{D} := \{z \in \mathbf{C} \mid |z| < 1\}$.

Identification of a domain $\{w \in \mathbf{D}^n \mid |f_k(w)| \leq 1 \ (1 \leq k \leq m)\}$, defined for holomorphic functions $f_1, \ldots, f_m$ on $\mathbf{D}^n$, with a submanifold $\Sigma = \{(w, z) \in \mathbf{D}^{m+n} \mid z = f_k(w) \text{ for } 1 \leq k \leq m\}$ of $\mathbf{D}^{m+n}$ had been introduced by A. Weil (1906-98) [We] to obtain a generalized Cauchy's integral formula. The point is that Oka's reduction of the problem is based on the solution of an interpolation problem on the polydiscs, or more specifically, an extension problem from Σ.

Let's see below how he formulated the strategy for that.

1. Définitions. Dans l'espace $((x))$ des n variables complexes $x_1, x_2, \ldots, x_n$, considérons la région[1] Δ définie par

$$(\Delta) \qquad x_i \in X_i, \qquad R_j((x)) \in Y_j, \qquad (i=1, 2, \ldots, n;\ j=1, 2, \ldots, \nu),$$

où X_i, Y_j sont des domaines[2] bornés et univalents (schlicht) sur le plan, et $R_j((x))$ des fonctions rationnelles. Toute région que l'on peut exprimer en forme pareille, sera dite pour simplifier la langue, *d'appartenir à* (Ω_0). Etant donnée une région de (Ω_0) dans l'espace $((x))$, nous appelons *l'ordre de la région* le minimum des nombres de fonctions rationnelles des n variables x_i, par lesquelles on puisse définir la région, les fonctions x_i étant exclues. L'ordre de la région Δ est $\leqq \nu$; toute région de (Ω_0) de l'ordre nul est un domaine cylindrique. Il est commode de donner ici les problèmes à traiter en forme concrète.

Problème I. En introduisant les nouvelles variables complexes $y_1, y_2, \ldots, y_\nu$, considérons l'espace $((x, y))$, et dans lequel la multiplicité Σ définie par

$$(\Sigma) \qquad y_j = R_j((x)), \qquad ((x)) \in \Delta, \qquad (j=1, 2, \ldots, \nu).$$

La frontière de la multiplicité Σ se situe entièrement sur le contour du domaine cylindrique,

$$(C) \qquad x_i \in X_i, \qquad y_j \in Y_j, \qquad (i=1, 2, \ldots, n;\ j=1, 2, \ldots, \nu).$$

Soit $f((x))$ une fonction holomorphe des n variables x_i dans Δ[3]; regardons la de nouveau comme fonction des $n+\nu$ variables x_i, y_j, elle est holomorphe en tout point M, $((x^0, y^0))$ sur Σ, puisque $((x^0))$ est nécessairement un point de Δ.

Dans ces conditions, *construire une fonction holomorphe dans (C'), admettant la valeur donnée $f(M)$ pour tout point M sur la portion de Σ dans (C')*, (C') étant un domaine donné dans l'intérieur à (C).

Nous l'appelons *problème I d'ordre ν*, dont ν est la moitié de la différence entre le nombre des dimensions de l'espace $((x, y))$ est celui de la multiplicité Σ. L'ordre commence cette fois-ci, de 1.

Problème II. Nous donnons dans la région Δ des pôles $(\mathfrak{p})$ par la méthode habituelle, à savoir que: En faisant correspondre à tout point P de Δ, un polycylindre (γ) et une fonction $g((x))$ qui y est méromorphe, de façon que, pour toute partie commune des (γ), les fonctions attachées $g((x))$ soient équivalentes mutuellement par rapport à soustraction; nous définissons les pôles $(\mathfrak{p})$ localement par ceux des fonctions $g((x))$. Alors, voici le problème:

Trouver une[1] *fonction méromorphe admettant les pôles* $(\mathfrak{p})$ *donnés, pour une région Δ' donnée à priori dans l'intérieur à Δ.*

Ceci sera appelé *problème II d'ordre μ*, μ étant l'ordre de Δ. On sait bien grâce à M. P. Cousin,[2] que le problème II de l'ordre nul est résoluble.

from "Sur Les Fonctions Analytiques de Plusieurs Variables, par Kiyoshi Oka, IWANAMI SHOTEN Tokyo Japan 1961".

