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Abstract Let $n > 1$ be an integer. We prove that holomorphic maps from Stein manifolds X of dimension $<n$ to the complement $\mathbb{C}^n \setminus L$ of a compact convex set $L \subset \mathbb{C}^n$ satisfy the basic Oka property with approximation and interpolation. If L is polynomially convex then the same holds when $2 \dim X \leq n$. We also construct proper holomorphic maps, immersions and embeddings $X \to \mathbb{C}^n$ with additional control of the range, thereby extending classical results of Remmert, Bishop and Narasimhan.

Keywords Oka principle · Holomorphic flexibility · Oka manifold · Polynomial convexity

Mathematics Subject Classification (2010) Primary 32E10 · 32E20 · 32E30 · 32H02; Secondary 32Q99

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1 Introduction

The results in the present paper were motivated by the question whether the complement of a compact convex set L in a Euclidean space $\mathbb{C}^n$ for $n > 1$ is an *Oka manifold*; that is, whether maps $X \to \mathbb{C}^n \backslash L$ from Stein manifolds X satisfy the Oka principle. (See [18, §5.4] or [19] for these notions.) In particular, what is the answer when L is the closed ball $\overline{\mathbb{B}} \subset \mathbb{C}^n$? Using the characterization of Oka manifolds by the *Convex Approximation Property* (cf. [18, Theorem 5.4.4]) the question can be phrased as follows. We adopt the convention that a map is holomorphic on a compact set K in a complex manifold X if it is holomorphic on an unspecified open neighborhood of K in X.

$(*)$ *Let K be a compact convex set in $\mathbb{C}^N$. Is every holomorphic map $K \to \mathbb{C}^n \backslash \overline{\mathbb{B}}$ a uniform limit on K of entire maps $\mathbb{C}^N \to \mathbb{C}^n \backslash \overline{\mathbb{B}}$?*

This fundamental problem of Oka theory seems out of reach with the current methods when $N \geq n$. However, we give positive results for source Stein manifolds of dimension $< n$. Specifically, we prove the following result.

Theorem 1 *Let L be a compact convex set in $\mathbb{C}^n$ for some $n > 1$. Assume that X is a Stein manifold of dimension $< n$, K is a compact $\mathscr{O}(X)$-convex set in X, and X' is a closed complex subvariety of X. For any holomorphic map $f : K \cup X' \to \mathbb{C}^n \backslash L$ and number $\epsilon > 0$ there exists a holomorphic map $F : X \to \mathbb{C}^n \backslash L$ satisfying*

$$(i)\ ||F - f||_K < \epsilon \quad \text{and} \quad (ii)\ F|_{X'} = f|_{X'}.$$

The same is true if L is polynomially convex and $2 \dim X \leq n$. If the map $f|_{X'} : X' \to \mathbb{C}^n$ is proper or $X' = \emptyset$ then F can also be chosen proper. If $2 \dim X \leq n$ then F can be chosen an immersion (an embedding if $2 \dim X + 1 \leq n$) provided that $f|_{X'}$ is such.

The precise assumption on f in the theorem is that it is holomorphic on an open neighborhood of K and the restriction $f|_{X'}$ is holomorphic on the subvariety X'.

Theorem 1 is proved in §4 as a part of Theorem 15. The latter result pertains to the more general situation when the initial map $f : K \cup X' \to \mathbb{C}^n$ may intersect the compact set L (only) in the interior of K; in this case we construct a (proper) holomorphic map $F : X \to \mathbb{C}^n$ satisfying the conclusion of Theorem 1 and also the condition $F(X \backslash K) \subset \mathbb{C}^n \backslash L$. As explained in Remark 16 in §4, there are no topological obstructions to extending a continuous map $K \cup X' \to \mathbb{C}^n$, which sends $bK \cup (X' \backslash K)$ to $\mathbb{C}^n \backslash L$, to a continuous map $X \to \mathbb{C}^n$ sending $X \backslash \mathring{K}$ to $\mathbb{C}^n \backslash L$.

Two new ideas are introduced here for the first time. One of them, which depends on the Andersén-Lempert theory, provides the separation of certain pairs of disjoint compact sets in $\mathbb{C}^n$ by Fatou-Bieberbach domains (Proposition 9 in §2). The other one concerns the separation of a compact convex set $L \subset \mathbb{C}^n$ from certain big pieces of a non-closed Stein variety in $\mathbb{C}^n \backslash L$ by Fatou-Bieberbach domains (Lemma 12 in §3). This combines Proposition 9 with a result of Dor [9] and of Drinovec Drnovšek and Forstnerič [12] (see Theorem 6 in §2). The latter result enables us to perturb the variety so that the union of a big compact piece of the new variety and the set L is polynomially convex.

Granted these new techniques, the proof of Theorem 1 proceeds as in the Oka theory by inductively enlarging the domain on which the map is holomorphic. Here is a brief description of the main step. Assume that $B \subset X$ is a compact convex bump attached to a compact strongly pseudoconvex domain $A \subset X$. Let $C = A \cap B$, and assume that $f_0 : A \to \mathbb{C}^n$ is a holomorphic map such that $f_0(C) \cap L = \emptyset$. We show that f_0 can be approximated uniformly on C by a holomorphic map $f : C \to \mathbb{C}^n \backslash L$ whose image $f(C)$ can be separated from L by a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^n$, in the sense that $f(C) \subset \Omega \subset \mathbb{C}^n \backslash L$ (see Lemma 12 in §3). Since Ω is biholomorphic to $\mathbb{C}^n$, the Oka-Weil theorem allows us to approximate the map $f|_C$, uniformly on C, by a holomorphic map $g\colon B \to \Omega$. Finally we glue f_0 and g into a holomorphic map $\tilde{f}\colon A \cup B \to \mathbb{C}^n$ such that $\tilde{f}(B) \subset \mathbb{C}^n \backslash L$. The last statement concerning immersions and embeddings is an immediate consequence of the general position arguments.

Applying Theorem 1 with X' a countable discrete set gives the following corollary.

Corollary 2 *Let L be a compact convex set in $\mathbb{C}^n$ for some $n > 1$. Assume that X is a Stein manifold of dimension $< n$ and $\{a_j\}_{j\in\mathbb{N}} \subset X$ is a discrete sequence without repetition. For every sequence $\{b_j\}_{j\in\mathbb{N}} \subset \mathbb{C}^n \backslash L$ there exists a holomorphic map $F\colon X \to \mathbb{C}^n \backslash L$ satisfying $F(a_j) = b_j$ for all $j = 1, 2, \ldots$. In particular, there exists a holomorphic map $X \to \mathbb{C}^n \backslash L$ with everywhere dense image. If $L \subset \mathbb{C}^n$ is polynomially convex then the above statements hold for Stein manifolds X with $2\dim X \le n$.*

In particular, if L is a compact polynomially convex set in $\mathbb{C}^n$ for some $n > 1$ then there exists a holomorphic map $\mathbb{C} \to \mathbb{C}^n \backslash L$ with a dense image.