In this way, Oka released the interpolation theory from solutions of (C-II), and put it in a much broader context.

Oka continued the study of the interpolation problem and established that any holomorphic function defined on a closed complex analytic subset (of arbitrary codimension) of a domain $\Omega \subset \mathbf{C}^n$ can be extended holomorphically to Ω if Ω is a domain of holomorphy. For that, he introduced the notion of ideals with indeterminate domains in [O-5] and proved the celebrated coherency theorems. Furthermore, Oka characterized the domains of holomorphy over **C** by pseudoconvexity (cf. [O-4, 6]), by which he laid a foundation for exploring more refined analysis and geometry on multi-dimensional domains (cf. [Kn-1, 2], [H-1, 2], [Fe-1, 2], [Hi]). Concerning more detail of the latter work of Oka, see [O-7] and [Li], for instance.

Let's proceed next to Oka principle. After the solution of Cousin's first problem, Cousin's second problem must have been completely under Oka's control, since the difference of the first and the second is only additive or multiplicative and the topological character of the logarithm was well understood (the argument principle). Perhaps a nontrivial thing here is the choice of a condition that pinpoints the obstruction in transforming the multiplicative question to the additive one. After analyzing a counterexample of T. H. Gronwall (1877-1932) [Gn], Oka found a homotopical condition. The result was unexpectedly beautiful even to Oka himself. In fact, he wrote about this article in a letter (dated 1937/12/19) to his friend Ukichiro Nakaya (1900-62, a physicist who made artificial snowflakes in 1936) expressing his delight : "As for the other article, on which I started to work in Novembre, I could finish writing it in a wonderfully perfect form." (cf. [T]).

One may recall here that Cauchy's theory of complex integration is homological and one of Gauss's proofs of the fundamental theorem of algebra is homotopical.

Anyway, let's see how Oka formulated it in [O-3].

Introduisons la notion auxiliaire suivante : Des zéros $(\mathfrak{z})$ donnés dans D seront appelés *balayables*, s'il correspond pour tout point P de D, une hypersphère (γ) de centre P et une fonction continue $f((x),t)$ sur $[(\gamma),\ 0\leqq t\leqq 1]$, de telle façon que :

1° $f((x),0)$ admette $(\mathfrak{z})$ pour zéros, et $f((x),1)$ soit non-nulle ;

2° $f((x),t)$ ne s'annule identiquement dans aucune portion à $(2n+1)$ dimensions ;

3° Pour toute paire de hypersphères contiguës $(\gamma_1), (\gamma_2)$, les fonctions correspondantes $f_1((x),t)$, $f_2((x),t)$ soient équivalentes sur $[(\delta), 0\leqq t\leqq 1]$, (δ) étant la partie commune de (γ_1) et (γ_2).

from "Sur Les Fonctions Analytiques de Plusieurs Variables, ibid"

As above, Oka calls a locally principal analytic subset $(\mathfrak{z})$ of the domain D *balayable* (=sweepable) if a system of its local defining functions is homotopically equivalent, in an appropriate sense, to that of nowhere vanishing functions. It means that $(\mathfrak{z})$ can be continuously deformed towards the boundary so that it is swept out from the interior of D. A straightforward example of non-balayable $(\mathfrak{z})$ is given by the set defined by $z_1 = (z_2)^i$ in $(\mathbf{C}^*)^2$, where $\mathbf{C}^* := \{z \in \mathbf{C} \mid z \neq 0\}$. Notice that it intersects with $|z_1| = |z_2| = 1$ transversally at (1,1).

Oka showed that $(\mathfrak{z})$ is globally principal if and only if it is balayable. This choice of language suggests something related to the potential theory.

Anyway, as we shall see later, [O-3] was a sprouting of the seed (C-II) that became a big tree.

Let's proceed towards the cohomological formulation of [O-3], which has a great merit of abstraction. First, (C-I) and (C-II) are naturally generalized as follows.