Theorem 1 extends classical theorems of Remmert [26], Bishop [7] and Narasimhan [25] concerning the existence of proper holomorphic maps of Stein manifolds of dimension $< n$ into $\mathbb{C}^n$; here we provide additional control on the range of the map. When $2\dim X \le n$, we get similar improvements in theorems of Acquistapace et al. [1] concerning the interpolation of proper holomorphic immersions $X \to \mathbb{C}^n$ (embeddings when $2\dim X + 1 \le n$) on closed complex subvarieties of X. Our proof is conceptually different from those in the mentioned papers, mainly because of our novel use of Fatou-Bieberbach domains and the gluing techniques.

Two other lines of related results deserve to be mentioned. One of them was developed by Dor [10] and Drinovec Drnovšek and Forstnerič [11,12], and it pertains to the following situation. Let X be a smoothly bounded, relatively compact, strongly pseudoconvex Stein domain in another complex manifold $\widetilde{X}$. Let Z be a complex manifold with $\dim Z > \dim X$, and assume that $f\colon \overline{X} \to Z$ is a continuous map that is holomorphic on X. Under suitable geometric conditions on Z (depending on $\dim X$) that can be expressed in terms of q-convexity and Morse indices of a Morse exhaustion function, f can be approximated uniformly on compact subsets of X by proper holomorphic maps $X \to Z$. A simplified version of the main result from [12], which is stated as Theorem 6 in §2 below, is an important ingredient in the proof of our main theorem. The novelty of the results in the present paper is that X can be an arbitrary Stein manifold of dimension $< n$ (not just a strongly pseudoconvex domain), and we construct global holomorphic maps $X \to Z$ with approximation on compact $\mathscr{O}(X)$-convex sets. The latter condition necessitates that Z be 'holomorphically flexible' in a suitable sense; for example, an Oka manifold. Of course the Euclidean space $\mathbb{C}^n$ is such, but the problem becomes nontrivial when trying to avoid a compact subset of $\mathbb{C}^n$ as we do here.

Another recent line of results concerns maps to Stein manifolds satisfying Varolin's *density property* (see [31,32] or [18, §4.10]). Recall that a complex manifold Z has the (holomorphic) density property if the Lie algebra of all holomorphic vector fields on Z is densely generated by the Lie subalgebra generated by all the $\mathbb{C}$-complete holomorphic vector fields on Z. On a Stein manifold this condition implies the Andersén-Lempert-Forstnerič-Rosay theorem on approximation of isotopies of injective holomorphic maps on Runge domains in X by holomorphic automorphisms of X. (See [2,3,20] for the case $X = \mathbb{C}^n$ and Theorem 4.10.6 in [18, p. 132] or [27, Appendix] for the general case.) By using this approach, Andrist and Wold [5] constructed immersions and embeddings of open Riemann surfaces to any Stein manifold Z of dimension $\dim Z \ge 2$ ($\dim Z \ge 3$ for embeddings) enjoying the density property. Any such manifold Z is Oka (see [18, p. 206]). This line of work has been extended recently in [4] where the authors constructed proper holomorphic embeddings of any Stein manifold X to an arbitrary Stein manifold Z with the density property, provided that $2\dim X + 1 \le \dim Z$. The techniques in [4], which differ from those in the present paper in certain main aspects, do not apply when $\dim X > 1$ and $2\dim X \ge \dim Z$. In particular, we do not see how to use them to obtain the main results of this paper.

We do not know whether Theorem 1 still holds for Stein manifolds X of dimension $\dim X \ge n$. In particular, the following remains an open problem.

Problem 3 Is the complement of a compact convex set in $\mathbb{C}^n$ an Oka manifold?

The most useful geometric sufficient conditions for being Oka are *ellipticity* in the sense of Gromov [22] and *subellipticity* [18, p. 203]. We have the following inclusions of the respective classes of complex manifolds (see [18, p. 237] for a more complete picture):

$$\text{elliptic} \subset \text{subelliptic} \subset \text{Oka} \subset \text{strongly dominable} \subset \text{dominable}.$$

Recall that a connected complex manifold Z of dimension n is *dominable* if it admits an entire map $f : \mathbb{C}^n \to Z$ which has maximal rank n at $0 \in \mathbb{C}^n$ (and hence at a generic point $z \in \mathbb{C}^n$), and is *strongly dominable* if for every point $z \in Z$ there exists an f as above with $f(0) = z$.

It is not known which of the above inclusions are proper. The following result follows directly from Proposition 9 below by taking L to be polynomially convex and M to be a point $p \in \mathbb{C}^n \backslash L$. The special case when L is convex is due to Rosay and Rudin ([28], Theorem 8.5 on p. 72).

Proposition 4 *If L is a compact polynomially convex set in $\mathbb{C}^n$ for $n > 1$, then for every point $p \in \mathbb{C}^n \backslash L$ there exists an injective holomorphic map $f : \mathbb{C}^n \to \mathbb{C}^n \backslash L$ (a* Fatou-Bieberbach map*) such that $f(0) = p$. In particular, $\mathbb{C}^n \backslash L$ is* strongly dominable.

On the other hand, it has been recently shown by Andrist and Wold [6] that $\mathbb{C}^n \backslash \overline{\mathbb{B}}$ for $n \ge 3$ fails to be subelliptic, so at least one of the inclusions *subelliptic* $\subset$ *Oka* $\subset$ *strongly dominable* is a proper inclusion.

We also wish to mention a connection between our Corollary 2 and the *universal domination property* of a complex manifold Z which has been introduced recently by Chen and Wang [8] (and was inspired by the work of Winkelmann [34]). Assuming that Z is connected, this property boils down to the existence of a holomorphic map $\mathbb{C} \to Z$ with everywhere dense image. It is immediate that any Oka manifold Z is universally dominable: simply choose a countable dense set $\{b_j\}_{j \in \mathbb{N}}$ in Z and apply the Oka property with interpolation to find a holomorphic map $f : \mathbb{C} \to Z$ with $f(j) = b_j$ for all $j = 1, 2, \ldots$ (cf. [18, Theorem 5.4.4]). The last statement in Corollary 2 above says that the complement $\mathbb{C}^n \backslash L$ of any compact polynomially convex set L in $\mathbb{C}^n$ for $n > 1$ is universally dominable in the sense of Chen and Wang [8].

It is not clear whether strong dominability implies universal dominability. The answer is negative if we omit the word strong: there exist pairs of domains $\Omega \subset \Omega' \subset \mathbb{C}^n$ such that Ω is strongly dominable by $\mathbb{C}^n$ but Ω' is not universally dominable. An example is given by [8, Proposition 5.7]. Another class of examples is obtained as follows.

Example 5 Let Ω be a Fatou-Bieberbach domain in $\mathbb{C}^n$ whose boundary is smooth (at least of class $\mathcal{C}^1$) near some point $p \in b\Omega$. (Such domains were constructed by Globevnik [21] and Stensønes [29].) By bumping out Ω near p we obtain a domain $\Omega' \subset \mathbb{C}^n$ containing

$\Omega \cup \{p\}$ which agrees with Ω outside a small neighborhood of p and contains a strongly pseudoconvex boundary point $p' \in b\Omega'$ near p. By taking the maximum of a local negative strongly plurisubharmonic peak function near p' and a negative constant we find a negative plurisubharmonic function ρ on Ω' which is strongly plurisubharmonic near p'. Since $\mathbb{C}$ does not admit any nonconstant negative subharmonic functions, the image of any holomorphic line $\mathbb{C} \to \Omega'$ must be contained in the set where ρ is constant, and hence no such line can approach the boundary point p' of Ω'.

It seems unknown whether there exists a compact complex manifold which is dominable, but is not strongly dominable or universally dominable.

2 Preliminaries

In this section we gather some of the main tools used in the paper.

We adopt the convention that a map f is holomorphic on a compact set K in a complex manifold X if it is holomorphic on an unspecified open neighborhood of that set. If in addition X' is a closed complex subvariety of X, then saying that f is holomorphic on $K \cup X'$ will mean that f is holomorphic on an open neighborhood of K and the restriction of f to X' is holomorphic on X'.

The following result is a special case of Theorem 1.1 in [12]; we record it here for reference. Similar results for maps to domains of holomorphy in $\mathbb{C}^n$ were proved earlier by Dor [9,10].

Theorem 6 *Assume that Z is a Stein manifold of dimension $\dim Z \geq 2$ and $\sigma : Z \to \mathbb{R}$ is a strongly plurisubharmonic Morse exhaustion function. Let X be a Stein manifold, $D \Subset X$ be a smoothly bounded strongly pseudoconvex domain in X, K be a compact set contained in D, c be a real number, and $f_0 : \overline{D} \to Z$ be a continuous map that is holomorphic in D and satisfies $f_0\big(\overline{D\backslash K}\big) \subset \{\sigma > c\}$. Assume that one of the following two conditions holds:*

(a) $2 \dim X \leq \dim Z$.
(b) $\dim X < \dim Z$ *and σ has no critical points of index > 2 in the set $\{\sigma > c\}$.*

Given a constant $c' > c$, the map f_0 can be approximated uniformly on K by holomorphic maps $f : \overline{D} \to Z$ satisfying $f(\overline{D\backslash K}) \subset \{\sigma > c\}$ and $f(bD) \subset \{\sigma > c'\}$. It can also be approximated uniformly on K by proper holomorphic maps $f : D \to Z$ satisfying the condition $f(D\backslash \mathring{K}) \subset \{\sigma > c\}$.

The second statement is obtained from the first one by a standard limiting argument. Examples in [12] show that conditions (a) and (b) in Theorem 6 can not be relaxed. We shall apply this theorem with different exhaustion functions on $Z = \mathbb{C}^n$.

Remark 7 Given a map f_0 as in Theorem 6 and a compact set L in $\{\sigma \leq c\}$ such that $f_0(\overline{D}) \cap L = \emptyset$, there exists a proper holomorphic map $f : D \to Z$ as in Theorem 6 such that $f(D) \cap L = \emptyset$. Indeed, since $\overline{D\backslash K}$ is mapped to $\{\sigma > c\}$ while $L \subset \{\sigma \leq c\}$, it suffices to choose f to be close enough to f_0 on K.

We recall the following well known result (see e.g. [17, Lemma 6.5]).

Lemma 8 *Let L be a compact polynomially convex set in $\mathbb{C}^n$ and V be a closed complex subvariety of $\mathbb{C}^n$. For any compact $\mathscr{O}(V)$-convex set $A \subset V$ such that $L \cap V \subset A$, the union $A \cup L$ is polynomially convex. The analogous result holds if L is a compact $\mathscr{O}(Z)$-convex set in a Stein manifold Z.*

The following result on separating pairs of compact sets by Fatou-Bieberbach domains is one of the new ingredients introduced in this paper to the construction of proper holomorphic maps. It will be applied in the proof of Lemma 12 below, and we are hoping that it will be of independent interest.

Proposition 9 *Assume that* $L, M \subset \mathbb{C}^n$ $(n > 1)$ *are disjoint compact sets such that one of them is holomorphically contractible and the union* $L \cup M$ *is polynomially convex. Then there exists a Fatou-Bieberbach domain* $\Omega \subset \mathbb{C}^n$ *satisfying*

$$M \subset \Omega \subset \mathbb{C}^n \backslash L. \tag{2.1}$$

Recall that a compact set L in $\mathbb{C}^n$ is said to be *holomorphically contractible* if there exists a smooth 1-parameter family of holomorphic maps $\theta_t : U \to \mathbb{C}^n$ $(0 \le t \le 1)$ on an open neighborhood of L such that $\theta_0 = \mathrm{Id}$, θ_t is a biholomorphism of L onto a subset of L for $0 \le t < 1$, and θ_1 is a constant map $L \to p \in L$. For example, every compact convex (or starshaped) set is holomorphically contractible by a family of dilations.

Before continuing we state the following special case of Proposition 9; note that the union of two disjoint compact convex sets in $\mathbb{C}^n$ is polynomially convex [30, p. 62].

Corollary 10 *Any pair of disjoint compact convex sets* L, M *in* $\mathbb{C}^n$ *for* $n > 1$ *can be separated by Fatou-Bieberbach domains as in* (2.1).

Proof of Proposition 9 We first consider the case when the set L is convex. We shall find a Fatou-Bieberbach domain Ω satisfying (2.1) as the domain of convergence of a random iteration of holomorphic automorphisms of $\mathbb{C}^n$, applying the so called *push-out method* (see [18, §4.4]).

Recall that $\mathbb{B}$ denotes the unit ball in $\mathbb{C}^n$. Pick a number $N_1 \in \mathbb{N}$ such that $M \cup L \subset B_1 := N_1\mathbb{B}$. Choose an affine linear automorphism ψ_1 of $\mathbb{C}^n$ such that $\psi_1(L) \subset \mathbb{C}^n \backslash \overline{B}_1$ and the set $\overline{B}_1 \cup \psi_1(L)$ is polynomially convex. (The latter property holds whenever $\psi_1(L)$ is contained in a closed ball disjoint from $\overline{B}_1$.) By Corollary 4.12.4 in [18, p. 145] (which uses the Andersén-Lempert theory [3,20]) there exists a holomorphic automorphism ϕ_1 of $\mathbb{C}^n$ that is uniformly close to the identity on M, and is uniformly close to the map ψ_1 on L. Set $L_1 = \phi_1(L)$. If the approximations are close enough then $\phi_1(M) \subset B_1$, $\overline{B}_1 \cap L_1 = \emptyset$, and $\overline{B}_1 \cup L_1$ is still polynomially convex.

Next we pick a number $N_2 \ge N_1 + 1$ such that $L_1 \subset B_2 := N_2\mathbb{B}$. By repeating the above argument (with M replaced by $\overline{B}_1$ and L replaced by L_1) we can find a holomorphic automorphism ϕ_2 of $\mathbb{C}^n$ that approximates the identity map on $\overline{B}_1$ and sends L_1 to a set $L_2 := \phi_2(L_1) \subset \mathbb{C}^n \backslash \overline{B}_2$ such that $\overline{B}_2 \cup L_2$ is polynomially convex.

Continuing inductively we find a sequence of holomorphic automorphisms ϕ_k of $\mathbb{C}^n$ for $k = 1, 2, \ldots$ such that the sequence of their compositions $\Phi_k = \phi_k \circ \phi_{k-1} \circ \cdots \circ \phi_1$ converges on a domain $\Omega \subset \mathbb{C}^n$ to a Fatou-Bieberbach map $\Phi : \Omega \to \mathbb{C}^n$ onto $\mathbb{C}^n$. (See Corollary 4.4.2 in [18, p. 115].) The domain Ω consists precisely of the points $z \in \mathbb{C}^n$ with bounded orbits $\{\Phi_k(z) : k \in \mathbb{N}\}$. By the construction we have $M \subset \Omega$ and $\Omega \cap L = \emptyset$, so the proof is complete. This also proves Corollary 10.