(C-I') Given an open covering $\{U_\alpha\}_{\alpha \in A}$ of Ω and a system of holomorphic functions $u_{\alpha\beta}$ defined on $U_\alpha \cap U_\beta$ such that $u_{\alpha\beta} + u_{\beta\gamma} + u_{\gamma\alpha} = 0$ holds whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, find a system of holomorphic functions u_α on U_α such that $u_{\alpha\beta} = u_\beta - u_\alpha$ is satisfied on $U_\alpha \cap U_\beta$.

(C-II') Given $\{U_\alpha\}$ as above and a system of nowhere vanishing holomorphic functions $v_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ such that $v_{\alpha\beta} v_{\beta\gamma} = v_{\alpha\gamma}$ holds on $U_\alpha \cap U_\beta \cap U_\gamma$, find holomorphic functions v_α without zeros on U_α satisfying $v_{\alpha\beta} = v_\beta / v_\alpha$ on $U_\alpha \cap U_\beta$.

In shorter notations of sheaf cohomology, assertions implying affirmative answers to(C-I') and (C-II') are expressed respectively as $H^1(\Omega, \mathcal{O}) = 0$ and $H^1(\Omega, \mathcal{O}^*) = 1$.

Here $\mathcal{O}$ denotes the structure sheaf of Ω, i.e. the sheaf over Ω of the rings of germs of holomorphic functions, and $\mathcal{O}^*$ denotes the sheaf of the multiplicative groups of germs of nowhere vanishing holomorphic functions.

Oka's theorem in [O-2] says that $H^1(\Omega, \mathcal{O}) = 0$ holds if Ω is a domain of holomorphy. This was generalized as Cartan's Thoerem B on Stein manifolds (cf. §3).

In [O-3], Oka explicitly describes a nontrivial element of $H^1(\Omega, \mathcal{O}^*)$ when Ω is the product of two annuli.

In the thirties, while Oka was busy in completing the picture of his first discovery, important new ideas were brought into analysis and topology. They are Morse theory and fiber bundles (cf. [M], [St]). Moreover, Cartan [Ca-3] started to study local bases of the ideals of holomorphic functions and matrices transforming them to each other (cf. §3). Based on such a development, certain "non-abelian generalization" of (C-II) became tractable. After the cohomological formulation of [O-3] by Cartan and Serre, Frenkel [Fr-1, 2, 3] extended it to the case of solvable groups. Finally, Grauert succeeded in completely generalizing Oka's solution of (C-II) in this context.

§2. Towards Oka principle with growth conditions Before entering the topic of Oka manifolds, let us make a small detour and review some refinements of Oka principle, because there seems to remain interesting open questions in this direction.

A motivation in refining Oka principle comes from Weierstrass's σ-function:

$$\sigma(z) = z \prod_{\gamma\in\Gamma\setminus\{0\}}(1 - z/\gamma)\exp(z/\gamma+(1/2)(z/\gamma)^2)$$

where Γ is a lattice in $\mathbf{C}$. This transcendental function is related to algebraic geometry via Weierstrass's $\wp$-function $\wp(z) := -(\log\sigma(z))''$ in such a way that the ratio $(1 : \wp : \wp')$ embeds $\mathbf{C}/\Gamma$ into $\mathbf{CP}^2$.

A general question arising from this part of Weierstrass theory is, roughly speaking, to find systems of defining functions with "slow growth" for those analytic subsets that are "small" at infinity. To be more specific, let us consider the class of functions of finite order (cf. [C-G], [AA]).

Let X be a compact complex manifold of dimension n, let $D \subset X$ be a complex analytic subset of codimension one, let ds^2 be a Hermitian metric on X, and let $\delta(z)$ be the distance from $z \in X$ to D with respect to ds^2.

Definition 1.

$$\mathcal{O}_{\text{f.o.}}(X, D) = \{f \in \mathcal{O}(X-D) \mid \log^+|f(z)| < k\delta(z)^{-l} \quad \text{for some } k, l \in \mathbf{N}\},$$

where $\log^+ r := \max\{\log r, 0\}$ for any $r \geq 0$.

Clearly, $\mathcal{O}_{\text{f.o.}}(X, D)$ is a subring of $\mathcal{O}(X-D)$. For any $f \in \mathcal{O}_{\text{f.o.}}(X, D)$, the *order* of f is defined as

$$\inf\{ l \mid \log^+|f(z)| < k\delta(z)^{-l} \text{ holds for some } k\in\mathbf{N}\}.$$

It is easy to see that $\sigma(z)$ is of order 2.

Let $Z \subset X - D$ be a complex analytic subset of pure dimension m. We put

$$Z[r] = \{z \in Z \mid \delta(z) > r\}$$

and

$$N(Z, r) = \int_r^1 \left(\int_{Z[t]} \omega^m\right)(dt/t),$$

where ω denotes the fundamental form of ds^2.

Let $\overline{Z}$ denote the closure of Z in X. Then a theorem of Bishop-Stoll says

$$\overline{Z} \text{ is complex analytic} \Leftrightarrow \int_Z \omega^m < \infty \Leftrightarrow N(Z, r) = O(\log r^{-1}) \quad (r \to 0)$$

(cf. [Stz]).

Definition 2. Z is of *finite order* if $N(Z, r) = O(r^{-l})$ holds for some $l \in \mathbf{N}$. The infimum of such l is called the *order* of Z.

Clearly, lattices in $\mathbf{C}^n$ are of order 2n. In view of that lattices in $\mathbf{C}$ are defined by holomorphic functions of order 2, one may ask whether every lattice in $\mathbf{C}^n$ is the common zero set of n holomorphic functions of order 2. It is certainly ture if the lattice is the product of n lattices in $\mathbf{C}$, but nothing more seems to be known in the literature. The point is that there are no periodic hypersurfaces when the quotient of $\mathbf{C}^n$ by the lattice is of algebraic dimension zero. Nevertheless, it is likely that there exist n "quasi-periodic" hypersurfaces whose intersection is the given lattice.

By the way, concerning the defining functions of hypersurfaces, P. A. Griffiths [Gr-1, 2] showed that, if $X = \mathbf{CP}^n$, D is a hyperplane and Z is a hypersurface in $X - D$, Z is of finite order if there exists $f \in \mathcal{O}_{f.o.}(X, D)$ such that $f^{-1}(0) = Z$. The difficulty of generalizing Griffiths's result to higher codimensional Z was shown by a counterexample of M. Cornalba and B. Shiffman [C-S], which is a discrete subset of $\mathbf{C}^2$ of infinite order that can be defined by two functions of order zero.

On the other hand, for any smooth algebraic curve C in $\mathbf{C}^n$, it is proved in [F-Oh] that the ideal sheaf of C admits $n - 1$ generators that are of finite order. This solves a question posed by Cornalba and Griffiths in [C-G].

§3. Modern theory of Oka principle In 2001/November, in a conference held in Kyoto university in honor of Oka, the author had a chance to be at the same lunch table with Grauert and Forstnerič. After lunch we sat in a cozy café in which Oka could have haunted with his students. Grauert did not talk so much, but listened to us carefully. At some point he seemed to be delighted and said, "Yes, vector bundles over noncompact Riemann surfaces are trivial." It was when Forstnerič was explaining to us a pleasantly short proof of the following theorem of R. C. Gunning and R. Narasimhan.

Theorem 1. (cf. [G-N]) Any noncompact Riemann surface admits a holomorphic function without critical points.

Theorem 1 also realizes Oka principle because it was known before [G-N] that noncompact Riemann surfaces are Stein on one hand (cf. [B-S-2]) and they admit smooth submersions to $\mathbf{C}$ on the other (cf. [Ph]). This nice result gave a motivation to improve the bound for the dimension of the euclidean space into which every n-dimensional Stein manifold is properly embeddable (cf. [F-4], [E-G]).

We shall reproduce Forstnerič's new proof of Theorem 1 below because it illustrates well the modern theory of Oka principle initiated by Gromov [Gm-1, 2]. (See also a paper of G. Henkin and J. Leiterer [H-L].)