The general case follows by finding a holomorphic automorphism ψ of $\mathbb{C}^n$ which separates L and M, in the sense that their images $\psi(L)$ and $\psi(M)$ are contained in disjoint closed balls $B_1, B_2 \subset \mathbb{C}^n$, respectively. If $\widetilde{\Omega}$ is a Fatou-Bieberbach domain containing B_2 and not intersecting B_1, then $\Omega = \psi^{-1}(\widetilde{\Omega})$ satisfies (2.1). To find such ψ, Corollary 4.12.4 in [18, p. 145] applies directly if one of the sets L, M is starshaped. However, the proof given there also applies when one of the two sets, say L, is holomorphically contractible. □

Remark 11 The proof of Proposition 9 easily adapts to the case when one of the two sets, say L, is a finite union of pairwise disjoint compact holomorphically contractible sets L_j. Pick a pair of disjoint closed balls $B_1, B_2 \subset \mathbb{C}^n$ such that $M \subset \mathring{B}_1$. Choose an isotopy of biholomorphic maps in a neighborhood of L which contracts each component L_j of L almost to a point in L_j and then moves it along a path in $\mathbb{C}^n \backslash M$ to a small neighborhood of a point $p_j \in \mathring{B}_2$. This isotopy can be chosen such that the main result of [20] (also stated as Theorem 4.9.2 in [18, p. 125]) applies and gives an automorphism ψ of $\mathbb{C}^n$ which is almost the identity on M, so $\psi(M) \subset B_1$, while $\psi(L) \subset B_2$. At this point we may apply Corollary 10.

3 Fatou-Bieberbach domains separating a variety from a convex set

The following separation lemma is the main ingredient in the proof of Theorem 15. Its proof combines Theorem 6 and Proposition 9 from the previous section.

Lemma 12 *Let $n > 1$. Assume that L is a compact convex set in $\mathbb{C}^n$, X is a Stein manifold with $\dim X < n$, and C is a compact $\mathscr{O}(X)$-convex set in X. Every holomorphic map $f_0 : C \to \mathbb{C}^n \backslash L$ can be approximated uniformly on C by holomorphic maps $f : C \to \mathbb{C}^n \backslash L$ such that there exists a Fatou-Bieberbach domain Ω in $\mathbb{C}^n$ satisfying*

$$f(C) \subset \Omega \subset \mathbb{C}^n \backslash L.$$

The same holds if L is polynomially convex, C is a compact convex set in some local holomorphic coordinates on X, and $2 \dim X + 1 \le n$.

Proof Pick an open neighborhood $U \subset X$ of C such that f_0 is holomorphic on U and $f_0(U) \cap L = \emptyset$. Since C is $\mathscr{O}(X)$-convex, there exists a smooth strongly plurisubharmonic exhaustion function $\rho : X \to \mathbb{R}$ such that $\rho < 0$ on C and $\rho > 0$ on $X \backslash U$ [24, p. 116]. If $c \in \mathbb{R}$ is a regular value of ρ chosen sufficiently close to zero, then the set $D = \{\rho < c\}$ is a smoothly bounded strongly pseudoconvex domain in X with $C \subset D \subset \overline{D} \subset U$.

Assume first that the set $L \subset \mathbb{C}^n$ is compact and convex. There is a smooth strongly convex exhaustion function $\sigma : \mathbb{C}^n \to \mathbb{R}$ with a single critical point in L such that $\sigma < 0$ on L and $\sigma > 0$ on $f_0(\overline{D})$. (Note that such σ is strongly plurisubharmonic.) Here is a brief argument for the sake of completeness. By the Hahn-Banach theorem we can approximate L by a polyhedron P (an intersection of finitely many closed half-spaces) so that $L \subset \mathring{P}$ and $P \cap f_0(\overline{D}) = \emptyset$. Let ϕ denote the Minkowski functional of P with respect to some interior point. By convolving ϕ with a smooth approximate identity, for example, with the Gaussian kernel $C_t \exp^{-t|x|^2}$ for a sufficiently large $t > 0$, and subtracting the constant 1, we get a function σ with the stated properties. (We wish to thank E. L. Stout for this suggestion. Alternatively, one can convexify the supporting hyperplanes of P and smooth the corners as in [13] to find a smooth strongly convex domain approximating L; its Minkowski functional can then be used to define σ.)

Theorem 6-(b) and Remark 7 give a proper holomorphic map $f : D \to \mathbb{C}^n$ that approximates f_0 as closely as desired on C and satisfies $f(D) \cap L = \emptyset$. By Remmert's proper mapping theorem, the image $V = f(D)$ is a closed complex subvariety of $\mathbb{C}^n$, and we have $V \subset \mathbb{C}^n \backslash L$ by the construction.

Let L' be a closed ball in $\mathbb{C}^n$ containing $f(C) \cup L$, and set $C' = f^{-1}(L' \cap V) \subset X$. The compact set $f(C') = L' \cap V$ is then disjoint from L and is $\mathscr{O}(V)$-convex, hence the union $f(C') \cup L$ is polynomially convex by Lemma 8. Proposition 9, applied with $M = f(C')$, furnishes a Fatou-Bieberbach domain Ω in $\mathbb{C}^n$ such that $f(C) \subset f(C') \subset \Omega \subset \mathbb{C}^n \backslash L$. This completes the proof when the set L is convex.

It remains to prove the second case. Now C is a compact convex set in some holomorphic coordinate system on a neighborhood $U \subset X$ of C. By shrinking U we may assume that f_0 is holomorphic on U and $f_0(U) \cap L = \emptyset$. Pick an open, smoothly bounded, strongly convex domain $D \Subset U$ with $C \subset D$. As L is polynomially convex, there is a smooth strongly plurisubharmonic exhaustion function $\sigma\colon \mathbb{C}^n \to \mathbb{R}$ such that $\sigma < 0$ on L and $\sigma > 0$ on $f_0(\overline{D})$. Since $2 \dim X + 1 \le n$, Theorem 6-(a) furnishes a proper holomorphic embedding $f\colon D \hookrightarrow \mathbb{C}^n$ which approximates f_0 uniformly on C and such that $V = f(D) \subset \{\rho > 0\}$; hence $V \cap L = \emptyset$. Clearly $f(C)$ is $\mathscr{O}(V)$-convex, and hence $f(C) \cup L$ is polynomially convex by Lemma 8. Since C is convex and f is an embedding, $f(C)$ is holomorphically contractible in $\mathbb{C}^n$, so Proposition 9 applies.

Example 13 Lemma 12 is false if $\dim X \ge n$, and this is one of the principal reasons why our technique does not apply in that case. Indeed, the image of a holomorphic map $C \to \mathbb{C}^n \backslash L$ from a compact convex set $C \subset \mathbb{C}^n$ need not be contained in any pseudoconvex domain (and hence in any Fatou-Bieberbach domain) in $\mathbb{C}^n \backslash L$. To give an example, recall that Fornæss and Stout proved that every complex manifold of dimension n is the image of a locally biholomorphic map f from the polydisc $P \subset \mathbb{C}^n$ [14] or the ball $\mathbb{B} \subset \mathbb{C}^n$ [15]. Choose a ball $B \subset \mathbb{C}^n$ containing L. Let $f\colon P \to \mathbb{C}^n \backslash L$ be a surjective holomorphic map furnished by [14]. There is a slightly smaller closed polydisc $C \subset P$ such that $bB \subset f(\mathring{C})$. By the Hartogs extension theorem any pseudoconvex domain containing $f(C)$ must also contain the ball B, and hence the set L.