Proof of Theorem 1 after Forstnerič: Let U_1 and U_2 be relatively compact open subsets of a noncompact Riemann surface X such that

1) $U_1 \cap U_2$ is $\mathcal{O}(U_2)$-conex, i. e. the restriction map $\mathcal{O}(U_2) \rightarrow \mathcal{O}(U_1 \cap U_2)$ has dense image with respect to the topology of uniform convergence on compact subsets.

and

2) U_2 is simply connected.

Suppose that there exist $f \in \mathcal{O}(U_1)$ such that df vanishes nowhere. Then, by 1) and 2), and since $\dim X = 1$, for any compact subset $K \subset U_1 \cap U_2$ and for any $\varepsilon > 0$, one can find $g \in \mathcal{O}(U_2)$ without critical points such that $\sup_K |f - g| < \varepsilon$, by the Runge theorem applied to the logarithm of the coefficient of df. Let us fix a Hermitian metric ds^2 on X. Then, for any compact subset K_0 of K, there exists a neighborhood $V \supset K_0$ such that, for any $\varepsilon > 0$ one can find the above g in such a way that $f = g \circ \iota$ holds on V for some holomorphic embedding $\iota: V \longrightarrow X$ satisfying $\sup_K \mathrm{dist}(z, \iota(z)) < \varepsilon$. Here $\mathrm{dist}(z, \iota(z))$ denotes the distance between z and $\iota(z)$ with respect to ds^2.(Remember that f and g have no critical points.)

Hence, for any compact subsets $K_i \subset U$ $(i = 1, 2)$ such that $K_1 \cap K_2 \subset K$, there exist neighborhoods $V_i \supset K_i (i = 1, 2)$ such that, for any sufficiently small $\varepsilon > 0$ with g and ι as above, one can find holomorphic embeddings $\sigma_i : V_i \longrightarrow X$ such that $\sup_{K_i} \text{dist}(z, \sigma_i(z)) < C\varepsilon$, $\sigma_1(V_1) \supset K_1 \cap K_2$, and $\iota = \sigma_2 \circ \sigma_1^{-1}$ holds on a neighborhood of $K_1 \cap K_2$. Here C denotes a constant depending only on V_1 and V_2. This part is nontrivial but more or less standard since [G-2, 3, 4].

Since $f \circ \sigma_1 = g \circ \sigma_2$ holds on $K_1 \cap K_2$, one has a function $\tilde{f} = f \circ \sigma_1 = g \circ \sigma_2$ which approximates f and without critical points on a neighborhood of $K_1 \cup K_2$.

X is an increasing union of open subsets $X_1 \subset X_2 \subset \cdots$ such that $X_k = U_1$ and $X_{k+1} = U_1 \cup U_2$ with U_i as above (cf. [B-S-2]. See also [Oh] and [Dm]). Hence we obtain a desired function by taking the limit of $\tilde{f}$ as above. □

By a similar argument as above, Forstnerič generalized Theorem 1 as follows.

Theorem 2. (cf. [Fn-3]) An n-dimensional Stein manifold admits $[(n+1)/2]$ holomorphic functions with pointwise independent differentials.

Theorem 2 is sharp because there exist n-dimensional Stein manifolds whose tangent bundles do not have trivial quotient bundles of rank $[(n+1)/2] + 1$. The latter had already been observed by Forster in [F-3].

Forster's counterexample: Let

$$Y = \{(x:y:z) \in \mathbf{CP}^2 \mid x^2 + y^2 + z^2 \neq 0\}.$$

Put

$$X = \begin{cases} Y^m & \text{if } n = 2m \\ Y^m \times \mathbf{C} & \text{if } n = 2m+1. \end{cases}$$

Since Y is Stein, X is a Stein manifold of dimension n. The point is

"The first Chern class $c_1(Y)$ of the tangent bundle TY of Y is not zero" .

Proof. The real projective plane $\mathbf{RP}^2$, which is naturally embedded in Y, is a deformation retract of Y. A deformation $F_t : Y \longrightarrow Y$, $1 \geq t \geq 0$ to $\mathbf{RP}^2$ is given by

$$F_t(x{:}y{:}z) = ((\mathrm{Re}\, x + i\, t\, \mathrm{Im}\, x) : (\mathrm{Re}\, y + i\, t\, \mathrm{Im}\, y) : (\mathrm{Re}\, z + i\, t\, \mathrm{Im}\, z)),$$

where $x^2 + y^2 + z^2 > 0$. Since $TY|\mathbf{RP}^2$ is the complexification of the tangent bundle τ of $\mathbf{RP}^2$, $TY|\mathbf{RP}^2 \cong \tau\oplus\tau$. The total Stiefel-Whitney class $w(\tau)$ of τ is $(1 + \alpha)^3$, where $\alpha \in H^1(\mathbf{RP}^2, \mathbf{Z}_2)$ is the generator of the cohomology ring $H^*(\mathbf{RP}^2, \mathbf{Z}_2)$ (cf. [Hu]). Therefore $w(\tau\oplus\tau) = (1 + \alpha)^6 = 1 + \alpha^2$. Since the second Stiefel-Whitney class $w_2(TY)$ is the modulo 2 reduction of the first Chern class $c_1(Y)$ of TY, it follows that $c_1(Y) \neq 0$. □

Hence $c_m(X)$ is the nonzero element of $H^{2m}(X, \mathbf{Z}) = \mathbf{Z}_2$, so that TX cannot have the trivial bundle of rank $[(n+1)/2] + 1$ as a quotient.

Forster [F-4] noted that the above example X is not holomorphically embeddable into $\mathbf{C}^{n+[n/2]}$. Recall that, by Remmert-Narasimhan-Bishop's theorem, an n-dimensional Stein manifold can be properly embedded in $\mathbf{C}^q$ for $q = 2n + 1$. The main theorem in [F-4] asserts that $q = [5n/3] + 2$ suffices.

In 1992, Y. Eliashberg and Gromov published [E-G] which proved that the proper embeddability is true for $q > (3n + 1)/2$. (See also [Schr].) The method of [Gm-2] made the proof even simpler than [F-4].

In spite of this remarkable progress, it is still an open problem whether or not any open Riemann surface is properly embeddable into $\mathbf{C}^2$. We note that a highly transcendental proper embedding of the unit disc $\mathbf{D} \subset \mathbf{C}$ into $\mathbf{C}^2$ was discovered by T. Nishino (1931-1985) in the study of holomorphic functions on $\mathbf{C}^2$ (cf. [Ki]). On the other hand, very explicit embeddings into symmetric bounded domains are known for homogeneous bounded domains (cf. [I]).

For the remainder of this survey article, we shall try to catch a glimpse of [Gm-2] and subsequent developments made by Forstnerič, J. Prezelj, F. Lárusson and other people which culminated in the introduction of Oka manifolds and Oka maps in [Fn-7, 8]. Recall that Grauert's Oka principle is essentially to say that, given a complex analytic fiber bundle P over a Stein manifold M, each connected component of the space of continuous sections $C^0(M, P)$ contains precisely one connected component of the space $\mathcal{O}(M, P)$ of holomorphic sections, provided that P is a principal bundle (cf. (G-1')). Gromov generalized Grauert's Oka principle to the bundles with much more general fibers. Namely, he required the fibers only to satisfy the following property.

Definition 3. A complex manifold X is said to be *elliptic* if there exist a holomorphic vector bundle $\pi : E \longrightarrow X$ and a holomorphic map $s : E \longrightarrow X$ such that $\pi = s$ holds on the zero section and $s \mid E_x$ is a submersion at $0 \in E_x$ for all $x \in X$.

Complex homogeneous manifolds are obviously elliptic in this sense.

Theorem 3. (cf. [Gm-2] and [Fn-P-2]) The inclusion $\mathcal{O}(M, P) \subset C^0(M, P)$ is a weak homotopy equivalence if M is a Stein manifold and the fibers of P are elliptic.

Here, weak homotopy equivalence of a map means that it induces isomorphisms of homotopy groups.