Remark 14 The proof of Lemma 12 exposes the following interesting question. Assume that L is a compact convex set in $\mathbb{C}^n$, $V \subset \mathbb{C}^n \backslash L$ is a non-closed Stein variety of dimension $< n$, and $K \subset V$ is a compact $\mathscr{O}(V)$-convex subset. Can we make K and $K \cup L$ polynomially convex after a small deformation of K and V? When V is a closed subvariety of $\mathbb{C}^n$, the answer is affirmative by Lemma 8, and in this case no perturbation is necessary. The problem seems nontrivial for non-closed subvarieties. In our case V is the image of a strongly pseudoconvex domain, so it is rather special and we can apply Theorem 6 to make it proper. Examples of non-polynomially convex complex curves in $\mathbb{C}^n$, and Wermer's famous example [33] of an embedded holomorphic bidisc in $\mathbb{C}^3$ which fails to be polynomially convex, show that one must consider generic varieties.

4 The main result

We are now ready to prove the following main result of this paper which clearly includes Theorem 1; the latter concerns maps avoiding the set L.

Theorem 15 *Let L be a compact set in $\mathbb{C}^n$ for some $n > 1$. Let X be a Stein manifold, $K \subset X$ be a compact $\mathscr{O}(X)$-convex set, $U \subset X$ be an open set containing K, $X' \subset X$ be a closed complex subvariety, and $f\colon U \cup X' \to \mathbb{C}^n$ be a holomorphic map such that $f(bK \cup (X' \backslash K)) \cap L = \emptyset$. Suppose also that either*

(i) L is convex and $\dim X < n$*, or*
(ii) L is polynomially convex and $2 \dim X \le n$.

Then for every $\epsilon > 0$ there exists a holomorphic map $F\colon X \to \mathbb{C}^n$ satisfying

$$(a)\ F(X \backslash K) \subset \mathbb{C}^n \backslash L, \qquad (b)\ ||F - f||_K < \epsilon, \qquad (c)\ F|_{X'} = f|_{X'}.$$

If the map $f|_{X'}: X' \to \mathbb{C}^n$ is proper (in particular, if $X' = \emptyset$) then F can also be chosen proper. If $2\dim X \le n$ then F can be chosen an immersion (an embedding if $2\dim X + 1 \le n$) provided that $f|_{X'}$ is such.

Remark 16 There are no topological obstructions to extending a continuous map $K \cup X' \to \mathbb{C}^n$, which sends $bK \cup (X' \backslash K)$ to $\mathbb{C}^n \backslash L$, to a continuous map $X \to \mathbb{C}^n$ sending $X \backslash K$ to $\mathbb{C}^n \backslash L$. This is because the complement $\mathbb{C}^n \backslash L$ of a compact convex set is homotopy equivalent to the sphere S^{2n-1}, while the pair $(X, K \cup X')$ is a relative CW complex of dimension at most $\dim X < n$ (see [23] or [18, p. 96]). Similarly, if L is polynomially convex, then $\mathbb{C}^n \backslash L$ admits a CW decomposition containing only cells of dimension $\ge n$ (see [16] or [18, p. 98]), so again there are no obstructions if $\dim X < n$. The stronger inequality $2\dim X \le n$ is used in Theorem 6 which is one of the main ingredients in the proof of Theorem 1.

Proof of Theorem 15 We shall follow the general scheme used in Oka theory (see Chapter 5 in [18] for further details), applying also Lemma 12 at each step of the inductive construction.

We consider three cases: (1) $X' = \emptyset$; (2) $X' \ne \emptyset$ and the restricted map $f: X' \to \mathbb{C}^n$ is proper; (3) $X' \ne \emptyset$ and the restricted map $f: X' \to \mathbb{C}^n$ is not proper.

Case 1: $X' = \emptyset$. We shall construct a proper holomorphic map $F: X \to \mathbb{C}^n$ satisfying conditions (a) and (b) in Theorem 15.

The initial map $f: K \to \mathbb{C}^n$ is holomorphic on an open set $U \subset X$ containing K. Since $f(bK) \cap L = \emptyset$ by assumption, we can shrink U around K if necessary to ensure that $\{x \in U: f(x) \in L\} \subset \mathring{K}$.

Since the set K is $\mathscr{O}(X)$-convex, there is a smooth strongly plurisubharmonic Morse exhaustion function $\rho: X \to \mathbb{R}$ such that $\rho < 0$ on K and $\rho > 0$ on $X \backslash U$ [24, p. 116]. We may assume that 0 is a regular value of ρ. Let $p_1, p_2, p_3, \ldots$ be the critical points of ρ in $\{\rho > 0\}$, ordered so that $0 < \rho(p_1) < \rho(p_2) < \rho(p_3) < \cdots$ (the case in which ρ has finitely many critical points also follows easily from the following argument). Choose a sequence of numbers $0 = c_0 < c_1 < c_2 < \cdots$ with $\lim_{j\to\infty} c_j = +\infty$ such that $c_{2j-1} < \rho(p_j) < c_{2j}$, and such that c_{2j-1} and c_{2j} are sufficiently close to $\rho(p_j)$, for every $j = 1, 2, \ldots$ (this condition will be specified later). Each of the sets

$$D_j = \{x \in X: \rho(x) \le c_j\}, \qquad j = 0, 1, 2, \ldots$$

is a smoothly bounded compact strongly pseudoconvex domain in X, and these domains exhaust X. Note that $K \subset \mathring{D}_0 \subset D_0 \subset U$ and $f(D_0 \backslash \mathring{K}) \subset \mathbb{C}^n \backslash L$.

On the target side we pick an increasing sequence of closed balls

$$L_1 \subset L_2 \subset \cdots \subset \bigcup_{k=1}^{\infty} L_k = \mathbb{C}^n, \tag{4.1}$$

where L_1 is chosen such that $f(K) \cup L \subset L_1$.

If the set L is convex, there exists a smooth strongly convex exhaustion function $\sigma: \mathbb{C}^n \to \mathbb{R}$ such that

$$L \subset \{\sigma < 0\}, \qquad f(D_0 \backslash \mathring{K}) \subset \{\sigma > 0\}, \tag{4.2}$$

and such that σ has no critical points in the set $\{\sigma > 0\}$. (See the proof of Lemma 12.) Theorem 6 furnishes a holomorphic map $f_0: D_0 \to \mathbb{C}^n$ that approximates f as closely as desired uniformly on K and satisfies

$$f_0(D_0 \backslash \mathring{K}) \subset \mathbb{C}^n \backslash L, \qquad f_0(bD_0) \subset \mathbb{C}^n \backslash L_1. \tag{4.3}$$

Similarly, if L is polynomially convex, there exists a smooth strongly plurisubharmonic exhaustion function $\sigma : \mathbb{C}^n \to \mathbb{R}$ satisfying (4.2). If $2 \dim X \le n$ then Theorem 6 gives a holomorphic map $f_0 \colon D_0 \to \mathbb{C}^n$ that approximates f as closely as desired uniformly on K and satisfies (4.3). We replace f by f_0 as our initial map.

We shall now inductively construct a sequence of holomorphic maps $f_j \colon D_j \to \mathbb{C}^n$ $(j = 1, 2, \ldots)$ such that f_j approximates f_{j-1} uniformly on D_{j-1} and satisfies

$$f_j(D_j \backslash \mathring{D}_{j-1}) \subset \mathbb{C}^n \backslash L_j, \quad j = 1, 2, \ldots. \tag{4.4}$$

Assuming as we may that the approximations are close enough, the sequence f_j converges uniformly on compact subsets of X to a holomorphic map $F = \lim_{j\to\infty} f_j \colon X \to \mathbb{C}^n$ that satisfies condition (b) of Theorem 15. Condition (4.4) guarantees that F is proper and also satisfies condition (a).

We begin by explaining how to approximate the map $f_0 \colon D_0 \to \mathbb{C}^n$ by a holomorphic map $f_1 \colon D_1 \to \mathbb{C}^n$ satisfying (4.4) for $j = 1$. This is the so called *noncritical case* (cf. [18, p. 222]). In fact, every step from $f_{2j} \colon D_{2j} \to \mathbb{C}^n$ to $f_{2j+1} \colon D_{2j+1} \to \mathbb{C}^n$ in the inductive construction will be of this kind.

Since D_1 is a noncritical strongly pseudoconvex extension of D_0, it is obtained from D_0 by attaching finitely many convex bumps (see [18, §5.10] for the details). More precisely, there exists a finite sequence of compact strongly pseudoconvex domains

$$D_0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_m = D_1$$

such that for every $j = 0, \ldots, m-1$ we have $A_{j+1} = A_j \cup B_j$, where B_j and $C_j = A_j \cap B_j$ are smoothly bounded strongly convex domains in some local holomorphic coordinates on X in a neighborhood of $\overline{B}_j$, and $\overline{A_j \backslash B_j} \cap \overline{B_j \backslash A_j} = \emptyset$. (Such a pair (A_j, B_j) is called a *special Cartan pair*.) Furthermore, in view of (4.3), we may assume that for every attaching set C_j we have

$$f_0(D_0 \cap C_j) \subset \mathbb{C}^n \backslash L_1, \quad j = 0, \ldots, m-1. \tag{4.5}$$

Of course we may have $D_0 \cap C_j = \emptyset$ for some of the attaching sets.

We now successively extend our map to each bump in the sequence, always approximating the previous map on its domain. All steps are of the same kind, so it suffices to explain how to approximately extend f_0 from $A_0 = D_0$ to $A_1 = A_0 \cup B_0$.

Note that $C_0 = A_0 \cap B_0 = D_0 \cap B_0$ and $f_0(C_0) \subset \mathbb{C}^n \backslash L_1$ by our choice of the bumps. Since the set $f_0(C_0)$ is compact, we can choose a slightly larger closed ball $L_1' \subset \mathbb{C}^n$, containing L_1 in its interior, such that $f_0(C_0) \subset \mathbb{C}^n \backslash L_1'$.

By Lemma 12, applied with the compact sets $C = C_0 \subset X$ and $L = L_1' \subset \mathbb{C}^n$, we can approximate the map $f_0|_{C_0}$ as close as desired by a holomorphic map $\tilde{f}_0 \colon C_0 \to \mathbb{C}^n \backslash L_1'$ such that $\tilde{f}_0(C_0) \subset \Omega \subset \mathbb{C}^n \backslash L_1'$ for some Fatou-Bieberbach domain Ω. (By Example 13 this is false if $\dim X \ge n$, and this is the main reason why our proof fails in this case.)

Since Ω is biholomorphic to $\mathbb{C}^n$, we can use the Oka-Weil theorem to approximate $\tilde{f}_0$ as close as desired on C_0 by a holomorphic map $g_0 \colon B_0 \to \Omega$. The holomorphic maps $f_0 \colon A_0 \to \mathbb{C}^n$ and $g_0 \colon B_0 \to \Omega \subset \mathbb{C}^n$ are then uniformly close to each other on $C_0 = A_0 \cap B_0$. Since (A_0, B_0) is a Cartan pair, we can glue f_0 and g_0 into a holomorphic map $h_1 \colon A_1 = A_0 \cup B_0 \to \mathbb{C}^n$ that is close to f_0 on A_0 and to g_0 on B_0. This amounts to solving an additive Cousin problem with bounds, a classical problem whose solution uses a sup-norm bounded linear solution operator for the $\overline{\partial}$-equation at the level of $(0, 1)$-forms on a strongly pseudoconvex Stein domain. (See e.g. Lemma 5.8.2 in [18, p. 212]. Since we are gluing maps to Euclidean spaces, we do not need the more advanced gluing lemma furnished by

Proposition 5.8.1 in [18, p. 211].) Recall that $g_0(B_0) \subset \Omega \subset \mathbb{C}^n \backslash L_1'$. Assuming as we may that the approximation of f_0 by g_0 was close enough on C_0, the map h_1 is so close to g_0 on B_0 that it satisfies $h_1(B_0) \subset \mathbb{C}^n \backslash L_1$. Furthermore, in view of (4.5) we may assume that h_1 is so close to f_0 on $A_0 = D_0$ that

$$h_1(A_0 \cap C_j) \subset \mathbb{C}^n \backslash L_1, \qquad j = 1, \ldots, m-1.$$

As $C_1 \subset A_1 = A_0 \cup B_0$, it follows that $h(C_1) \subset \mathbb{C}^n \backslash L_1$. By applying the same construction to the map $h_1 \colon A_1 \to \mathbb{C}^n$ we find a holomorphic map $h_2 \colon A_2 \to \mathbb{C}^n$ that approximates h_1 uniformly on A_1 and satisfies

$$h_2(B_1) \subset \mathbb{C}^n \backslash L_1 \quad \text{and} \quad h_2(A_1 \cap C_j) \subset \mathbb{C}^n \backslash L_1, \quad j = 2, \ldots, m-1.$$

After m steps of this kind we find a holomorphic map $f_1 = h_m \colon D_1 \to \mathbb{C}^n$ that approximates f_0 as close as desired on D_0 and satisfies (4.4) for $j = 1$.

Before proceeding, we apply Theorem 6 to approximate f_1 uniformly on D_0 by a holomorphic map $\tilde{f}_1 \colon D_1 \to \mathbb{C}^n$ such that

$$\tilde{f}_1(D_1 \backslash \mathring{D}_0) \subset \mathbb{C}^n \backslash L_1, \qquad \tilde{f}_1(bD_1) \subset \mathbb{C}^n \backslash L_2.$$

To simplify the notation we now replace f_1 by $\tilde{f}_1$.