A great advantage of this generalization is that, for any complex analytic subset $A \subset \mathbf{C}^n$ of codimension ≥ 2 whose closure $\langle A \rangle$ in $\mathbf{CP}^n$ does not contain the hyperplane at infinity, $\mathbf{C}^n - A$ becomes elliptic. Here is the main observation for that (cf. [Fn-9, Chapter 4]). Assume that $\pi : \mathbf{C}^n \longrightarrow \mathbf{C}^{n-1}$ is a linear projection such that the restriction $\pi | A : A \longrightarrow \mathbf{C}^{n-1}$ is proper; then $\pi(A)$ is a closed analytic subset of $\mathbf{C}^{n-1}$. If $\dim A < n - 1$ then $\pi(A)$ is a proper subset of $\mathbf{C}^{n-1}$, and hence there exist nonconstant holomorphic functions on $\mathbf{C}^{n-1}$ that vanish on A. Let f be such a function. Choose a nonzero vector $v \in \ker \pi$ and consider it as a constant holomorphic vector field on $\mathbf{C}^n$. Consider a vector field on $\mathbf{C}^n$ vanishing on A and generating the $\mathbf{C}$+ action $F\ (z) = z + t f(\pi(z))v$, $t \in \mathbf{C}$. This action preserves $\pi^{-1}(\pi(A)) = A + \mathbf{C}v$ pointwise. This suggests to consider those A such that, in suitable coordinates on $\mathbf{C}^n$, there exist sufficiently many linear projections $\pi : \mathbf{C}^n \to \mathbf{C}^{n-1}$ that are proper when restricted to A.

Definition 4. A closed analytic subset $A \subset \mathbf{C}^n$ is *tame* if there exists a holomorphic automorphism $\Phi \in \mathrm{Aut}\, \mathbf{C}^n$ such that $\langle \Phi(A) \rangle$ does not contain the hyperplane at infinity.

Every closed complex subvariety of $\mathbf{C}^n$ contained in a proper algebraic subvariety is tame. A hypersurface A is tame if and only if there exists a $\Phi \in \mathrm{Aut}\, \mathbf{C}^n$ such that $\Phi(A)$ is algebraic.

Proposition 1. If $A \subset \mathbf{C}^n$ is a tame analytic subset of codimension at least two, then there exist finitely many holomorphic vector fields on $\mathbf{C}^n$ that vanish on A and that span the tangent space $T_z\mathbf{C}^n$ at every point $z \in \mathbf{C}^n - A$.

Corollary. $\mathbf{C}^n - A$ is elliptic if A is tame.

After [Gm-2], a lot of so-called Oka properties have been studied. These concern the task of deforming a continuous map f from a Stein manifold into a holomorphic map. The study of Oka properties are partly motivated by the model theory (cf. [Lá-1, 2]).

BOPI: If the deformation can always be done for any continuous map f from a Stein manifold M to X in such a way that f is kept fixed on the way on a closed complex submanifold of M on which f is holomorphic, then X is said to have the *basic Oka property with interpolation* (BOPI).

BOPA: If f is always deformable to a holomorphic map so that the deformed maps stay arbitrarily close to f on an $\mathcal{O}(M)$-convex compact subset K of M on which f is holomorphic and are holomorphic on a common neighborhood of K, then X is said to have the *basic Oka property with approximation* (BOPA).

CAP: If every holomorphic map to X from a compact convex subset K of $\mathbf{C}^n$ can be approximated uniformly on K by holomorphic maps from $\mathbf{C}^n$ to X, then X is said to have the *convex approximation property* (CAP).

CIP: Let us call a submanifold T of $\mathbf{C}^n$ *special* if T is the graph of a proper holomorphic embedding of a convex domain Ω in $\mathbf{C}$, $k \geq 1$, as a submanifold of $\mathbf{C}^n$, i.e. $T = \{(x, \phi(x)) \in \mathbf{C}^k \times \mathbf{C}^{n-k} \mid x \in \Omega\}$, where $\phi : \Omega \to \mathbf{C}^{n-k}$ is a proper holomorphic embedding. Then X is said to satisfy the *convex interpolation property* (CIP) if each holomorphic map to X from a special submanifold T of $\mathbf{C}^n$ extends holomorphically to $\mathbf{C}^n$, i.e. the restriction map $\mathcal{O}(\mathbf{C}^n, X) \to \mathcal{O}(T, X)$ is surjective.