The next task is to approximate f_1 on D_1 by a holomorphic map $f_2 \colon D_2 \to \mathbb{C}^n$ satisfying (4.4) for $j = 2$. On $\mathring{D}_2 \backslash D_1$ the function ρ has a unique critical point p_1 where the topology of the sublevel set changes. To find f_2 we follow the *critical case* explained in [18, §5.11], proceeding in three substeps as follows:

(i) Extend $f_1|_{D_1}$ smoothly across the stable manifold E of the critical point p_1 of ρ, with values in $\mathbb{C}^n \backslash L_2$. (By Remark 16 there are no topological obstructions.)
(ii) Approximate the extended map $f_1 \colon D_1 \cup E \to \mathbb{C}^n$ by a holomorphic map in a neighborhood of $D_1 \cup E$. (For this we use a version of Mergelyan's theorem; see Theorem 3.7.2 in [18, p. 81].)
(iii) Reduce to the noncritical case for a different strongly plurisubharmonic function. (Here we must assume that the number c_1 is close enough to $\rho(p_1)$.)

All substeps can be accomplished exactly as in the cited source. In this way we obtain a constant $c_2 > \rho(p_1)$ close to $\rho(p_1)$ and a holomorphic map $f_2 \colon D_2 = \{\rho \le c_2\} \to \mathbb{C}^n$ satisfying the required properties. Applying again Theorem 6 we approximate f_2 uniformly on D_1 by a holomorphic map $\tilde{f}_2 \colon D_2 \to \mathbb{C}^n$ satisfying $\tilde{f}_2(D_2 \backslash \mathring{D}_1) \subset \mathbb{C}^n \backslash L_2$ and $\tilde{f}_2(bD_2) \subset \mathbb{C}^n \backslash L_3$. To simplify the notation we replace f_2 by $\tilde{f}_2$.

Next we construct a holomorphic map $f_3 \colon D_3 \to \mathbb{C}^n$ that approximates f_2 on D_2 and satisfies (4.4) for $j = 3$. This is exactly the same as the construction of the map f_1 from f_0 (the noncritical case). Similarly, the construction of f_4 from f_3 is analogous to the construction of f_2 from f_1 (the critical case). The induction may proceed. If ρ has only finitely many critical points, then at some point all subsequent steps are noncritical.

This completes the proof of Theorem 15 in Case 1 when $X' = \emptyset$.

Case 2: $X' \neq \emptyset$ and the restricted map $f \colon X' \to \mathbb{C}^n$ is proper. We shall find a proper holomorphic map $F \colon X \to \mathbb{C}^n$ satisfying conditions (a), (b) and (c) in Theorem 15.

Choose an exhausting sequence of closed balls (4.1) in $\mathbb{C}^n$. The initial map f is now defined and holomorphic on an open neighborhood of K and on the subvariety X'. By standard techniques of Cartan theory (see e.g. Theorem 3.4.1 in [18, p. 68]) we can approximate f uniformly on K by a holomorphic map on an open set $U \supset K \cup X'$ that agrees with f on the subvariety X'; we still denote the new map by f. After shrinking U we may assume that $\{x \in U \colon f(x) \in L\} \subset \mathring{K}$.

Since the map $f|_{X'}\colon X' \to \mathbb{C}^n$ is proper, we can find a compact $\mathscr{O}(X)$-convex set $K' \subset X$ such that $K \subset K'$ and $f(X' \setminus \mathring{K}') \subset \mathbb{C}^n \setminus L_1$. (For instance, K' could be a sublevel set of a strongly plurisubharmonic exhaustion function on X.) The set $S := K \cup (K' \cap X') \subset U$ is then $\mathscr{O}(X)$-convex by Lemma 8. Hence there is a smooth strongly plurisubharmonic Morse exhaustion function $\rho\colon X \to \mathbb{R}$ such that $\rho < 0$ on S and $\rho > 0$ on $X \setminus U$. We may assume that 0 is a regular value of ρ. Let $D_0 = \{\rho \le 0\}$; then $S \subset \mathring{D}_0 \subset D_0 \subset U$ and $f(bD_0 \cap X') \subset \mathbb{C}^n \setminus L_1$. By applying a minor extension of Theorem 6 we can approximate f uniformly on S by a holomorphic map $f_0\colon D_0 \to \mathbb{C}^n$ satisfying (4.3) such that f_0 agrees with f on the subvariety X'. (This is obtained by a straightforward modification of the construction in [12]: we need not do anything near the set $bD_0 \cap X'$ which is already mapped to the complement of the ball L_1, while the rest of the boundary of D_0 can be pushed out of L_1 by the lifting procedure described in [12].) This gives a new holomorphic map $f_0\colon D_0 \to \mathbb{C}^n$ such that $f_0(bD_0) \subset \mathbb{C}^n \setminus L_1$ and $\{x \in D_0\colon f_0(x) \in L\} \subset \mathring{K}$. We take f_0 as our new initial map. As before, we may assume that f_0 is holomorphic in an open neighborhood V of $D_0 \cup X'$.

Pick a sequence $0 = c_0 < c_1 < c_2 \ldots$ with $\lim_{j\to\infty} c_j = +\infty$ such that every c_j is a regular value of ρ and, setting $D_j = \{\rho \le c_j\} \Subset X$, we have

$$f_0(X' \setminus \mathring{D}_j) \subset \mathbb{C}^n \setminus L_{j+1}, \quad j = 0, 1, 2, \ldots.$$

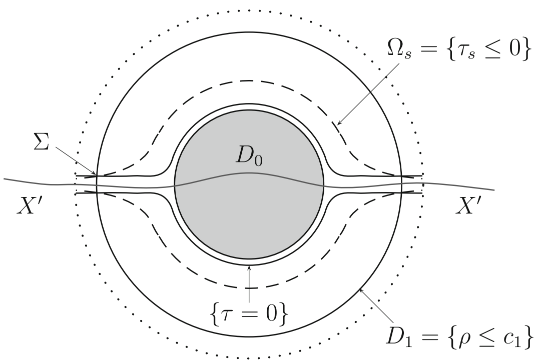
We now inductively construct a sequence of holomorphic maps $f_j\colon D_j \to \mathbb{C}^n$ such that f_{j+1} both approximates f_j on D_j and agrees with f_j (and hence with f) on X', and such that (4.4) holds for every $j = 1, 2, \ldots$. The limit map $F = \lim_{j\to\infty} f_j\colon X \to \mathbb{C}^n$ then satisfies the stated properties provided that all approximations were close enough.

Since the inductive steps are all of the same kind, it suffices to explain the construction of the map $f_1\colon D_1 \to \mathbb{C}^n$. We follow the proof of Proposition 5.12.1 in [18, p. 224]; see in particular Fig. 5.4 in [18, p. 226] which explains the underlying geometry. We reproduce it here with the appropriate notation adapted to our situation.

The compact set $K_1 := D_0 \cup (D_1 \cap X') \subset V \subset X$ is $\mathscr{O}(X)$-convex by Lemma 8. Hence there exists a smooth strongly plurisubharmonic exhaustion function $\tau\colon X \to \mathbb{R}$ such that $\tau < 0$ on K_1 and $\tau > 0$ on $X \setminus V$. By a general position argument we may assume that 0 is a regular value of τ and that the hypersurfaces $\{\rho = c_1\} = bD_1$ and $\{\tau = 0\}$ intersect transversely along the submanifold $\Sigma = \{\rho = c_1\} \cap \{\tau = 0\}$. (See Fig. 1.) For each $s \in [0, 1]$ we set

$$\tau_s = (1 - s)\tau + s(\rho - c_1), \qquad \Omega_s = \{\tau_s \le 0\}.$$

Fig. 1 The sets Ω_s



Since τ_s is a convex linear combination of two strongly plurisubharmonic functions, it is strongly plurisubharmonic. As the parameter $s \in [0, 1]$ increases from 0 to 1, the pseudoconvex domains $\Omega_s \cap D_1$ monotonically increase from $\{\tau \le 0\} \cap D_1$ to $\Omega_1 = \{\rho \le c_1\} = D_1$. All hypersurfaces $\{\tau_s = 0\}$ intersect along Σ. The hypersurface $b\Omega_s \cap D_1 = \{\tau_s = 0\} \cap D_1$ is strongly pseudoconvex at every point where $d\tau_s \neq 0$. As explained in [18, §5.12], a generic choice of the function τ ensures that the topology of the sublevel set $\{\tau_s \le 0\}$ changes at only finitely many points of $D_1 \cap \{\tau > 0\}$, and at any of those points the change corresponds to passing a Morse critical point of index $\le n$ of a strongly plurisubharmonic function. It is then possible to use the techniques explained in the first part of the proof (extension to convex bumps, reduction of the critical case to the noncritical case) to approximately extend f_0 to a map f_1 on $D_1 \cup X'$ with the stated properties. For further details we refer to [18, §5.11].

Case 3: $X' \neq \emptyset$ and the restricted map $f\colon X' \to \mathbb{C}^n$ is not proper. We need to find a holomorphic map $F\colon X \to \mathbb{C}^n$ satisfying properties (a) and (b) in Theorem 15.

The construction is the same as above, but is even simpler since we do not need to worry about properness. We always work in the complement $\mathbb{C}^n \backslash L$ of the initial compact set L, and we just need to ensure that our sequence of maps f_j satisfies condition (4.4) with $L_j = L$ for all $j \in \mathbb{Z}_+$. Interpolation on X' (condition (c) in Theorem 15) is obtained just as in Case 2; we only need to observe that none of the attaching sets for our bumps intersect the subvariety X', so the existing proof applies without changes. □

References

1. Acquistapace, F., Broglia, F., Tognoli, A.: A relative embedding theorem for Stein spaces. Ann. Sc. Norm. Sup. Pisa, Cl. Sci. **2**(4), 507–522 (1975)
2. Andersén, E.: Volume-preserving automorphisms of $\mathbb{C}^n$. Complex Var. Theory Appl. **14**(1–4), 223–235 (1990)
3. Andersén, E., Lempert, L.: On the group of holomorphic automorphisms of $\mathbb{C}^n$. Invent. Math. **110**, 371–388 (1992)
4. Andrist, R., Forstneric, F., Ritter, T., Wold, E.F.: Proper holomorphic embeddings into Stein manifolds with the density property. arxiv.org/abs/1309.6956
5. Andrist, R.B., Wold, E.F.: Riemann surfaces in Stein manifolds with density property. Ann. Inst. Fourier (to appear). arXiv:1106.4416
6. Andrist, R.B., Wold, E.F.: The complement of the closed unit ball in $\mathbb{C}^3$ is not Subelliptic. arXiv:1303.1804
7. Bishop, E.: Mappings of partially analytic spaces. Am. J. Math. **83**, 209–242 (1961)
8. Chen, B.-Y., Wang, X.: Holomorphic maps with large images. arxiv:1303.5242
9. Dor, A.: Approximation by proper holomorphic maps into convex domains. Ann. Sc. Norm. Sup. Pisa Cl. Sci. **20**(4), 147–162 (1993)
10. Dor, A.: Immersions and embeddings in domains of holomorphy. Trans. Am. Math. Soc. **347**, 2813–2849 (1995)
11. Drinovec Drnovšek, B., Forstnerič, F.: Holomorphic curves in complex spaces. Duke Math. J. **139**, 203–254 (2007)
12. Drinovec Drnovšek, B., Forstnerič, F.: Strongly pseudoconvex Stein domains as subvarieties of complex manifolds. Am. J. Math. **132**, 331–360 (2010)
13. Fornæss, J.E.: Embedding strictly pseudoconvex domains in convex domains. Am. J. Math. **98**, 529–569 (1976)
14. Fornæss, J.E., Stout, E.L.: Spreading polydiscs on complex manifolds. Am. J. Math. **99**, 933–960 (1977)
15. Fornæss, J.E., Stout, E.L.: Regular holomorphic images of balls. Ann. Inst. Fourier (Grenoble) **32**, 23–36 (1982)
16. Forstnerič, F.: Complements of Runge domains and holomorphic hulls. Mich. Math. J. **41**, 297–308 (1994)
17. Forstnerič, F.: Interpolation by holomorphic automorphisms and embeddings in $\mathbb{C}^n$. J. Geom. Anal. **9**, 93–118 (1999)
18. Forstnerič, F.: Stein Manifolds and Holomorphic Mappings (The Homotopy Principle in Complex Analysis). Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, 56. Springer, Berlin (2011)
19. Forstnerič, F., Lárusson, F.: Survey of Oka theory. N.Y. J. Math. **17a**, 11–38 (2011)

20. Forstnerič, F., Rosay, J.-P.: Approximation of biholomorphic mappings by automorphisms of $\mathbb{C}^n$. Invent. Math. **112**, 323–349 (1993). Erratum. Invent. Math. **118**, 573–574 (1994)
21. Globevnik, J.: On Fatou-Bieberbach domains. Math. Z. **229**, 91–106 (1998)
22. Gromov, M.: Oka's principle for holomorphic sections of elliptic bundles. J. Am. Math. Soc. **2**, 851–897 (1989)
23. Hamm, H.: Zum Homotopietyp Steinscher Räume. J. Reine Ang. Math. **338**, 121–135 (1983)
24. Hörmander, L.: An Introduction to Complex Analysis in Several Variables, 3rd edn. North-Holland Mathematical Library, vol. 7. North-Holland, Amsterdam (1990)
25. Narasimhan, R.: Imbedding of holomorphically complete complex spaces. Am. J. Math. **82**, 917–934 (1960)
26. Remmert, R.: Sur les espaces analytiques holomorphiquement séparables et holomorphiquement convexes. C. R. Acad. Sci. Paris **243**, 118–121 (1956)
27. Ritter, T.: A strong Oka principle for embeddings of some planar domains into $\mathbb{C} \times \mathbb{C}^*$. J. Geom. Anal. **23**, 571—597 (2013)
28. Rosay, J.-P., Rudin, W.: Holomorphic maps from C^n to C^n. Trans. Am. Math. Soc. **310**, 47–86 (1988)
29. Stensønes, B.: Fatou-Bieberbach domains with $\mathcal{C}^\infty$-smooth boundary. Ann. Math. **145**(2), 365–377 (1997)
30. Stout, E.L.: Polynomial Convexity. Birkhäuser, Boston (2007)
31. Varolin, D.: The density property for complex manifolds and geometric structures. J. Geom. Anal. **11**, 135–160 (2001)
32. Varolin, D.: The density property for complex manifolds and geometric structures II. Int. J. Math. **11**, 837–847 (2000)
33. Wermer, J.: An example concerning polynomial convexity. Math. Ann. **139**, 147–150 (1959)
34. Winkelmann, J.: Non-degenerate maps and sets. Math. Z. **249**, 783–795 (2005)