CAP and CIP are remarkable because the sources of maps are restricted to very special sets. They were introduced in [Fn-6] and [Lá-3], respectively. (See also [Lá-4].)

The above properties all have parametric versions where instead of a single map f we have a family of maps depending continuously on a parameter.

F-POP (Forstnerič's Parametric Oka Property): X is said to satisfy the *parametric Oka property* if the following is true: Given a compact $\mathcal{O}(X)$-convex subset K of X, a closed complex analytic subset X' of X, compact sets $P_0 \subset P$ in a Euclidean space $\mathbf{R}^m$, and a continuous map $f : P \times X \to Z$ such that

(a) for every $p \in P$, $f(p, \ ): X \to Z$ is a section of $Z \to X$ that is holomorphic on a neighborhood of K (independent of p) and such that $f(p, \ )|X'$ is holomorphic on X',

and

(b) $f(p, \ )$ is holomorphic on X for every $p \in P_0$,

there is a homotopy $f_t : P \times X \to Z$ $(t \in [0, 1])$, with $f_0 = f$, such that f_t enjoys properties (a) and (b) for all $t \in [0, 1]$, and also the following hold:

(i) $f_1(p, \ )$ is holomorphic on X for all $p \in P$,

(ii) f_t is uniformly close to f on $P \times K$ for all $t \in [0, 1]$,

and

(iii) $f_t = f$ on $(P_0 \times X) \cup (P \times X')$ for all $t \in [0, 1]$.

If, in addition to (a) and (b), every section $f(p, \ ): X \longrightarrow Z$ is holomorphic on a neighborhood U of X', then for every $s \in \mathbf{N}$, f_t is holomorphic on U for every $t \in [0, 1]$ and $p \in P$, and it agrees with f_0 to order s on X'. In particular, f_t may be chosen tangent to f to a given finite order along X'.

It is remarkable that more than a dozen Oka properties including the above mentioned ones are all equivalent (cf. [Fn-9]).

Definition 5. X is called an *Oka manifold* if it satisfies either BOPI, BOPA, CAP, CIP, or F-POP.

The strongest implication is CAP $\Rightarrow$ F-POP which was essentially conjectured by Gromov [Gm-2, p. 881] and established by Forstnerič [Fn-6, 7].

In [Fn-9], F-POP is adopted as the defining property of Oka manifolds.

A holomorphic map is said to be an *Oka map* if it is a topological (Serre) fibration and it enjoys "*Forstnerič's parametric Oka property for maps*". Instead of describing this strong property, we shall be contented with mentioning another one which turned out to be equivalently strong (cf. [Fn-8]).

BOP for maps: Let E and B be reduced complex spaces. A holomorphic map $\pi : E \longrightarrow B$ is said to enjoy the *basic Oka property* (BOP) if, given a holomorphic map $f : X \longrightarrow B$ from a reduced Stein space X and a continuous map $F_0 : X \longrightarrow E$ satisfying $\pi F_0 = f$ (a lifting of f) such that F_0 is holomorphic on a closed complex analytic subset X' of X and in a neighborhood of a compact $\mathcal{O}(X)$-convex subset K of X, for any $\varepsilon > 0$ there is a homotopy of liftings $F_t : X \longrightarrow E$ $(t \in [0, 1])$ of f to a holomorphic lifting F_1 such that for every $t \in [0, 1]$, F_t is holomorphic in a neighborhood of K (independent of t), $\sup_{x \in K} \mathrm{dist}(F_t(x), F_0(x)) < \varepsilon$, and $F_t|X' = F_0|X'$ (the homotopy is fixed on X').

For further information on the modern Oka principle, the reader is referred to the forthcoming book [Fn-9] which contains introduction to Stein spaces, a chapter on automorphisms of $\mathbf{C}^n$, Gromov's Oka principle, proof of the equivalence of Oka properties, and many interesting applications.

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